Midterm Exam

Apr 2026

1. Solve the ODE

$$\frac{dN}{dt} = -kN^3$$

with the initial condition: $N(0) = N_0$.

Sol:

$$\frac{dN}{dt} = -kN^3$$

Collect factors of N on the left hand side and factors of t on the right hand side:

$$\frac{dN}{N^3} = -kdt$$

Integrate and add a constant C:

$$\int \frac{dN}{N^3} = -\int k \cdot dt + C'$$

$$-\frac{1}{2N^2} = -kt + C'$$

$$N = \sqrt{\frac{1}{2kt + C}}$$

The constant C can be obtained by the initial condition: $N(0) = N_0$

$$N_0 = \sqrt{\frac{1}{C}}$$

$$N = \sqrt{\frac{1}{2kt + \frac{1}{N_0^2}}} = N_0 \sqrt{\frac{1}{1 + 2kN_0^2 t}}$$

2. Solve the equation of $x(t)$ by the method of integrating factor

$$x' + \frac{1}{t+4} x = t^5$$

Remember to have an unspecified constant in your solution.

Sol: The integrating factor would be $\alpha(t) = \exp\left[\int \frac{1}{t+4} dt\right] = \exp \ln(t+4) = t + 4$.

Multiply the whole equation by the integrating factor $\alpha(x)$:

$$(t+4)x' + x = \frac{d}{dt}[(t+4)x] = t^6 + 4t^5$$

Integrate and add a constant C:

$$(t+4)x = \int dt \cdot (t^6 + 4t^5) + C = \frac{t^7}{7} + \frac{2}{3}t^6 + C$$

$$x = \frac{\frac{t^7}{7} + \frac{2}{3}t^6 + C}{(t+4)}$$

3. Find the solution $x(t)$ of the equation of

$$x'' + 4x' + 3x = 0$$

with the initial condition: $x(0) = 0, x'(0) = 2$.

Hint: The two solution you find using standard complex number procedure are both real and hence the general solutions are simply their linear combinations $ax_1 + bx_2$.

Sol: Guess the solution is :$z = z_0 e^{\alpha t}$ and plug it into the equation:

$$\alpha^2 e^{\alpha t} + 4\alpha e^{\alpha t} + 3e^{\alpha t} = 0$$

The unknown α satisfies the algebraic equation:

$$\alpha^2 + 4\alpha + 3 = 0$$

There are two real solutions:

$$\alpha = 1{,}3$$

Both solutions are real:

$$x = ae^{-t} + be^{-3t}$$

Using $x(0) = 0$, we find $a = -b. x = ae^{-t} - ae^{-3t}$

Using $x'^{(0)} = 2$, we find $a = 1$.

$$x = e^{-t} - e^{-3t}$$

4. First consider the homogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 4x = 0$$

A. Find the general solutions x_{h} of this ODE.

B. Consider the inhomogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 4x = 2e^{-t}$$

Find one solution. Hint: $x \propto e^{-t}$ would work.

C. Find the general solutions x_{inh} of the inhomogeneous ODE.

Sol:

A. 取$x = ce^{\alpha t}$，代入微分方程式，得到α的代數方程式：$\alpha^2 + 3\alpha + 4 = 0$：

求解：$\alpha = \frac{-3\pm i\sqrt{7}}{2}$，取$e^{\alpha t}$的實數部與虛數部，就能得到兩個實數解，取線性組合，就得到一般解：

$$x_{\text{h}} = c_1 e^{-3t/2}\cos\frac{\sqrt{7}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$$

B. 取$x = de^{-t}$，代入微分方程式：$(1-3+4)de^{-t} = 2e^{-t}$，$d = 1$

得到一個特解：$x = e^{-t}$

C. 一般解：

$$x_{inh} = c_1 e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{3}}{2}t + e^{-t}$$