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Re-establishing Kepler's first two laws for planets in a concise way through the non-stationary Earth

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Abstract

In this we design a simple and insightful way to achieve Kepler's first two laws for planets. The approach is quite different from what we have done for the Earth before. It is because the planet–Sun distance can be determined only through the Earth–Sun distance in the analysis. By applying the law of equal areas for the Earth and the observed angular speeds of a planet over the Sun, the law of equal areas for planets can be re-constructed. Furthermore, for the periodicity of a planet to the Sun, the distance from each planet to the Sun may be expressed as an angular periodic function. By coordinating with the observed data, this periodic distance function depicts an exact elliptical path. Here, we apply relatively easy mathematical skills to illustrate the invariant forms of planetary motions and indicate the key factors used to analyse the motions in complicated planetary systems.

Keywords: Kepler, law of equal areas, law of ellipses, planet

1. Introduction

Mathematical models were widely employed to describe natural phenomena during Plato's era (427–347 BC). Spherical geometry was also applied to astronomy. Geometry is a part of cosmology, and its theory represents a realization of the structure of the entire universe. Therefore, a knowledge of geometry is crucial for understanding astronomy [1].

Almagest, the cosmology literature written by Ptolemy (85–165 AD), firmly established the Greek trigonometry theory that was in place for more than a thousand years. He took the concept of epicycle and deferent to depict the motion of planets, which was widely accepted by the public to save the appearances at that time, namely, to have a mere confident geometrical model to describe the phenomenon. Until his pursuit of the notion of mathematical harmony and symmetry as perfection, Nicolaus Copernicus (1473–1543) strongly suspected human manipulation and complexity in the epicycle, deferent, and equant. The heliocentric doctrine was established and led to the revolution of astronomy [2, 3]. Nonetheless, he did not fully reject epicycles, which were necessary for him to describe non-uniform motion [4].

Johannes Kepler (1571–1630) was deeply enlightened by Copernicus' heliocentric theory and fully supported the doctrine throughout his life. He rejected epicycles as non-physical and kept equants. In particular, he considered the Sun as the dynamical centre for the motion of planets, and further proposed three laws for the planets [5]. His theories provided a concrete basis for Isaac Newton's (1642–1727) dynamics. Kepler's laws of equal areas and ellipses were published in *New Astronomy* in 1609 [6]. The book's content is very obscure for a reader today because of his use of complicated geometry rather than simple mathematical forms.

Most researchers have rediscovered and obtained Kepler's laws of planetary motion, either from Newton's laws of motion and universal gravitation [7–9], or from the principles of the conservation of energy and angular momentum [10–12]. To emphasize Kepler's important influence, others derived, algebraically or graphically, the inverse-square law of gravitation from Kepler's first two laws [13–15]. However, very few articles have discussed how Kepler originally derived his laws of planetary motion [16–19].

In Copernicus' and Kepler's astronomical system, the Earth is no longer stationary, which makes the determination of planet position more complicated. Fortunately, the establishment of the rules for the motion of the Earth, which is the law of equal areas [6, 20], presents the Earth as the starting point for depicting the positions of other planets. This rule has become the primary basis and powerful tool for discovering the laws for other planets. This process, in fact, was reflected in chapter 32 of *New Astronomy*, where Kepler declared the proposition: the elapsed times of the Earth over equal arcs of the eccentric were proportional to the Earth's distances from the centre whence the eccentricity originated. This law was also named by the followers as the 'distance rule' [18], which was equivalent to the law of equal areas for the Earth.

To reveal the fundamental spirit of Kepler's first two laws, the present study uses simple approaches, such as trigonometric functions and the law of sines, to re-establish the laws of equal areas and ellipses for planets other than the Earth. It clarifies and simplifies the development of planet laws, which were originally difficult to interpret, enabling researchers to understand the intimate relations and analytical methods among geometry, astronomy, and physics. Moreover, this study allows researchers to practically realize the plentiful insights in major scientific developments, immerse in the joy of rediscovering scientific theories by previous great scientists, and cultivate the extensive and deep scientific prospects of these theories.

