# 習題五

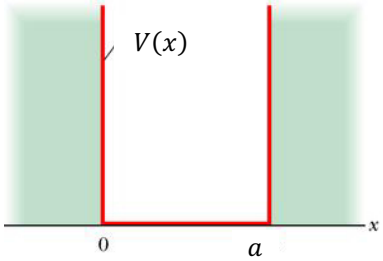
1. Consider the wavefunction of an electron at a certain moment:

$$\psi(x) = \sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}, 0 < x < a,$$

$$\psi(x) = 0, x > a, x < 0$$

This is the stationary state wavefunction of an electron located within an infinite potential box, with boundaries at $x = 0$ and $x = a$:

(The potential is $V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$. We will discuss this wavefunction in details in class later.)



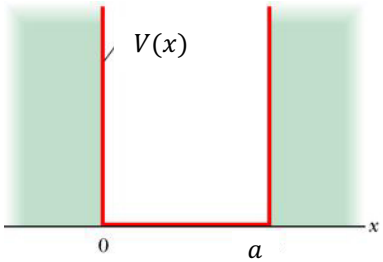

A. Prove that it is normalized: $\int_0^a |\psi(x)|^2 \cdot dx = 1$.(15)

B. Calculate the Probability of finding the electron between $0 < x < \frac{a}{12}$.(15)

Hint: You might need this formula: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

2. Consider an infinite potential box, with boundaries at $x = 0$ and $x = a$:

$V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$.




As we have shown in class, in this potential the energy eigenstate can be written as $\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ with eigenvalues $E_n = \left(\frac{\hbar^2}{2m}\right)\frac{\pi^2}{a^2}n^2$ (you can use the notation $E_n$ to simplify your answers) . Assume the wavefunction of a particle at $t = 0$ (probability already normalized to one) is:

$$\Psi(x,0) = \sqrt{\frac{4}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right) + \sqrt{\frac{1}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right) \quad 0 < x < a,$$

$$= 0 \quad x < 0, x > a$$

A. For a later time $t$, write down the wave function $\psi(x,t)$. There is no need to simplify the answer. (15)

B. At $x = \frac{a}{6}$, calculate the probability density $P\left(\frac{a}{6}, t\right)$ as a function of time $t$.
Hint: $|A+B|^2 = (A^* + B^*)(A+B) = |A|^2 + |B|^2 + A^*B + B^*A$

3. The Heat Equations is the equation for the temperature distribution in a heat conductivity thin ring. The ring is thin and its temperature depends only on $x$ $(0 < x < 2\pi R)$, the location along the circumference of the ring. Denote the temperature difference from a constant temperature $T_0$ at $x$ and time $t$ as $u(x,t)$. $R$ is the radius of the ring. Using the law of conductivity and the first law of thermodynamics, it can be shown that $u(x,t)$ satisfies the following partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t}$$



$\sigma$ is a fixed real positive constant. It is almost identical to the Schrodinger Wave Equation except that the coefficient $\sigma$ in front of $\frac{\partial u}{\partial t}$ is real now instead of imaginary. It is reasonable to think that there exist solutions that are separable just like in the wave equations we discussed in class:

$$u(x,t) = X(x) \cdot T(t)$$

Since it is a ring, the temperature satisfies a Periodic Boundary condition at all times:

$$u(t,0) = u(t,2\pi R)$$

This condition is widely used in solid state physics.

A. Find the Ordinary Differential Equation satisfied by $T(t)$. Assume that $u(x,t)$ will not grow indefinitely as $t \to \infty$. Show that the exponentially decreasing function:

$$T(t) = e^{-\frac{\lambda}{\sigma}t}$$

is a solution for $T(t)$. $\lambda$ is a positive constant to be determined.

B. Find the Ordinary Differential Equation satisfied by $X(x)$. This equation is identical to the time independent Schrodinger Equation for free space and hence the solution can be written as:

$$X(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x$$

Let's concentrate on $X(x) = D\sin\sqrt{\lambda}x$ and ignore the Cosine part for now. The periodic boundary condition would be satisfied if

$$X(x) = X(x + 2\pi R)$$

The condition for $\sqrt{\lambda}$ to satisfy the periodic condition is

$$\lambda_n = cn^2$$

Calculate the constant $c$. Write down the $u(x,t)$ for $\lambda_n$. Observe that it will vanish $u(x,t) \to 0$ as $t \to \infty$, meaning the temperature distribution on the ring will eventually become uniform ($= T_0$).

Afterword 後話: To put $\cos\sqrt{\lambda}x$ back into consideration, we need one more periodic condition: $X'(x) = X'(x + 2\pi R)$. Apparently, the first derivative is continuous along the whole ring. Then the condition for $\sqrt{\lambda}$ will be the same.