數物 I 期末考

June 2026

1. The electron spin 自旋 corresponds to a 2×2 matrix. When the spin is along the direction $\hat{n} = (\sin\theta, 0, \cos\theta)$, the matrix is: $S_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Find the eigenvalues of the matrix $\frac{S_n}{\frac{\hbar}{2}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and the eigenvector (normalized to length one) that corresponds to the positive eigenvalue. (20)

Sol: 特徵方程式：$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$，$\lambda^2 - 1 = 0$，$\lambda = \pm 1$

For $\lambda = 1$: $(\cos\theta - 1)a_1 + \sin\theta\, a_2 = 0$

$$\boldsymbol{a} = \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix} \frac{1}{\sqrt{2 - 2\cos\theta}} = \begin{pmatrix} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\sin\frac{\theta}{2} \end{pmatrix} \frac{1}{2\sin\frac{\theta}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

2. We mentioned in class that through the diagonalization of a matrix, we can represent a matrix by its eigenvectors and eigenvalues:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^{-1}$$

$$\boldsymbol{U} \equiv \begin{pmatrix} \boldsymbol{a}^{(1)} & \boldsymbol{a}^{(2)} \end{pmatrix}$$

With any one column vector $\boldsymbol{a}$ and any two numbers λ_1, λ_2 we can construct a matrix which has $\boldsymbol{a}$ as one of the eigenvectors and λ_1, λ_2 as eigenvalues.

Let's start with $\boldsymbol{a}^{(1)} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. (30)

A. For a symmetric matrix, eigenvectors are orthogonal to each other. Use this property to find a normalized $\boldsymbol{a}^{(2)}$ such that $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(1)} = 0, \boldsymbol{a}^{T(2)}\boldsymbol{a}^{(2)} = 1$.

B. Construct $\boldsymbol{U}$ and calculate $\boldsymbol{U}^{-1}$ using $\boldsymbol{U}^{-1} = \frac{1}{\det \boldsymbol{U}}\begin{pmatrix} \boldsymbol{U}_{22} & -\boldsymbol{U}_{12} \\ -\boldsymbol{U}_{21} & \boldsymbol{U}_{11} \end{pmatrix}$.

C. Choose the two eigenvalues as $\lambda_1 = 10, \lambda_2 = 20$. Find the matrix $\boldsymbol{A}$.

D. Check indeed $\boldsymbol{A} \cdot \boldsymbol{a}^{(1)} = \lambda_1 \boldsymbol{a}^{(1)}$.

Sol:

A. Assume that $\boldsymbol{a}^{(2)} = \begin{pmatrix} b \\ c \end{pmatrix}$. Since $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(1)} = 0$, $b - 2c = 0$. Hence $\boldsymbol{a}^{(2)} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Using $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(2)} = 1$, $\boldsymbol{a}^{(2)} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

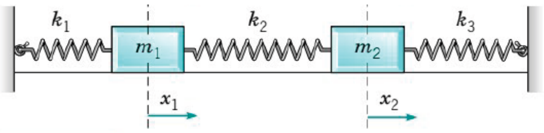
B. $\boldsymbol{U} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \boldsymbol{U}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

C. $\boldsymbol{A} = \boldsymbol{U\Lambda U^{-1}} = \boldsymbol{U}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\boldsymbol{U}^{-1} = \frac{1}{5}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}10 & 0\\ 0 & 20\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$

$$\frac{1}{5}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}10 & -20\\ 40 & 20\end{pmatrix} = \frac{1}{5}\begin{pmatrix}90 & 20\\ 20 & 60\end{pmatrix} = \begin{pmatrix}18 & 4\\ 4 & 12\end{pmatrix}$$

D. $\begin{pmatrix}18 & 4\\ 4 & 12\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\ -2\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}10\\ -20\end{pmatrix} = 10 \times \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ -2\end{pmatrix}$

3. Consider a coupled oscillation of two particles as shown below (all quantities are in SI unit):



with equations of motion:

$$\frac{d^2x_1}{dt^2} = -\frac{k_1 + k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2, \qquad \frac{d^2x_2}{dt^2} = \frac{k_2}{m_2}x_1 - \frac{k_2 + k_3}{m_2}x_2$$

which can be written in the notations of matrices:

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

Assume that $m_1 = m_2 = m$ and

$$\frac{k_2}{m} = 0.5 \ll \frac{k_1}{m} = \frac{k_3}{m} = 4$$

The matrix $\boldsymbol{A}$ equals:

$$\boldsymbol{A} \equiv \begin{pmatrix}4.5 & 0.5\\ 0.5 & 4.5\end{pmatrix}$$

The general solutions can be written as:

$$\boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = c_1\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}\cos(\omega_1 t + \phi_1) + c_2\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}\cos(\omega_2 t + \phi_2)$$

A. Find the numbers $\omega_1, \begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}$ and $\omega_2, \begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}$. (15)
Hint: These are, of course, just eigenvalues and eigenvectors of matrix $\boldsymbol{A}$.

$$4.5^2 = 20.25, \sqrt{5} \sim 2.2$$

B. If the initial condition is $x_1(0) = 0, x_2(0) = 2, x_1'(0) = x_2'(0) = 0$, find the solution $x_1(t), x_2(t)$. (8) Hint: $\phi_1 = \phi_2 = 0$.

