$$\Gamma(p) = \int_0^\infty dx \cdot x^{p-1} e^{-x}$$

Let's start with this simple calculation of $\Gamma(p)$ for integer p:

$$\int_0^\infty dx \cdot e^{-\alpha x} = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = 0 + \frac{1}{\alpha} = \frac{1}{\alpha}$$

Now differentiate the integral with respect to $-\alpha$ once:

$$-\frac{d}{d\alpha}\int_0^\infty dx \cdot e^{-\alpha x} = \int_0^\infty dx \cdot x e^{-\alpha x} = \frac{1}{\alpha^2}$$

Differentiate the original integral with respect to $-\alpha$ n times:

$$(-1)^n \frac{d^n}{d\alpha^n}\int_0^\infty dx \cdot e^{-\alpha x} = \int_0^\infty dx \cdot x^n e^{-\alpha x} = \frac{n!}{\alpha^n}$$

$\alpha = 1$

$$\Gamma(n+1) = \int_0^\infty dx \cdot x^n e^{-x} = n!$$

Gamma Function of positive integral n is n factorial! What a surprise!

Indeed, we can prove from definition for any p (not just integer n):

$\Gamma(p+1) = p\Gamma(p)$ The key equation of Gamma function.

Using integration by parts to compute the left-hand side:

$$\int_a^b dx \cdot u(x)v'(x) = u(x)v(x)\Big|_a^b - \int_a^b dx \cdot u'(x)v(x)$$

$$\Gamma(p+1) = \int_0^\infty dx \cdot x^p e^{-x} = -x^p e^{-x}\Big|_0^\infty + \int_0^\infty dx \cdot px^{p-1}e^{-x} = p\Gamma(p)$$

Note that if we choose p as a positive integer:

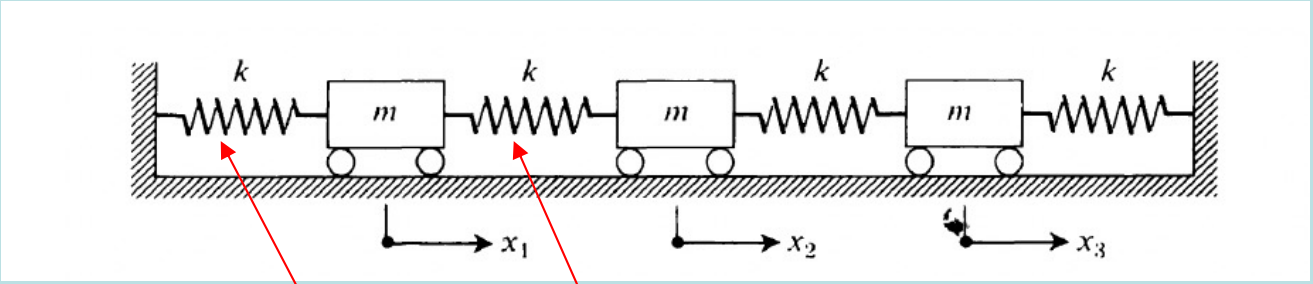
$$\Gamma(1) = \int_0^\infty dx \cdot e^{-x} = 1$$

Now apply the above formula, we can easily show:

$\Gamma(2) = 1\Gamma(1) = 1$ $\Gamma(3) = 2\Gamma(2) = 2!$ $\Gamma(n+1) = n!$

Gamma Function is n factorial in disguise.

The concept of Oscillation Modes could be extended to more than two masses.



Equations of motion can be written as a system of ODEs:

$$\frac{d^2x_1}{dt^2} = -\frac{k}{m}x_1 + \frac{k}{m}(x_2 - x_1) = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\frac{d^2x_2}{dt^2} = \frac{k}{m}(x_3 - x_2) - \frac{k}{m}(x_2 - x_1) = -2\omega_0^2 x_2 + \omega_0^2 x_3 + \omega_0^2 x_1$$

$$\frac{d^2x_3}{dt^2} = -\frac{k}{m}(x_3 - x_2) - \frac{k}{m}x_3 = -2\omega_0^2 x_3 + \omega_0^2 x_2$$

Define column vector $\boldsymbol{x}$ and matrix $\boldsymbol{A}$. Equations of motion is ODE of vector $\boldsymbol{x}$:

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x} = -\omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Assume there are mode solutions. Their time dependence is a factor $e^{i\omega t}$ as SHM.

$$\boldsymbol{x} = ce^{i\omega t} \cdot \boldsymbol{a}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = ce^{i\omega t} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Time evolution can be isolated.

Since the time factor can be separated, these are called separable solutions.

We understand, eventually we will only keep real part of this complex form.

$$\boldsymbol{x} = \mathrm{Re}\, ce^{i\omega t} \cdot \boldsymbol{a} = a_m \cos(\omega t + \phi) \cdot \boldsymbol{a}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a_m \cos(\omega t + \phi) \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Take this form as a guess of solutions and plug it into the equation:

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A} \cdot \boldsymbol{x}$$

We keep complex number form because exponentials are easier to differentiate.

$$\boldsymbol{a}\frac{d^2}{dt^2}e^{i\omega t} = -\omega^2 \boldsymbol{I}\boldsymbol{a}e^{i\omega t} = -\boldsymbol{A} \cdot \boldsymbol{a} \cdot e^{i\omega t}$$

$$(\boldsymbol{A} - \omega^2 \boldsymbol{I}) \cdot \boldsymbol{a} = 0$$

$$(\boldsymbol{A} - \omega^2 \boldsymbol{I}) \cdot \boldsymbol{a} = 0$$

We transform system of ODE into system of algebraic equations for unknown $a_{1,2,3}$.

It has nonzero solutions only when the determinant equals zero.

$$|\boldsymbol{A} - \omega^2 \boldsymbol{I}| = 0$$ This is called characteristic equation CE.

Only certain ω^2's that satisfy this CE will allow a solution for unknown $a_{1,2,3}$.

To find these ω^2's : $\omega^2 \equiv \lambda\omega_0^2$

$$\boldsymbol{A} - \omega^2 \boldsymbol{I} = \omega_0^2 \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix}$$

$$|\boldsymbol{A} - \omega^2 \boldsymbol{I}| = 0 \Rightarrow (2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)(2-4\lambda+\lambda^2) = 0$$

λ has three solutions. Order and label the solutions of ω from small to large.

$$\omega_1 \sim \sqrt{2-\sqrt{2}}\omega_0 \qquad \omega_2 \sim \sqrt{2}\omega_0 \qquad \omega_3 \sim \sqrt{2+\sqrt{2}}\omega_0$$

$\omega_{1,2,3}$ are eigenvalues of $\boldsymbol{A}$.

Let's take ω_2 as an example for a more detailed study.

First solve $a_{1,2,3}$ for ω_2.

$\lambda = 2, \qquad \omega = \omega_2 \sim \sqrt{2}\omega_0$

$$(\boldsymbol{A} - 2\omega_0^2 \boldsymbol{I}) \cdot \boldsymbol{a} = \omega_0^2 \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} \boldsymbol{a} = \omega_0^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$-a_2 = 0$

$-a_1 - a_3 = 0$

$-a_2 = 0$

$$\boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This solution is denoted as $\boldsymbol{a}^{(2)}$, determined up to a constant.

$\boldsymbol{a}^{(2)}$ is called an eigenvector of $\boldsymbol{A}$.

An eigenvector gives a solution!

$$\boldsymbol{x} = c\boldsymbol{a}^{(2)} e^{i\sqrt{2}\omega_0 t} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} a_m e^{i(\sqrt{2}\omega_0 t + \phi)}$$

Take the real part of this complex form.

$$\mathrm{Re}\,\boldsymbol{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \left[f_2 \cos\left(\sqrt{2}\omega_0 t\right) + g_2 \sin\left(\sqrt{2}\omega_0 t\right) \right]$$

$x_{1,2,3}$ oscillate as SHM by the same frequency, with amplitudes $\propto a_{1,2,3}$.

$\boldsymbol{x}(t) \propto \boldsymbol{a}^{(2)}$ $\quad x_1(t) : x_2(t) : x_3(t) = a_1 : a_2 : a_3 = -1 : 0 : 1$

In fact, proportion among $x_{1,2,3}$ is fixed as $a_1 : a_2 : a_3$ at all t.

