Homework III

1.

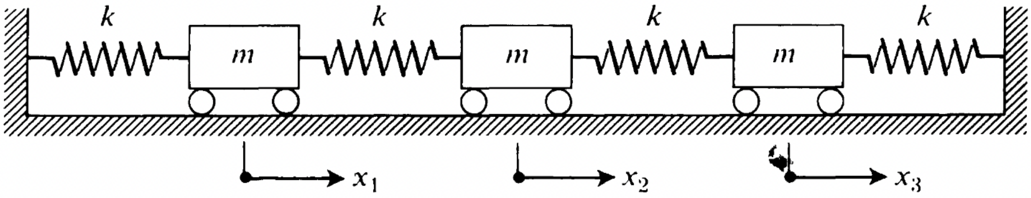


Figure 10.13

Consider the problem of three statically coupled masses (Fig. 10.13) analogous to those in Section 10.1, Example 1. Let x_1, x_2, and x_3 be their displacements from equilibrium positions.

a) Set up the equations of motion on the basis of Newton's second law.
b) Evaluate the potential energy of the system and show that it reduces to

$$V = kx_1^2 + kx_2^2 + kx_3^2 - kx_1x_2 - kx_2x_3.$$

d) Show that the characteristic frequencies can be obtained by solving

$$\det \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

e) Find the characteristic frequencies and the normal modes.

Hint:

$$m\frac{d^2x_1}{dt^2} = k(x_2 - x_1) - kx_1 = -2kx_1 + kx_2$$

$$m\frac{d^2x_2}{dt^2} = k(x_3 - x_2) - k(x_2 - x_1) = -2kx_2 + kx_3 + kx_1$$

$$m\frac{d^2x_3}{dt^2} = -k(x_3 - x_2) - kx_3 = -2kx_3 + kx_2$$

and the determinant of a 3×3 matrix is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

Sol:

A. 運動方程式：

$$\frac{d^2x_1}{dt^2} = \omega_0^2(x_2 - x_1) - \omega_0^2 x_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\frac{d^2x_2}{dt^2} = \omega_0^2(x_3 - x_2) - \omega_0^2(x_2 - x_1) = -2\omega_0^2 x_2 + \omega_0^2 x_3 + \omega_0^2 x_1$$

$$\frac{d^2x_3}{dt^2} = -\omega_0^2(x_3 - x_2) - \omega_0^2 x_3 = -2\omega_0^2 x_3 + \omega_0^2 x_2$$

B. 計算彈簧的彈力位能：

$$V = \frac{k}{2}[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2] = k(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

C. In matrix notation, the system of ODE's can be written as:

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \boldsymbol{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A} \cdot \boldsymbol{x}$$

Take $\boldsymbol{x} \equiv \boldsymbol{a}e^{i\omega t}$，Plug into the equation：$\boldsymbol{A} \cdot \boldsymbol{a} = \omega^2 \boldsymbol{a} \equiv \lambda\omega_0^2 \boldsymbol{a}$。

$$(\boldsymbol{A} - \lambda\omega_0^2 \boldsymbol{I}) \cdot \boldsymbol{a} = 0$$

The characteristic equation is $|\boldsymbol{A} - \lambda\omega_0^2 \boldsymbol{I}| = 0$ or

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

It is simplified to an algebraic equation for λ:

$$(2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)(2 - 4\lambda + \lambda^2) = 0$$

Solved as

$$\lambda = 2, 2 \pm \sqrt{2}, \text{ or } \omega^2 = 2\omega_0^2, \left(2 \pm \sqrt{2}\right)\omega_0^2$$

D. Compute the eigenvectors：

For $\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_{\boldsymbol{0}}$: Solve $(\boldsymbol{A} - \lambda\omega_0^2) \cdot \boldsymbol{a} = 0$

$$\omega_0^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Written explicitly

$$-a_2 = 0$$

$$-a_1 - a_3 = 0$$

$$-a_2 = 0$$

We find the eigenvector up to a constant:

$$\boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 2 - \sqrt{2}, \omega = \omega_1 \sim \sqrt{2 - \sqrt{2}}\omega_{\mathbf{0}}$:

$$\omega_0^2 \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Hence

$$\sqrt{2}a_1 - a_2 = 0$$

$$-a_1 + \sqrt{2}a_2 - a_3 = 0$$

$$-a_2 + \sqrt{2}a_3 = 0$$

$$\boldsymbol{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

For $\lambda = 2 + \sqrt{2}, \omega = \omega_3 \sim \sqrt{2 + \sqrt{2}}\omega_{\mathbf{0}}$:

$$\omega_0^2 \begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

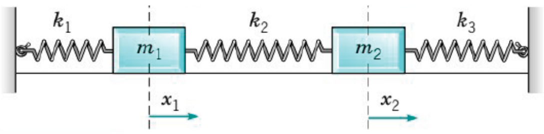
$$\sqrt{2}a_1 + a_2 = 0$$

$$a_1 + \sqrt{2}a_2 + a_3 = 0$$

$$a_2 + \sqrt{2}a_3 = 0$$

$$\boldsymbol{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

2. Consider a coupled oscillation of two particles as shown below:



with equations of motion:

$$\frac{d^2x_1}{dt^2} = -\frac{k_1 + k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2, \qquad \frac{d^2x_2}{dt^2} = \frac{k_2}{m_2}x_1 - \frac{k_2 + k_3}{m_2}x_2$$

which can be written in the notations of matrices:

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A} \cdot \boldsymbol{x}$$

Assume that the matrix $\boldsymbol{A}$ equals:

$$\boldsymbol{A} \equiv \omega_0^2 \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

The general solutions can be written as:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \cos(\omega_1 t + \phi_1) + c_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

A. Find the numbers $\omega_1, \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $\omega_2, \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$.

B. If the initial condition is $x_1(0) = a_m, x_2(0) = 0, x_1'(0) = x_2'(0) = 0$, find the solution. Hint: $\phi_1 = \phi_2 = 0$.

Sol:

A. Guess the solutions are $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$, $\boldsymbol{A} \cdot \boldsymbol{a} = \omega^2 \boldsymbol{a}$. This is eigenvalue problem of $\boldsymbol{A}$. The characteristic equation:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \omega_0^2 \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$\lambda = 1 \text{ or } \lambda_2 = 9$$

$$\omega_1 = \omega_0, \ \omega_2 = \sqrt{3}\omega_0$$

For $\omega_1 = \omega_0$, $(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a_1} = \omega_0^2 \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$, $\boldsymbol{a_1} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\omega_2 = \sqrt{3}\omega_0$, $(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a_2} = \omega_0^2 \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$, $\boldsymbol{a_2} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_0 t + \phi_1) + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{3}\omega_0 t + \phi_2)$$

B. $x_1(0) = c_1 + c_2 = a_m, x_2(0) = c_1 - c_2 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{a_m}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \omega_0 t + \frac{a_m}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \sqrt{3}\omega_0 t$$