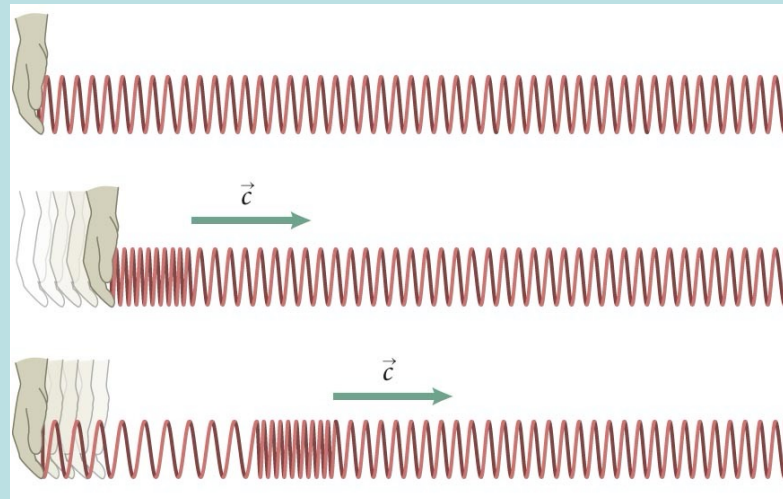
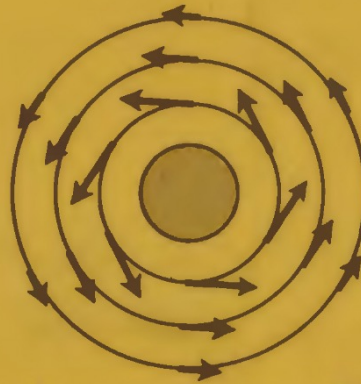


Wave Equation for continuous medium

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$



MATHEMATICAL PHYSICS



Eugene Butkov

Undergraduate Texts in Mathematics

UTM

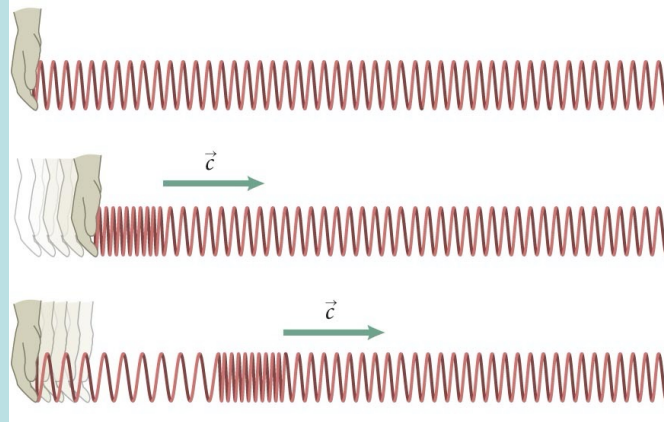
Peter J. Olver

Introduction to Partial Differential Equations

 Springer

Wave Equation

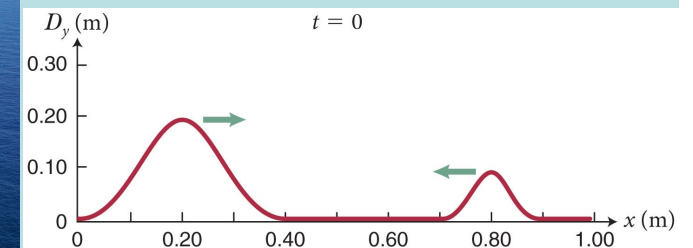
$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$



If the size of the medium is infinite or its boundaries are far away,
we have already d'Alembert solution of a **travelling or propagating** wave from :

$$\phi(x, t) = f(x - vt) + g(x + vt)$$

行進傳播



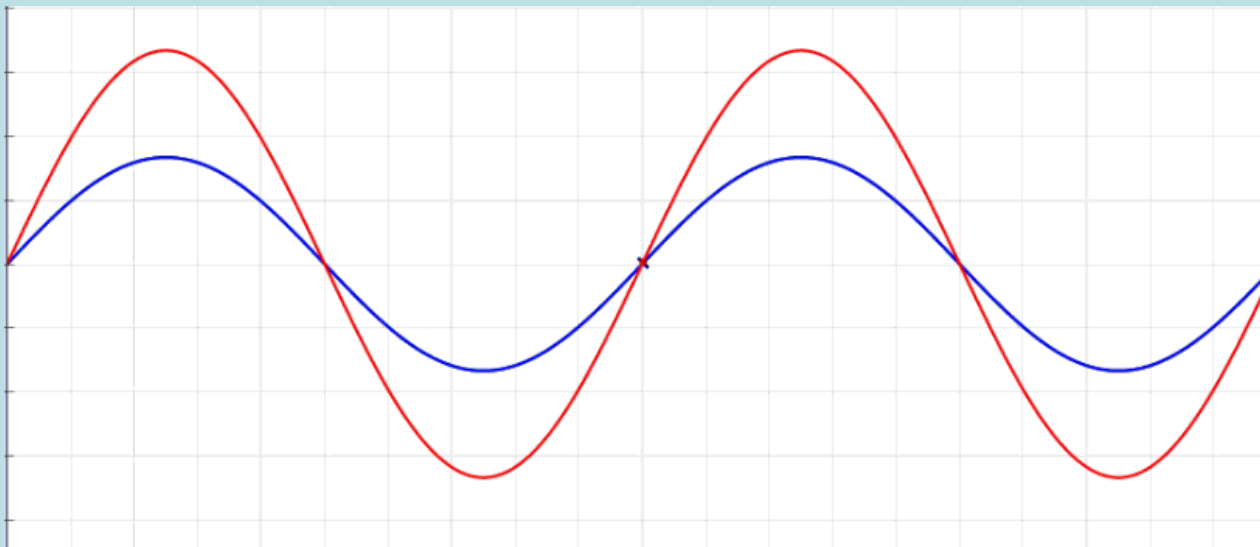


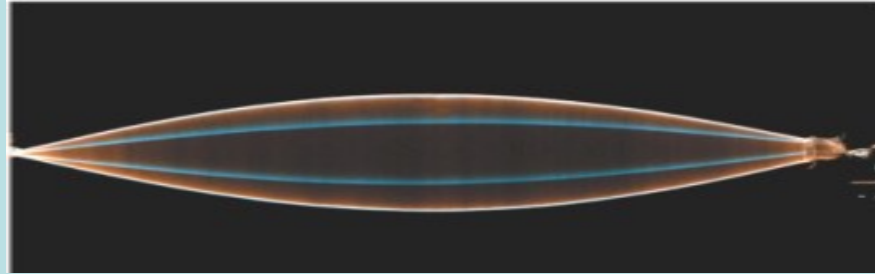
If the size of the medium is finite, its boundaries are nearby.

Reflection from boundaries would make wave behave more like **vibrations**!

It resembles a system of coupled springs!

振盪

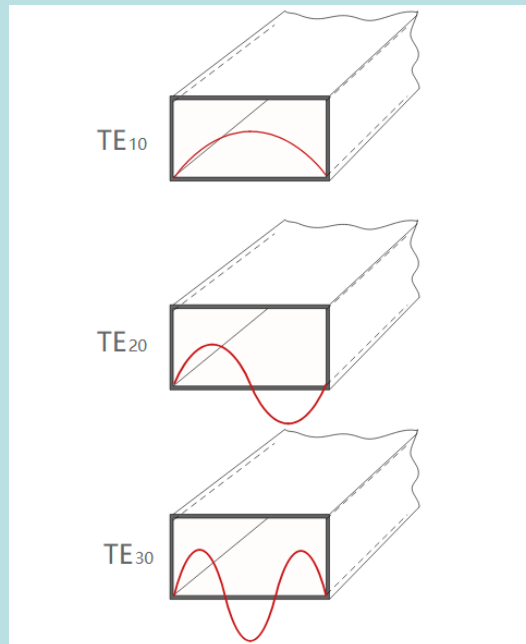




The medium with finite size usually comes with Boundary Condition 邊界條件, such as:

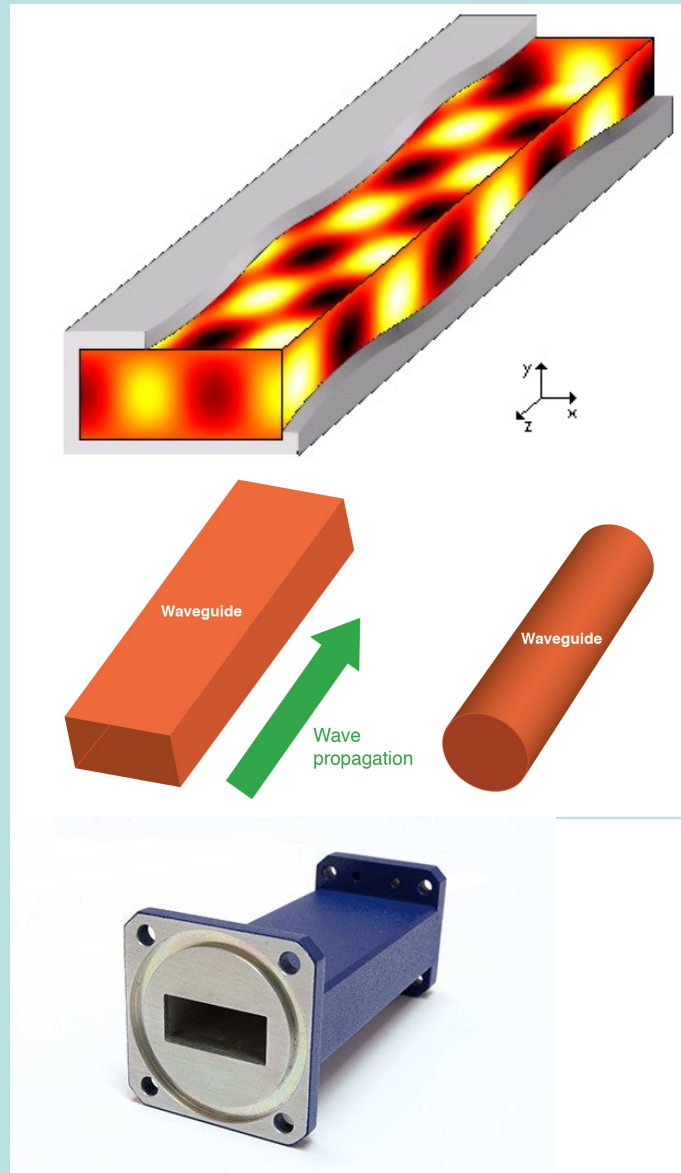
$$\phi(0, t) = \phi(L, t) = 0$$

Conducting boundaries impose the same condition on $\vec{E}$.



For 3D medium, boundaries could come in different shapes.

It could be rectangular, cylinder, sphere etc. Vibrating waves are very different.



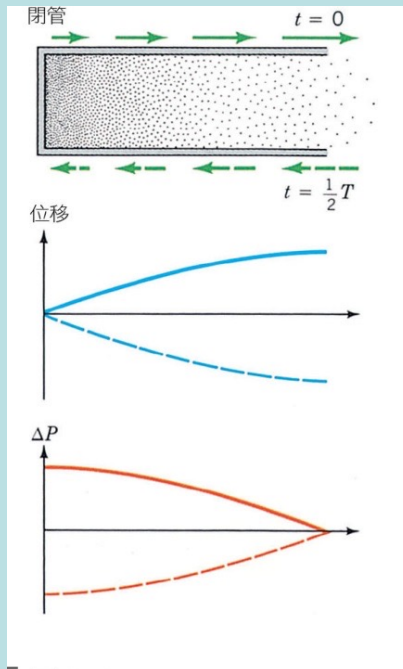


圖 17.4

閉管的基本模式。封閉端為一位移的節點，而且是壓力的腹點。實線說明在 $t = 0$ 的情形，而虛線則說明在 $t = T/2$ 之情形。

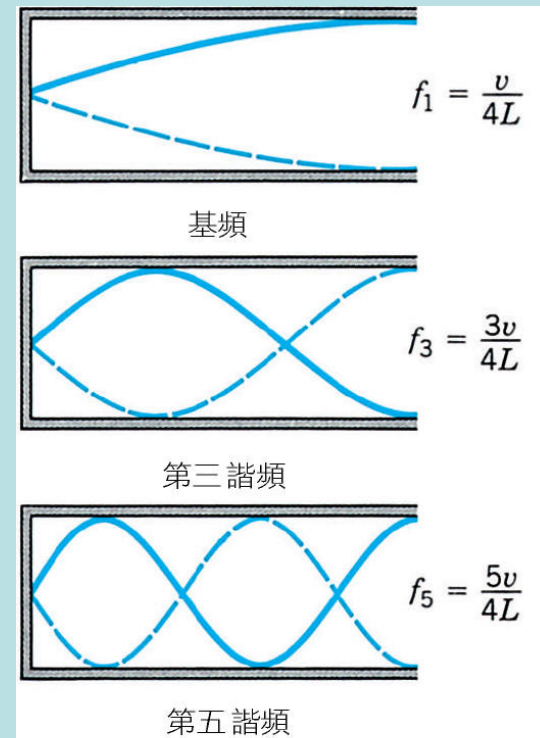
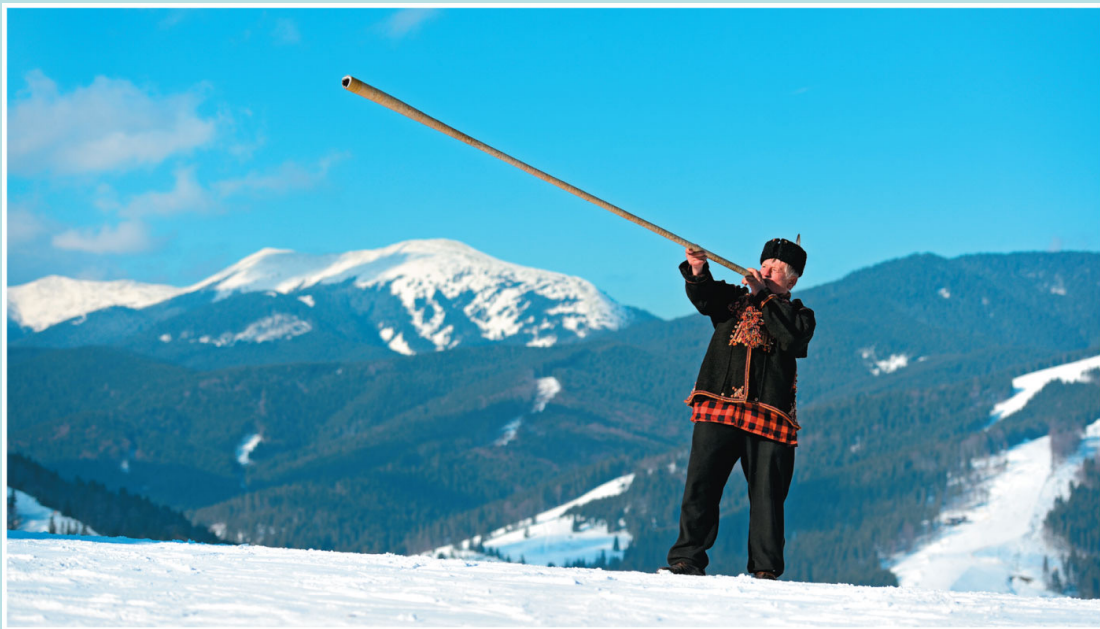