This is expected since time evolution can be isolated.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} [f_2 \cos(\sqrt{2}\omega_0 t) + g_2 \sin(\sqrt{2}\omega_0 t)]$$

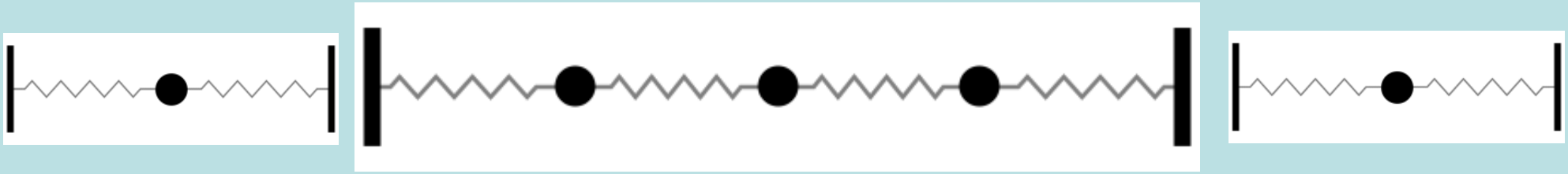
If the initial conditions start with proportion $-1:0:1$, oscillation mode will continue.

$$\boldsymbol{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \sim f_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{x}'(0) = \begin{pmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{pmatrix} \propto \omega_2 g_2 \boldsymbol{a}^{(2)}$$

Note that constant f_2, g_2 is determined by initial conditions $x_{1,2,3}(0), x'_{1,2,3}(0)$.

This is one oscillation mode that would oscillate independently.

$$\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_0 \qquad \boldsymbol{x}(t) \propto \boldsymbol{a}^{(2)} \propto \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



The middle particle does not move. It is called a node 節點。

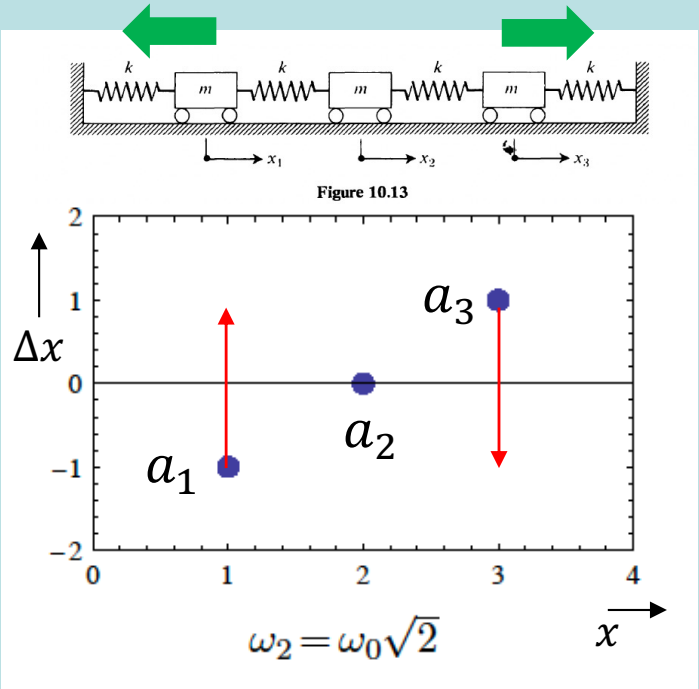
Treat node as a fixed point, it is like two systems of double springs. $\omega = \sqrt{\frac{2k}{m}}$

$$x_1(t) : x_2(t) : x_3(t) = a_1 : a_2 : a_3 = -1 : 0 : 1$$ at all t.

Therefore $a_1 : a_2 : a_3$ is a good illustrations of $x_{1,2,3}$.

We can draw $a_1 : a_2 : a_3$ versus original locations of blocks to represent the mode.

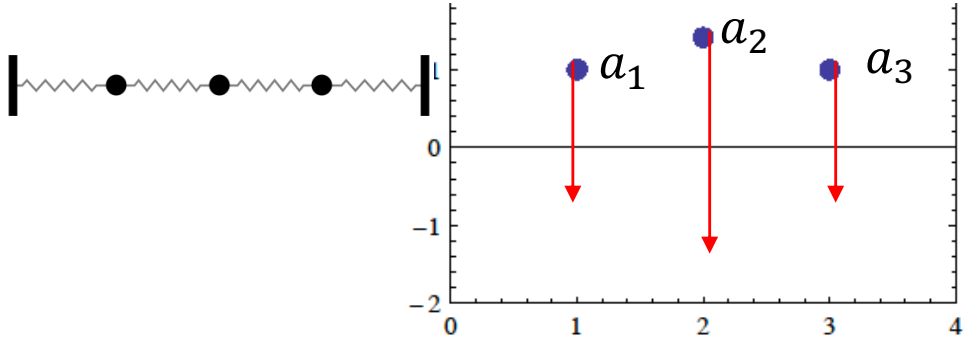
Red arrows indicate the SHM of individual blocks.



Normal Modes of oscillation

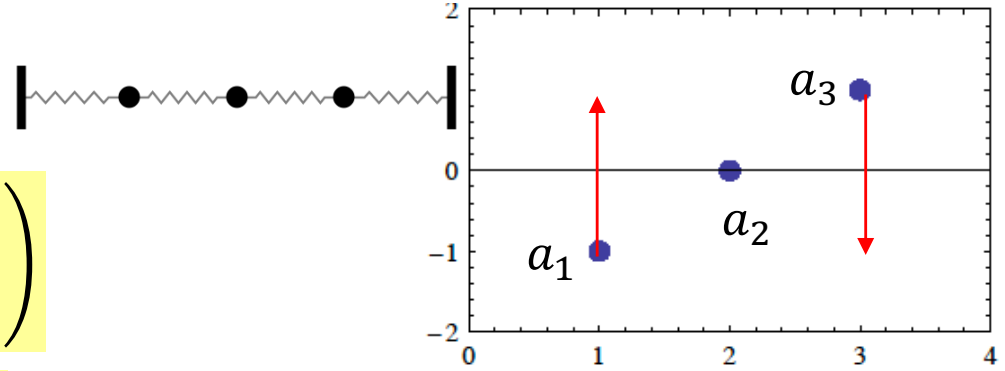
$$\omega_1 \sim \sqrt{2-\sqrt{2}}\,\omega_0 \sim 0.77\omega_0$$

$$\boldsymbol{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$



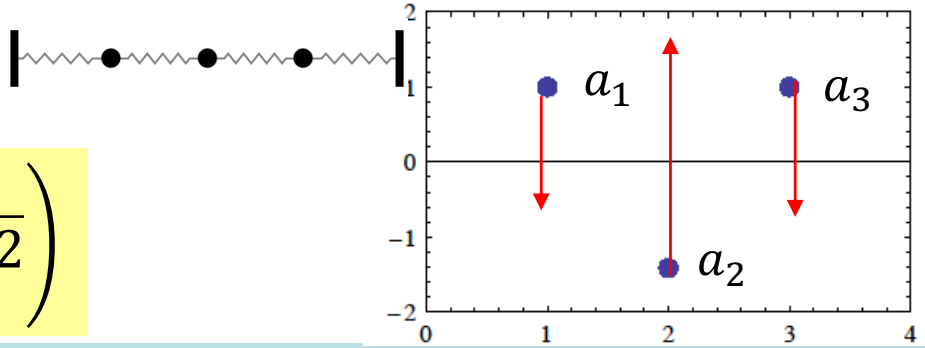
$$\omega_2 \sim \sqrt{2}\,\omega_0 \sim 1.41\omega_0$$

$$\boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\omega_3 \sim \sqrt{2+\sqrt{2}}\,\omega_0 \sim 1.85\omega_0$$

$$\boldsymbol{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$



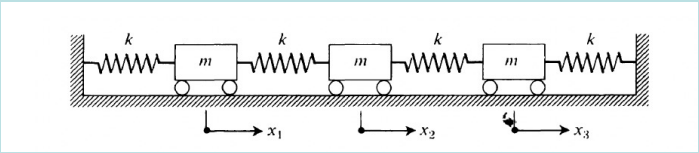
One eigenvalue, one eigenvector of the matrix corresponds one oscillation mode.

Any $n \times n$ symmetric matrix has n eigenvalues and they all are real.

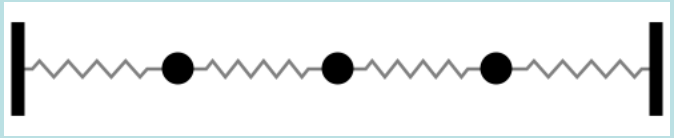
$$\boldsymbol{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

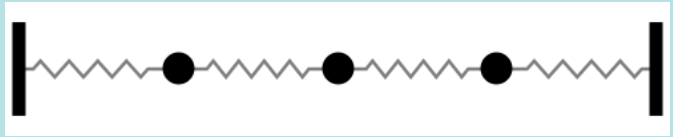
$\boldsymbol{A}$ is a symmetric matrix.

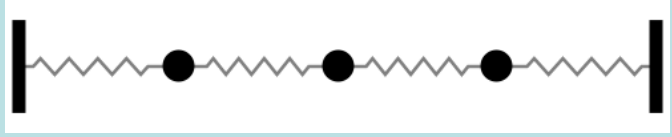
The number of modes equals the number of particles.

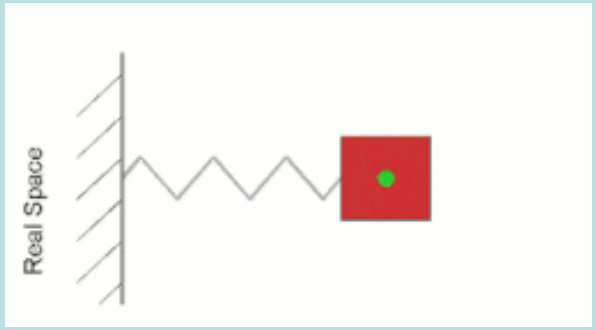


In modes, coupled springs act like one spring!

