數物期中考

Nov 6 2026

1. Calculate $\int_{-\infty}^{\infty} dy \cdot y^6 e^{-\alpha y^2}$. (20)

   Hint: Put $\alpha y^2 = x$. $\Gamma(p+1) = \int_0^{\infty} dx \cdot x^p e^{-x} = p!$. $\Gamma(p+1) = p\Gamma(p)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

   Sol: $\alpha y^2 = x,\ \ 2\alpha y dy = dx$

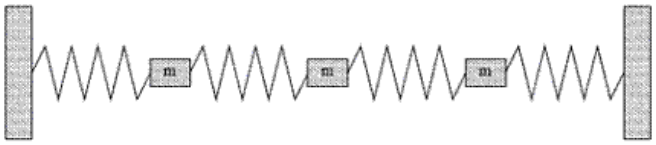
$$\int_{-\infty}^{\infty} dy \cdot y^6 e^{-\alpha y^2} = \int_0^{\infty} dy \cdot y^6 e^{-\alpha y^2} + \int_{-\infty}^{0} dy \cdot y^6 e^{-\alpha y^2}$$

$$= 2\int_0^{\infty} dy \cdot y^6 e^{-\alpha y^2} = \frac{1}{\alpha^{7/2}} \int_0^{\infty} dx \cdot x^{7/2} e^{-x} = \frac{1}{\alpha^{7/2}} \Gamma\left(\frac{7}{2}\right) = \frac{1}{\alpha^{7/2}} \frac{5}{2} \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15\sqrt{\pi}}{8\alpha^{7/2}}$$

2. Consider the system of $N = 3$ spring-coupled masses. Assume that the spring constants are all equal to k and the masses are all equal to m. $\omega_0 = \sqrt{\frac{k}{m}}$.

   If we move particles so that the initial position is proportional to an eigenvector, the corresponding oscillation mode will be excited. For example, we can set initial conditions: $x_1(0) = -a$, $x_2(0) = 0, x_3(0) = a$, $x_1'(0) = x_2'(0) = x_3'(0) = 0$, the second mode will be excited.



The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

$$\omega_1 \sim \sqrt{2-\sqrt{2}}\omega_0 \sim 0.77\omega_0 \qquad \boldsymbol{a}^{(1)} = c_1\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

$$\omega_2 \sim \sqrt{2}\omega_0 \sim 1.41\omega_0 \qquad \boldsymbol{a}^{(2)} = c_2\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

$$\omega_3 \sim \sqrt{2+\sqrt{2}}\omega_0 \sim 1.85\omega_0 \qquad \boldsymbol{a}^{(3)} = c_3\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

If the initial conditions start with proportion $-1:0:1$, oscillation mode will continue.

$$\boldsymbol{x}(0) = \begin{pmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{pmatrix} \sim f_2\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \qquad \boldsymbol{x}'(0) = \begin{pmatrix}x_1'(0)\\ x_2'(0)\\ x_3'(0)\end{pmatrix} \propto \omega_2 g_2 \boldsymbol{a}^{(2)}$$

Note that constant f_2, g_2 is determined by initial conditions $x_{1,2,3}(0), x'_{1,2,3}(0)$.

This is one oscillation mode that would oscillate independently.

$$\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_0 \qquad \boldsymbol{x}(t) \propto \boldsymbol{a}^{(2)} \propto \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

Take the real part of this complex form.

$$\mathrm{Re}\,\boldsymbol{x} = \begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix} = \begin{pmatrix}1\\ 0\\ -1\end{pmatrix}\left[f_2\cos\left(\sqrt{2}\omega_0 t\right) + g_2\sin\left(\sqrt{2}\omega_0 t\right)\right]$$

A. Write down $x_1(t), x_2(t), x_3(t), v_1(t) \equiv \frac{dx_1}{dt}(t)$ in terms of a, ω_0. (10)

B. The total potential energy of the system is: $V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}k_3x_3^2$. Calculate the kinetic energy K_1, potential energy V_1 (the part of V related to x_1) and total mechanical energy E_1 of the **first** particle. (10)

Sol:

$$\begin{pmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{pmatrix} = \begin{pmatrix}a\\ 0\\ -a\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}f_2$$

$$f_2 = \sqrt{2}a$$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}f_2\cos(\omega_2 t) = \begin{pmatrix}a\\ 0\\ -a\end{pmatrix}\cos(\omega_2 t)$$

$$x_1 = a\cos\sqrt{2}\omega_0 t$$

$$x_2 = 0$$

$$x_3 = -a\cos\sqrt{2}\omega_0 t$$

$$v_1(t) \equiv \frac{dx_1}{dt}(t) = a\sqrt{2}\omega_0 \ \sin\sqrt{2}\omega_0 t$$

$$V_1 \supset \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 = kx_1^2 = k\,a^2\cos^2\sqrt{2}\omega_0 t$$

