Homework IV

1. Calculate $\int_{-\infty}^{\infty} dy \cdot y^7 e^{-\alpha y^2}$.

   Hint: Put $\alpha y^2 = x$. $\Gamma(p) = \int_0^{\infty} dx \cdot x^{p-1} e^{-x}$

   Sol: $\alpha y^2 = x,\ \ 2\alpha y dy = dx$

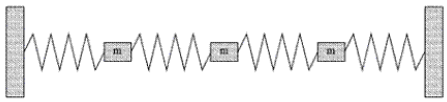
$$\int_{-\infty}^{\infty} dy \cdot y^7 e^{-\alpha y^2} = \int_0^{\infty} dy \cdot y^7 e^{-\alpha y^2} + \int_{-\infty}^{0} dy \cdot y^7 e^{-\alpha y^2}$$

$$= 2\int_0^{\infty} dy \cdot y^7 e^{-\alpha y^2} = \frac{1}{\alpha}\int_0^{\infty} dx \cdot x^3 e^{-x} = \frac{1}{\alpha}\Gamma(4) = \frac{1}{\alpha}3! = \frac{6}{\alpha}$$

2. Consider the spring system of $N = 3$ coupled masses. Assume that the spring constants are all equal to k and the masses are all equal to m. $\omega_0 = \sqrt{\frac{k}{m}}$.

   We move the second particle to the right by a and let go as all three particles are at rest, that is, the initial condition is $x_2(0) = a$, $x_1(0) = x_3(0) = 0$, $x_1'(0) = x_2'(0) = x_3'(0) = 0$.

   Solve $x_1(t)$. The solution can be simplified as $ax_{1m} \cos \alpha\omega_0 t \cos \beta\omega_0 t$. Find numerical x_{1m}, α, β.



The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

$\omega_1 \sim \sqrt{2-\sqrt{2}}\omega_0 \sim 0.77\omega_0$ $\quad \boldsymbol{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$\omega_2 \sim \sqrt{2}\omega_0 \sim 1.41\omega_0$ $\quad \boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\omega_3 \sim \sqrt{2+\sqrt{2}}\omega_0 \sim 1.85\omega_0$ $\quad \boldsymbol{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

Indeed, eigenvectors are orthogonal to each other.

$\boldsymbol{a}^{(2)T}\boldsymbol{a}^{(1)} = \boldsymbol{a}^{(2)} \cdot \boldsymbol{a}^{(1)} = 0$

We could choose the length of all eigenvectors to be equal to one.

$\boldsymbol{a}^{(m)T}\boldsymbol{a}^{(n)} = \boldsymbol{a}^{(m)} \cdot \boldsymbol{a}^{(n)} = \delta_{mn}$ They are called orthonormal.

You will see this orthogonality property is extremely important.

Linear Combinations of the 3 mode solutions are solutions, too.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

Determine 3 coefficients $f_{1,2,3}$ by 3 initial conditions $x_{1,2,3}(0)$. Plug in $t = 0$.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i$$

I have chosen $\boldsymbol{a}^{(i)}$ to be orthonormal. $\boldsymbol{a}^{(m)T}\boldsymbol{a}^{(n)} = \delta_{mn}$ Taking inner product:

$$\boldsymbol{a}^{(i)T}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{j=1}^{3} \boldsymbol{a}^{(i)T}\boldsymbol{a}^{(j)} f_j = \sum_{j=1}^{3} \delta_{ij} f_j = f_i$$

Inner product of $\boldsymbol{a}^{(i)}$ and initial column vector equals f_i:

$$\boldsymbol{a}^{(i)T}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

Hint: $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

Sol:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

$$\boldsymbol{a}^{(i)T}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

$$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} = f_1 = \frac{a}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}(1,0,-1)\begin{pmatrix}0\\a\\0\end{pmatrix} = f_2 = 0$$

$$\left(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2}\right)\begin{pmatrix}0\\a\\0\end{pmatrix} = f_3 = -\frac{a}{\sqrt{2}}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t)]$$

$$= \frac{a}{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\1\\\frac{1}{\sqrt{2}}\end{pmatrix}\cos(\omega_1 t) - \frac{a}{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-1\\\frac{1}{\sqrt{2}}\end{pmatrix}\cos(\omega_3 t)$$

$$x_1 = \frac{a}{2}\frac{1}{\sqrt{2}}(\cos 0.77\omega_0 t + \cos 1.85\omega_0 t) = \frac{a}{\sqrt{2}}\cos(2.62\omega_0 t)\cos(1.08\omega_0 t)$$

3. One of the first Schrodinger Wave Equations to be written down and solved is that of an electron in the simple harmonic potential

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{\hbar^2}{2m}\alpha^2 x^2\Psi$$

α is actually $\frac{m\omega}{\hbar} = \frac{\sqrt{mk}}{\hbar}$. But calculation would be easier to keep α, as above. $\hbar$ is the Planck Constant h divided by 2π. $\Psi(x,t)$ is the wave function. As you can see from the equation, the left-hand side contains an i and hence the wavefunction must be complex valued. It is still reasonable to think that there exist solutions that are separable just like in the classical wave equation we discussed in class:

$$\Psi(x,t) = \psi(x)\cdot\phi(t).$$

A. Find the Ordinary Differential Equations satisfied respectively by $\psi(x)$ and $\phi(t)$. Show that:

$$\phi(t) = e^{-i\frac{E}{\hbar}t}$$

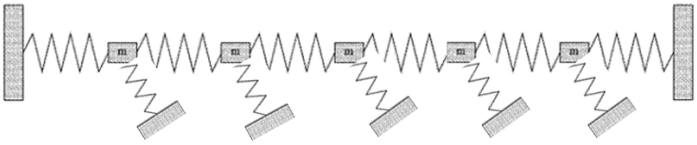
is the solution for $\phi(t)$. E is the separation constant to be determined.