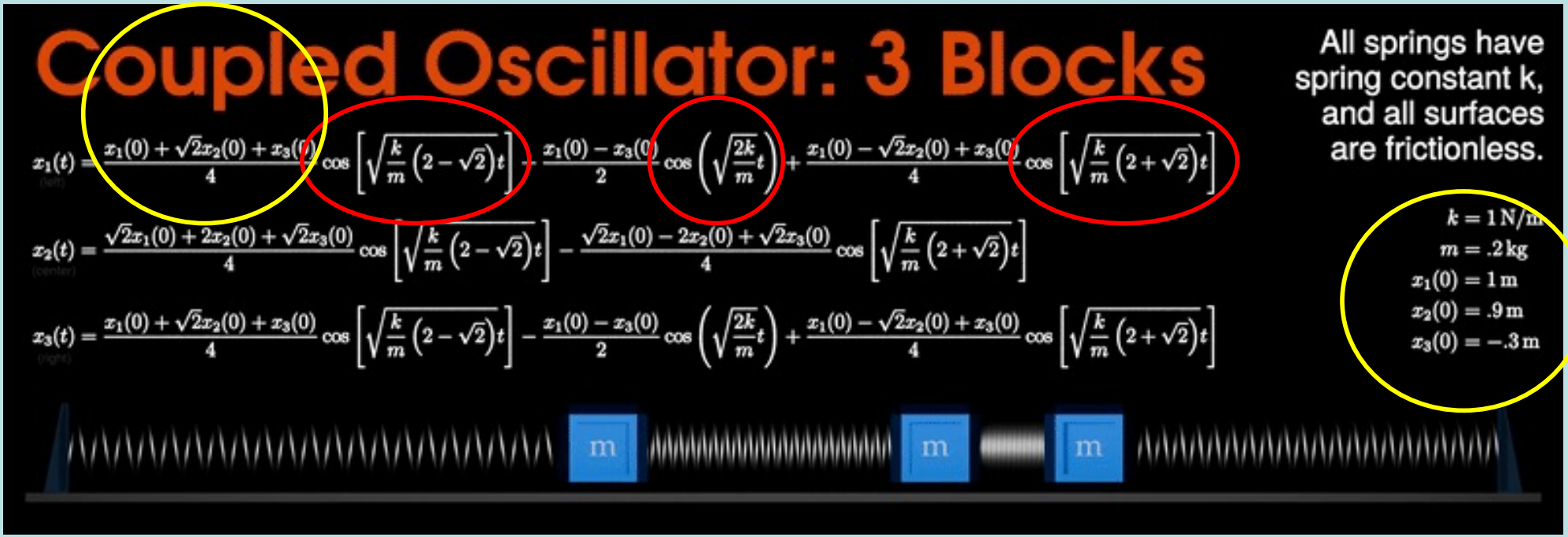








ω_1 ω_2 ω_3



In general, more than one modes can be excited. Then it is not a pure oscillation.

All three modes, multiplied by 3 coefficients, add up, forming a linear combination.

The equation of motions are linear ODE.

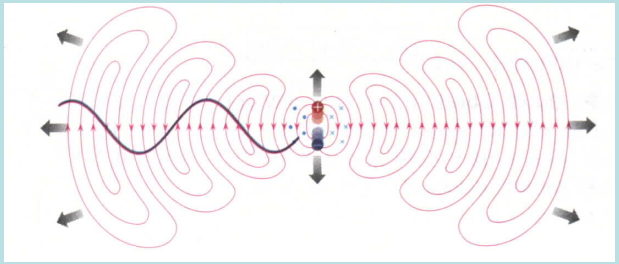
Linear Combinations of the 3 mode solutions are solutions, too.

Key Question: Is any solution of system a linear combination of 3 mode solutions?

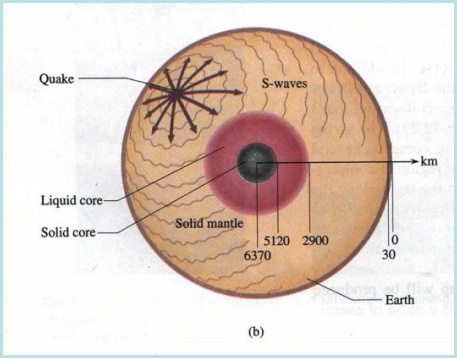
解波方程式 **Wave Equation** Equation of motion satisfied by all wave phenomena.

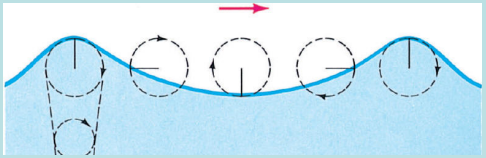
$$\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ 3D version

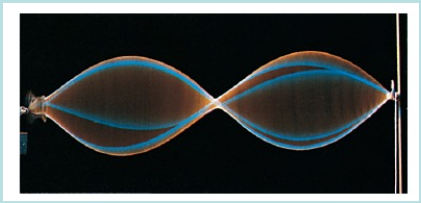
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ 1D version

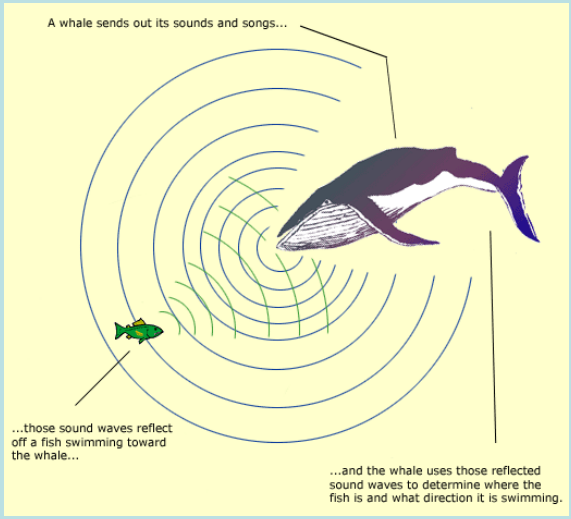


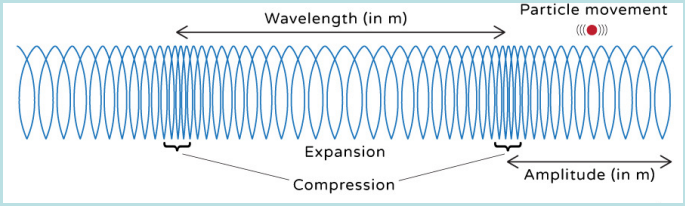












First step of solving wave equation:

Find solutions that can be separated into a time factor $T(t)$ and a space factor $X(x)$.

$$\phi(x,t) = X(x) \cdot T(t)$$

These will be called separable solutions,

same name we gave to oscillation modes in coupled springs.

Time factor $T(t)$ is separate from space $X(x)$ and independent of coordinate x.

Benefit: $T(t)$ and $X(x)$ can be solved more easily.

the equilibrium configuration of a planar body, e.g., the deformations of a membrane. Separation of variables seeks special solutions that can be written as the product of functions of the individual variables, thereby reducing the partial differential equation to a pair of ordinary differential equations. More-general solutions can then be expressed as infinite series in the appropriate separable solutions. For the two-variable equations considered here, this results in a Fourier series representation of the solution. In the case of the wave equation, separation of variables serves to focus attention on the vibrational character of the solution, whereas the earlier d'Alembert approach emphasizes its particle-like aspects.

Find separable solutions:

$$\phi(x,t) = X(x) \cdot T(t)$$

Plug it into the wave equation: $\dfrac{\partial^2 \phi}{\partial t^2} = v^2 \dfrac{\partial^2 \phi}{\partial x^2}$

$$X\frac{\partial^2 T}{\partial t^2} = Tv^2 \frac{\partial^2 X}{\partial x^2} \quad \Rightarrow \quad X\frac{d^2 T}{dt^2} = Tv^2 \frac{d^2 X}{dx^2}$$

Note that partial differentiation becomes ordinary differentiation.

Divide both sides by $v^2 X(x) T(t)$.

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2}$$

Now left-hand side is only t related while right-hand side only x related.

But x, t are independent variables. General equality is impossible.

The only exception: the two formulae are both constants. Call it λ: separation constant

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} \equiv \lambda \quad \Rightarrow \quad \frac{d^2 X}{dx^2} = \lambda \cdot X \qquad \frac{d^2 T}{dt^2} = v^2 \lambda \cdot T$$

We find separate equations for time factor $T(t)$ and space factor $X(x)$。

Even better, equations are both ordinary differential equation instead of partial DE.

The time factor $T(t)$ can be determined right away

$$\frac{d^2T}{dt^2} = v^2\lambda T \equiv -\omega^2 T$$

If $\lambda < 0$, $T(t)$ is a SHM with angular frequency $\omega = v\sqrt{-\lambda}$.