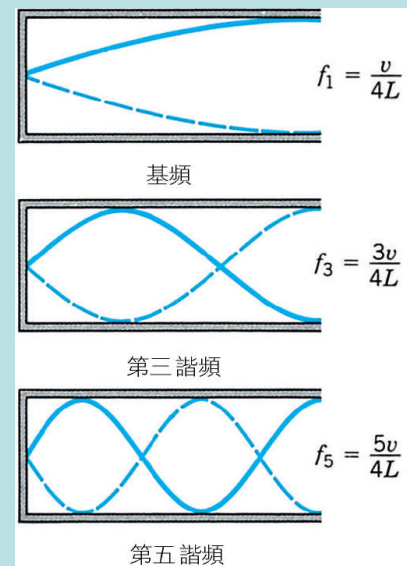
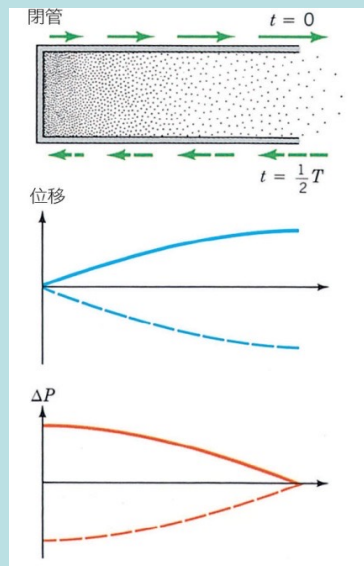
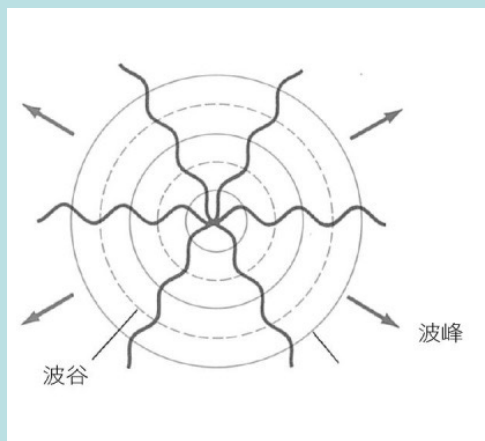
圖 17.5

閉管中前三種共振模。實線及虛線表示時差半週期的兩組縱波。

樂器空腔內的空氣的振盪就是邊界條件下的模式。



廣大空間的聲音，就是無邊界的聲波傳播。

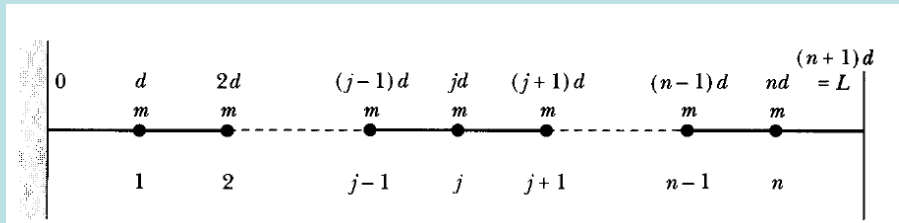


樂器空腔內的空氣的振盪就是邊界條件下的模式。

Solving wave equation with boundary conditions

對照表：Wave system is continuous limit of coupled spring system.

The solution here also applies to electromagnetic wave in vacuum.



dj

$N \rightarrow \infty, d \rightarrow 0$

Continuous limit

$x_j(t)$

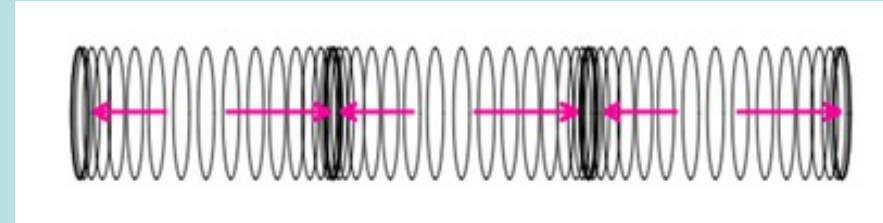
Equations of Motion

$$\frac{d^2 \mathbf{X}}{dt^2} = \mathbf{A} \cdot \mathbf{X}$$

System of Ordinary Differential E.

Solutions: Sum of Oscillation Modes $\longrightarrow$?

Let's first review Oscillation Modes for $N = 3$.



x

$\phi(dj, t) = \phi(x, t)$

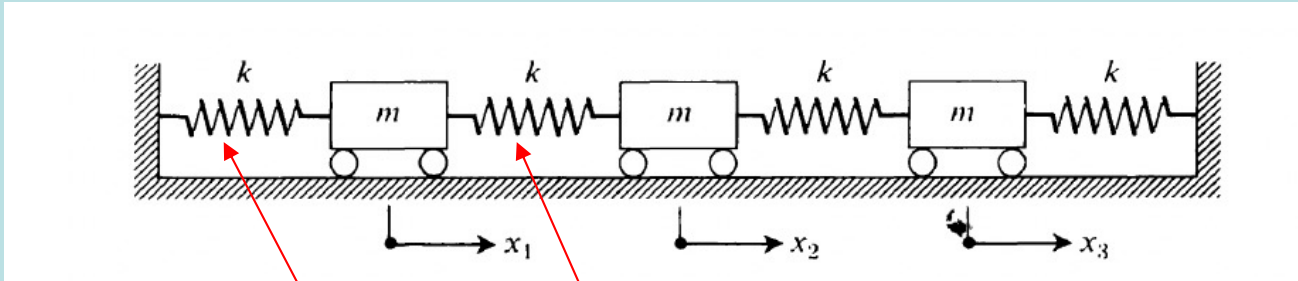
Wave function

Wave Equation

$$\frac{\partial^2 \phi}{\partial t^2}(x, t) = v^2 \frac{\partial^2 \phi}{\partial x^2}(x, t)$$

This is a **partial differential equation PDE**

The concept of **Oscillation Modes** could be extended to more than two masses.



Equations of motion can be written as a system of ODEs:

$$\frac{d^2 x_1}{dt^2} = -\frac{k}{m} x_1 + \frac{k}{m} (x_2 - x_1) = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\frac{d^2 x_2}{dt^2} = \frac{k}{m} (x_3 - x_2) - \frac{k}{m} (x_2 - x_1) = -2\omega_0^2 x_2 + \omega_0^2 x_3 + \omega_0^2 x_1$$

$$\frac{d^2 x_3}{dt^2} = -\frac{k}{m} (x_3 - x_2) - \frac{k}{m} x_3 = -2\omega_0^2 x_3 + \omega_0^2 x_2$$

Define column vector $\mathbf{x}$ and matrix $\mathbf{A}$. Equations of motion is ODE of vector $\mathbf{x}$:

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x} = -\omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Assume there are **mode solutions**. Their time dependence is a factor $e^{i\omega t}$ as SHM.

$$\mathbf{x} = ce^{i\omega t} \cdot \mathbf{a}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = ce^{i\omega t} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Time evolution can be isolated.

Since the time factor can be separated, these are called **separable solutions**.

We understand, eventually we will only keep real part of this complex form.

$$\mathbf{x} = \text{Re } ce^{i\omega t} \cdot \mathbf{a} = a_m \cos(\omega t + \phi) \cdot \mathbf{a}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a_m \cos(\omega t + \phi) \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Take this form as a guess of solutions and plug it into the equation:

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

We keep complex number form because exponentials are easier to differentiate.

$$\mathbf{a} \frac{d^2}{dt^2} e^{i\omega t} = -\omega^2 \mathbf{I} \mathbf{a} e^{i\omega t} = -\mathbf{A} \cdot \mathbf{a} \cdot e^{i\omega t}$$

$$(\mathbf{A} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$$

$$(A - \omega^2 I) \cdot a = 0$$

We transform system of ODE into system of algebraic equations for unknown $a_{1,2,3}$.

It has nonzero solutions only when the **determinant equals zero**.

$$|A - \omega^2 I| = 0$$

This is called characteristic equation CE.

Only certain ω^2 's that satisfy this CE will allow a solution for unknown $a_{1,2,3}$.

To find these ω^2 's :

$$\omega^2 \equiv \lambda \omega_0^2$$

$$A - \omega^2 I = \omega_0^2 \begin{pmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{pmatrix}$$

$$|A - \omega^2 I| = 0 \quad \rightarrow \quad (2 - \lambda)^3 - 2(2 - \lambda) = (2 - \lambda)(2 - 4\lambda + \lambda^2) = 0$$

λ has three solutions. Order and label the solutions of ω from small to large.

$$\omega_1 \sim \sqrt{2 - \sqrt{2}} \omega_0$$

$$\omega_2 \sim \sqrt{2} \omega_0$$

$$\omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

$\omega_{1,2,3}$ are eigenvalues of A .

Let's take ω_2 as an example for a more detailed study.

First solve $a_{1,2,3}$ for ω_2 .

$$\lambda = 2, \quad \omega = \omega_2 \sim \sqrt{2}\omega_0$$

$$(\mathbf{A} - 2\omega_0^2 \mathbf{I}) \cdot \mathbf{a} = \omega_0^2 \begin{pmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{pmatrix} \mathbf{a} = \omega_0^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$-a_2 = 0$$

$$-a_1 - a_3 = 0$$

$$-a_2 = 0$$

$$\mathbf{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This solution is denoted as $\mathbf{a}^{(2)}$, determined up to a constant.

$\mathbf{a}^{(2)}$ is called an eigenvector of $\mathbf{A}$.

An eigenvector gives a solution!

$$\mathbf{x} = c \mathbf{a}^{(2)} e^{i\sqrt{2}\omega_0 t} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} a_m e^{i(\sqrt{2}\omega_0 t + \phi)}$$

Take the real part of this complex form.

$$\text{Re } \mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} [f_2 \cos(\sqrt{2}\omega_0 t) + g_2 \sin(\sqrt{2}\omega_0 t)]$$

$x_{1,2,3}$ oscillate as SHM by the same frequency, with amplitudes $\propto a_{1,2,3}$.

$$\mathbf{x}(t) \propto \mathbf{a}^{(2)} \quad x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = -1:0:1$$

In fact, proportion among $x_{1,2,3}$ is fixed as $a_1:a_2:a_3$ at all t .

This is expected since time evolution can be isolated.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} [f_2 \cos(\sqrt{2}\omega_0 t) + g_2 \sin(\sqrt{2}\omega_0 t)]$$

If the initial conditions start with proportion $-1:0:1$, oscillation mode will continue.

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \sim f_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

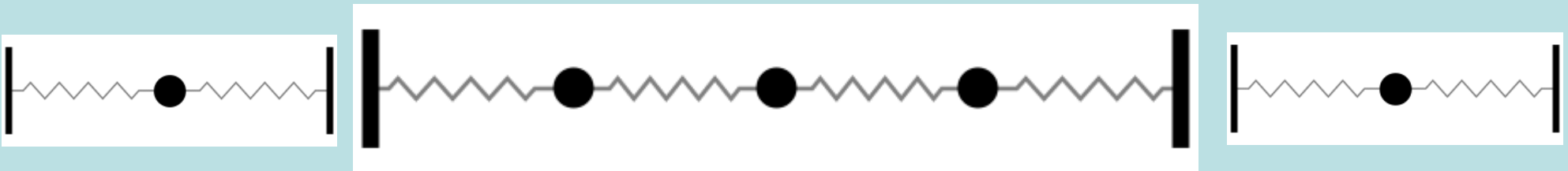
$$\mathbf{x}'(0) = \begin{pmatrix} x'_1(0) \\ x'_2(0) \\ x'_3(0) \end{pmatrix} \propto \omega_2 g_2 \mathbf{a}^{(2)}$$

Note that constant f_2, g_2 is determined by initial conditions $x_{1,2,3}(0), x'_{1,2,3}(0)$.

