Homework II

1. Find the general solution $x(t)$ of the equation of

$$x'' + 2x' + 5x = 0$$

using the damp oscillation formula we gave in class.

Sol: Guess the solution is :$z = z_0 e^{\alpha t}$ and plug it into the equation:

$$\alpha^2 e^{\alpha t} + 2\alpha e^{\alpha t} + 5e^{\alpha t} = 0$$

The unknown α satisfies the algebraic equation:

$$\alpha^2 + 2\alpha + 5 = 0$$

There are two real solutions:

$$\alpha = \frac{-2 \pm 4i}{2}$$

For $\alpha = \frac{-2+4i}{2}$, give us one complex solution:

$$z = z_0 e^{-t} e^{4it} = z_0 e^{-t} \cos 4t + i z_0 e^{-t} \sin 4t$$

The real part and the imaginary part are two real solutions. Note that $\alpha = \frac{-2-4i}{2}$ gives us the same real solutions.

Hence The general solutions are their linear combination:

$$x = c_1 e^{-t} \cos 4t + c_2 e^{-t} \sin 4t$$

2. Solve the equation of

$$x'' + 4x' + 3x = 0$$

And initial conditions: $x(0) = 1, x'(0) = 0$.

Sol: Guess the solution is :$z = z_0 e^{\alpha t}$ and plug it into the equation:

$$\alpha^2 e^{\alpha t} + 4\alpha e^{\alpha t} + 3e^{\alpha t} = 0$$

The unknown α satisfies the algebraic equation:

$$\alpha^2 + 4\alpha + 3 = 0$$

There are two real solutions:

$$\alpha = -1, -3$$

We get two solutions and the general solutions are their linear combination:

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

Put in the initial conditions: $x(0) = 1, x'(0) = 0$:

$$c_1 + c_2 = 1, c_1 + 3c_2 = 0$$

$$x = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

3. First consider the homogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 3x = 0$$

A. Find the general solutions x_{h} of this ODE.

B. Consider the inhomogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 3x = 12e^{-2t}$$

Find one solution. Hint: $x \propto e^{-2t}$ would work.

C. Find the general solutions x_{inh} of the inhomogeneous ODE.

D. Assume the initial condition $x(0) = 0, \frac{dx}{dt}(0) = 0$, find the solution $x(t)$.

Sol:

A. 取$x = ce^{\alpha t}$，代入微分方程式，得到α的代數方程式：$\alpha^2 + 3\alpha + 3 = 0$：

求解：$\alpha = \frac{-3 \pm i\sqrt{3}}{2}$，取$e^{\alpha t}$的實數部與虛數部，就能得到兩個實數解，取線性組合，就得到一般解：

$$x_{\mathrm{h}} = c_1 e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{3}}{2}t$$

B. 取$x = de^{-2t}$，代入微分方程式：$(4 - 6 + 3)de^{-2t} = 12e^{-2t}$，$d = 12$

得到一個特解：$x = 12e^{-2t}$

C. 一般解：

$$x_{inh} = c_1 e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{3}}{2}t + 12e^{-2t}$$

D. 代入起始條件$x(0) = c_1 + 12 = 0$，$c_1 = -12$。

$\frac{dx}{dt}(0) = -\frac{3}{2}c_1 + \frac{\sqrt{3}}{2}c_2 - 24 = 0$，$c_2 = \frac{12}{\sqrt{3}} = 4\sqrt{3}$。

$$x_{inh} = -12e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + 4\sqrt{3}e^{-3t/2}\sin\frac{\sqrt{3}}{2}t + 12e^{-2t}$$