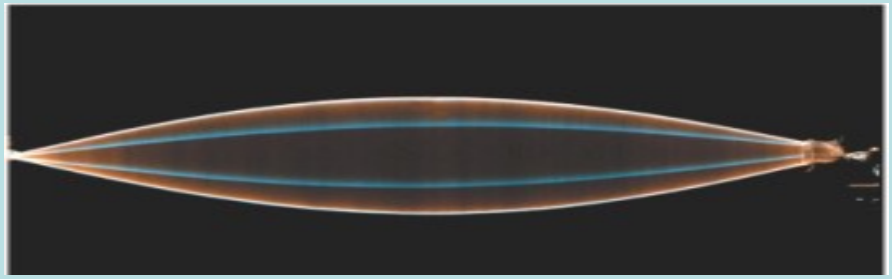
$$T(t) = ce^{i\omega t} = a_m e^{i\phi} e^{i\omega t}$$

or

$$T(t) = \text{Re}\, a_m e^{i\phi} e^{i\omega t} = a_m \cos(\omega t + \phi) = f\cos(\omega t) + g\sin(\omega t)$$

Hence a separable solutions of wave equation is also an oscillation mode solution.

$X(x)$ satisfies a standard 2nd order ODE. $\frac{d^2X}{dx^2} = \lambda X$



The medium with finite size usually comes with Boundary Condition 邊界條件,

$$\phi(0,t) = \phi(L,t) = 0$$

Fixed End Boundary Condition

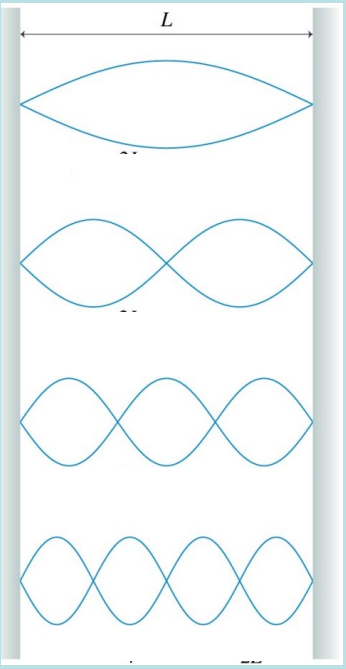
$$\frac{d^2X}{dx^2} = \lambda X$$

$$k \equiv \sqrt{-\lambda}$$

If $\lambda < 0$ $\quad X(x) = \mathrm{Re}\left(A_m e^{i\phi'} e^{\pm ikx}\right) = A_m \cos(kx + \phi') = A\cos kx + B\sin kx$

Choosing B equal to zero, X automatically satisfy one of boundary conditions:

$\phi(0,t) = 0$ ➡ $X(0) = 0$ $\quad X(x) = A\sin kx$



The other boundary condition will restrict k we can choose:

$\phi(L,t) = 0$ ➡

$X(L) \sim \sin(kL) = 0$ ➡ $k_n = \frac{n\pi}{L} = \sqrt{-\lambda_n}$ $\quad$ Wavelength $= \frac{2L}{n}$

These $\lambda_n = -\frac{n^2\pi^2}{L^2}$ that allow a solution for X are called eigenvalues.

The solutions $X_n(x)$ for X are called eigenfunctions.

$$X_n(x) = A\sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,\cdots\infty$$

These names remind us X_n are analogs of eigenvectors of matrix in coupled springs.

$$\frac{d^2X_n}{dx^2} = \lambda_n X_n$$

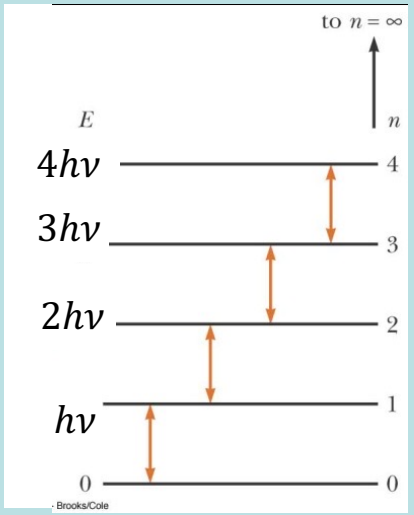
With λ_n, k_n, the angular frequency of T corresponds to ω_n:

$$\frac{d^2T}{dt^2} = v^2\lambda_n T \equiv -\omega_n^2 T$$

$T(t)$ is a SHM with angular frequency $\omega_n = v\sqrt{-\lambda_n} = vk_n$.

$$T(t) = f_n \cos(\omega_n t) + g_n \sin(\omega_n t)$$

$$\omega_n = v\frac{n\pi}{L}$$

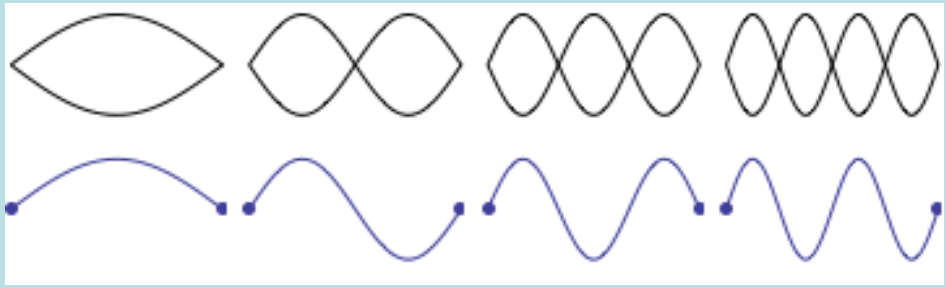


These separable solutions we will call $u^{(n)}$:

$$u^{(n)}(x,t) = X_n(x) \cdot T(t) = \sin(k_n x) \cdot (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$k_n = \frac{n\pi}{L}$$

$$\omega_n = vk_n = \frac{n\pi}{L}v$$



The time factor $T(t)$ is an oscillating function of angular frequency ω_n,

and the space factor $X(x)$ is a periodic function of angular wave number k_n.

These are typical standing waves.

Solving Wave Equation:

求解策略 I：

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

Search for separable solutions, products of a function of x and a function of t.

They correspond to oscillation modes in coupled spring system.

$$\phi(x,t) = X(x) \cdot T(t)$$

$$\frac{d^2 X}{dx^2} = \lambda \cdot X \qquad \frac{d^2 T}{dt^2} = v^2 \lambda \cdot T$$

$T(t)$ is solved right away as SHM.

With boundary condition, allowed $X_n(x), n = 1, \cdots \infty$ would be countable. ◦

We found a series of separable solutions:

$$u_n(x,t) = X_n(x) \cdot T_n(t) = \sin\left(\frac{n\pi}{L} x\right)\left(\sqrt{\frac{2}{L}} A_n \cos \omega_n t + \sqrt{\frac{2}{L}} B_n \sin \omega_n t\right)$$

We redefined the coefficients A_n, B_n to have an extra factor of $\sqrt{\frac{2}{L}}$.

It would prove convenient when we calculate coefficients by projection.

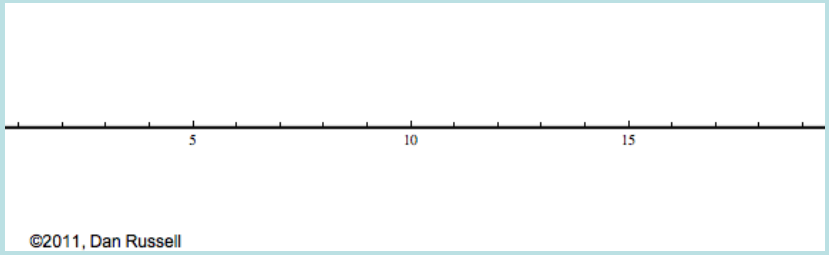
Separable solutions are really special!

$$u^{(n)}(x,t) = \sin\left(\frac{n\pi}{L}x\right)\left(\sqrt{\frac{2}{L}}A_n \cos\omega_n t + \sqrt{\frac{2}{L}}B_n \sin\omega_n t\right)$$

All particles oscillate as SHM by identical ω_n and phase, amplitudes $\propto \sin\left(\frac{n\pi}{L}x\right)$.