2. The law of equal areas for Mars

The period of Mars orbiting around the Sun is approximately 687 days, which indicates that Mars will return back to the same position after 687 days. This period was determined by both Copernicus and Kepler from the data that the time interval from the one opposition—where the Sun, the Earth, and Mars were aligned—to the next opposition was about 780 days [6, 21]. The period of the Earth orbiting around the Sun is 365 days. The periods of these two

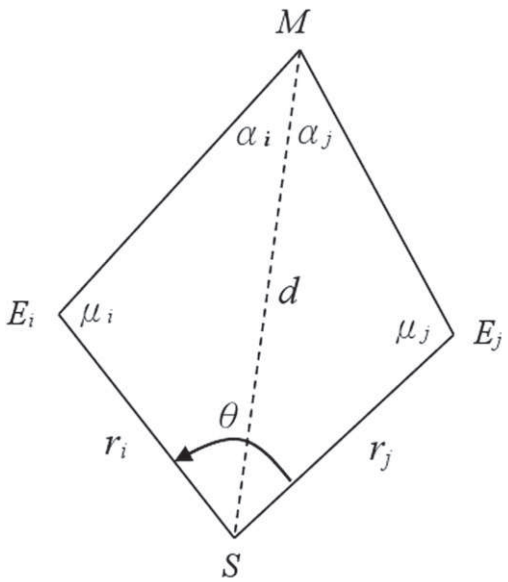

Figure 1. The illustration of the Sun (S), Mars (M), and two corresponding positions of the Earth, (E_i) and (E_j), in an adjacent Martian year. Notice that the middle line SM is not the opposition of Mars.

planets revolving around the Sun do not have a common value and are mutually prime numbers. This suggests the corresponding Earth position is different for every Martian year, or when Mars returns to its original position. As indicated by figure 1, if S represents the Sun, M is Mars or the position of Mars on the next Martian year, and E_i and E_j are the corresponding positions of the Earth before and after one Martian year, respectively. Therefore, a square (SE_iME_j) can be formed by the Sun, Mars, and two other positions of the Earth. The SE_i and SE_j lines represent the distance between the Sun and the different positions of the Earth, r_i and r_j, respectively; this is known as the Earth–Sun distance. The SM line represents the distance between the Sun and Mars, d, known as the Mars–Sun distance.

An approximation has been made in figure 1. Point M actually does not locate on the same plane depicted by three points E_i, E_j and S because the inclination of Mars' orbit with respect to the Earth's is $1°53'$. For simplicity, we neglect the effect due to this small inclination, and project the Mars' orbit onto the ecliptic plane.

2.1. Mars–Sun distance d

We randomly selected the date of E_i as 5 a.m. on 13 May 1950, and the corresponding date of position E_j as 4 a.m. on 30 March 1952. The time difference between E_i and E_j is a Martian year. The angle between S and M from the E_i position is $\angle SE_iM = \mu_i$. Because lots of observational data need to be applied in the study, the more convenient approach without actual measurements is available from the astronomical data by the Multiyear Interactive Computer Almanac (MICA) software [22], which is quite consistent with and can be regained from observations. One found by MICA that Mars' ecliptic longitude was 172.557°, and the Sun's longitude was 51.901°, which indicated that $\mu_I = 172.557° - 51.901° = 120.656°$. Similarly, the angle between S and M at E_j was $\angle SE_jM = \mu_j = 9.473° + 360° - 228.333° = 141.140°$.

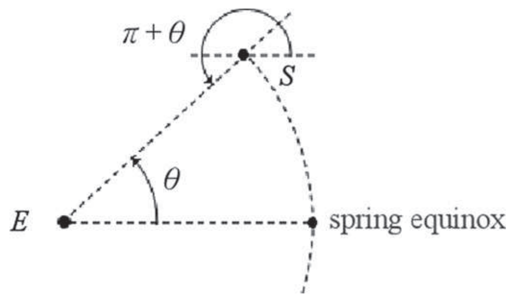

Figure 2. The Sun and planets can be directly or indirectly observed from the Earth at any moment.

The corresponding Sun's longitude to the Earth is the projection point of the Sun on a celestial sphere and is marked as an ecliptic longitude while observing the Sun from the Earth. The longitude is set as 0° while observing the Sun from the Earth on the vernal equinox and the Sun's longitude is set as 90° on the summer solstice as shown in figure 2. The longitudes of the Sun and any other planets can be determined from the Earth.

The $\angle E_iSE_j = \theta$ in figure 1 is observable because the observed longitude of S as seen from the Earth at E_i and E_j were 51.901° and 9.473°, respectively. Thus

$$E_iSE_j = \theta = 51.901° - 9.473° = 42.428°. \tag{1}$$

Hence, θ can be confirmed.