This is one oscillation mode that would oscillate independently.

$$\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_0$$

$$\mathbf{x}(t) \propto \mathbf{a}^{(2)} \propto \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



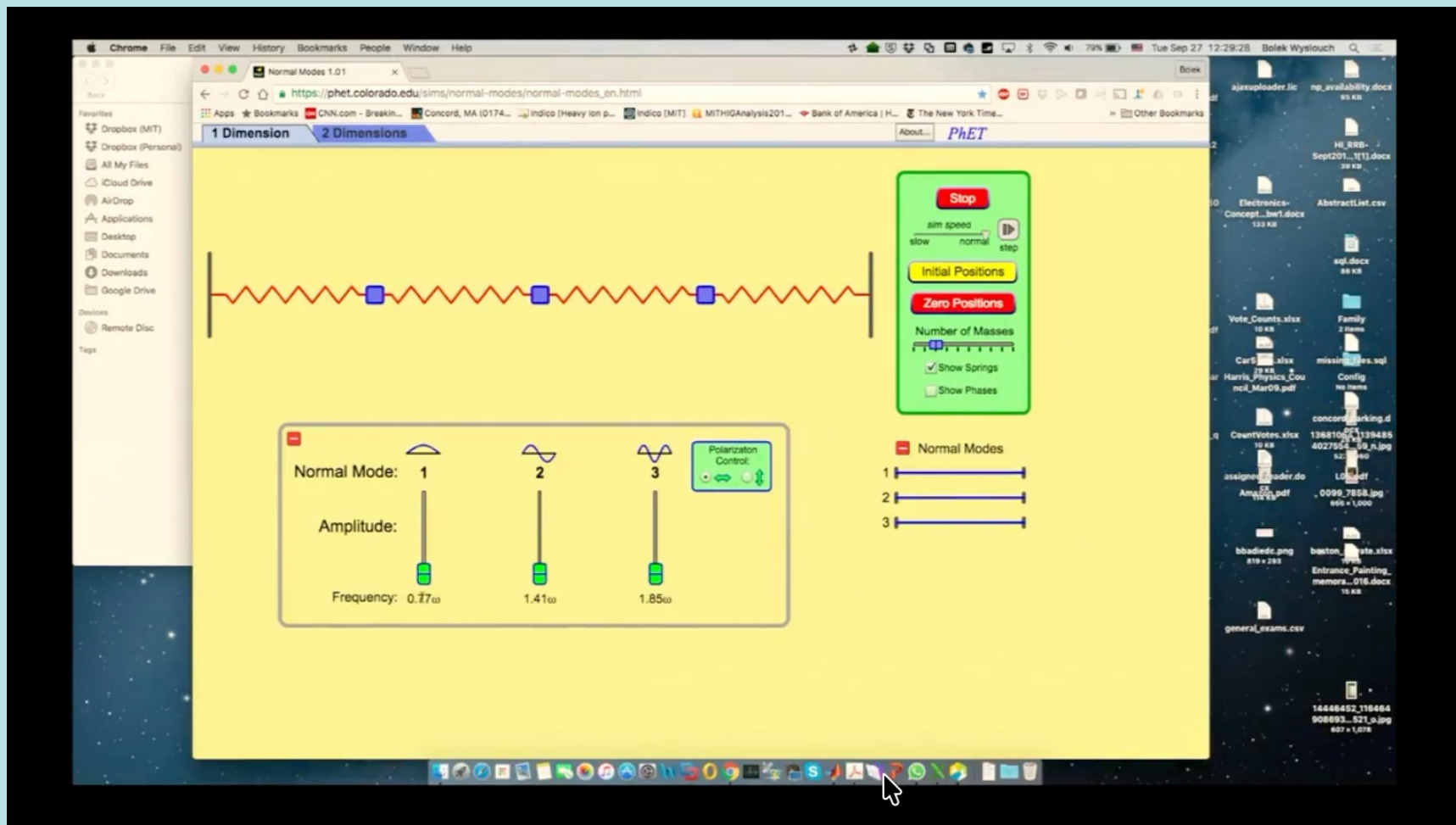
The middle particle does not move. It is called a node 節點。

Treat node as a fixed point, it is like two systems of double springs.

$$\omega = \sqrt{\frac{2k}{m}}$$

$$\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_0$$

$$x(t) \propto \mathbf{a}^{(2)} \propto \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$x_{1,2,3}$ oscillate by the same ω , with amplitudes $\propto a_{1,2,3}$.

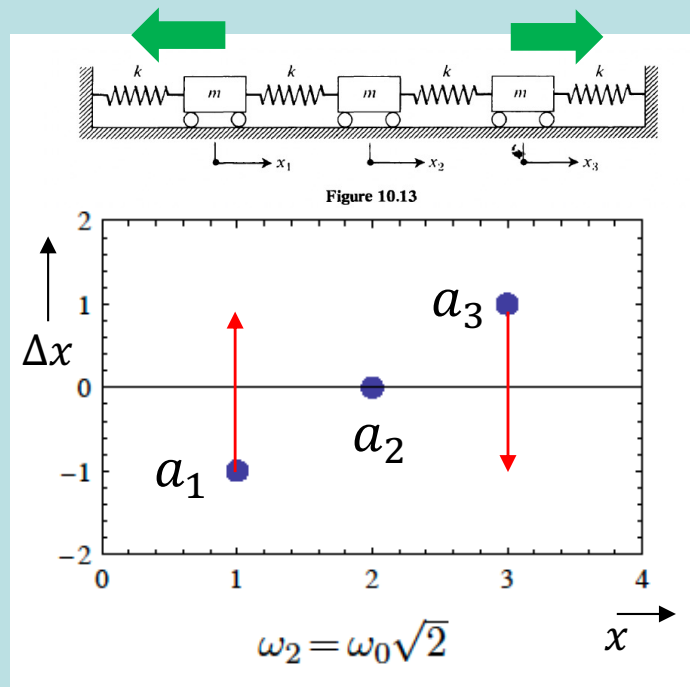
Looked individually every particle is doing a simple harmonic motion.

$$x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = -1:0:1 \quad \text{at all } t.$$

Therefore $a_1:a_2:a_3$ is a good illustration of $x_{1,2,3}$.

We can draw $a_1:a_2:a_3$ versus original locations of blocks to represent the mode.

Red arrows indicate the SHM of individual blocks.



$$\lambda = 2 - \sqrt{2}, \omega = \omega_1 \sim \sqrt{2 - \sqrt{2}}\omega_0:$$

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\mathbf{a}^{(1)} \sim \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{a}^{(1)} c e^{i\sqrt{2-\sqrt{2}}\omega_0 t} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} a_m e^{i(\sqrt{2-\sqrt{2}}\omega_0 t + \phi)}$$

$$\text{Re } \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \mathbf{a}^{(1)} \left[f_1 \cos\left(\sqrt{2-\sqrt{2}}\omega_0 t\right) + g_1 \sin\left(\sqrt{2-\sqrt{2}}\omega_0 t\right) \right]$$

$$\mathbf{x}(t) \propto \mathbf{a}^{(1)}$$

$$x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = 1:\sqrt{2}:1$$

$x_{1,2,3}$ oscillate as SHM by same ω , with amplitudes $\propto a_{1,2,3}$.

$$\begin{aligned} \sqrt{2}a_1 - a_2 &= 0 \\ -a_1 + \sqrt{2}a_2 - a_3 &= 0 \\ -a_2 + \sqrt{2}a_3 &= 0 \end{aligned}$$

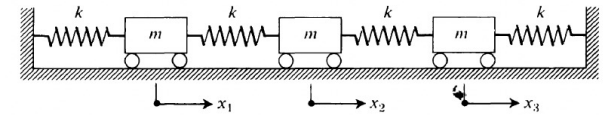
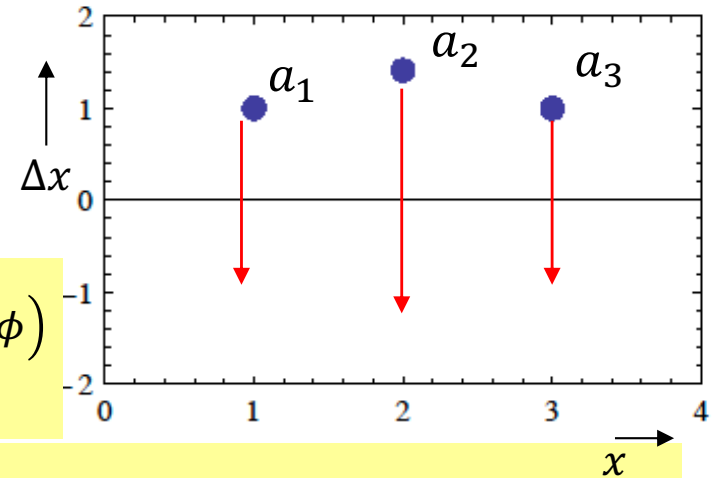
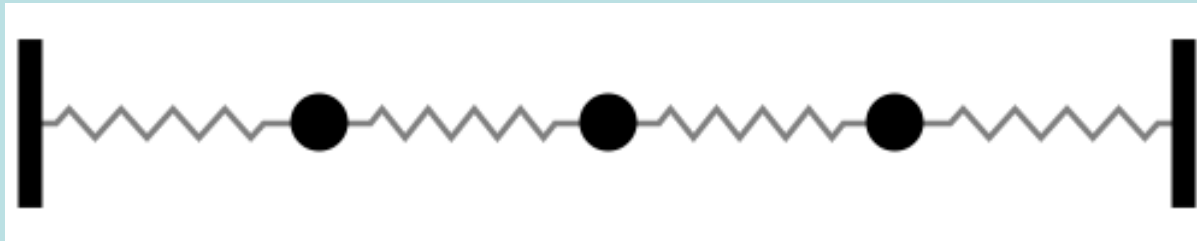


Figure 10.13



This is another oscillation mode.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} [f_2 \cos(\sqrt{2}\omega_0 t) + g_2 \sin(\sqrt{2}\omega_0 t)]$$



$$\lambda = 2 - \sqrt{2}, \omega = \omega_1 \sim \sqrt{2 - \sqrt{2}} \omega_0:$$

$$\mathbf{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

54:37 / 1:22:19 Task Equation

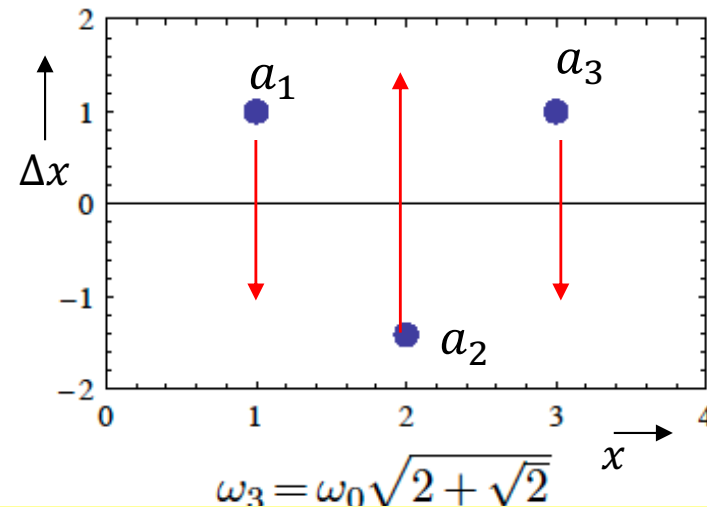
YouTube

$$\lambda = 2 + \sqrt{2}, \omega = \omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

$$\mathbf{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{a}^{(3)} e^{i\sqrt{2+\sqrt{2}}\omega_0 t}$$

$$\begin{aligned} \sqrt{2}a_1 + a_2 &= 0 \\ a_1 + \sqrt{2}a_2 + a_3 &= 0 \\ a_2 + \sqrt{2}a_3 &= 0 \end{aligned}$$

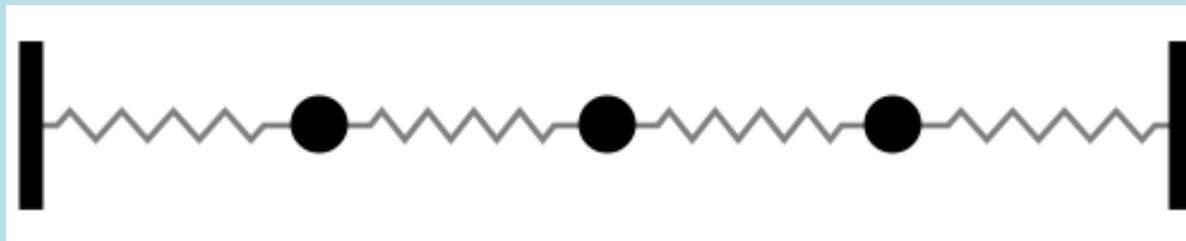


$$\text{Re } \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \left[f_1 \cos\left(\sqrt{2 + \sqrt{2}}\omega_0 t\right) + g_3 \sin\left(\sqrt{2 + \sqrt{2}}\omega_0 t\right) \right]$$

$$\mathbf{x}(t) \propto \mathbf{a}^{(3)}$$

$$x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = 1:-\sqrt{2}:1$$

This is the third oscillation mode.



$$\lambda = 2 + \sqrt{2}, \omega = \omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

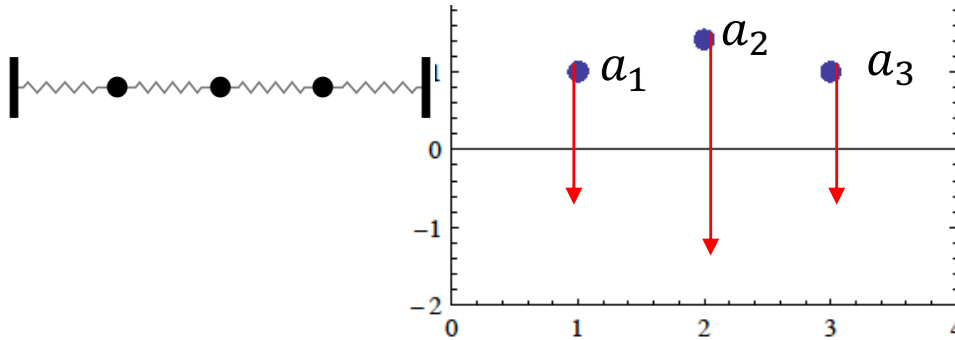
$$\mathbf{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

The screenshot displays the PhET simulation 'Normal Modes 1.01' in a web browser. The main interface shows a 1D chain of three masses connected by springs. The 'Normal Modes' panel on the right lists three modes with their respective frequencies: Mode 1 at 0.77ω , Mode 2 at 1.41ω , and Mode 3 at 1.85ω . A 'Polarization Control' panel is also present. The video player at the bottom indicates the video is at 54:58 / 1:22:19, with a 'Task Equation' label. The background shows a desktop with various files and a taskbar.