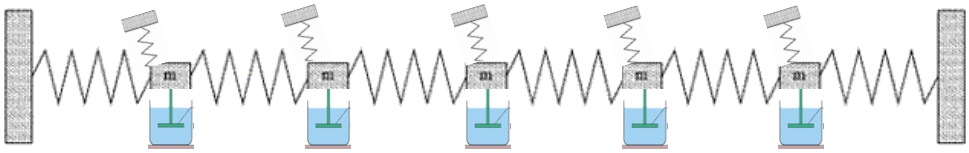
$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}m\cdot 2\omega_0^2\,a^2\sin^2\sqrt{2}\omega_0 t = k\,a^2\sin^2\sqrt{2}\omega_0 t$$

The sum is independent of time: $E_1 = ka^2$

3. Consider the following wave equation:

$$\frac{\partial^2\phi}{\partial t^2} + b\frac{\partial\phi}{\partial t} = v^2\frac{\partial^2\phi}{\partial x^2} - V(x)\phi$$

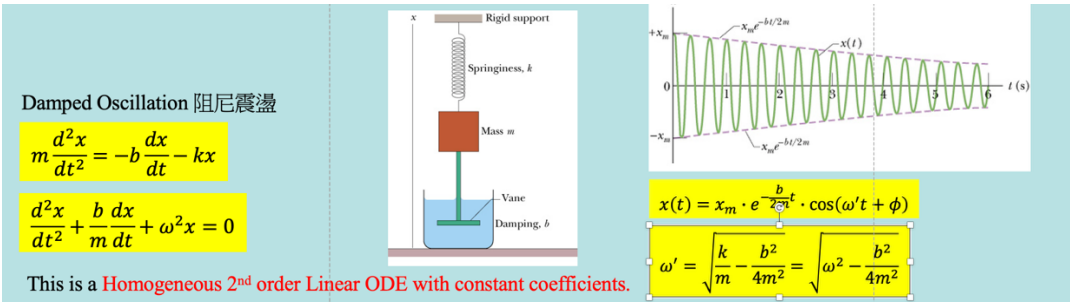
Here we add a term $V(x)\phi$ to the right-hand side of wave equation. Assume that $V(x) = v^2\alpha^2x^2$. α is a constant. This could be done by adding a position x dependent restoring force to particles in the coupled spring system. We also add a term $b\frac{\partial\phi}{\partial t}$ to the left-hand side of wave equation to account for damping, by adding a damping force to each particle before taking the continuous limit. b is also a constant.



It is reasonable to think that there exist solutions to this extended equation that are separable:

$$\phi(x,t) = X(x)\cdot T(t).$$

A. Find the Ordinary Differential Equations satisfied by $X(x)$ and $T(t)$. Solve $T(t)$. Hint: It is convenient to write the separation variable λ as $v^2\lambda = -\omega^2$. And use the following formulae.(20)



B. It was shown in Homework III that $e^{-\alpha\frac{x^2}{2}}$ is a solution for $X(x)$. Now show that $u_1(x) = c \cdot x \cdot e^{-\alpha\frac{x^2}{2}}$ is also a solution. c is constant coefficient not yet determined.. What is the value of ω for this solution, in terms of α and v? (15)

Sol:

A. Plug $\phi(x,t) = X(x) \cdot T(t)$ into $\frac{\partial^2\phi}{\partial t^2} + b\frac{\partial\phi}{\partial t} = v^2\frac{\partial^2\phi}{\partial x^2} - V(x)\phi$:

$$X\frac{\partial^2 T}{\partial t^2} + X\frac{\partial T}{\partial t} = Tv^2\frac{\partial^2 X}{\partial x^2} - VXT$$

Divide the whole equation by v^2XT:

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} + \frac{b}{v^2}\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} - \frac{V}{v^2} = \lambda$$

It can be equal to a constant λ. Therefore the equation for T:

$$\frac{d^2T}{dt^2} + b\frac{dT}{dt} = v^2\lambda \cdot T \equiv -\omega^2 T$$

$$\frac{d^2T}{dt^2} + b\frac{dT}{dt} + \omega^2 T = 0$$

This is the same equation as a damping SHM for $\omega = v\sqrt{-\lambda}$. The solution is the same:

$$T(t) = a_m e^{-\frac{b}{2}t}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\omega^2 - \frac{b^2}{4}}$$

The equation for X:

$$\frac{d^2X}{dx^2} - \frac{V}{v^2}X = \frac{d^2X}{dx^2} - \alpha^2x^2X = \lambda \cdot X$$