The three observable parameters, μ_i, μ_j, and θ, together with the law of equal areas for the Earth, will be applied hereafter to calculate the Mars–Sun distance d. If $\angle E_jMS = \alpha_j$, $\angle E_iMS = \alpha_i$ because the total internal angles are 360° for a quadrilateral; thus

$$\mu_i + \mu_j + \theta + \left(\alpha_i + \alpha_j\right) = 360°, \;\; \alpha_i = 360° - \mu_i - \mu_j - \theta - \alpha_j.$$

Let $\beta = 360° - \mu_i - \mu_j - \theta$, β is an observable value, and

$$\alpha_i = \beta - \alpha_j\,. \tag{2}$$

On the other hand, the quadrilateral SE_iME_j can be considered as the combination of ΔSE_iM and ΔSE_jM, where SM is a common side. By applying the law of sines

$$\frac{d}{\sin\mu_i} = \frac{r_i}{\sin\alpha_i}, \quad \frac{d}{\sin\mu_j} = \frac{r_j}{\sin\alpha_j},$$

the Mars–Sun distance d can be expressed as

$$d = \frac{\sin\mu_j}{\sin\alpha_j} r_j. \tag{3}$$

and the ratio between the sines of α_i and α_j may be written as

$$\frac{\sin\alpha_i}{\sin\alpha_j} = \frac{r_i}{r_j}\frac{\sin\mu_i}{\sin\mu_j}. \tag{4}$$

In (4) the ratio of Earth–Sun distances r_i/r_j can be expressed by four observable angles μ_i, μ_j, α_i and α_j if SM in figure 1 is the opposition of Mars, and E_i and E_j represent the positions of the Earth one Martian year before and after SM. This relationship has assisted us to achieve Kepler's law of equal areas for the Earth before [20]. However, we no longer have the similar relationship for the Mars–Sun distance d, as indicated in (3). This difference signifies that the method of re-establishing Kepler's laws for planets is not a mere generalization from the Earth.

It is truly fortunate and helpful that the law of equal areas or the distance rule for the Earth, as mentioned in chapter 32 of *New Astronomy*, has been constructed. Kepler also set out the distances of the Sun from the Earth at 180 different positions in a table in chapter 30. He then spoke out clearly: physicists, pick up your ears! For here is raised a deliberation involving an inroad to be made into your province [6]. The crucial law of equal areas for the Earth can be depicted as follows [20]

$$r_i^2/r_j^2 = \omega_j/\omega_i.$$

This relation implies that the ratios of the Earth–Sun distances, which were originally difficult to measure, can be calculated by the angular speeds ω_i and ω_j of the Earth at different positions, which may be measured from daily observations. Therefore, r_i/r_j can be obtained. Because μ_i and μ_j are also known, the ratio in (4) can be set as an observable value k. Then

$$\sin\alpha_i = k\sin\alpha_j. \tag{5}$$

Replacing (5) with (2), we have

$$\sin\beta\cos\alpha_j - \cos\beta\sin\alpha_j = k\sin\alpha_j.$$

Dividing by $\cos\alpha_j$ on both sides of the above equation, α_j can be found as follow

$$\alpha_j = \tan^{-1}\left(\frac{\sin\beta}{k + \cos\beta}\right). \tag{6}$$

Hence, α_j turns out to be an observable value because β and k are all observable. Combining (3) and (6), the Mars–Sun distance d can be directly represented by the Earth–Sun distance r_j. For five randomly selected observation dates, the Mars–Sun distances from (6) and (3) are listed in table 1. The Earth–Sun distance is set as $r_{j1} = r_0 = 100\,000$ on 30 March 1952 (figure 3).

2.2. Angular speed of Mars ω

The angular speed of Mars revolving around the Sun represents the angular change of Mars with respect to the Sun within a certain period of time, such as within one day. This value cannot be directly achieved by observation; the indirect relations with observable values must be determined.

Two quadrilaterals $SE_{i1}M_1E_{j1}$ and $SE_{i2}M_2E_{j2}$ are formed by the Sun, the Earth and Mars within two days, as shown in figure 4. The value of daily angular speed of Mars ω with respect to the Sun is the value of angle φ swept by Mars moving from M_1 to M_2. The angles $\angle SE_{i1}M_1$, $\angle SE_{j1}M_1$, and $\angle E_{i1}SE_{j1i}$ in quadrilateral $SE_{i1}M_1E_{j1}$ denoted by μ_i, μ_j, and θ in figure 1 are also observable. The angle $\angle E_{j1}M_1S$ can further be approached from (6) like α_j in figure 1. As for the $\Delta SE_{j1}M_1$, using the relation of interior angles, as shown in table 1, we obtained

$$\angle M_1SE_{j1} = a = 180° - \angle SE_{j1}M_1 - \angle SM_1E_{j1} = 180° - 141.140° - 22.973° = 15.887°.$$

Similarly, $\Delta SE_{j2}M_2$ formed by the Sun, the Earth, and Mars on the second day gave

$$\angle M_2SE_{j2} = b = 180° - \angle SE_{j2}M_2 - \angle SM_2E_{j2} = 180° - 142.194° - 22.438° = 15.368°.$$

Table 1. Five different randomly selected dates used to calculate the Mars–Sun distance d. (The quoted and non-quoted dates have a difference of one Martian year.)

