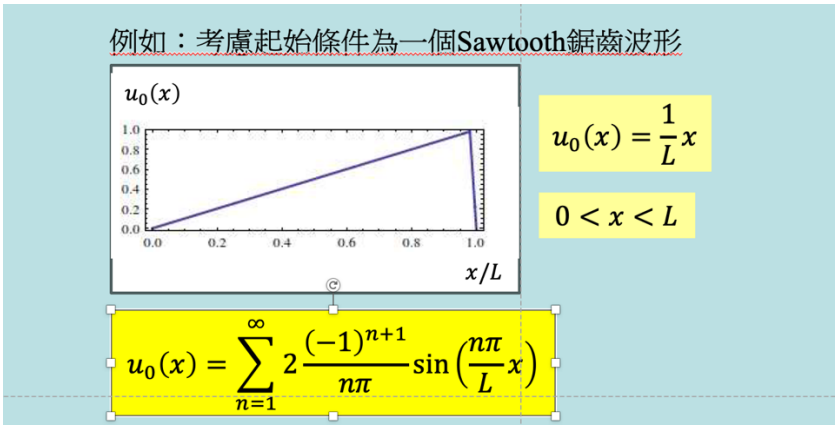
Homework III

1. Consider a string with fixed ends of length L. The initial condition is a Sawtooth wave: $\phi(x,0) = u_0(x)$ as we studied in class and initial velocities are zero: $v_0(x) = 0$. The following is the Fourier Series expansion of $u_0(x)$.



Calculate numerically wavefunction $\phi(x,t)$ and its time derivative $\frac{\partial\phi}{\partial t}$ at $x = \frac{L}{4}, t = \frac{L}{4v}$, using Fourier series with $n = 1,2,\cdots 8$ as an approximation.

Sol:

From the formula, the coefficients are:

$$A_n = \sqrt{2L}\frac{(-1)^{n+1}}{n\pi}$$

Plug it into the Bernoulli's solution:

$$u_0(x) = \sum_{n=1}^{\infty} A_n\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{L}x\right)$$

Adding the leading eight terms:

$$\phi(x,t) \sim \sum_{n=1}^{8} A_n\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)\cdot\cos\omega_n t = \sum_{n=1}^{8} 2\frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\cdot\cos\left(v\frac{n\pi}{L}t\right)$$

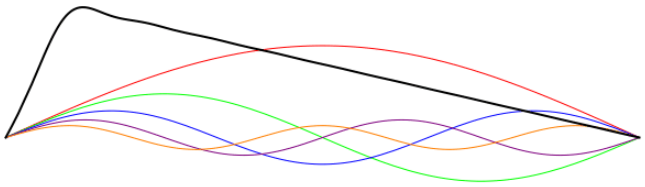
$$\phi\left(\frac{L}{4},\frac{L}{4v}\right) \sim \sum_{n=1}^{8} 2\frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{4}\right)\cdot\cos\frac{n\pi}{4} = \frac{1}{\pi} - \frac{1}{3\pi} + \frac{1}{5\pi} - \frac{1}{7\pi} = \frac{0.72}{\pi}$$

The time partial will give the velocity of the string:

$$\frac{\partial\phi}{\partial t} = -\sum_{n=1}^{8} 2\frac{(-1)^{n+1}}{n\pi}v\frac{n\pi}{L}t\cdot\sin\left(\frac{n\pi}{L}x\right)\cdot\sin\left(v\frac{n\pi}{L}t\right)$$

$$\frac{\partial \phi}{\partial t}\left(\frac{L}{4},\frac{L}{4v}\right)=\sum_{n=1}^{8}(-1)^{n+2}\frac{2v}{L}\cdot\sin^2\left(\frac{n\pi}{4}\right)=\frac{v}{L}(-1+2-1-1+2-1)=0$$

This is a bit ambiguous since the series does not quite converge. The reason is that at the end point $x = L$ the function $f(x)$ is not differentiable. Therefore, the derivative of its Fourier series is not convergent. However, the answer turns out to be correct. Look at a similar plot to sawtooth wave below. Most points on the string indeed do not move at all most of the time.



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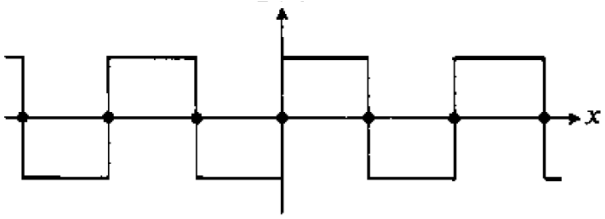
2. Consider a periodic function of x of period 2π: $f(x) = 1,\ 0 < x < \pi$ and $f(x) = -1, -\pi < x < 0$.



Write down the Fourier series of $f(x)$:

完整的傅立葉級數公式：

$$f(x)=a_0+\sum_{n=1}^{\infty}[a_n\cdot\cos nx+b_n\cdot\sin nx],-\pi<x<\pi$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\,f(x)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\,f(x)\cos nx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\,f(x)\sin nx$$

Sol:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\sin(nx) = \frac{1}{\pi}\int_{0}^{\pi} dx\ \sin(nx) + \frac{1}{\pi}\int_{-\pi}^{0} dx\ \sin(nx)$$

$$= -\frac{1}{\pi}\left[\frac{1}{n}[\cos(nx)]|_0^{\pi} + \frac{1}{n}[\cos(nx)]|_{-\pi}^{0}\right] = -\frac{2}{n\pi}[\cos(n\pi) - 1] = \frac{2}{n\pi}$$

$$= \frac{4}{n\pi} \quad n \text{ is odd}$$

$$= 0 \quad n \text{ is even}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\cos(nx) = \frac{1}{\pi}\int_{0}^{\pi} dx\ \cos(nx) + \frac{1}{\pi}\int_{-\pi}^{0} dx\ \cos(nx)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}[\sin(nx)]|_0^{\pi} + \frac{1}{n}[\sin(nx)]|_{-\pi}^{0}\right] = 0$$

$$f(x) = \frac{4}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right]$$

$$f(x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\left[\frac{1}{2m-1}\cdot\sin(2m-1)x\right]$$