Normal Modes of oscillation

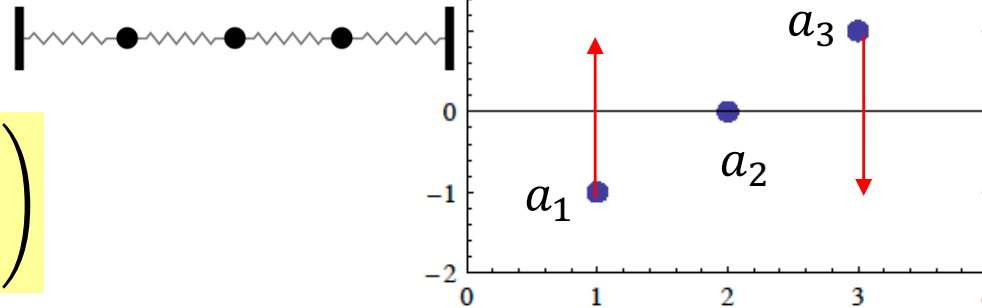
$$\omega = \omega_1 \sim \sqrt{2 - \sqrt{2}}\omega_0:$$

$$\mathbf{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$



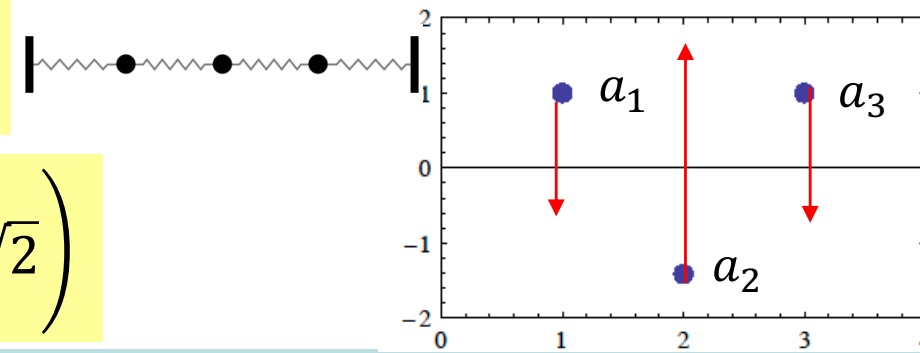
$$\omega = \omega_2 \sim \sqrt{2}\omega_0$$

$$\mathbf{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\omega = \omega_3 \sim \sqrt{2 + \sqrt{2}}\omega_0$$

$$\mathbf{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$



簡正模 [編輯]

文 23 種語言 ▾

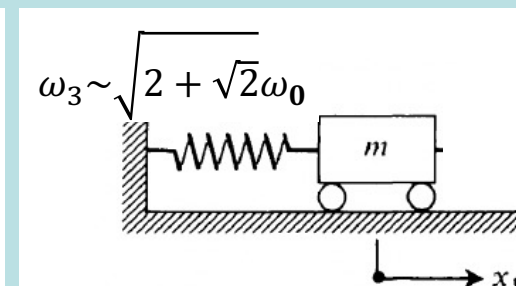
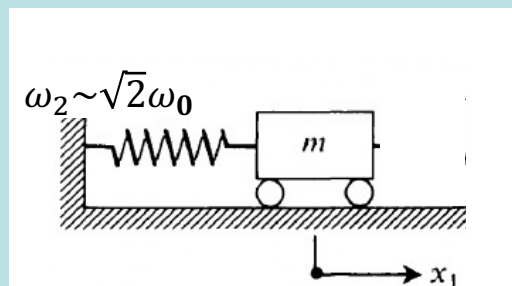
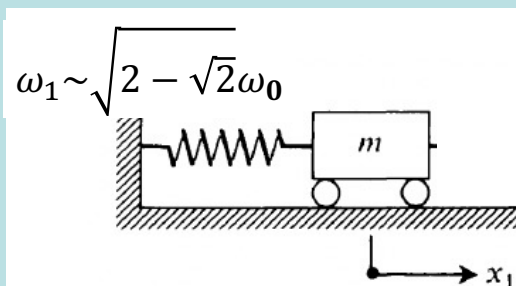
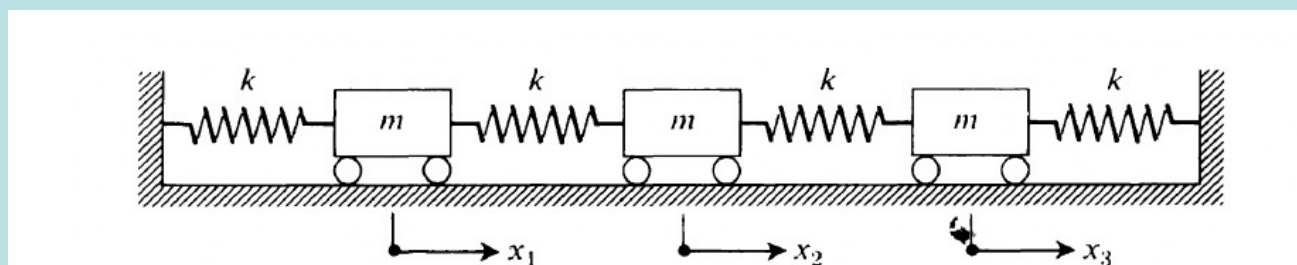
條目 討論 臺灣正體 ▾

閱讀 編輯 檢視歷史 工具 ▾

維基百科，自由的百科全書

簡正模（英語：Normal mode）是一個振盪系統中所有部分都以相同的頻率以正弦函數形式運動的模式^[1]。由簡正模描述的自由運動發生在固定的頻率上。一個系統的簡正模對應的振動頻率被稱為其固有頻率或共振頻率。任何物體，如建築物、橋梁或分子，都具有一組簡正模及對應的固有頻率，取決於其結構，材料和邊界條件^[2]。

一個系統一般的運動可以寫成其簡正模的疊加。之所以簡正模被稱作「簡正的（normal）」，是因為它們可以獨立運動，即給物體施加一種模的激發，永遠不會導致物體以另一個模運動。簡正模彼此正交。

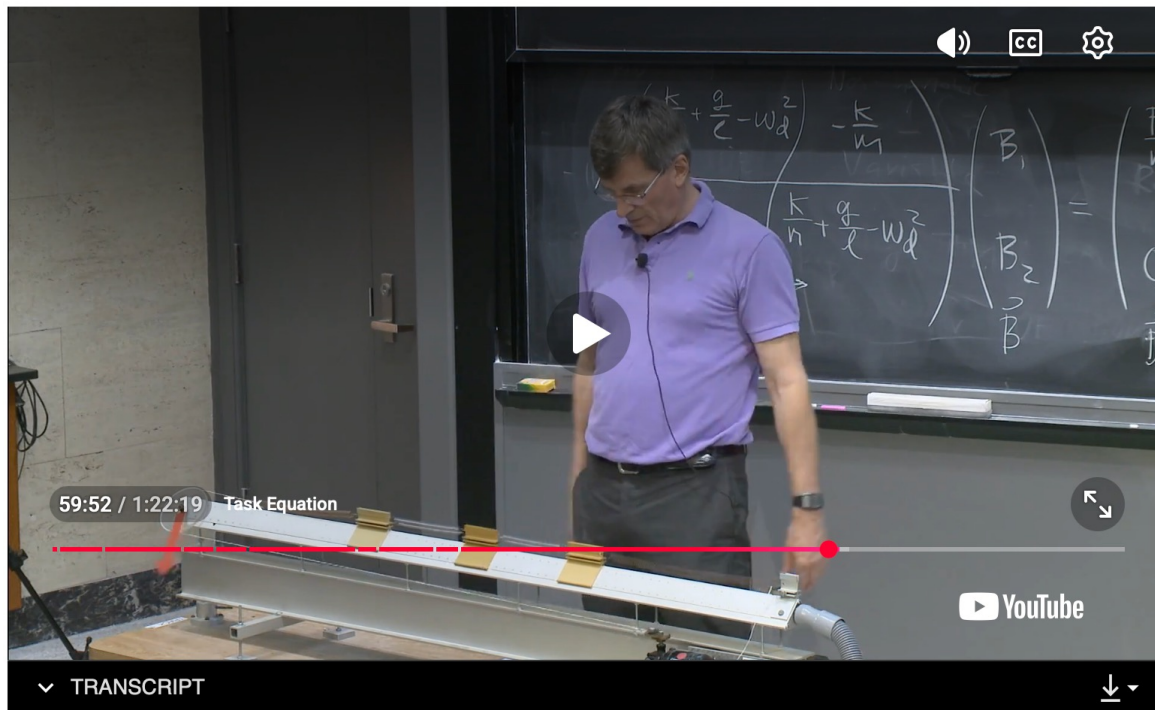


System of 3 coupled spring equals 3 independent springs.

The coordinates of the 3 independent springs are linear combinations of $x_{1,2,3}$.

Lecture 6: Driven Oscillators, Resonance

Coupled Air Carts



59:52 / 1:22:19 Task Equation

YouTube

TRANSCRIPT

59:23 I'm not exactly at the right place, but I'm close.

Course Info



INSTRUCTOR

[Prof. Yen-Jie Lee](#)

DEPARTMENTS

[Physics](#)

AS TAUGHT IN

Fall 2016

LEVEL

[Undergraduate](#)

TOPICS

✓ [Science](#)

✓ [Physics](#)

[Atomic, Molecular, Optical Physics](#)

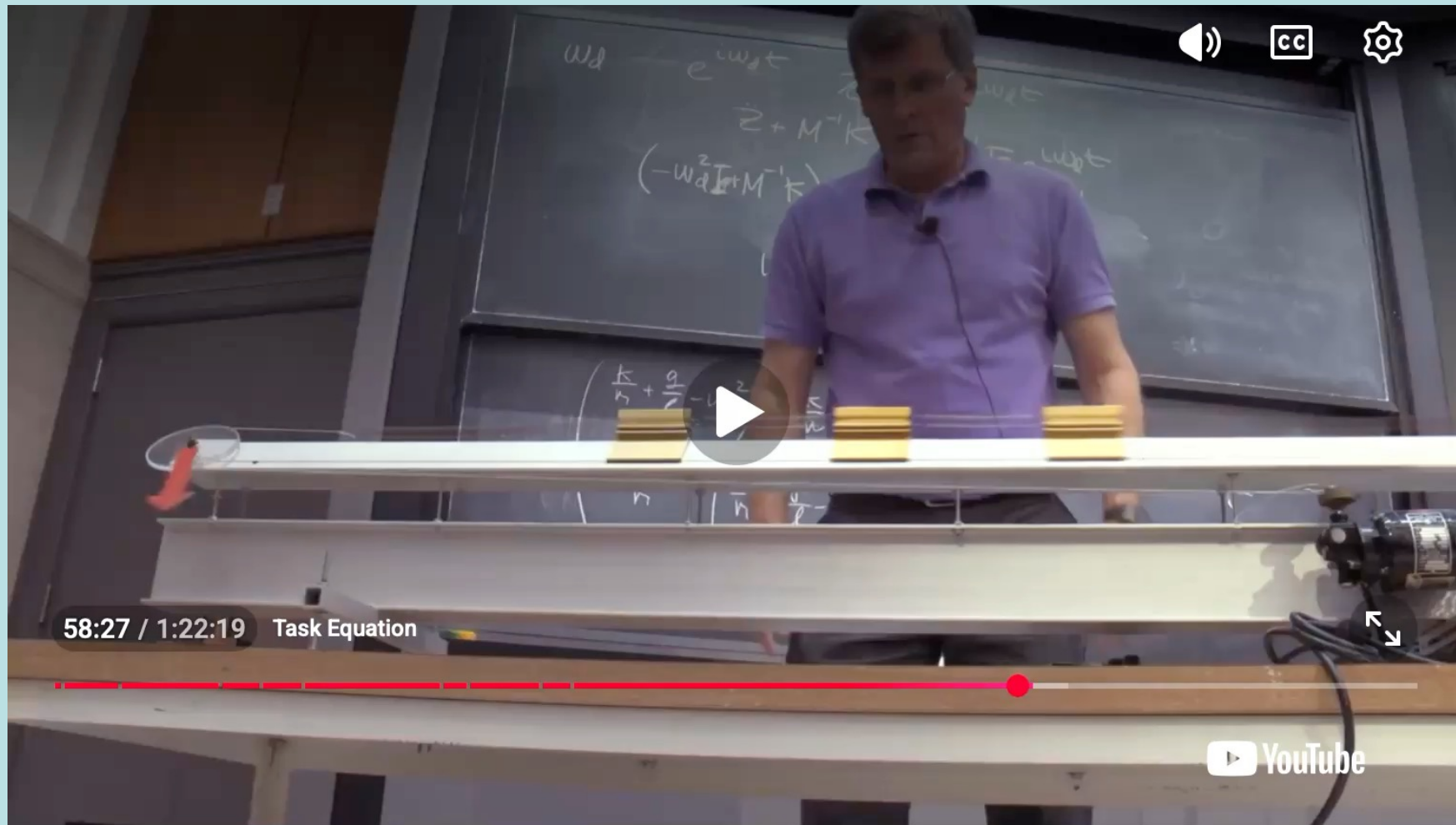
[Classical Mechanics](#)

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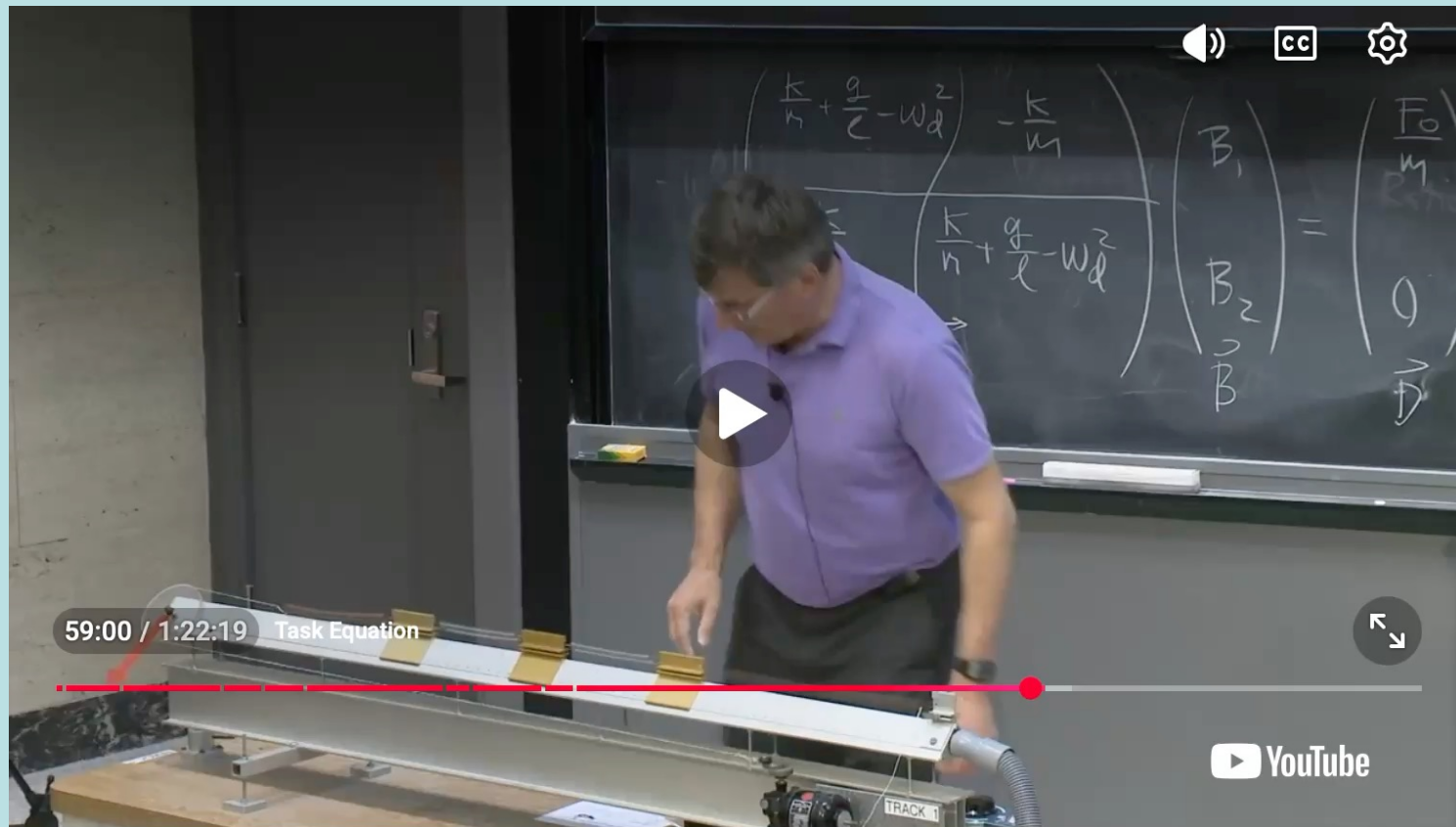
LEARNING RESOURCE TYPES

[Exams](#)

1st Mode



$$\mathbf{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$



$$\mathbf{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Adjusting external force frequency close to $\omega_{2,3}$ to generate 2nd 3rd oscillation modes.