Time	μ_i	μ_j	θ	β	k	α_j	r_j	d
13 May1950 (30 March 1952)	120.656°	141.140°	42.456°	304.253°	1.387	22.973°	100 000	160 750
21 June 1952 (9 May 1954)	122.319°	130.592°	41.744°	294.655°	1.12	30.593°	101 072	150 805
15 August 1954 (2 July 1956)	125.251°	116.179°	41.593°	283.024°	0.906	40.722°	101 770	139 993
1 November 1956 (19 September 1958)	126.837°	116.786°	43.041°	286.665°	0.885	39.256°	100 518	141 804
7 January1959 (24 November 1960)	122.044°	133.781°	44.249°	300.073°	1.17	27.383°	98 793	155 079

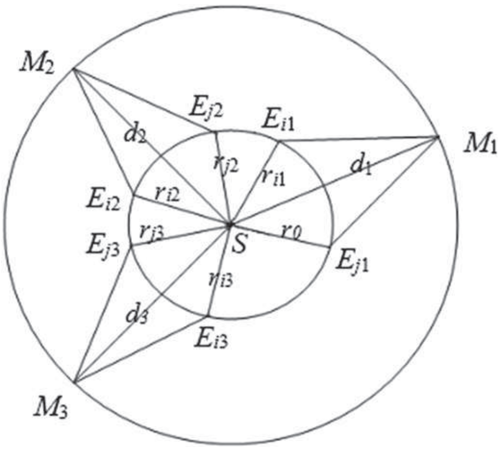

Figure 3. The Earth–Sun distance r can be used to represent the Mars–Sun distance d. ($r_{j1} = r_0 = 100\,000$ is set to be a normalized value.)

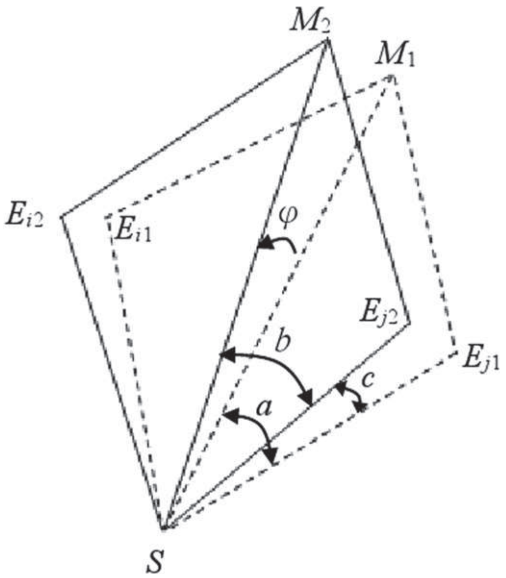

Figure 4. Two quadrilaterals $SE_{i1}M_1E_{j1}$ and $SE_{i2}M_2E_{j2}$ are formed by the Sun, Earth, and Mars within two days. The angle swept by Mars within one day from M_1 to M_2 is $\varphi = b + c - a$.

The angle $\angle E_{j1}SE_{j2}$ formed by the Sun to E_{j1} and the Sun to E_{j2} is similar to θ in figure 1. It could be obtained from (1) as follows:

$$\angle E_{j1}SE_{j2} = c = 10.461^\circ - 9.473^\circ = 0.988^\circ.$$

Therefore, the angles swept by Mars with respect to the Sun within one day was

$$\varphi = b + c - a = 15.368^\circ + 0.988^\circ - 15.887^\circ = 0.469^\circ.$$

Table 2. The calculated angular speed ω of Mars on five different dates from the observed data shown in table 1.

Time	ω
13 May 1950	0.469
21 June 1952	0.534
15 August 1954	0.620
1 November 1956	0.603
7 January 1959	0.504

Table 3. The ratios of the square of the Mars–Sun distance d_j^2/d_i^2 and the corresponding ratios of angular speeds ω_i/ω_j on five different dates obtained from tables 1 and 2.

Time	d	ω	d_j^2/d_i^2	ω_i/ω_j
13 May 1950	160 750	0.469	1.000	1.000
21 June 1952	150 805	0.534	0.880	0.878
15 August 1954	139 993	0.620	0.758	0.756
1 November 1956	141 804	0.603	0.778	0.778
7 January 1959	155 079	0.504	0.931	0.931

Furthermore, its value was the same as that of angular speed ω of Mars on that day. Table 2 lists the calculated angular speeds ω per day of Mars on five different dates from table 1.