B. Space part $X(x)$ satisfies:

$$\frac{d^2X}{dx^2} - \alpha^2x^2X = \lambda \cdot X$$

Plug in

$$u_1(x) = c \cdot x \cdot e^{-\alpha\frac{x^2}{2}}$$

And simplifies:

$$-\frac{d^2u_1}{dx^2} + \alpha^2x^2u_1 = \frac{\omega^2}{v^2} \cdot u_1$$

Calculate the differentiation:

$$\frac{du_1}{dx} = ce^{-\alpha\frac{x^2}{2}} - c\alpha x^2 e^{-\alpha\frac{x^2}{2}}$$

$$-\frac{d^2u_1}{dx^2} = c\alpha x e^{-\alpha\frac{x^2}{2}} - c\alpha^2 x^3 e^{-\alpha\frac{x^2}{2}} + 2c\alpha x e^{-\alpha\frac{x^2}{2}}$$

Plug in the equation:

$$-\frac{d^2u_1}{dx^2} + \alpha^2 x^2 u_1 = c\alpha x e^{-\alpha\frac{x^2}{2}} - c\alpha^2 x^3 e^{-\alpha\frac{x^2}{2}} + 2c\alpha x e^{-\alpha\frac{x^2}{2}} + c\alpha^2 x^3 e^{-\alpha\frac{x^2}{2}}$$

$$= 3\alpha \cdot cxe^{-\alpha\frac{x^2}{2}} = 3\alpha \cdot u_1 = \frac{\omega^2}{v^2} \cdot u_1$$

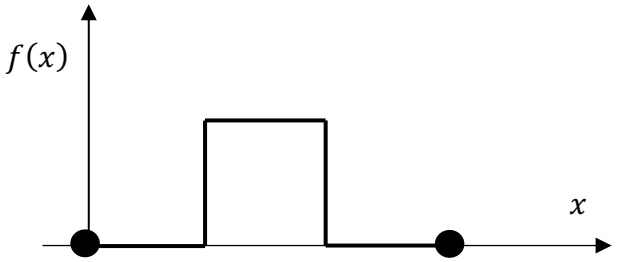
If $\frac{\omega^2}{v^2} = 3\alpha$, $cxe^{-\alpha\frac{x^2}{2}}$ is a solution.

$$\omega^2 = 3\alpha v^2$$

4. Consider a function $f(x)$ of x:

$$f(x) = 0, \qquad 0 < x < \frac{1}{3}L \text{ and } \frac{2}{3}L < x < L$$

$$f(x) = a, \qquad \frac{1}{3}L < x < \frac{2}{3}L$$



It satisfies fixed end boundary condition.

Write down the first 4 nonzero terms of Fourier expansion series of $f(x)$. (25)

任一滿足邊界條件$f(0) = f(L) = 0$的函數$f(x)$都可分解為一正弦函數級數：

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi}{L}x\right) \qquad a_n = \frac{2}{L}\int_0^L dx \ \sin\left(\frac{n\pi}{L}x\right) f(x)$$

Sol:

$$a_n = \frac{2}{L}\int_0^L dx\, f(x)\sin\left(\frac{n\pi}{L}x\right) = \frac{2}{L}\int_{\frac{1}{3}L}^{\frac{2}{3}L} dx\ \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{2}{L}\frac{L}{n\pi}\left[-\cos\left(\frac{n\pi}{L}x\right)\right]\Big|_{\frac{1}{3}L}^{\frac{2}{3}L} = \frac{2}{n\pi}\left[-\cos\left(\frac{2n}{3}\pi\right)+\cos\left(\frac{n}{3}\pi\right)\right]$$

$$a_1 = \frac{2}{\pi}\left[-\cos\left(\frac{2}{3}\pi\right)+\cos\left(\frac{1}{3}\pi\right)\right] = \frac{2}{\pi},$$

$$a_2 = \frac{2}{2\pi}\left[-\cos\left(\frac{4}{3}\pi\right)+\cos\left(\frac{2}{3}\pi\right)\right] = 0,$$

$$a_3 = \frac{2}{3\pi}\left[-\cos\left(\frac{6}{3}\pi\right)+\cos\left(\frac{3}{3}\pi\right)\right] = -\frac{4}{3\pi}$$

$$a_4 = 0, a_5 = +\frac{2}{5\pi}, a_6 = 0,$$

$$a_7 = +\frac{2}{7\pi}$$

$$f(x) = \frac{2}{\pi}\sin\left(\frac{\pi}{L}x\right) - \frac{4}{3\pi}\sin\left(\frac{3\pi}{L}x\right) + \frac{2}{5\pi}\sin\left(\frac{5\pi}{L}x\right) + \frac{2}{7\pi}\sin\left(\frac{7\pi}{L}x\right)\cdots$$

$$= \frac{2}{\pi}\left[\sin\left(\frac{\pi}{L}x\right) - \frac{2}{3}\sin\left(\frac{3\pi}{L}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{L}x\right) + \frac{1}{7}\sin\left(\frac{7\pi}{L}x\right)\cdots\right]$$