This is called to **excite the second and third mode by resonance** in jargon.

物體的振動

<http://www.youtube.com/watch?v=17tqXgvCN0E&NR=1>

Breaking Glass with Sound

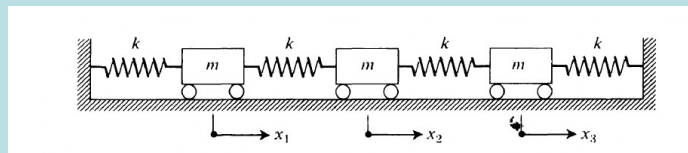
**MIT Department of Physics
Technical Services Group**

One eigenvalue, one eigenvector of the matrix corresponds one oscillation mode.

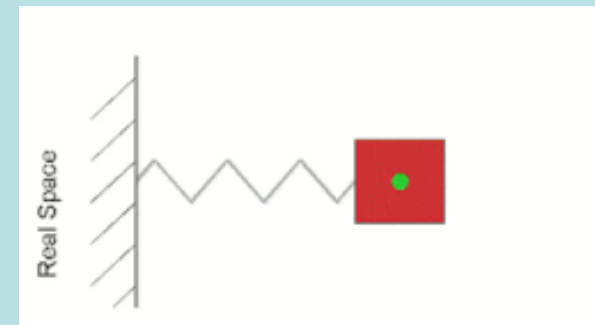
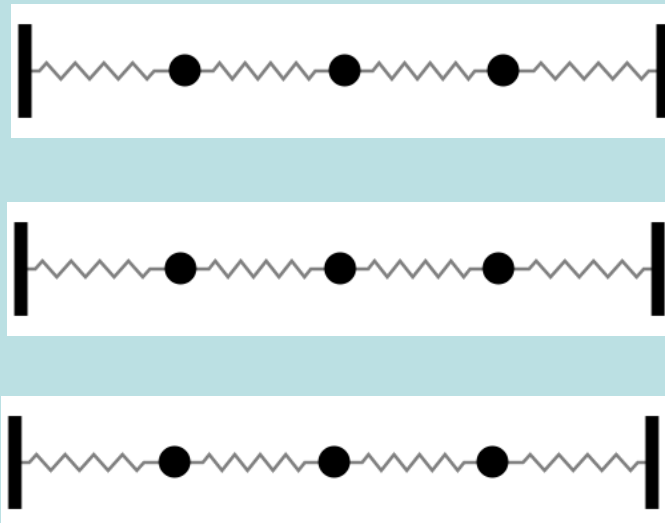
Any $n \times n$ symmetric matrix has n eigenvalues and they all are real.

$$\mathbf{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \mathbf{A} \text{ is a symmetric matrix.}$$

The number of modes equals the number of particles.



In modes, coupled springs act like one spring!



The n eigenvectors $\mathbf{u}$ can be chosen to be orthogonal.

$$\omega_1 \sim \sqrt{2 - \sqrt{2}} \omega_0:$$

$$\mathbf{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\omega_2 \sim \sqrt{2} \omega_0$$

$$\mathbf{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

$$\mathbf{a}^{(3)} = c_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Indeed, eigenvectors are orthogonal to each other.

$$\mathbf{a}^{(2)T} \mathbf{a}^{(1)} = \mathbf{a}^{(2)} \cdot \mathbf{a}^{(1)} = 0$$

We could choose the length of all eigenvectors to be equal to one.

$$\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \mathbf{a}^{(m)} \cdot \mathbf{a}^{(n)} = \delta_{mn} \quad \text{They are called orthonormal.}$$

You will see this orthogonality property is extremely important.

ω_1 ω_2 ω_3

Coupled Oscillator: 3 Blocks

All springs have spring constant k , and all surfaces are frictionless.

$$x_1(t) = \frac{x_1(0) + \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2}) t \right] - \frac{x_1(0) - x_3(0)}{2} \cos \left(\sqrt{\frac{2k}{m}} t \right) + \frac{x_1(0) - \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2}) t \right]$$

$$x_2(t) = \frac{\sqrt{2}x_1(0) + 2x_2(0) + \sqrt{2}x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2}) t \right] - \frac{\sqrt{2}x_1(0) - 2x_2(0) + \sqrt{2}x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2}) t \right]$$

$$x_3(t) = \frac{x_1(0) + \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2}) t \right] - \frac{x_1(0) - x_3(0)}{2} \cos \left(\sqrt{\frac{2k}{m}} t \right) + \frac{x_1(0) - \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2}) t \right]$$

$$\begin{aligned} k &= 1 \text{ N/m} \\ m &= .2 \text{ kg} \\ x_1(0) &= 1 \text{ m} \\ x_2(0) &= .9 \text{ m} \\ x_3(0) &= -.3 \text{ m} \end{aligned}$$



In general, more than one modes can be excited. Then it is not a pure oscillation.

All three modes, multiplied by 3 coefficients, add up, forming a linear combination.

The equation of motions are linear ODE.

Linear Combinations of the 3 mode solutions are solutions, too.

Key Question: Is any solution of system a linear combination of 3 mode solutions?

Linear Combinations of the 3 mode solutions are solutions, too.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

Determine 3 coefficients $f_{1,2,3}$ by 3 initial conditions $x_{1,2,3}(0)$. Plug in $t = 0$.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} f_i$$

I have chosen $\mathbf{a}^{(i)}$ to be orthonormal. $\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \delta_{mn}$ Taking inner product:

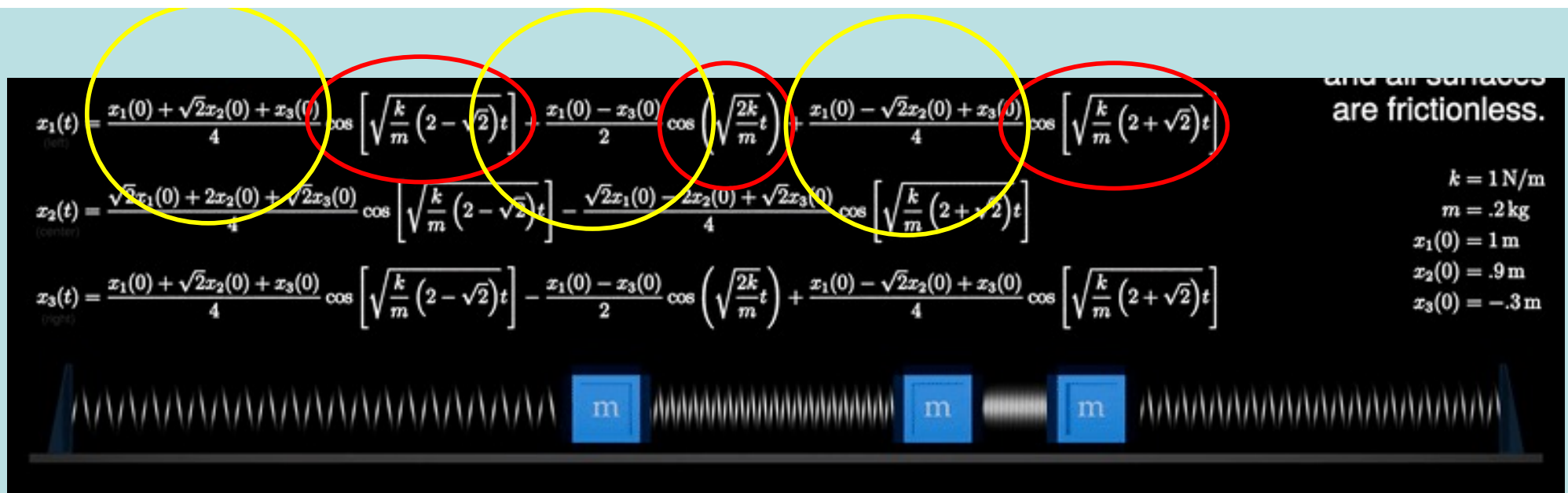
$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{j=1}^3 \mathbf{a}^{(i)T} \mathbf{a}^{(j)} f_j = \sum_{j=1}^3 \delta_{ij} f_j = f_i$$

$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

Inner product of $\mathbf{a}^{(i)}$ and initial column vector equals f_i :

For example,

$$f_1 = \mathbf{a}^{(1)} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \frac{1}{2} [x_1(0) + \sqrt{2}x_2(0) + x_3(0)]$$



$$f_1 = \frac{1}{2} [x_1(0) + \sqrt{2}x_2(0) + x_3(0)]$$

$$f_2 = \frac{1}{\sqrt{2}} [x_1(0) - x_3(0)]$$

$$f_3 = \frac{1}{2} [x_1(0) - \sqrt{2}x_2(0) + x_3(0)]$$

If initial velocities are all zero, $g_{1,2,3} = 0$, use numbers $f_{1,2,3}$ to build a solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} f_1 \cos(\omega_1 t) + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} f_2 \cos(\omega_2 t) + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} f_3 \cos(\omega_2 t)$$

If initial velocities are not all zero,

determined coefficients $g_{1,2,3}$ similarly by initial velocities $x'_{1,2,3}(0)$.

$$\begin{pmatrix} x'_1(0) \\ x'_2(0) \\ x'_3(0) \end{pmatrix} = 0 + \sum_{i=1}^3 \omega_i \mathbf{a}^{(i)} g_i$$

$$\mathbf{a}^{(i)T} \begin{pmatrix} x'_1(0) \\ x'_2(0) \\ x'_3(0) \end{pmatrix} = \omega_i g_i$$

Use the numbers $f_{1,2,3}$, $g_{1,2,3}$ to build a solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

It satisfies both the system of equations of motion and initial conditions.

There is one and only one function that satisfies both EOM and initial conditions.

We prove: any solution of the system is a linear combination of 3 mode solutions.

Remember the key of the proof is: eigenvectors are orthonormal!

求解策略：

Solving Equation of motion:

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

首先時間變化為簡諧振盪的模式解：

$$\mathbf{x} = e^{i\omega t} \mathbf{a}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = e^{i\omega t} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

時間變化與足標 $i = 1, 2, 3$ 無關！

模式解滿足矩陣的本徵向量方程式：

$$(\mathbf{A} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$$

本徵向量數量與粒子數目相等： $\mathbf{a}^{(i)}, i = 1, \dots, 3$ ！

方程式組是線性方程式，模式解的線性組合，還是方程式組的解。

$$\mathbf{x} = \sum_{i=1}^3 [\mathbf{a}^{(i)} f_i \cdot \cos(\omega_i t) + \mathbf{a}^{(i)} g_i \cdot \sin(\omega_i t)]$$

將 $t = 0$ 的起始條件分解為本徵向量 $\mathbf{a}^{(i)}$ 的線性組合，這永遠能作到。

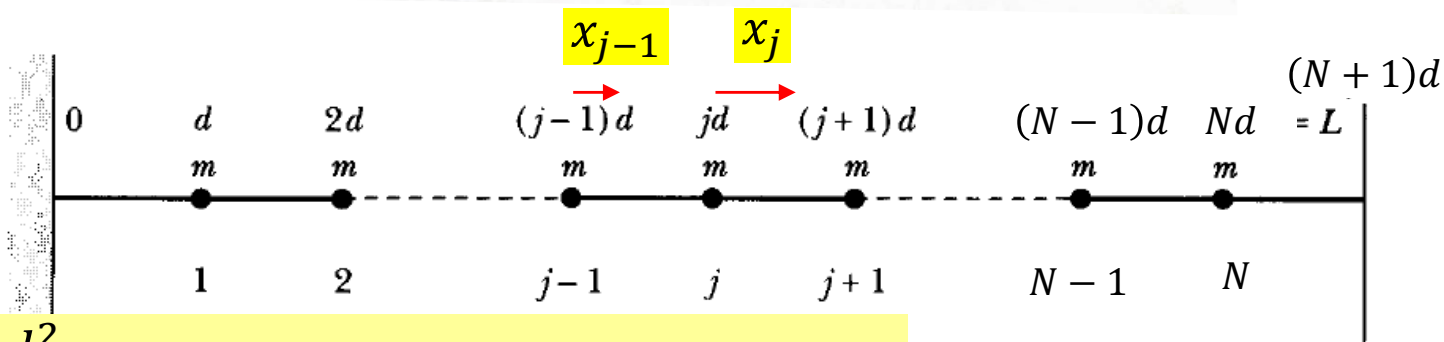
$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} f_i$$

$$\begin{pmatrix} x'_1(0) \\ x'_2(0) \\ x'_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} g_i$$

讓組合中的 $\mathbf{a}^{(i)}$ 各自作時間演化 (簡諧振盪) 後，到時間 t 再組合！

解矩陣的本徵向量方程式是關鍵，其餘都是自動完成。

Now extend the formalism to a large number N of particles.



$$m \frac{d^2 x_j}{dt^2} = -k(x_j - x_{j-1}) + k(x_{j+1} - x_j)$$

$$j = 1, 2, \dots, N$$

$$\frac{d^2 x_j}{dt^2} = \frac{k}{m} x_{j-1} - \frac{2k}{m} x_j + \frac{k}{m} x_{j+1}$$

x_1, x_N 二次微分的方程式中包含 x_0, x_{N+1} 。

邊界是固定的，就稱為 **Boundary Condition**.