2.3. The law of equal areas for planets

The area ΔA with spanned angle $\Delta\theta$ swept out by the line joining a planet and the Sun with distance d in a small periods of time can be expressed as

$$\Delta A = \frac{1}{2}d^2\Delta\theta.$$

The area velocity is then

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}d^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}d^2\omega,$$

where ω is the instantaneous angular speed of a planet around the Sun. Hence, to prove the law of equal areas for a planet, it is necessary to only show that the product of the square of the distance from a different planet to the Sun and the corresponding angular speed of that planet is a constant. Namely, for a planet at any two arbitrary positions i and j

$$d_i^2\omega_i = d_j^2\omega_j. \tag{7}$$

This method is the same as that employed to inspect the equivalence of d_j^2/d_i^2 and ω_i/ω_j. Combining the Mars–Sun distances and angular speeds of Mars on different dates from tables 1 and 2, the corresponding values of the ratios of d_j^2/d_i^2 and ω_i/ω_j can be obtained, as shown in table 3, where the referenced date is set as 13 May 1950.

The ratios of d_j^2/d_i^2 and ω_i/ω_j for Mars differ up to only 0.3% from the last two columns in table 3. The difference arises from the fact that the values ω_i and ω_j used in table 3 are

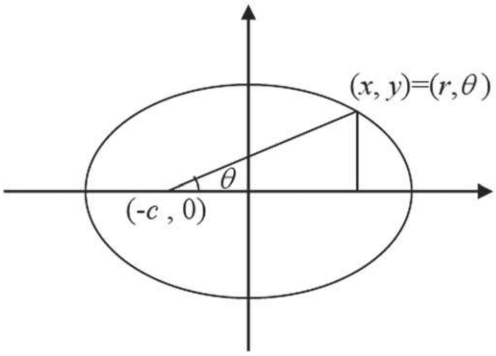

Figure 5. The relationships between the Cartesian coordinates (x, y) and polar coordinates (r, θ) for an ellipse.

average angular speeds per day instead of instantaneous angular speeds per second. This small deviation indicates that Mars obeys Kepler's law of equal areas from the acknowledged astronomical data. This is an exciting and unsurprising result. Furthermore, since any region on the Mars' orbital plane projected onto the ecliptic plane is constantly reduced by the same factor of $\cos(1°53') = 0.9995 \approx 1$, the ratio of any two areas on the Mars' orbital plane and that of the corresponding areas on the ecliptic plane are identical. Hence, the law of equal areas for Mars works on the Mars' orbit plane as well as on the ecliptic plane.

3. The law of ellipses for Mars

From the perspective of analytical geometry, the relationships between the Cartesian coordinates (x, y) and polar coordinates (r, θ) for an ellipse with one of the foci located at ($-c$, 0), as shown in figure 5, are

$$x = r\cos\theta - c, \quad y = r\sin\theta. \tag{8}$$

The equation of an ellipse in Cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{9}$$

where a and b are the semi-major and semi-minor axis respectively, or

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

Substituting (9) with (8) gives

$$r(a - c\cos\theta) - b^2 = 0; \quad r(a + c\cos\theta) + b^2 = 0.$$

Taking the positive value of r

$$\frac{1}{r} = \frac{a - c\cos\theta}{b^2} = \frac{a}{b^2}(1 - e\cos\theta),$$

where $e = c/a = \sqrt{1 - (b/a)^2}$ is the eccentricity [23]. For general situations, where $\theta \neq 0$ along the x-axis, the equation of the ellipse in polar coordinates can then be expressed as

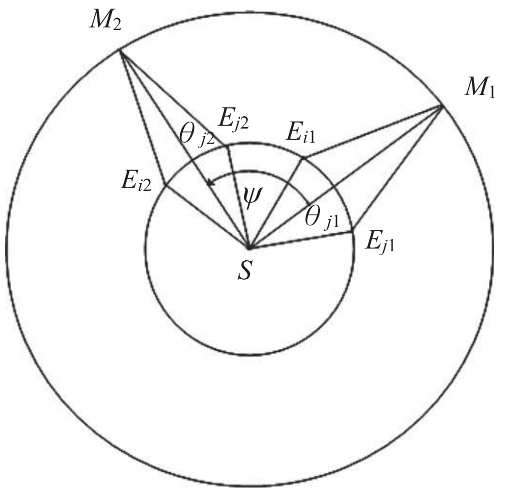

Figure 6. The angle ψ is defined as the angle spanned by two sides connecting Mars and the Sun on two different Mars positions. The line connecting M_1 to S was set to be the horizontal axis.

$$\frac{1}{r} = \frac{a}{b^2}[1 - e\cos(\theta + \alpha)] = c_0 + c_1\cos\theta + c_2\sin\theta, \tag{10}$$

where

$$c_1^2 + c_2^2 = a^2e^2/b^4$$

or

$$e = b^2\sqrt{c_1^2 + c_2^2}/a = \sqrt{c_1^2 + c_2^2}/c_0. \tag{11}$$

Therefore, (10) has the same form of ellipse as that of (9).

