

習題四

1. Diagonalization of a matrix: We have calculated in class that the eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ are $c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 10$. The diagonalizing matrix can be written as: $\mathbf{U} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

A. Calculate \mathbf{U}^{-1} using $\mathbf{U}^{-1} = \frac{1}{\det \mathbf{U}} \begin{pmatrix} \mathbf{U}_{22} & -\mathbf{U}_{12} \\ -\mathbf{U}_{21} & \mathbf{U}_{11} \end{pmatrix}$.

B. Calculate $\mathbf{A} \cdot \mathbf{U}$ and $\mathbf{U} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and show they are equal.

C. Check $\mathbf{U}^{-1} \cdot \mathbf{A} \cdot \mathbf{U} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Sol:

A. $\mathbf{U}^{-1} = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

B. $\mathbf{A} \cdot \mathbf{U} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix}$,

$$\mathbf{U} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix}$$

C. $\mathbf{U}^{-1} \cdot \mathbf{A} \cdot \mathbf{U} = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

2. Find the eigenvalues and eigenvectors (normalized to length one) of the following matrices:

A. $\mathbf{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$

B. $\mathbf{C} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$

C. You can find the eigenvectors are the same as

$$\mathbf{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

which we discussed in detail in class. And checking the corresponding eigenvalues, we will find that $\mathbf{B} = \mathbf{A}^m$ and $\mathbf{C} = \mathbf{A}^n$. Find m, n .

D. Find the matrix \mathbf{U} that will diagonalize $\mathbf{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$:

$$\mathbf{U}^{-1} \cdot \mathbf{B} \cdot \mathbf{U} = \mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_{1,2}$ the two eigenvalues of \mathbf{B} .

Sol:

A. $\mathbf{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$, 特徵方程式: $\begin{vmatrix} 0.7 - \lambda & 0.45 \\ 0.3 & 0.55 - \lambda \end{vmatrix} = 0$,

$$\lambda^2 - 1.25\lambda + 0.25 = 0, \text{ 本徵值 } \lambda_1 = 1, \lambda_2 = 0.25 = \frac{1}{4},$$

$$\lambda_1 = 1, \text{ 本徵向量滿足 } -0.3a_1 + 0.45a_2 = 0, \mathbf{a}_1 \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$$

$$\lambda_2 = 0.25, \text{ 本徵向量滿足 } 0.45a_1 + 0.45a_2 = 0, \mathbf{a}_2 \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

B. $\mathbf{C} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$, 特徵方程式: $\begin{vmatrix} 0.6 - \lambda & 0.6 \\ 0.4 & 0.4 - \lambda \end{vmatrix} = 0$

$$\lambda^2 - 1.0\lambda = 0, \text{ 本徵值 } \lambda_1 = 1, \lambda_2 = 0,$$

$$\lambda_1 = 1, \text{ 本徵向量滿足 } -0.3a_1 + 0.45a_2 = 0, \mathbf{a}_1 \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$$

$$\lambda_2 = 0, \text{ 本徵向量滿足 } 0.6a_1 + 0.6a_2 = 0, \mathbf{a}_2 \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

C. $\mathbf{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$ 的本徵值 $\lambda_1 = 1, \lambda_2 = \frac{1}{2}$,

$$\text{本徵向量: } \mathbf{a}_1 = c_1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}, \mathbf{a}_2 = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

可見 $\mathbf{A}, \mathbf{B}, \mathbf{C}$ 本徵向量相同, 根據本徵值: $\mathbf{B} = \mathbf{A}^2, \mathbf{C} = \mathbf{A}^\infty$ 。

D. $\mathbf{U} \equiv (\mathbf{a}_1 \quad \mathbf{a}_2) = \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix}$

$$\begin{aligned} \mathbf{U}^{-1} \cdot \mathbf{B} \cdot \mathbf{U} &= \begin{pmatrix} 1 & 1 \\ 0.4 & -0.6 \end{pmatrix} \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix} \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0.4 & -0.6 \end{pmatrix} \begin{pmatrix} 0.6 & 0.25 \\ 0.4 & -0.25 \end{pmatrix} = \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0.25 \end{pmatrix} \end{aligned}$$