$$x_0 = x_{N+1} = 0$$

$$\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix} \quad \mathbf{X} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

$$\omega_0^2 = \frac{k}{m}$$

$$\frac{d^2 \mathbf{X}}{dt^2} = -\mathbf{A} \cdot \mathbf{X}$$

$$\mathbf{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Elevate x_i to complex number and put in the following guess solution:

$$X = \mathbf{a}e^{i\omega t} \quad \frac{d^2 X}{dt^2} = -A \cdot X \quad \Rightarrow \quad -A \cdot \mathbf{a} = -\omega^2 \mathbf{a} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}$$

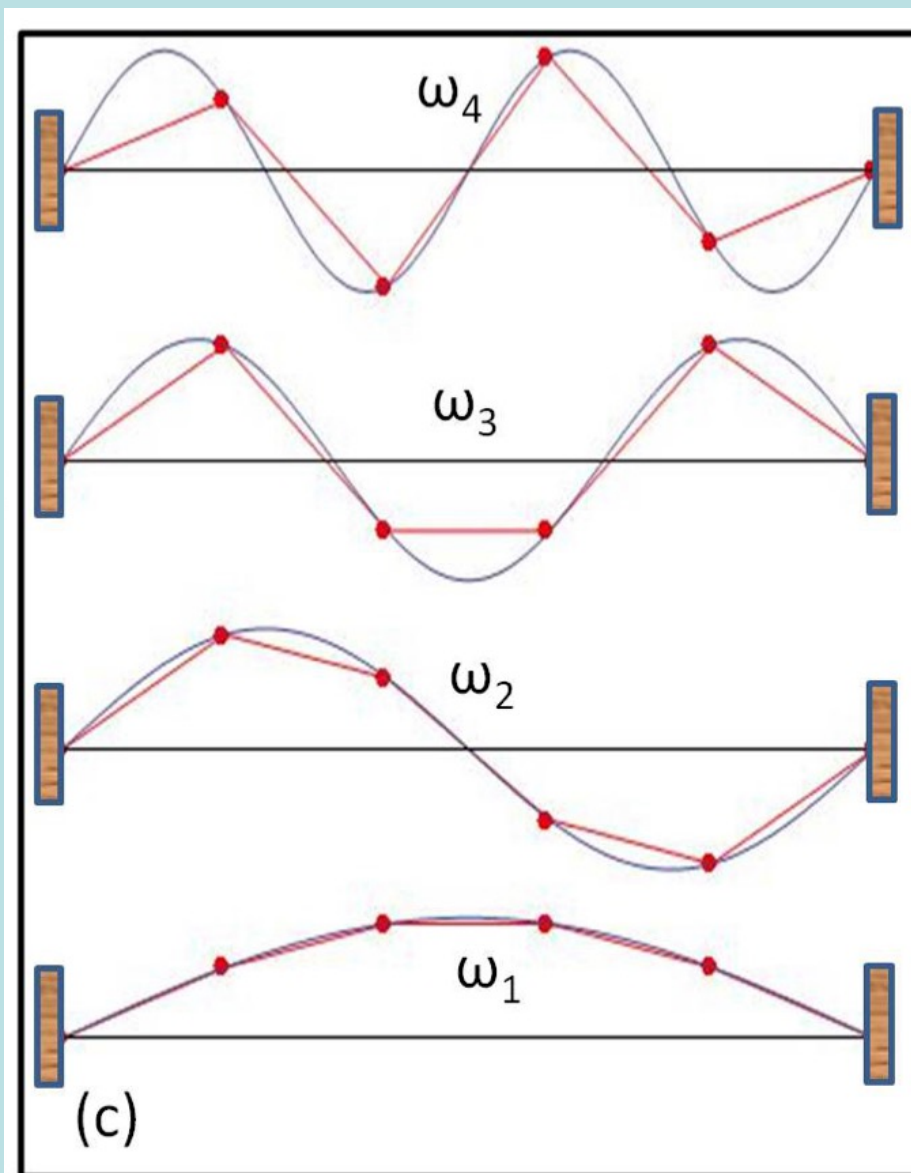
The system of differential equations is transformed into a matrix eigenvector problem.

$$-\omega_0^2 \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} = -\omega^2 \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}$$

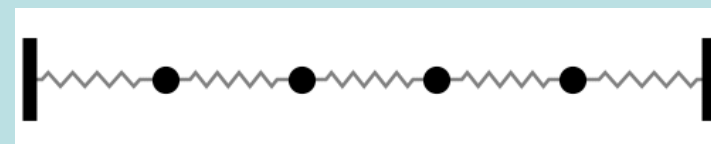
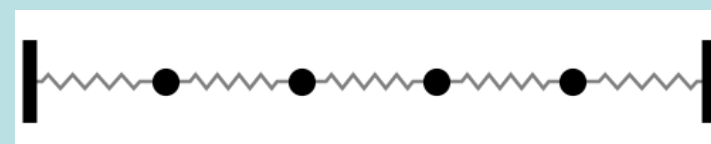
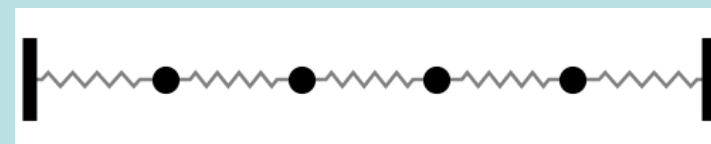
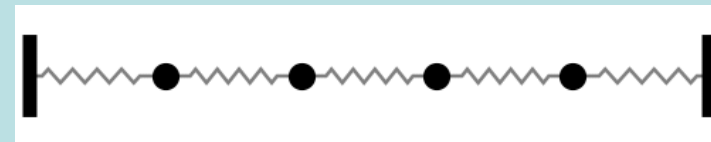
It is a bit impractical to solve it by the condition of vanishing determinant.

But we are sure symmetric positive definite $N \times N$ matrix will have N positive eigenvalues corresponding to N eigenvectors.

And hence there are N oscillation modes.



$$N = 4$$



本徵方程式其實就是 N 個方程式，其中第 j 個方程式可寫成：

$$\omega_0^2[-a_{j-1} + 2a_j - a_{j+1}] = \omega^2 a_j \quad \text{這次用個別式子想比用矩陣更容易。}$$

如果本徵向量的元素如上述是等比數列，左手邊與右手邊的比例是定值，與 j 無關。

$$a_j^{(p)} \sim e^{-ip \cdot j} \quad \frac{a_{j+1}}{a_j} \sim e^{-ip} \quad \frac{a_{j-1}}{a_j} \sim e^{ip} \quad \text{這裏嘗試比例為 } e^{-ip}, p \text{ 是一未定的常數。}$$

$$\text{整式除以 } a_j, \text{ 得: } \omega_0^2[-e^{ip} + 2 - e^{-ip}] = \omega^2$$

只要 ω^2 的值使此式正確，第 j 個方程式就滿足。

但注意，此式中沒有 j ，因此只要 ω^2 的值使此式正確，所有 N 個方程式都滿足。

$$\mathbf{a}^{(p)} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} \sim \begin{pmatrix} e^{-ip} \\ e^{-i2p} \\ \vdots \\ e^{-ip \cdot j} \\ \vdots \\ e^{-ip \cdot N} \end{pmatrix} \quad \text{的確就是 } N \times N \text{ 矩陣 } \mathbf{A} \text{ 的本徵向量！}$$

$$\text{滿足此式的 } \omega^2 \text{ 值就是對應的本徵值。} \quad \omega_0^2[-e^{ip} + 2 - e^{-ip}] = \omega^2$$

$$\omega_0^2 [2 - (e^{-ip} - e^{ip})] = \omega^2$$

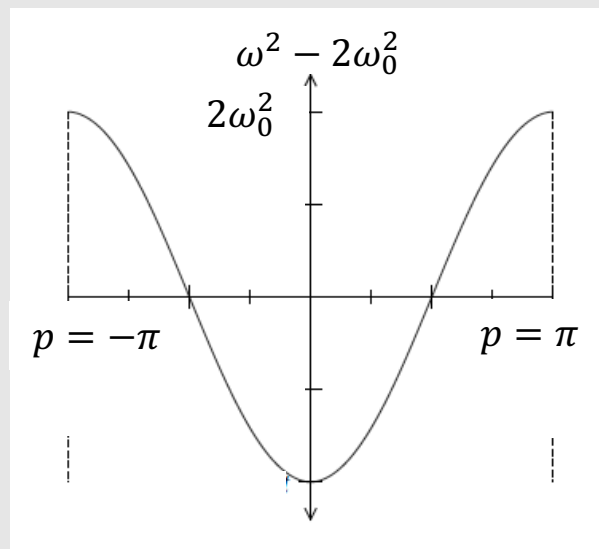
化簡此式就可以得到本徵值 ω^2 。

$$\omega^2 = \omega_0^2 (2 - 2 \cos p)$$

注意本徵值 ω^2 都是正的。

一個 p 對應一特定的本徵值， p 還待決定。

如果把等比數列的配方延伸到 x_0, x_{N+1} ，不是所有 p 都可滿足邊界條件。



$$x_0 = x_{N+1} = 0$$

$$\mathbf{X} = \mathbf{a}^{(p)} e^{i\omega t} = \begin{pmatrix} e^{-i0p} = 1 \\ e^{-ip} \\ \vdots \\ e^{-ijp} \\ \vdots \\ e^{-iNp} \\ e^{-i(N+1)p} \end{pmatrix} e^{i\omega t}$$

$$a_j^{(p)} \sim e^{-ip \cdot j} = \cos pj + i \sin pj$$

因矩陣 $\mathbf{A}$ 是實數矩陣，本徵向量 $\mathbf{a}$ 可取其實數部或虛數部都滿足本徵向量方程式：

取其虛數部則可以直接滿足左側的邊界條件： $x_0 = 0 \Rightarrow a_0 = 0$

$$a_j^{(p)} \sim e^{-ip \cdot j} \Rightarrow a_j^{(p)} = \sin pj \Rightarrow a_0 = 0$$

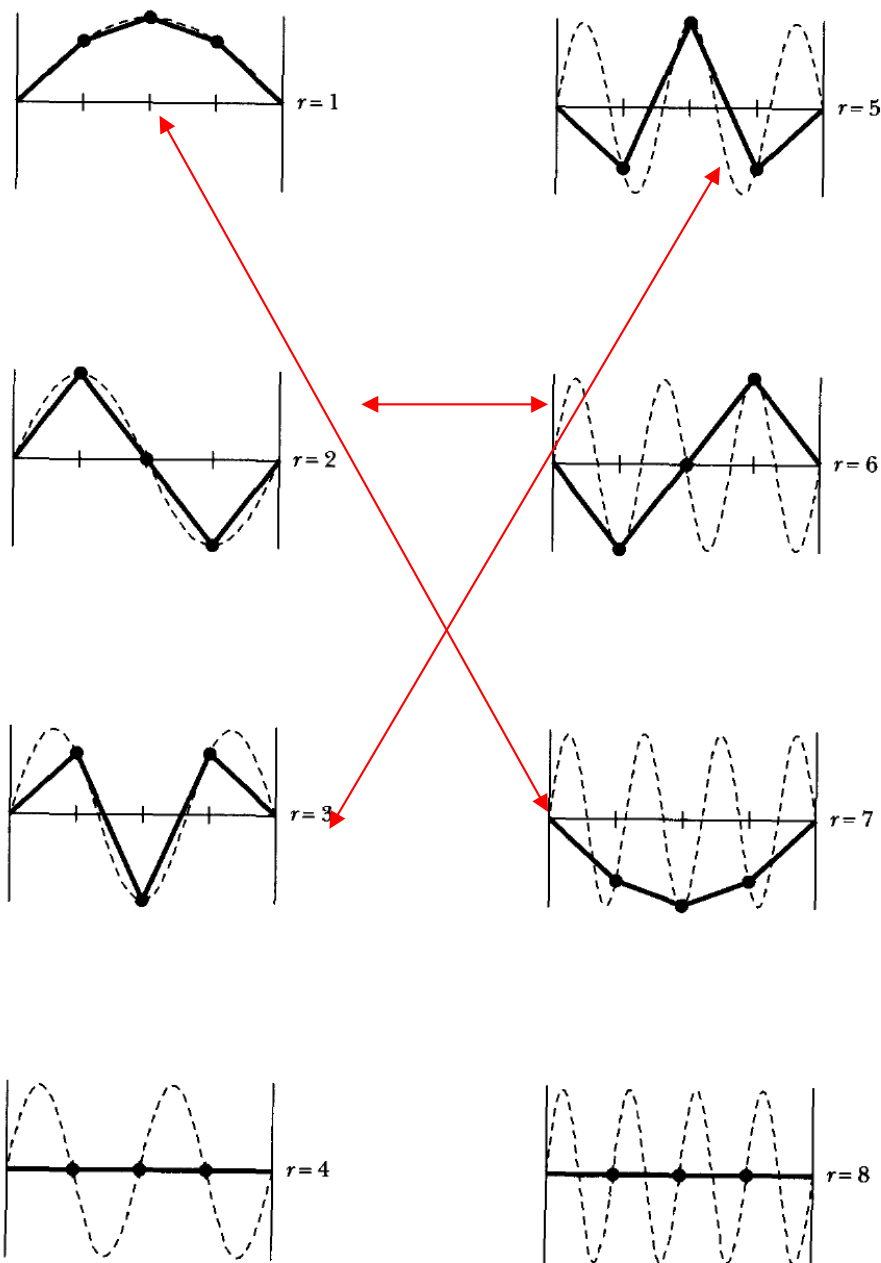
$$\mathbf{a}^{(p)} = \sim \begin{pmatrix} 0 \\ \sin p \\ \vdots \\ \sin jp \\ \vdots \\ \sin Np \\ \sin(N+1)p \end{pmatrix}$$

模式本徵向量的元素都是正弦函數值，角度以等差 p 增加。

若要滿足右側的邊界條件： $a_{N+1} = \sin(N+1)p = 0$

$p \cdot (N+1) = m\pi$ m 是任一自然數。

$$p = \frac{m\pi}{N+1} \quad \text{此角度的等差 } p \text{ 是離散的，以 } m \text{ 標定：} \quad a_j^{(m)} = \sin \frac{m\pi}{N+1} \cdot j$$



