

Homework II

1. A field $\vec{E}$ equals :

$$\vec{E} = (x^2 + y^2 + z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

Calculate $\vec{\nabla} \cdot \vec{E}$

$$\text{Sol: } \frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x} [x(x^2 + y^2 + z^2)] = (x^2 + y^2 + z^2) + 2x^2$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 3(x^2 + y^2 + z^2) + 2(x^2 + y^2 + z^2) = 5r^2$$

2. A field $\vec{E}$ equals :

$$\vec{E} = (x^2 + yz^2)\hat{i} + (2x - y^3)\hat{j}$$

Calculate $\vec{\nabla} \times \vec{E}$.

Along a circle $(x - 2)^2 + (y - 3)^2 = 9, z = 0$, calculate $\oint \vec{E} \cdot d\vec{s}$ using $\vec{\nabla} \times \vec{E}$.

Sol:

$$(\vec{\nabla} \times \vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

$$(\vec{\nabla} \times \vec{E})_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 2yz$$

$$(\vec{\nabla} \times \vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 2 - z^2 = 2 - z^2$$

According to Stokes' Law: $\oint \vec{E} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$. The surface is on x-y plane:

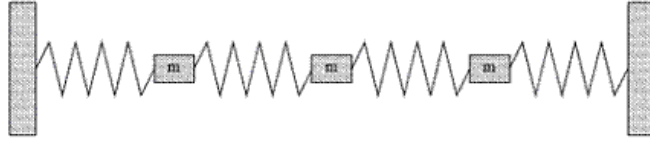
$$d\vec{a} = da \cdot \hat{k}$$

$$\int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \int (\vec{\nabla} \times \vec{E})_z da = \int 2da = 2A$$

3. Consider the spring system of $N = 3$ coupled masses. Assume that the spring

constants are all equal to k and the masses are all equal to m . $\omega_0 = \sqrt{\frac{k}{m}}$.

If we move the first particle to the right by a and let go as all three particles are at rest, that is, the initial condition is $x_1(0) = a, x_2(0) = x_3(0) = 0, x_1'(0) = x_2'(0) = x_3'(0) = 0$, solve the $x_1(t)$. Write the solution in terms of a, ω_0 .



有了以上三個模式的解，一般解將是三者的線性組合：

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

$$\omega_1 \sim \sqrt{2 - \sqrt{2}} \omega_0$$

$$\omega_2 \sim \sqrt{2} \omega_0$$

$$\omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

三個未定常數 $f_{1,2,3}$ 將由三個起使條件 $x_{1,2,3}(0)$ 決定：

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} f_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

根據對稱矩陣本徵向量的展開定理，三個本徵向量彼此正交： $\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \delta_{mn}$

可以如上選 $\mathbf{a}^{(n)}$ 使其向量長度為 1。

任一起始條件的行向量 $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$ ，都可以展開成三個本徵向量的線性組合。

$f_{1,2,3}$ 可以解出：

$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

另外三個未定常數 $g_{1,2,3}$ 則由另外三個起使條件 $x'_{1,2,3}(0)$ 決定。

Sol:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

$$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2} \right) \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_1 = \frac{a}{2}$$

$$\frac{1}{\sqrt{2}} (1, 0, -1) \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_2 = \frac{1}{\sqrt{2}} a$$

$$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_3 = \frac{a}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} \cdot [f_i \cos(\omega_i t)]$$

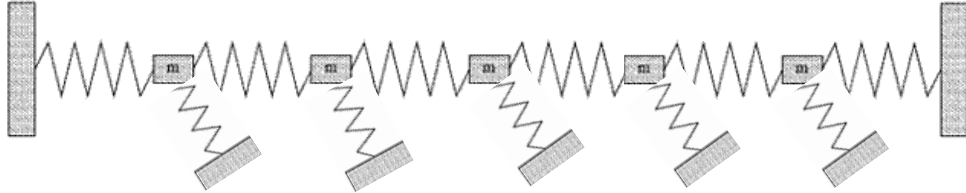
$$= \frac{a}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cos(\omega_1 t) + a \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\omega_2 t) + \frac{a}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cos(\omega_3 t)$$

$$x_1 = \frac{a}{2} \left(\frac{1}{2} \cos \sqrt{2 - \sqrt{2}} \omega_0 t + \cos \sqrt{2} \omega_0 t + \frac{1}{2} \cos \sqrt{2 + \sqrt{2}} \omega_0 t \right)$$

4. Consider the following wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} - V(x) \phi$$

Here we add a term $V(x)\phi$ to the right-hand side of the wave function we studied in class. $V(x) = v^2 \alpha^2 x^2$ is a known function. This could be done by adding a position dependent restoring force to the particles on the string.



It is reasonable to think that there exists solutions that are separable:

$$\phi(x, t) = X(x) \cdot T(t).$$

A. Find the Ordinary Differential Equation satisfied by $X(x)$ and $T(t)$. Show that $T(t)$ satisfies the ODE of a simple harmonic oscillator and the solution can be written as:

$$T(t) = a_m \cos(\omega t + \phi) \sim f \cos(\omega t) + g \sin(\omega t)$$

B. Check that $e^{-\alpha \frac{x^2}{2}}$ is a solution for $X(x)$. Write down the corresponding $T(t)$. What is the value of ω for this solution?

Sol:

A. Plug $\phi(x, t) = X(x) \cdot T(t)$ into $\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} - V(x)\phi$:

$$X \frac{\partial^2 T}{\partial t^2} = T v^2 \frac{\partial^2 X}{\partial x^2} - V X T$$

Divide the whole equation by $v^2 X T$:

$$\frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} - \frac{V}{v^2} = \lambda$$

It can be equal to a constant λ . Therefore the equation for T :

$$\frac{d^2 T}{dt^2} = v^2 \lambda \cdot T \equiv -\omega^2 T$$

This is the same equation as a SHM for $\omega = v\sqrt{-\lambda}$. The solution is the same as a SHM:

$$T(t) = a_m \cos(\omega t + \phi) \sim f \cos(\omega t) + g \sin(\omega t)$$

The equation for X :

$$\frac{d^2 X}{dx^2} - \frac{V}{v^2} X = \frac{d^2 X}{dx^2} - \alpha^2 x^2 X = \lambda \cdot X$$

B. Plug $e^{-\alpha \frac{x^2}{2}}$ into

$$\frac{d^2 X}{dx^2} - \alpha^2 x^2 X = \lambda \cdot X$$

$$-\alpha e^{-\alpha \frac{x^2}{2}} + (\alpha x)^2 e^{-\alpha \frac{x^2}{2}} - \alpha^2 x^2 e^{-\alpha \frac{x^2}{2}} = \lambda e^{-\alpha \frac{x^2}{2}}$$

If $\lambda = -\alpha$, $e^{-\alpha \frac{x^2}{2}}$ is a solution of X . The whole solution:

$$\phi(x, t) = X(x) \cdot T(t) = e^{-\alpha \frac{x^2}{2}} [f \cos(\omega t) + g \sin(\omega t)]$$

$$\omega = v\sqrt{\alpha}$$