From the other side of the perspective, because the motion of Mars around the Sun is periodic, the distance function d from a planet to the Sun, or its reciprocal $1/d$, can also be described as a period function of the polar angle θ. That is, it can be expressed as an infinite Fourier series of sines and cosines with different multiple angles as follows [24]:

$$\frac{1}{d} = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right].$$

In the ideal case, this function can be approximated by a single period of the trigonometric functions:

$$\frac{1}{d} = a_0 + a_1\cos\theta + b_1\sin\theta. \tag{12}$$

That is, this simplified periodic (12) is equivalent to the equation of the ellipse in polar coordinates as shown in (10).

To determine the three unknown a_0, a_1, and b_1 as shown in (12), three sets of data are required to set up simultaneous linear equations with three unknowns. After solving these sets of equations, the equation for the ellipse and its corresponding eccentricity can be obtained. The law of ellipses for a planet will be spontaneously revealed.

The position of Mars M_1 on 13 May 1950, is now selected as a reference point (figure 6; table 1). In $\Delta SE_{j1}M_1$, the angle $\angle SE_{j1}M_1 = \mu_{j1}$ is observable, and $\angle SM_1E_{j1} = \alpha_{j1}$ is

Table 4. The angle ψ swept by Mars moving from the referenced position, which was selected on 13 May 1950, to the other three positions shown in table 3.

Time	ψ
13 May 1950	0.000°
21 June 1952	41.337°
15 August 1954	98.458°
1 November 1956	174.976°

Table 5. Five randomly selected dates comparing the Mars–Sun distance d' from periodic (13) with the Mars–Sun distance d from (3) by the law of equal areas for the Earth.

Time	d	ψ	d'	$(d'-d)/d$ (%)
7 January 1959	155 090	235.697°	154 866	−0.144
21 February 1961	164 125	277.814°	163 935	−0.116
25 March 1963	166 678	310.942°	166 587	−0.055
29 April 1965	163 511	345.914°	163 462	−0.030
6 June 1967	155 039	24.493°	155 018	−0.014

calculable, which can be achieved from (6) and is listed in table 1. Therefore, $\theta_{j1} = 180° - \mu_{j1} - \alpha_{j1} - 180.000° - 141.140° - 22.973° = 15.887°$. Similarly, $\theta_{j2} = 180.000° - 130.592° - 30.593° = 18.815°$ from the observed μ_{j2} and calculated α_{j2} with respect to Mars M_2 on 21 June 1952, from table 1. Furthermore, the angle swept by the Earth from E_{j1} to E_{j2} was $\angle E_{j1}SE_{j2} = 38.394°$ by the two observable angular positions of the Earth to the Sun.

Finally, the angle swept by Mars from M_1 to M_2 is defined as ψ, which could be obtained by $\angle M_1SM_2 = \psi = \angle E_{j1}SE_{j2} + \theta_{j2} - \theta_{j1} = 38.394° + 18.815° - 15.887° = 41.322°$, where the line connecting M_1 to S was set to be the horizontal axis. The angles ψ swept by Mars on 15 August 1954, and 1 November 1956, shown in table 1 could also be determined in a similar manner (table 4).

By replacing (12) with the Mars–Sun distances d and the corresponding angles ψ swept by Mars on 21 June 1952, 15 August 1954, and 1 November 1956, respectively, we have

$$\frac{1}{d_1} = a_0 + a_1 \cos\psi_1 + b_1 \sin\psi_1,$$

$$\frac{1}{d_2} = a_0 + a_1 \cos\psi_2 + b_1 \sin\psi_2,$$

$$\frac{1}{d_3} = a_0 + a_1 \cos\psi_3 + b_1 \sin\psi_3.$$

Solving the simultaneous linear equations in three unknowns, we obtain

$$a_0 = 0.00\,000\,662, \quad a_1 = -0.00\,000\,040, \quad b_1 = 0.00\,000\,047.$$

Thus, the periodic equation of the reciprocal of the Mars–Sun distance is

$$\frac{1}{d'} = 0.00\,000\,662 - 0.00\,000\,040 \cos\psi + 0.00\,000\,047 \sin\psi. \tag{13}$$

To demonstrate the generality of the above periodic equation built by four Mars positions, we randomly selected five additional dates and calculated the corresponding angles ψ swept by Mars, as shown in the first two columns of table 5. The Mars–Sun distances d' were estimated from periodic (13). By comparing d' with d, which were calculated by the Mars–Sun distance (3), we found the relative error less than 0.2%. This small discrepancy mainly results from the fact that the periodic function $1/d'$ is expressed by only three terms, and the rest of terms of sines and cosines with higher multiple angles are omitted. The validity of periodic equation $1/d$ of ψ in (13) can hence be asserted.