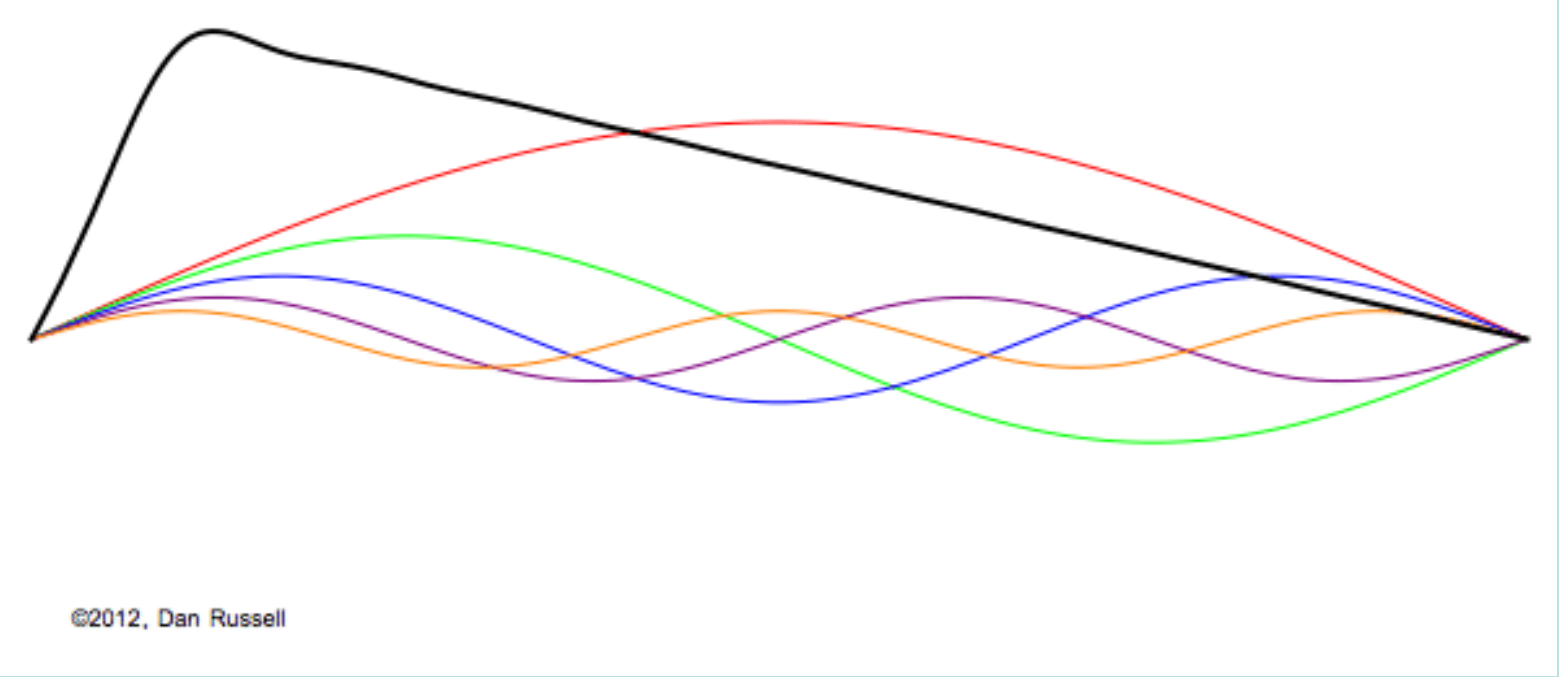
It could also be seen as a Sine function of x with time varying amplitude.

Not all solution acts like that.





Blackline is a solution when we pull string, with fixed ends, at one point and release.



D. Bernoulli, 1700-1782



求解策略II：

Since wave equation is linear, linear combinations of $X_n(x)T_n(t)$ are solutions, too.

$$\phi(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(\sqrt{\frac{2}{L}} A_n \cos\omega_n t + \sqrt{\frac{2}{L}} B_n \sin\omega_n t\right)$$

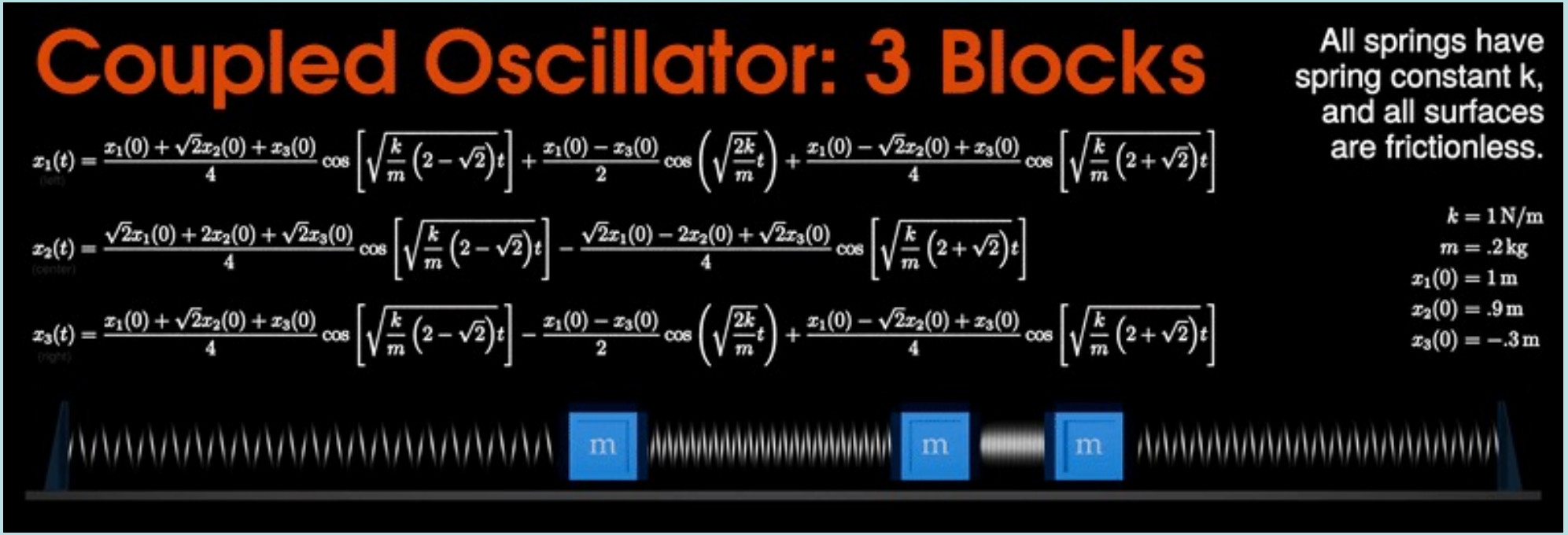
This is an infinite series of functions, summing up lots of separables with coefficients!

These are all solutions of wave equation.

That solutions of Wave E. can be written in this form was proposed by Bernoulli.

They are called Bernoulli Solutions. With A_n, B_n we can form lots of Bernoulli Solutions

This also happens in coupled spring system.

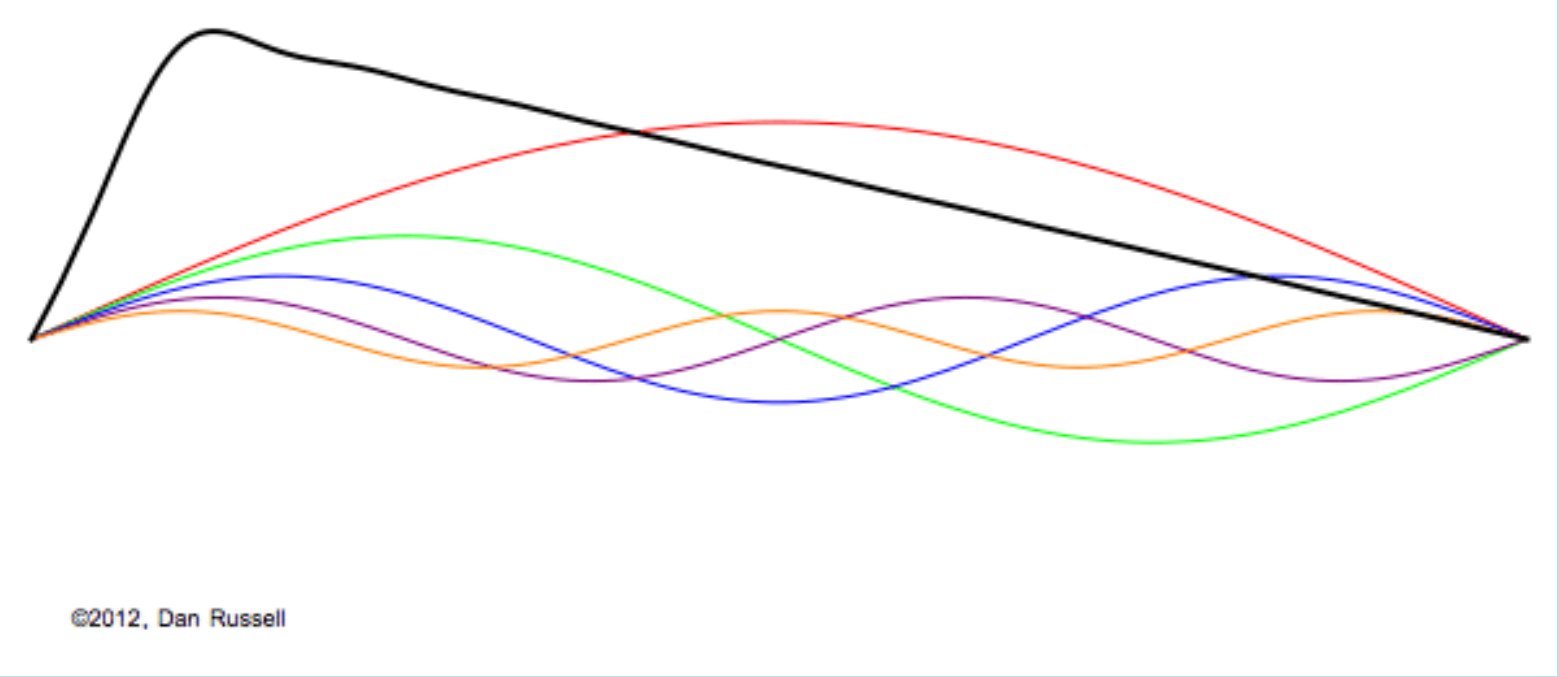


The equation of motions are linear ODE.