B. Observe that $\psi(x)$ satisfies the same equation as the space part $X(x)$ of Problem 4 in Homework II.

4. Consider the following wave equation:

$$\frac{\partial^2\phi}{\partial t^2} = v^2\frac{\partial^2\phi}{\partial x^2} - V(x)\phi$$

Here we add a term $V(x)\phi$ to the right-hand side of the wave function we studied in class. $V(x) = v^2\alpha^2x^2$ is a known function. This could be done by adding a position dependent restoring force to the particles on the string.



It is reasonable to think that there exists solutions that are separable:

$$\phi(x,t) = X(x)\cdot T(t).$$

$\psi(x)$ would have solutions for a discreet set of infinite number of positive values

Prove that $u_0(x) = ce^{-\alpha\frac{x^2}{2}}$ is a solution for the equation. c is constant coefficient not yet determined. Write down the corresponding E (this is the second smallest E and as we said Ecorresponds to energy.) in terms of $\alpha, m, \hbar$.

Sol:

A. 將$\Psi(x,t) = \psi(x)\cdot\phi(t)$代入波方程式：

$$-\frac{\hbar^2}{2m}\phi(t)\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\phi(t) = i\hbar\psi(x)\frac{d\phi(t)}{dt}$$

左右都除以 $\psi(x)\cdot\phi(t)$ ：

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\right] = i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}$$

唯一可能是左右兩式與兩個變數都無關，是一常數。設為E。

在左邊只與x有關，右邊只與t有關，兩者是獨立變數！

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\right] = i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} \equiv E$$

$$i\hbar\frac{d\phi(t)}{dt} = E\phi(t)$$

$$\phi(t) = e^{-i\frac{E}{\hbar}t}$$

B. 空間部分$\psi(x)$滿足：

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\right] = E$$

也就是：

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{\hbar^2}{2m}\alpha^2x^2\psi(x)=E\psi(x)$$

代入$\psi(x)=e^{-\alpha\frac{x^2}{2}}$

$$-\frac{d^2\psi}{dx^2}+\alpha^2x^2\psi=\frac{2m}{\hbar^2}E\cdot\psi$$

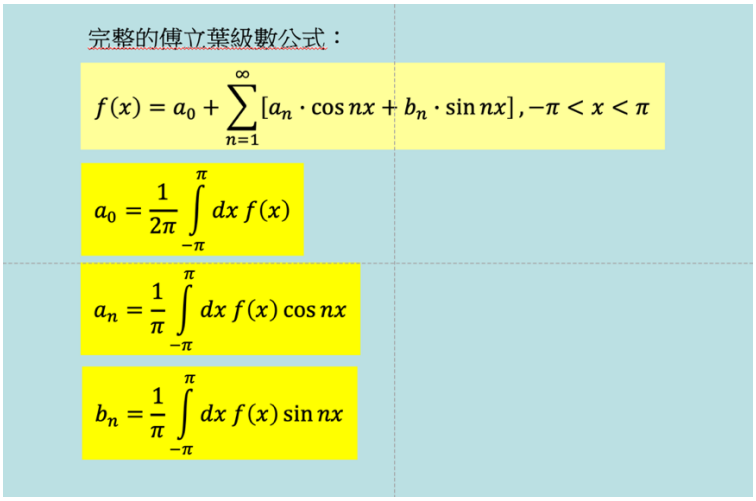
$$\alpha e^{-\alpha\frac{x^2}{2}}-(\alpha x)^2e^{-\alpha\frac{x^2}{2}}+\alpha^2x^2e^{-\alpha\frac{x^2}{2}}=\alpha e^{-\alpha\frac{x^2}{2}}=\frac{2m}{\hbar^2}E\cdot\psi$$

If $E=\frac{\hbar^2}{2m}\alpha$, $e^{-\alpha\frac{x^2}{2}}$ is a solution. The whole solution:

$$\Psi(x,t)=\psi(x)\cdot\phi(t)=e^{-\alpha\frac{x^2}{2}}e^{-i\frac{E}{\hbar}t}$$

$$E=\frac{\hbar^2}{2m}\alpha$$

4. Consider a periodic function of x of period 2π: $f(x)=1,\ 0<x<\pi$ and $f(x)=0,-\pi<x<0$. The Fourier series of $f(x)$ is as follows:



Calculate a_0 and a_3.

Sol: $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\,f(x)=\frac{1}{2\pi}\int_0^{\pi}dx\ =\frac{1}{2}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\,f(x)\sin(nx)=\frac{1}{\pi}\int_0^{\pi}dx\ \sin(nx)$$

$$=-\frac{1}{\pi}\left[\frac{1}{n}[\cos(nx)]|_0^{\pi}\right]=-\frac{1}{n\pi}[\cos(n\pi)-1]=\frac{2}{n\pi}$$

$$= \frac{2}{n\pi} \quad n \text{ is odd}$$

$$= 0 \quad n \text{ is even}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\cos(nx) = \frac{1}{\pi}\int_{0}^{\pi} dx\ \cos(nx) + \frac{1}{\pi}\int_{-\pi}^{0} dx\ \cos(nx)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}[\sin(nx)]|_0^{\pi} + \frac{1}{n}[\sin(nx)]|_{-\pi}^{0}\right] = 0$$