但還是有無限多個本徵值，這是不可能的。

$$p = \frac{m\pi}{N+1} \quad a_j^{(m)} = \sin pj$$

$m = N+1$ 對應 $a_j = 0$ 。非合理本徵向量。

$m = N+2$ 與 $m = N$ 對應的 p 之和為 2π ，
對應相似的 $a_j^{(N+2)} = -a_j^{(N)}$ 。

其餘 $m > N+2$ 都對應 $2N+2-m < N$ ，
其 p 和為 2π : $a_j^{(m)} = -a_j^{(2N+2-m)}$

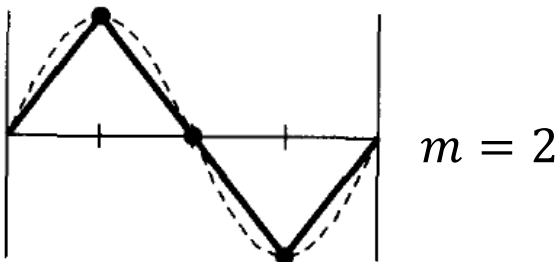
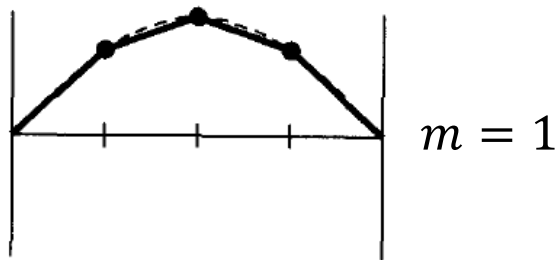
因此 m 只能取以下這些值，共 N 個。

$m = 1, 2, 3 \dots N$ 本徵向量才是獨立的！

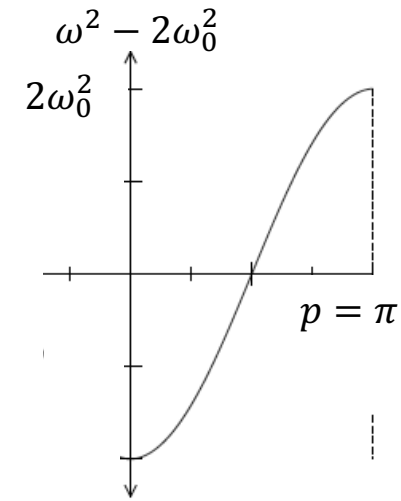
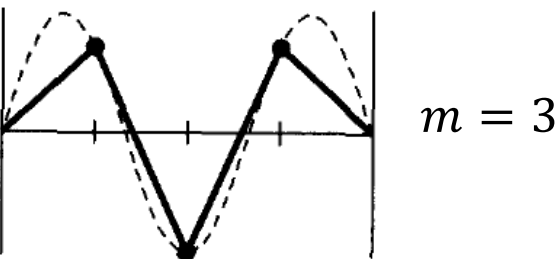
如預期恰有 N 個本徵值。

一個 p 對應一特定的本徵值及本徵向量 $\rightarrow$ 一個 m 對應一本徵值及本徵向量。

Summary



$n = 3$



一個自然數 m 對應一本徵值及一本徵向量。

$$p = \frac{m\pi}{N+1}$$

$$m = 1, 2, 3 \dots N$$

$$\mathbf{a}^{(p)} \text{ or } \mathbf{a}^{(m)} = \begin{pmatrix} \sin p \\ \vdots \\ \sin(j-1)p \\ \sin jp \\ \sin(j+1)p \\ \vdots \\ \sin Np \end{pmatrix}$$

$$\omega_m^2 = \omega_0^2(2 - 2 \cos p)$$

$$\mathbf{X} = \mathbf{a}^{(m)} C e^{i\omega_m t} \rightarrow \mathbf{a}^{(m)} \cdot [A_m \cos \omega_m t]$$

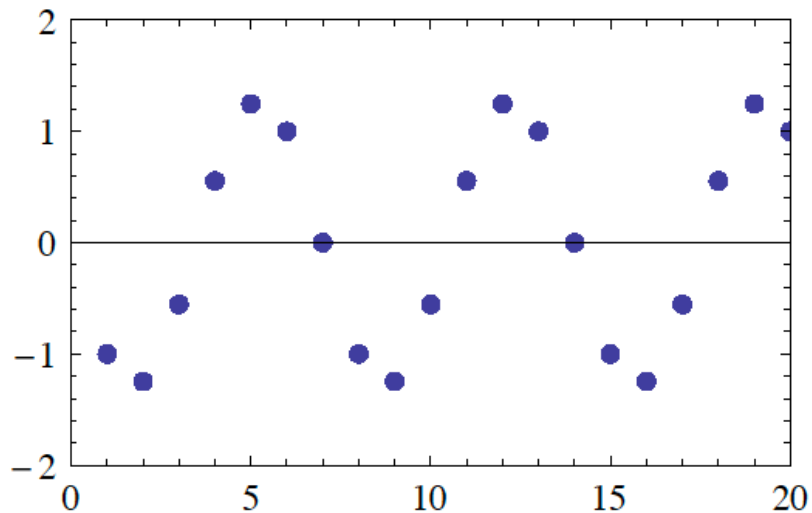
一個自然數 m 對應一個獨立振盪模式(駐波)。

$$a_j^{(p)} = \phi(dj, t = 0) \sim \sin jp \quad J\text{th component of the } p\text{th eigenvector}$$

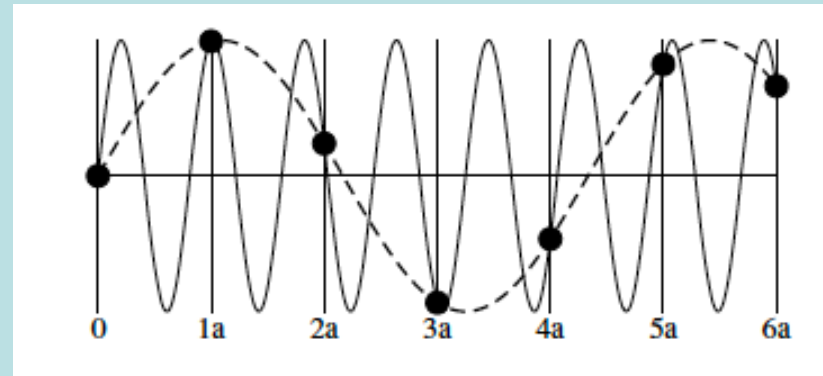
In continuous limit $N \rightarrow \infty, d \rightarrow 0$:

$$= \sin \frac{p}{d} \cdot jd \rightarrow \sin kx \quad \text{for } x = jd, j = 1 \cdots N$$

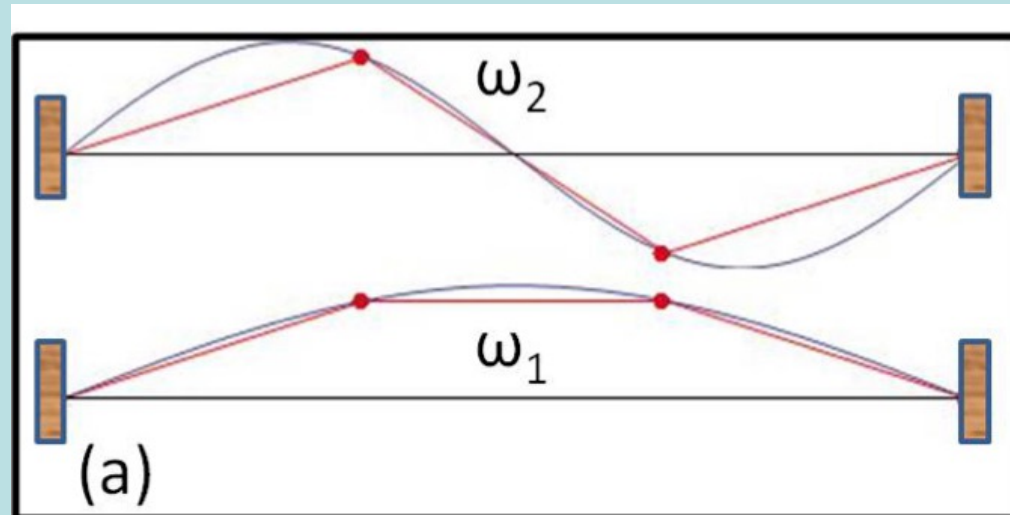
$$\frac{p}{d} \equiv k = \frac{m\pi}{(N+1)d} \rightarrow \frac{m\pi}{L}$$



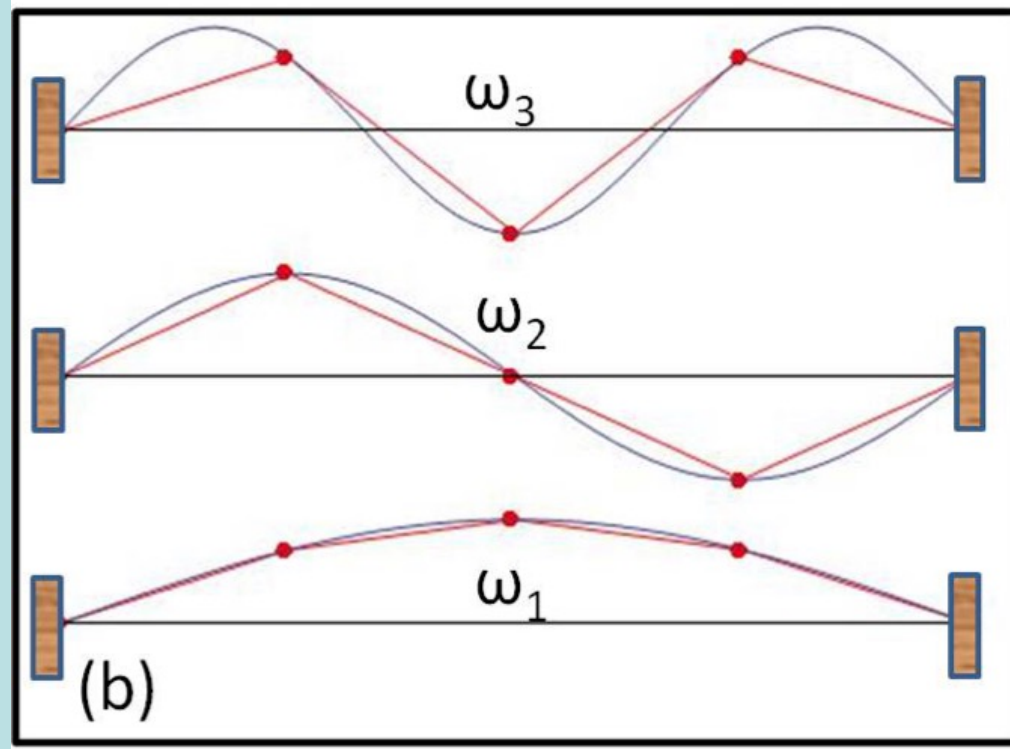
$$\phi(x, t = 0) \rightarrow \sin kx \quad \text{for } x = jd, j = 1 \cdots N$$



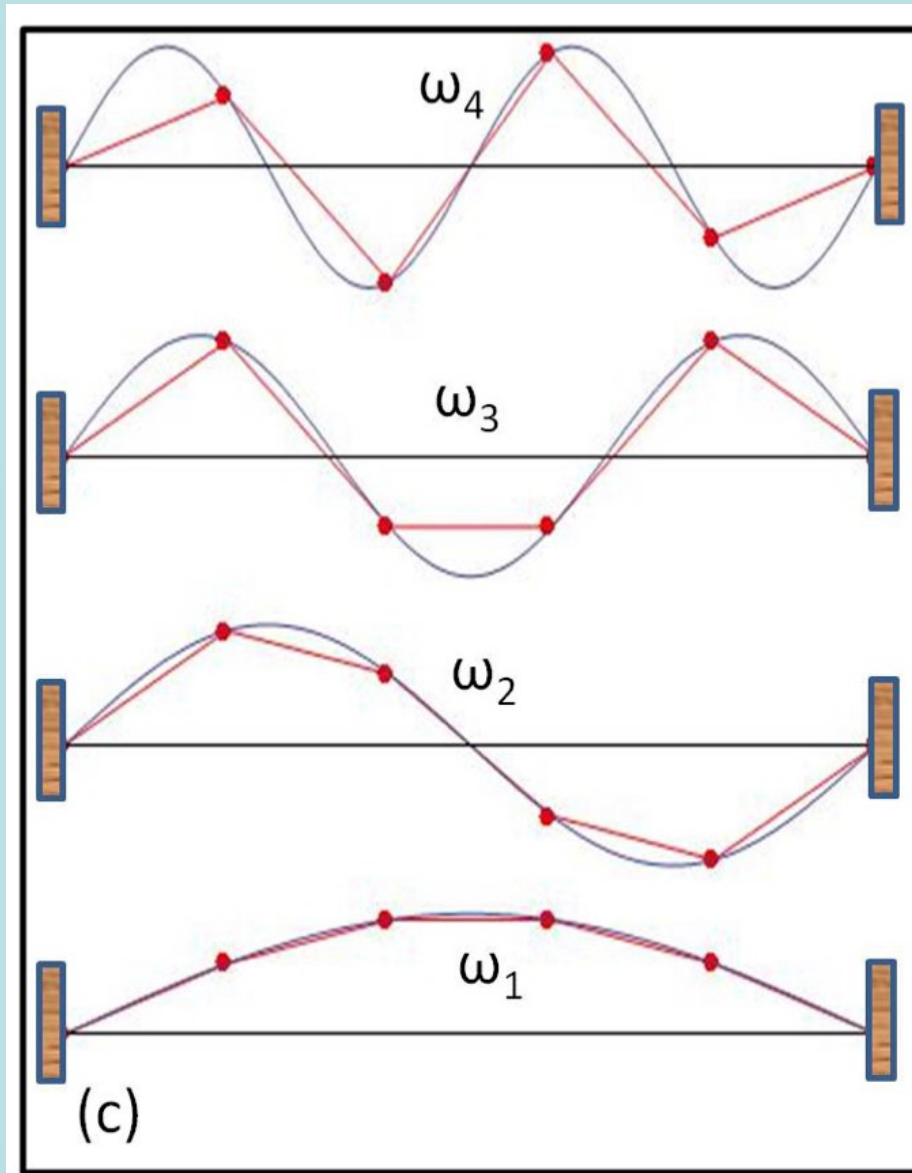
看起來就像一個正弦波的波形！



$N = 2$



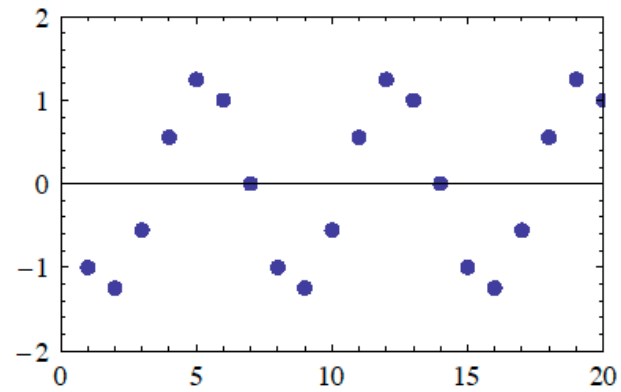
$N = 3$



$$N = 4$$

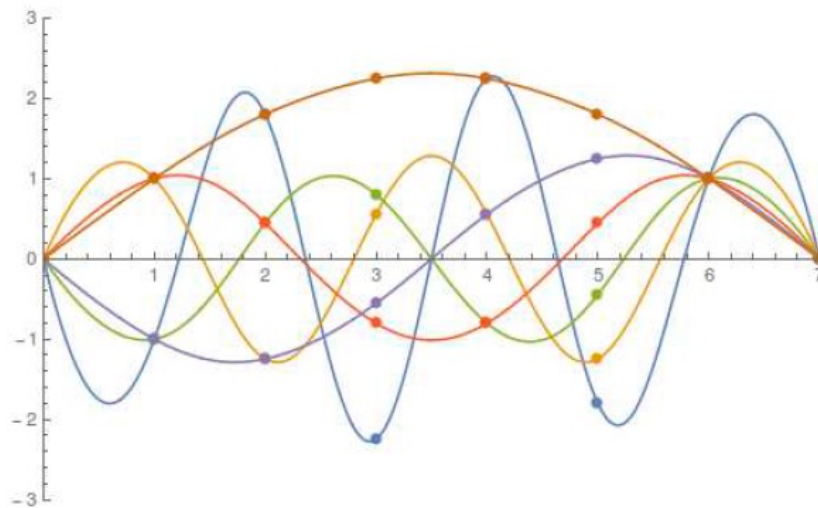
$$k = \frac{m\pi}{(N+1)d} = \frac{m\pi}{L}$$