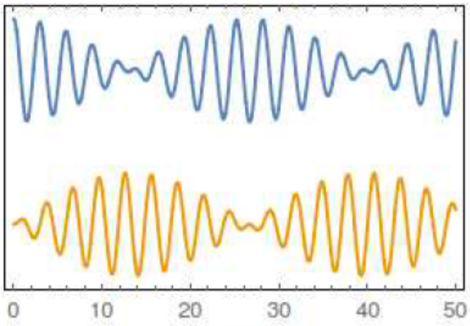
C. The solutions can be written as:

$$x_1(t) = c \cdot \sin\omega_{\text{fast}}t \cdot \sin\omega_{\text{slow}}t$$

$$x_2(t) = c \cdot \cos\omega_{\text{fast}}t \cdot \cos\omega_{\text{slow}}t$$

with $\omega_{\text{fast}} \gg \omega_{\text{slow}}$. Find constant c, ω_{fast} and ω_{slow}. (7)

Hint: $\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}, \cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$



$x_1(t)$ is the second function. Oscillation is alternating between the two particles. This phenomenon is called beat.

Sol:

A. Guess the solutions are $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$, $\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$**.** This is eigenvalue problem of $\boldsymbol{A}$**.**

The characteristic equation:

$$\mathbf{det}\,(\boldsymbol{A} - \lambda\boldsymbol{I}) = \mathbf{det}\begin{bmatrix} 4.5-\lambda & 0.5 \\ 0.5 & 4.5-\lambda \end{bmatrix} = \lambda^2 - 9\lambda + 20 = 0$$

$$\lambda = \omega^2 = \lambda_1 = \omega_1^2 = 4 \text{ or } \lambda_2 = \omega_2^2 = 5$$

$$\omega_1 = 2,\ \omega_2 = 2.2$$

For $\omega_1 = 2$, $(\boldsymbol{A} - \lambda\boldsymbol{I})\cdot\boldsymbol{a_1} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}\cdot\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$, $\boldsymbol{a_1} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\omega_2 = 2.2$, $(\boldsymbol{A} - \lambda\boldsymbol{I})\cdot\boldsymbol{a_2} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}\cdot\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$, $\boldsymbol{a_2} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2t + \phi_1) + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(2.2t + \phi_2)$$

B. $x_1(0) = c_1 + c_2 = 0, x_2(0) = -c_1 + c_2 = 2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2.2t$$

C. $x_1 = -\cos(2t) + \cos(2.2t) = -2\sin 0.1t \sin 2.1t$ or $2\sin(-0.1t)\sin 2.1t$

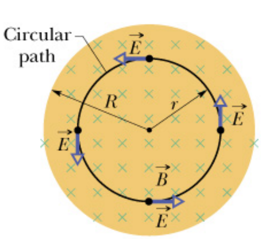
$c = 2$ or -2, $\omega_{\text{fast}} = 2.1$ and $\omega_{\text{slow}} = 0.1$ or -0.1.

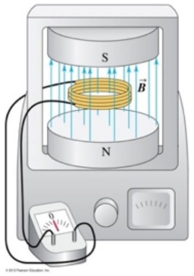
I'll give full credit to both answers.

4. The Maxwell Equation can be written as:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

A. We can generate a cylindrically symmetric time changing magnetic field pointing in the vertical z direction using the setup as below. The induced electric field in the area with magnetic field can be written as $\vec{E} = E_0(y, -x, 0)$.

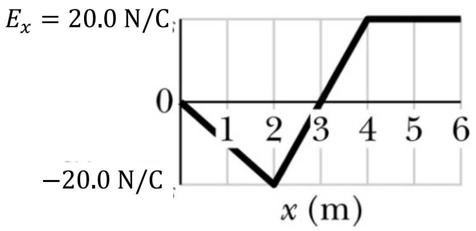




Calculate $\frac{\partial \vec{B}}{\partial t}$, using $\vec{\nabla} \times \vec{E}$. (10)

Hint: $\vec{\nabla} \times \vec{E} \equiv \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$

B. In a space with charge distribution density $\rho(x)$, the direction of the electric field is along the x axis. Its magnitude is independent of y, z and can be written as $E_x(x)$. $E_x(x)$ as a function of x is shown below:



Calculate the charge density $\rho(x = 3.0\text{ m})$ (in $\frac{\mathrm{C}}{\mathrm{m}^3}$), using $\vec{\nabla} \cdot \vec{E}$. (10)

Hint: Though $E_x(3.0) = 0$, its derivative is not zero.

Sol:

A. Use Faraday's Law

$$\vec{\nabla} \times \vec{E} \equiv \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= \left(0, 0, \frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}\right) = (0,0,2)$$

$$\frac{\partial \vec{B}}{\partial t} = (0,0,2)$$

B. Use Gauss's Law

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_x}{\partial x} = 20 \text{ C/m}^3$$

$\frac{\partial E_x}{\partial x}$ is the slope of the curve.