Equations (12) or (13) have the same form as (10). Therefore, the elliptical motion of Mars revolving around the Sun as one of the foci can be verified, and the eccentricity of Mars can also be obtained by (11) as

$$e = \frac{\sqrt{c_1^2 + c_2^2}}{c_0} = \frac{\sqrt{a_1{}^2 + b_1{}^2}}{a_0} = \frac{\sqrt{(-0.00\,000\,040)^2 + (0.00\,000\,047)^2}}{0.00\,000\,662} = 0.0932.$$

The result is almost exactly that of the well-known eccentricity of Mars, 0.0935, thus again confirming the law of ellipses for Mars. General speaking, the eccentricity $e = \sqrt{1 - (b/a)^2}$ for Mars on the Mars' orbital plane will be different from $e' = \sqrt{1 - (b'/a')^2}$ for its projection on the ecliptic plane. The length of projected semi-major axis a' may not strictly equal $a\ \cos(1°53')$, neither does that of b'. Nevertheless, the ratio of b'/a' will always be within two extreme values, namely, $0.9995\, b/a = b\cos\alpha/a \leqslant b'/a' \leqslant b/a\cos\alpha = 1.0005\, b/a$, where $\alpha = 1°53'$. It denotes that b'/a' is very close to b/a. The difference of the square of two eccentricities at the extreme case is $e'^2 - e^2 = \left(1 - \cos^2\alpha\right)(b/a)^2 = \sin^2\alpha\,(b/a)^2 \approx 10^{-3}(b/a)^2 \leqslant 10^{-3}$, which makes e' and e differ by a factor of 10^{-2} only.

The concise method for Mars can also be applied to the other four planets including Jupiter, Saturn, Mercury, and Venus to determine the laws of equal areas and ellipses. Here we propose the internal planet, Mercury, as an example to describe its conformity and completeness.

4. The laws of planet for Mercury

4.1. The law of equal areas

Referring to figure 7, which is similar to figure 1 for Mars, and combining (3) and (6), the Mercury–Sun distance d can also be represented by the Earth–Sun distance r_j, where the Earth–Sun distance is set to be $r_{j1} = r_0 = 100\,000$ on 27 April 1950. For the other four randomly selected observation dates, the calculated Mercury–Sun distances d from (3) and (6) are listed in table 6.

Applying the relationships as shown in figure 4, one may obtain the angular speed ω of Mercury at the different dates shown in table 6. By verifying whether the product $d^2\omega$ is a constant as shown in (7), or whether the identity $d_j^2/d_i^2 = \omega_i/\omega_j$ holds (table 7), we can establish the law of equal areas for Mercury. From table 7, the law of equal areas for Mercury can be certified.

4.2. The law of ellipses

After the position of Mercury on 27 April 1950 was selected as a reference point in figure 7, the angles ψ swept by Mercury on 6 August 1951, 21 November 1952, and 19 February 1954

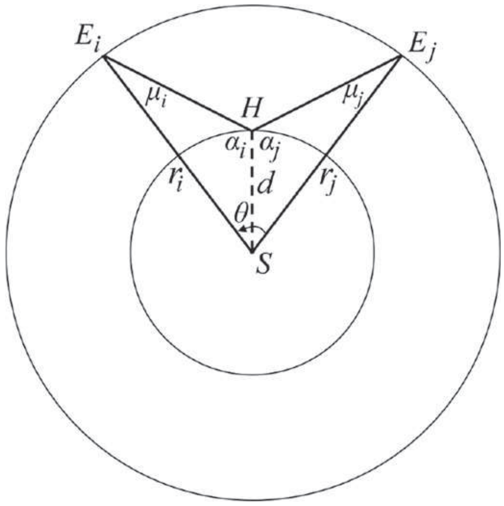

Figure 7. An illustration of the Sun (S), Mercury (H), and two corresponding positions of the Earth, (E_i) and (E_j), in an adjacent Mercurian year.

shown in table 7 could be determined by the same method as that employed in the case of Mars. The results are listed in table 8.