Linear Combinations of mode solutions are solutions, too.

For fixed end string: The following is one example of Bernoulli Solution.

The blackline solution is the sum of colored lines of separable oscillating solutions.



求解策略II：

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How to compute coefficients A_n, B_n for a specific problem.

Are all solutions of the form of Bernoulli Solutions?

Do Bernoulli Solutions exhaust all possible solutions of wave equation?

Equation of motion: 對照表 Wave Equation:

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$$

$$\frac{\partial^2\phi}{\partial t^2} = v^2\frac{\partial^2\phi}{\partial x^2}$$

x_j

$$\phi(jd) = \phi(x)$$

We usually choose linear combination coefficients to fit initial condition.

Initial Condition

$$\begin{pmatrix} x_1(0) \\ \vdots \\ x_j(0) \\ \vdots \\ x_N(0) \end{pmatrix}$$

$$\begin{pmatrix} x_1'(0) \\ \vdots \\ x_j'(0) \\ \vdots \\ x_N'(0) \end{pmatrix}$$

Time derivatives with fixed $j \to x$.

Initial Condition

$$\phi(x,0)$$

Wavefunction at $t = 0$

$$\frac{\partial\phi}{\partial t}(x,0)$$

Time partial of wavefunction at $t = 0$

Could we always find coefficients A_n, B_n to satisfy specific initial condition?

If yes, we are done, because we have found Bernoulli Solutions to all problems.

Every problem can have only one solution! So yes, all solutions are like Bernoulli's.

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(\sqrt{\frac{2}{L}}A_n \cos\omega_n t + \sqrt{\frac{2}{L}}B_n \sin\omega_n t\right)$$

Bernoulli's solution

The coefficients A_n, B_n are found, (or fixed) to satisfy initial conditions.

Plug in $t = 0$ to the expression above, we find the key equation:

$$\phi(x,0) = u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

Technically we'd say: function $u_0(x)$ is expanded as a series of Sine functions.

$$\frac{\partial}{\partial t}\phi(x,0) = v_0(x) = \sum_{n=1}^{\infty} B_n\omega_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

How to compute A_n, B_n?



1768-1830 Jean-Baptiste Joseph Fourier

The following method is proposed by Fourier 50 years after Bernoulli.

Expansion of a function as a series of Sine functions is called Fourier Series.

Decomposing $u_0(x)$ into linear combination of $X_n(x)$ could proceed similarly.

$$u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$$

$$\vec{a} = a_x \hat{\imath} + a_y \hat{\jmath}$$

Take inner product of u_0 and X_n in the sense of an integration: $a_x = \hat{\imath} \cdot \vec{a}$

$$\sqrt{\frac{2}{L}} \int_0^L dx\, u_0(x) \sin\left(\frac{n\pi}{L} x\right) = \frac{2}{L} \int_0^L dx \left[\sum_{m=1}^{\infty} A_m \cdot \sin\left(\frac{m\pi}{L} x\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

$$= \frac{2}{L} \sum_{m=1}^{\infty} \left[A_m \int_0^L dx \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right)\right] = \frac{2}{L} \sum_{m=1}^{\infty} \left[A_m \frac{L}{2} \delta_{mn}\right] = A_n$$

Jean-Baptiste Joseph Fourier



$$A_n = \sqrt{\frac{2}{L}} \int_0^L dx\; u_0(x) \sin\left(\frac{n\pi}{L} x\right) \quad \Longleftrightarrow \quad a_x = \hat{\imath} \cdot \vec{a}$$

Plug in these coefficients to Bernoulli's solution and we solve the problem.

Bernoulli's solution

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(\sqrt{\frac{2}{L}}A_n \cos\omega_n t + \sqrt{\frac{2}{L}}B_n \sin\omega_n t\right)$$

Fourier Series of initial condition.

$$\phi(x,0) = u_0(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right.$$

$$\frac{\partial}{\partial t}\phi(x,0) = v_0(x) = \sum_{n=1}^{\infty} B_n \omega_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

Fourier Series coefficient formula.

$$A_n = \sqrt{\frac{2}{L}}\int_0^L dx\, u_0(x)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$B_n = \frac{1}{\omega_n}\sqrt{\frac{2}{L}}\int_0^L dx\;\; v_0(x)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

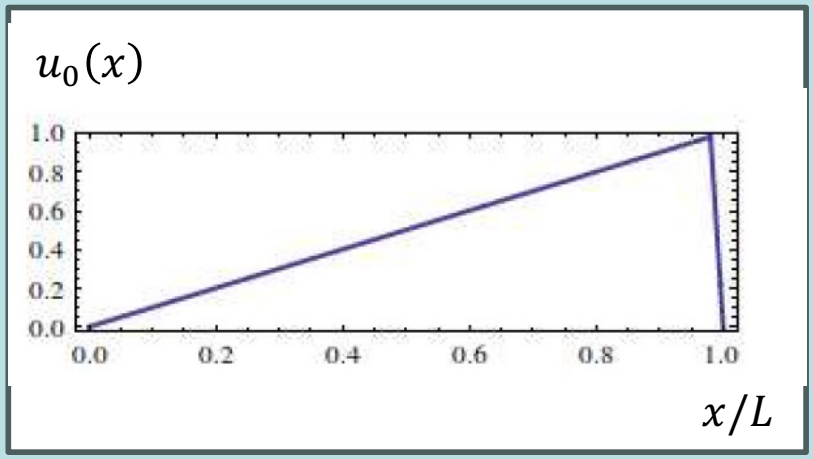
It satisfies both the system of equations of motion and initial conditions.

There is one and only one function that satisfies both EOM and initial conditions.

This is the only one solution.

We prove: any solution of Wave E. is a linear combination of separable solutions.

例如：考慮起始條件為一個Sawtooth鋸齒波形



$$u_0(x) = \frac{1}{L}x$$

$$0 < x < L$$

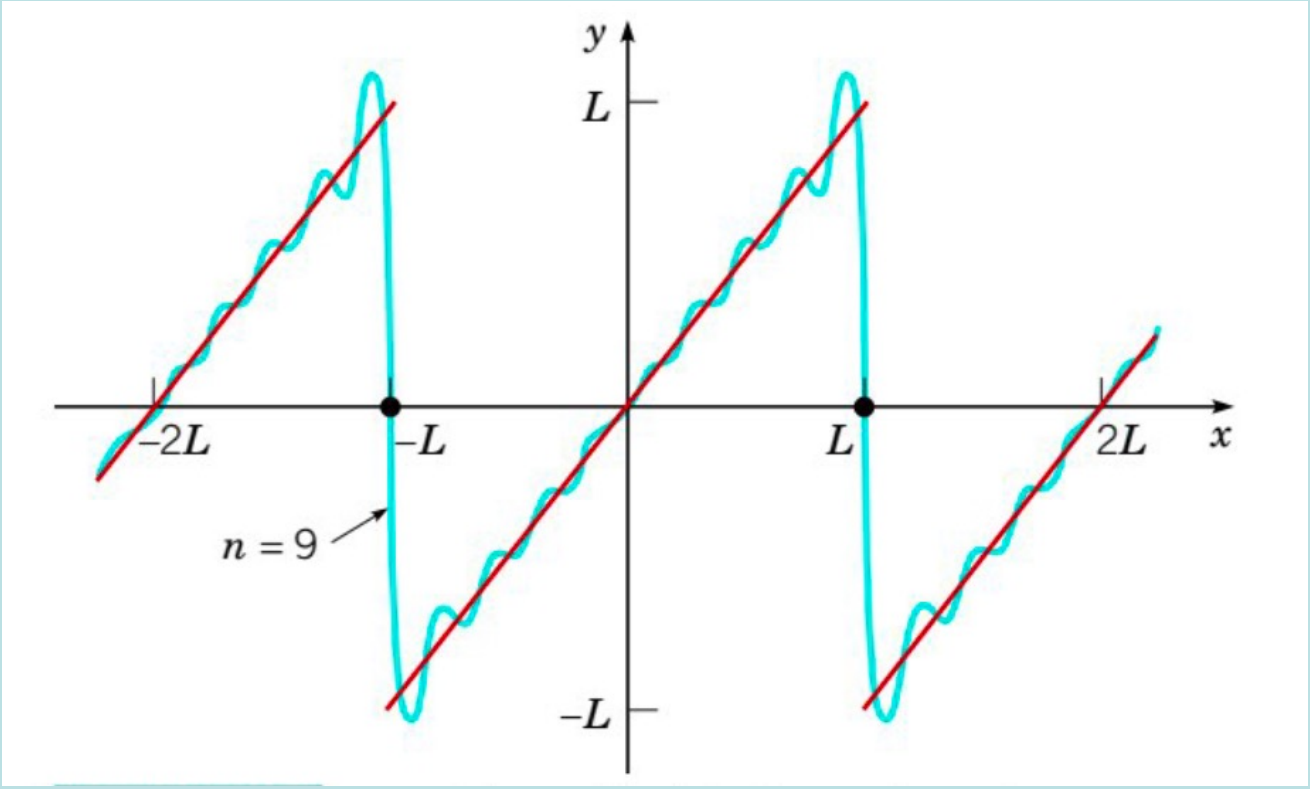
$$A_n = \sqrt{\frac{2}{L}}\int_0^L dx\ u_0(x)\sin\left(\frac{n\pi}{L}x\right) = \sqrt{\frac{2}{L}}\int_0^L dx\frac{1}{L}x\cdot\sin\left(\frac{n\pi}{L}x\right)$$

