Homework V

1. Find the eigenvalues of the following matrices: $\boldsymbol{A}$ and $\boldsymbol{B}$ and $\boldsymbol{AB}$ and $\boldsymbol{BA}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$$

Are the eigenvalues of $\boldsymbol{AB}$ equal the eigenvalues of $\boldsymbol{A}$ times the eigenvalue of $\boldsymbol{B}$?

Sol: $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$，特徵方程式：$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda + 2 = 0$, 本徵值 $\lambda_1 = 1$，$\lambda_2 = 2$，

$\lambda_1 = 1$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$，本徵向量滿足$-a_1 + a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$，特徵方程式：$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\lambda^2 - 7\lambda + 12 = 0$, 本徵值 $\lambda_1 = 3$，$\lambda_2 = 4$，

$\lambda_1 = 3$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 4$，本徵向量滿足$-a_1 + a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$$

$\boldsymbol{AB} = \boldsymbol{BA} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$特徵方程式：$\begin{vmatrix} 3-\lambda & 5 \\ 0 & 8-\lambda \end{vmatrix} = 0$

$\lambda^2 - 11\lambda + 24 = 0$, 本徵值 $\lambda_1 = 3$，$\lambda_2 = 8$，

$\lambda_1 = 3$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$，注意：$3 = 1 \times 3$

$\lambda_2 = 8$，本徵向量滿足$-5a_1 + 5a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$，$8 = 2 \times 4$

$\boldsymbol{AB} = \boldsymbol{BA}$與矩陣$\boldsymbol{B}$，與矩陣$\boldsymbol{A}$，具有相同的本徵向量，$\boldsymbol{AB} = \boldsymbol{BA}$的本徵值等於矩陣$\boldsymbol{A}$的本徵值乘矩陣$\boldsymbol{B}$的本徵值。

2. Prove that $\vec{\nabla} r^p = p r^{p-1} \hat{r}$, as p is an integer. You can use this formula in the following problems.

Sol:

$$\frac{\partial}{\partial x}r^p = \frac{\partial r}{\partial x}\frac{d}{dr}r^p = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}pr^{p-1} = pr^{p-1}\frac{x}{\sqrt{x^2+y^2+z^2}} = pr^{p-1}\hat{r}_x$$

Here $\hat{r} = \frac{\vec{r}}{r} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$

The same applied to $\frac{\partial}{\partial y,z}r^p = pr^{p-1}\hat{r}_{y,z}$. Hence $\vec{\nabla}r^p = pr^{p-1}\hat{r}$.

3. The electric potential of a point charge Q fixed at the origin can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Calculate $-\vec{\nabla}V$ and check it is the same as the electric field $\vec{E}$ of a point charge.

Sol: Use $\vec{\nabla}r^p = pr^{p-1}\hat{r}$ for $p = -1$:

$$-\vec{\nabla}V = -\frac{Q}{4\pi\varepsilon_0}\vec{\nabla}r^{-1} = \frac{Q}{4\pi\varepsilon_0}r^{-2}\hat{r} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

You can also calculate directly the components of $\vec{\nabla}V$:

$$\frac{\partial V}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{2}2x(x^2+y^2+z^2)^{-\frac{3}{2}} = -\frac{Q}{4\pi\varepsilon_0 r^2}\frac{x}{r}$$

The same is true for the other two components:

$$-\vec{\nabla}V = -\left(\frac{\partial V}{\partial x},\frac{\partial V}{\partial y},\frac{\partial V}{\partial z}\right) = \frac{Q}{4\pi\varepsilon_0 r^2}\frac{1}{r}(x,y,z) = \frac{Q}{4\pi\varepsilon_0 r^2}\frac{\vec{r}}{r} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

4. The electric field $\vec{E}$ of a point charge Q fixed at the origin can be written as:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

A. Calculate $\vec{\nabla}\times\vec{E}$.

B. Calculate $\vec{\nabla}\cdot\vec{E}$ and show it is equal to zero except at the origin $r = 0$.

Sol:

A. $\vec{\nabla}\times\vec{E} = \frac{Q}{4\pi\varepsilon_0}\vec{\nabla}\times\frac{1}{r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0}\vec{\nabla}\times\frac{1}{r^3}\vec{r} = \frac{Q}{4\pi\varepsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right)\times\vec{r} + \frac{1}{r^3}\vec{\nabla}\times\vec{r}\right]$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-3r^{-4}\hat{r}\times\vec{r} + \frac{1}{r^3}\vec{\nabla}\times\vec{r}\right]$$

Here we use the formula in problem 2. The first term equals zero.

The second term: $\vec{\nabla}\times\vec{r} = \left(\frac{\partial y}{\partial z} - \frac{\partial z}{\partial y}, \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}, \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right) = 0.$

$$\vec{\nabla} \times \vec{E} = 0$$

You can also calculate directly the components of $\vec{\nabla} \times \vec{E}$:

$$\left(\vec{\nabla} \times \vec{E}\right)_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{Q}{4\pi\varepsilon_0}\left[y\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - x\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right)\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[y\left(\frac{3x}{r^5}\right) - x\left(\frac{3y}{r^5}\right)\right] = 0$$

Same for the other two components.

B. $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right) \cdot \vec{r} + \frac{1}{r^3}\vec{\nabla} \cdot \vec{r}\right]$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-3r^{-4}\hat{r} \cdot \vec{r} + \frac{1}{r^3}\vec{\nabla} \cdot \vec{r}\right] == \frac{Q}{4\pi\varepsilon_0}\left[\frac{-3}{r^3} + \frac{3}{r^3}\right] = 0$$

Here we use $\vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$\vec{\nabla} \cdot \vec{E} = 0$ except for $r = 0$, where the two terms diverge $\rightarrow \infty$.

You can also calculate directly the components of $\vec{\nabla} \times \vec{E}$:

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\left[\left(\frac{3}{r^3}\right) - \frac{3}{r^5}(x^2 + y^2 + z^2)\right] = 0$$

5. You would have found in problem 3 that $\vec{\nabla} \cdot \vec{E}$ is infinite at $r = 0$. This divergence can be avoided if we replace an infinitely small point charge with a small sphere of radius R and constant charge density ρ. We learned from general physics outside the sphere $\vec{E}$ is just like outside a point charge but inside the sphere the electric field equals:

$$\vec{E} = \frac{\rho}{3\varepsilon_0}r\hat{r}$$

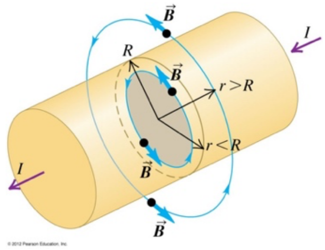
Calculate $\vec{\nabla} \cdot \vec{E}$ again and check it is consistent with Gauss's law.

Sol:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{\rho}{3\varepsilon_0}r\hat{r} = \frac{\rho}{3\varepsilon_0}\vec{\nabla} \cdot \vec{r} = \frac{\rho}{3\varepsilon_0}$$

$\vec{\nabla} \cdot \vec{E}$ is a costant for constant charge distribution.

6. The magnetic field inside a cylindrical current along the z axis with radius R and constant current density j can be written as ($r < R$):



$$\vec{B} = \frac{\mu_0 j}{2}(y, -x, 0)$$

Calculate $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$ and check it is consistent with Maxwell Equations.

Sol:

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0 j}{2}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0 j}{2}\left(-\frac{\partial x}{\partial z}, -\frac{\partial y}{\partial z}, \frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}\right) = \frac{\mu_0 j}{2}(0,0,2) = (0,0,\mu_0 j)$$