By replacing (12) with the Mercury–Sun distances d and the corresponding angles ψ swept by Mercury on 6 August 1951, 21 November 1952, and 19 February 1954, respectively, we obtain simultaneous linear equations with three unknowns. By solving them, the periodic equation of the reciprocal of the Mercury–Sun distance can then be achieved as

$$\frac{1}{d'} = 0.000\,02\,67+0.000\,000\,872\cos\psi - 0.00\,000\,544\sin\psi. \tag{14}$$

By comparing the Mercury–Sun distances d' estimated from periodic (14) with d calculated by the Mercury–Sun distance (3) at five different selected dates, we can determine that d' and d are approximately the same with relative error less than 0.6% as shown in table 9. Thus, the validity of periodic equation $1/d$ of ψ in (14) can be confirmed, and the elliptical motion of Mercury revolving around the Sun as a focus may also be asserted by combining (10) and (14).

The eccentricity of Mercury can be calculated by (11) as

$$e = \frac{\sqrt{{a_1}^2 + {b_1}^2}}{a_0} = \frac{\sqrt{(-0.000\,000\,872)^2 + (-0.00\,000\,544)^2}}{0.0000267} = 0.206.$$

The result is precisely the same as the well-known eccentricity of Mercury, 0.206. Thus again verifies the law of ellipses for Mercury.

5. Conclusions

In this study, we treat the Earth as a reference point to determine the law of motions for the other planets. The fact that the Earth has regular motion, which fulfills the law of equal area, enables us to establish the mathematical relation of planet–Sun distance and Earth–Sun

Table 6. Five randomly selected dates used to calculate the Mercury–Sun distance d. (Each date has one Mercury orbital period difference.)

Time	μ_i	μ_j	θ	β	k	α_j	r_j	d
27 April 1950	19.065°	21.155°	87.591°	232.189°	0.925	111.551°	100 000	38 802
6 August 1951	27.114°	20.249°	84.161°	228.477°	1.323	131.381°	102 489	47 274
21 November 1952	17.005°	16.952°	86.901°	239.143°	0.980	118.563°	102 548	34 043
19 February 1954	15.590°	16.538°	89.368°	238.505°	0.946	116.404°	100 211	31 847
7 June 1954	23.855°	18.041°	85.743°	232.320°	1.334	132.395°	100 858	42 296

Table 7. The ratios of the square of Mercury–Sun distance d_j^2/d_i^2 and the corresponding ratios of angular speeds ω_i/ω_j on five different dates obtained from table 6.

Time	d	ω	d_j^2/d_i^2	ω_i/ω_j
27 April 1950	38802	4.015	1.000	1.000
6 August 1951	47274	2.746	1.484	1.462
21 November 1952	34043	5.369	0.770	0.748
19 February 1954	31847	5.994	0.674	0.670
7 June 1955	42296	3.390	1.188	1.185

Table 8. The angle ψ swept by Mercury moving from the referenced position selected on 27 April 1950, to the other three positions shown in table 7.

Time	ψ
27 April 1950	0.000°
6 August 1951	80.512°
21 November 1952	200.370°
19 February 1954	291.629°

Table 9. Five randomly selected dates used to compare the Mercury–Sun distance d' from periodic (14) with the Mercury–Sun distance d from (3) by the law of equal areas for the Earth.

Time	d	ψ	d'	$(d' - d)/d$ (%)
6 June 1954	42 296	23.258°	42 104	−0.454
28 September 1955	42 235	113.923°	42 477	0.573
6 January 1957	31 173	268.060°	31 088	−0.273
4 April 1958	38 296	356.560°	38 232	−0.167
10 July 1959	46 560	59.121°	46 332	−0.49

distance, as shown in (3). The laws of equal areas for the other planets can be easily and naturally constructed by combining this relation with the angular speed of the planet around the Sun.

The periodicity of the planet around the Sun indicates that the planet–Sun distance can be represented as the periodic function of an angle. The angular position of the observing planet and the law of equal areas are used to determine the distance of the planet to the Sun and to build the periodic function of each planet. The trajectory equation of planet distance may thus be obtained. The planet orbits are proved to be ellipses, which take the Sun as a focus; thus, the law of ellipses for planets is rediscovered.

We have applied relatively simple geometry, trigonometry, and basic algebra to describe the invariant properties of planetary motion. These procedures allow researchers to comprehend the magnitude of the mathematical approaches for analysing the complicated and substantial planetary system, thus enabling an appreciation of the harmony and simplicity behind the natural phenomenon. The actual examples in this paper may be used by young students to establish and extend the essence and confidence applied toward scientific research.

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