$$= \sqrt{\frac{2}{L}}\left\{\frac{1}{n\pi}\left[x\cos\left(\frac{n\pi}{L}x\right)\right]\Big|_0^L - \frac{1}{n\pi}\int_0^L dx\ \cos\left(\frac{n\pi}{L}x\right)\right\}$$

Integration by part

$$= \sqrt{2L}\frac{1}{n\pi}\cos(n\pi) - \sqrt{2L}\left(\frac{1}{n\pi}\right)^2\left[\sin\left(\frac{n\pi}{L}x\right)\right]\Big|_0^L = \sqrt{2L}\frac{(-1)^{n+1}}{n\pi}$$

$$A_n = \sqrt{2L}\frac{(-1)^{n+1}}{n\pi}$$

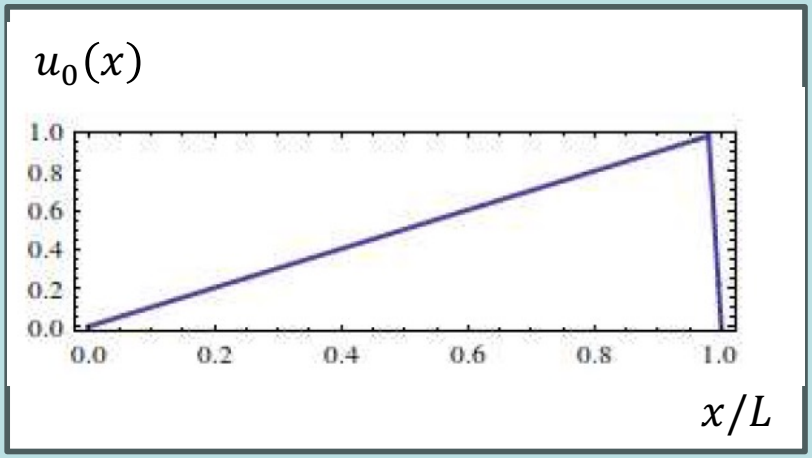


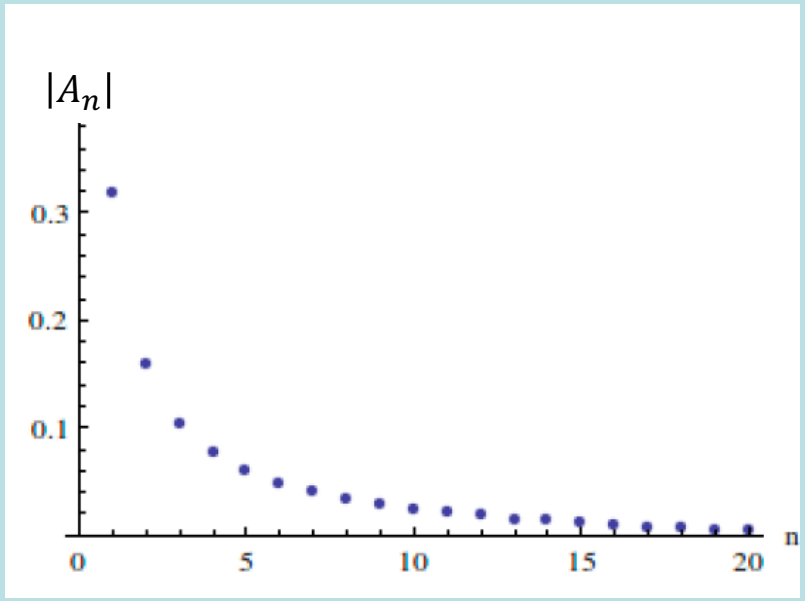
Sawtooth function $u_0(x)$ is expanded as a series of Sine functions.

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{2}{\pi}\left[\sin\left(\frac{\pi}{L}x\right) - \frac{1}{2}\sin\left(\frac{2\pi}{L}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{L}x\right) + \cdots\right]$$

If we start with string at rest: $v_0 = 0, B_i = 0$.





$$A_n = \sqrt{2L}\frac{(-1)^{n+1}}{n\pi}$$

Plug the coefficeints into the Bernoulli's solution:

$$\phi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\sqrt{\frac{2}{L}}\cos\omega_n t = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

It satisfies both the system of equations of motion and initial conditions.

This is the only one solution.

任一滿足邊界條件$f(0) = f(L) = 0$的函數$f(x)$都可分解為一正弦函數級數：

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$

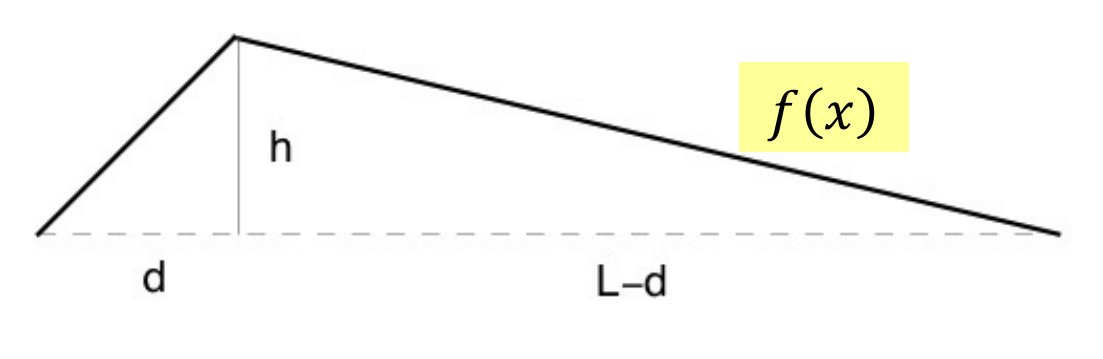
$$a_n = \frac{2}{L}\int_0^L dx \ \sin\left(\frac{n\pi}{L}x\right) f(x)$$

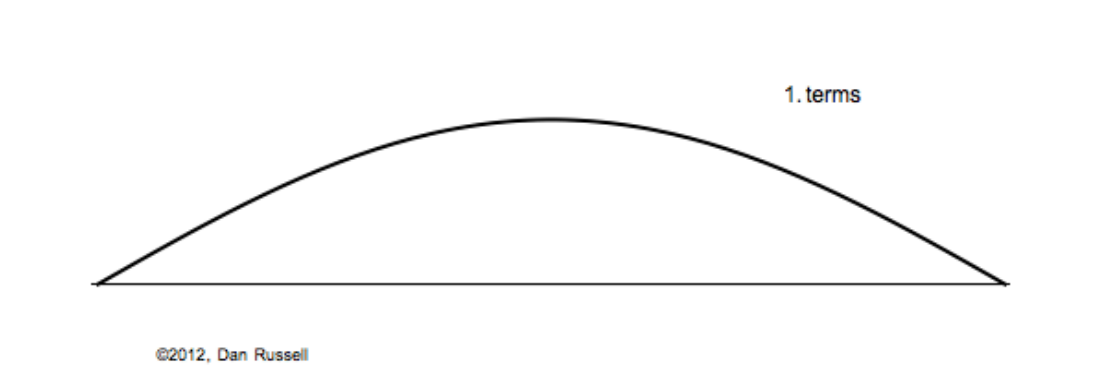
This is a special case of Fourier Series.



