習題四

1. Diagonalization of a matrix: We have calculated in class that the eigenvectors of the matrix $\boldsymbol{A} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ are $c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 10$. The diagonalizing matrix can be written as: $\boldsymbol{U} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

   A. Calculate $\boldsymbol{U}^{-1}$ using $\boldsymbol{U^{-1}} = \frac{1}{\det \boldsymbol{U}} \begin{pmatrix} \boldsymbol{U_{22}} & -\boldsymbol{U_{12}} \\ -\boldsymbol{U_{21}} & \boldsymbol{U_{11}} \end{pmatrix}$.

   B. Calculate $\boldsymbol{A} \cdot \boldsymbol{U}$ and $\boldsymbol{U} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and show they are equal.

   C. Check $\boldsymbol{U}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{U} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

2. Find the eigenvalues and eigenvectors (normalized to length one) of the following matrices:

   A. $\boldsymbol{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$

   B. $\boldsymbol{C} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$

   C. You can find the eigenvectors are the same as

   $$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

   which we discussed in detail in class. And checking the corresponding eigenvalues, we will find that $\boldsymbol{B} = \boldsymbol{A}^m$ and $\boldsymbol{C} = \boldsymbol{A}^n$. Find m, n.

   D. Find the matrix $\boldsymbol{U}$ that will diagonalize $\boldsymbol{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$:

   $$\boldsymbol{U}^{-1} \cdot \boldsymbol{B} \cdot \boldsymbol{U} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   with $\lambda_{1,2}$ the two eigenvalues of $\boldsymbol{B}$.