Jean-Baptiste Joseph Fourier

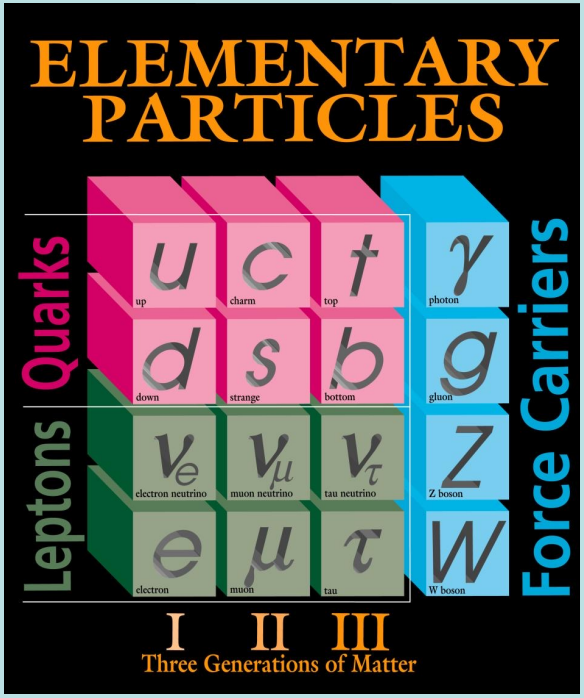
1768-1830





光子 **Photon** γ





The energy and momentum of one photon.

$$E = h\nu \qquad E = \hbar\omega$$

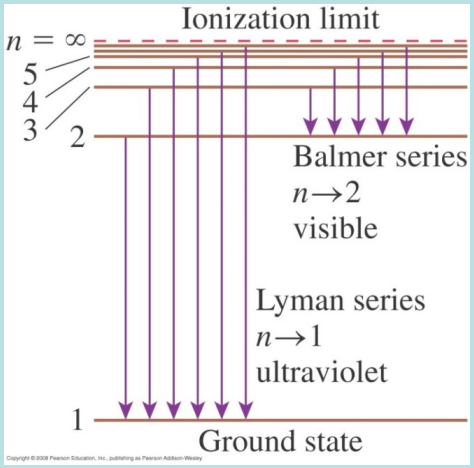
$$p = \frac{h}{\lambda} \qquad p = \hbar k$$

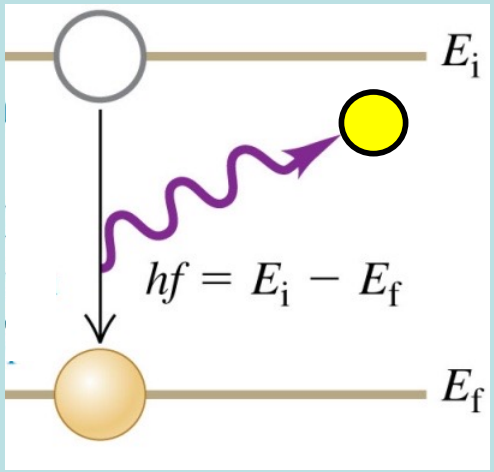
$$h = 6.626\times10^{-34}\ \mathrm{J\cdot s} = 4.136\times10^{-15}\ \mathrm{eV\cdot s}$$

The key assumption is photon is indivisible. There is no fraction of one photon.

Photon could be emitted and annihilated easily, usually one at a time.

波爾原子模型的基本假設 Basic Assumptions of Bohr Atomic Model





$$E_i \propto -\frac{R_H}{n^2}$$

$$h\nu \propto \frac{1}{\lambda} = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

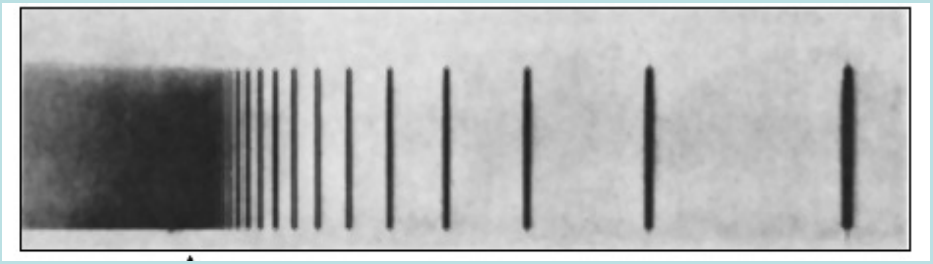
$$E_f \propto -\frac{R_H}{m^2}$$

Electron Quantum states inside atom are not continuously distributed, but discretely.

We'll say: Energy is quantized. Quantization of energy is a key feature of QM.

Atomic Emission is not generated as electron rotates, but as electron jumps.

Energy difference between two states are released by emitting one (just one) photon.

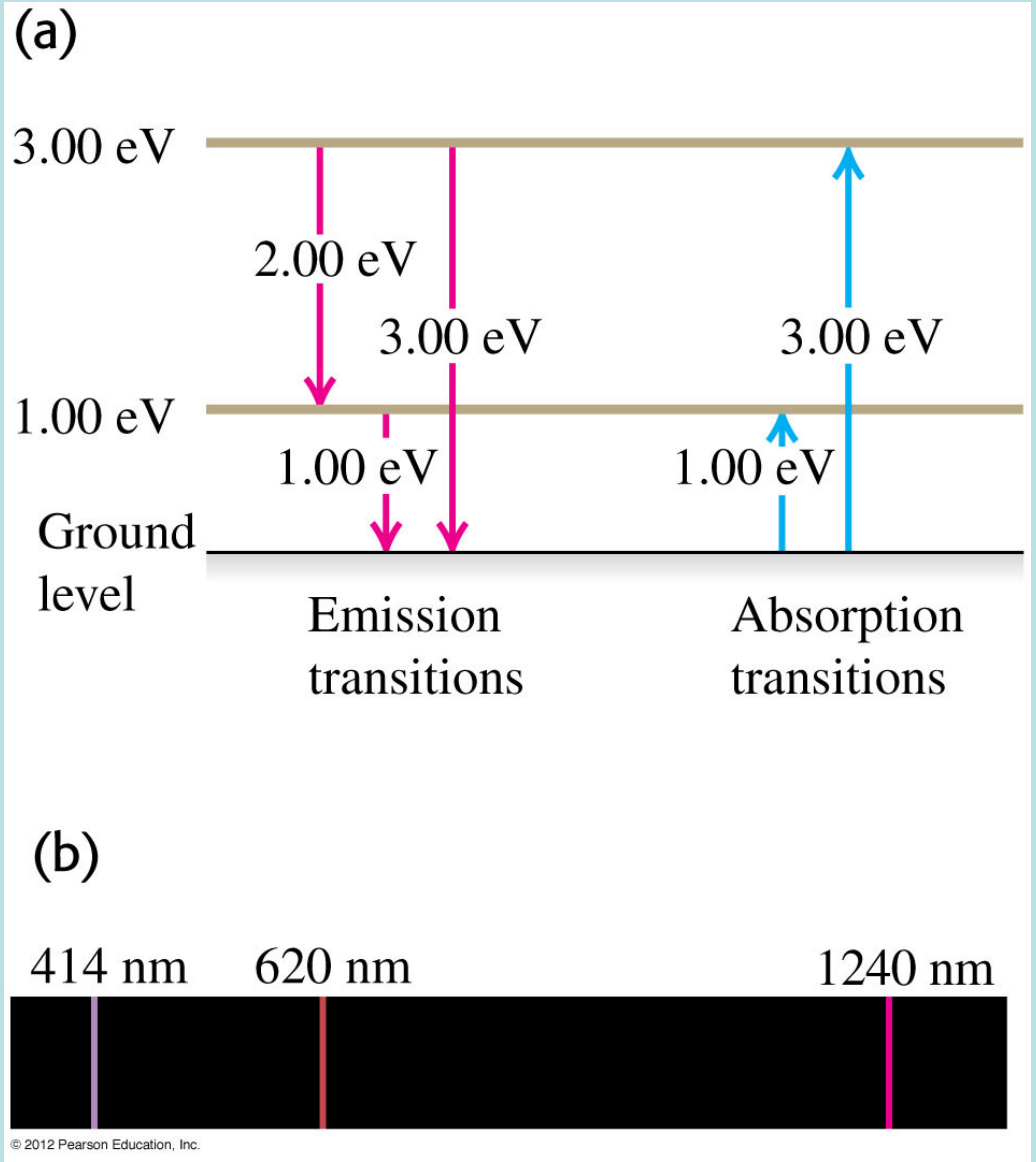


The sharp edge of spectrum indicates there is no states other than these discrete states.

Electron can not live in between two states.

我們現在知道：量子躍遷的機制也適用任何量子能階！

例如下圖的三個能階，對應有三條光譜：



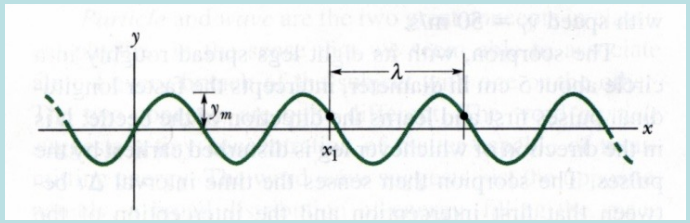
$h\nu = \Delta E$

山寨版電子波 Copycat matter wave：抄襲模仿光波與光子吧！

德布羅意假設物質波的頻率與能量，以及波長與動量的關係，就如光一樣！

de Broglie proposed to copy formulae from light to apply directly in case of electrons





$$E = hf = \frac{1}{2}mv^2$$

$$p = \frac{h}{\lambda} = mv$$