$$\phi(x, t = 0) \sim \sin kx \text{ for } x = jd, j = 1 \cdots N$$



$$N = 20$$

$$m = 15$$



$$N = 6$$

Figure 5. Same as Figure 4, but with interpolations extended to $n = 0$ and $n = 7$ to show the boundary conditions.

求解策略：

Solving Equation of motion:

$$\frac{d^2}{dt^2} \mathbf{X} = -\mathbf{A} \cdot \mathbf{X}$$

首先時間變化為簡諧振盪的**模式解**： $\mathbf{X} = e^{i\omega t} \mathbf{a}$

$$\mathbf{X} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix} = e^{i\omega t} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

時間變化與足標 $i = 1, \dots, n$ 無關！

模式解滿足矩陣的本徵向量方程式： $(\mathbf{A} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$

本徵向量數量與粒子數目相等： $\mathbf{a}^{(i)}, i = 1, \dots, n$ ！

關鍵：方程式組是線性方程式，模式解的線性組合，還是方程式組的解。

$$\mathbf{x} = \sum_{i=1}^n [\mathbf{a}^{(i)} f_i \cdot \cos(\omega_i t) + \mathbf{a}^{(i)} g_i \cdot \sin(\omega_i t)]$$

將 $t = 0$ 的起始條件分解為本徵向量 $\mathbf{a}^{(i)}$ 的線性組合，這永遠能作到。

$$\begin{pmatrix} x_1(0) \\ \vdots \\ x_j(0) \\ \vdots \\ x_N(0) \end{pmatrix} = \sum_{i=1}^n \mathbf{a}^{(i)} f_i$$

$$\begin{pmatrix} x'_1(0) \\ \vdots \\ x'_j(0) \\ \vdots \\ x'_N(0) \end{pmatrix} = \sum_{i=1}^n \mathbf{a}^{(i)} \omega_i g_i$$

讓組合中的 $\mathbf{a}^{(i)}$ 各自作時間演化 (簡諧振盪) 後，到時間 t 再組合！

解矩陣的本徵向量方程式是關鍵，其餘都是自動完成。

起始各個配料依配方 $f_{1,2,3}$ 收集



各個配料分離烹煮 $\cos(\omega_i t)$

個自簡諧振盪後，最後合體！



=



$$\begin{pmatrix} x_1(0) \\ \vdots \\ x_j(0) \\ \vdots \\ x_N(0) \end{pmatrix} = \sum_{i=1}^n \mathbf{a}^{(i)} f_i$$

$$\begin{pmatrix} x'_1(0) \\ \vdots \\ x'_j(0) \\ \vdots \\ x'_N(0) \end{pmatrix} = \sum_{i=1}^n \mathbf{a}^{(i)} g_i$$

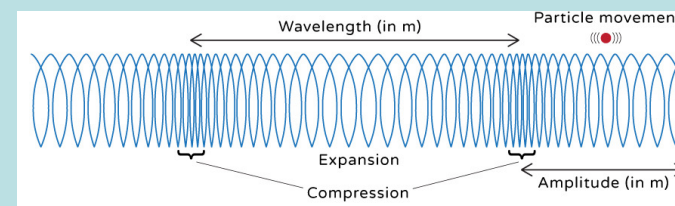
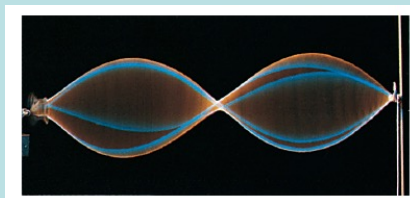
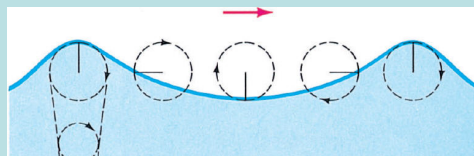
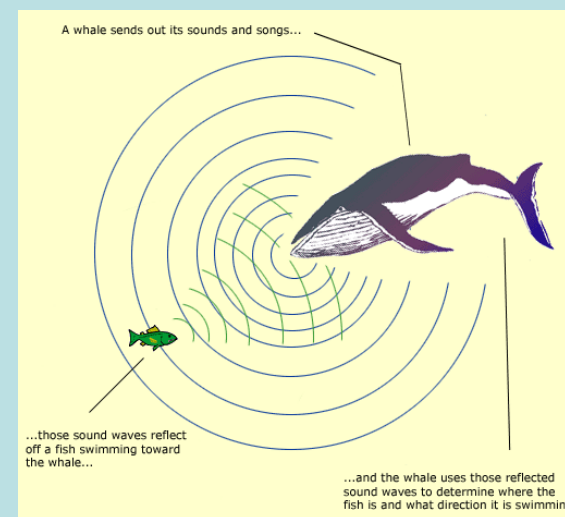
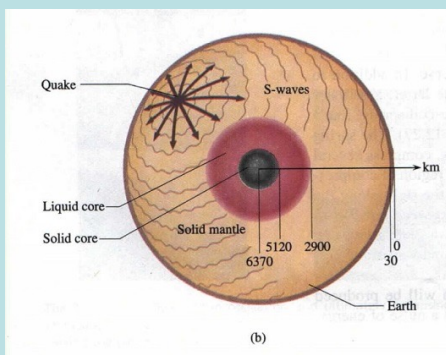
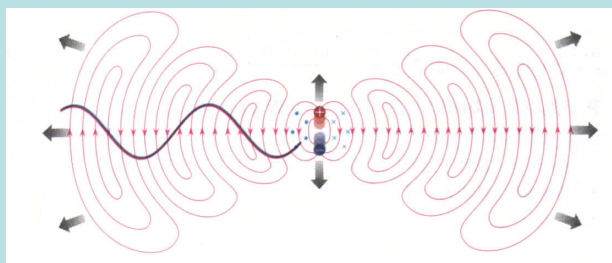
$$\mathbf{X} = \sum_{i=1}^3 [\mathbf{a}^{(i)} f_i \cdot \cos(\omega_i t) + \mathbf{a}^{(i)} g_i \cdot \sin(\omega_i t)]$$



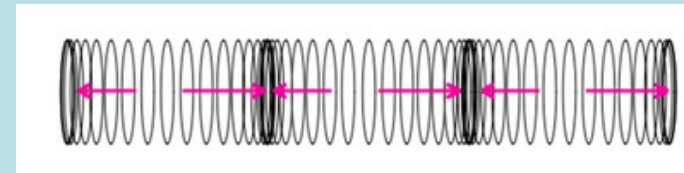
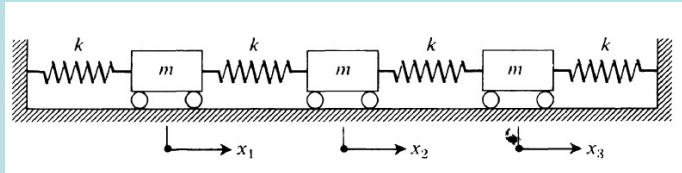
解波方程式 **Wave Equation** Equation of motion satisfied by all wave phenomena.

$$\nabla^2 y = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{3D version}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{1D version}$$



Equations of motion of coupled springs in continuous limit becomes wave equation.



$$\frac{d^2 \mathbf{X}}{dt^2} = \mathbf{A} \cdot \mathbf{X}$$

$$N \rightarrow \infty, \\ d \rightarrow 0$$

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

There is an analogy between coupled spring system and continuous elastic medium.

$$x_j$$



$$\phi(jd) = \phi(x)$$

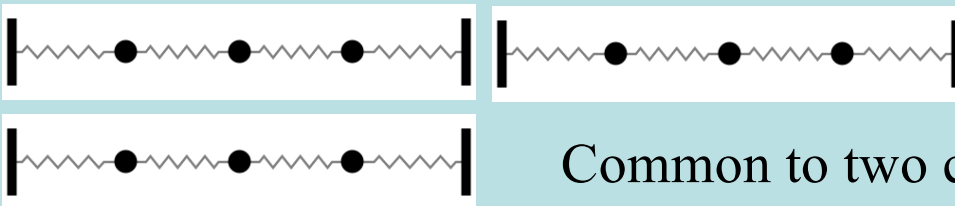
Displacement of the j th particles corresponds to the wavefunction at $jd = x$.

A series of mode solutions exist.

A series of similar solutions ϕ exist.

$$\mathbf{X} = \mathbf{a} e^{i\omega t}$$

$$\phi(x, t) = X(x) \cdot T(t)$$



Common to two cases: Time evolution can be isolated.

Time factor $e^{i\omega t}$ is separate from $\mathbf{a}$. Time factor $T(t)$ is separate from space $X(x)$.

All solutions are sums of modes.

First step of solving wave equation:

Find solutions that can be separated into a time factor $T(t)$ and a space factor $X(x)$.

$$\phi(x, t) = X(x) \cdot T(t)$$

These will be called **separable solutions**,

same name we gave to oscillation modes in coupled springs.

Time factor $T(t)$ is separate from space $X(x)$ and independent of coordinate x .

Benefit: $T(t)$ and $X(x)$ can be solved more easily.

the equilibrium configuration of a planar body, e.g., the deformations of a membrane. Separation of variables seeks special solutions that can be written as the product of functions of the individual variables, thereby reducing the partial differential equation to a pair of ordinary differential equations. More-general solutions can then be expressed as infinite series in the appropriate separable solutions. For the two-variable equations considered here, this results in a Fourier series representation of the solution. In the case of the wave equation, separation of variables serves to focus attention on the vibrational character of the solution, whereas the earlier d'Alembert approach emphasizes its particle-like aspects.

Chapter 4

Separation of Variables

Three cardinal linear second-order partial differential equations have collectively driven the development of the entire subject. The first two we have already encountered: The *wave equation* describes vibrations and waves in continuous media, including sound waves, water waves, elastic waves, electromagnetic waves, and so on. The *heat equation* models diffusion processes, including thermal energy in solids, solutes in liquids, and biological populations. Third, and in many ways the most important of all, is the *Laplace equation* and its inhomogeneous counterpart, the *Poisson equation*, which govern equilibrium mechanics. The latter two equations arise in an astonishing variety of mathematical and physical contexts, ranging through elasticity and solid mechanics, fluid mechanics, electromagnetism, potential theory, thermomechanics, geometry, probability, number theory, and many other fields. The solutions to the Laplace equation are known as harmonic functions, and the discovery of their many remarkable properties forms one of the most celebrated chapters in the history of mathematics. All three equations, along with their multi-dimensional kin, will appear repeatedly throughout this text.

The aim of the current chapter is to develop the method of separation of variables for solving these key partial differential equations in their two-independent-variable incarnations. For the wave and heat equations, the variables are time, t , and a single space coordinate, x , leading to initial-boundary value problems modeling the dynamical behavior of a one-dimensional medium. For the Laplace and Poisson equations, both variables represent space coordinates, x and y , and the associated boundary value problems model the equilibrium configuration of a planar body, e.g., the deformations of a membrane. Separation of variables seeks special solutions that can be written as the product of functions of the individual variables, thereby reducing the partial differential equation to a pair of ordinary differential equations. More-general solutions can then be expressed as infinite series in the appropriate separable solutions. For the two-variable equations considered here, this results in a Fourier series representation of the solution. In the case of the wave equation, separation of variables serves to focus attention on the vibrational character of the solution, whereas the earlier d'Alembert approach emphasizes its particle-like aspects. Unfortunately, for the Laplace equation, separation of variables applies only to boundary value problems in very special geometries, e.g., rectangles and disks. Further development of the separation of variables method for solving partial differential equations in three or more variables can be found in Chapters 11 and 12.

Undergraduate Texts in Mathematics

UTM

Peter J. Olver

Introduction to Partial Differential Equations

Find separable solutions:

$$\phi(x, t) = X(x) \cdot T(t)$$

Plug it into the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$X \frac{\partial^2 T}{\partial t^2} = T v^2 \frac{\partial^2 X}{\partial x^2}$$



$$X \frac{d^2 T}{dt^2} = T v^2 \frac{d^2 X}{dx^2}$$

Note that partial differentiation becomes ordinary differentiation.

Divide both sides by $v^2 X(x)T(t)$.

$$\frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

Now left-hand side is only t related while right-hand side only x related.

But x, t are independent variables. General equality is impossible.

The only exception: the two formulae are both constants. Call it λ : separation constant

$$\frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} \equiv \lambda$$



$$\frac{d^2 X}{dx^2} = \lambda \cdot X$$

$$\frac{d^2 T}{dt^2} = v^2 \lambda \cdot T$$

We find **separate** equations for time factor $T(t)$ and space factor $X(x)$ ◦

Even better, equations are both **ordinary differential equation** instead of partial DE.

The time factor $T(t)$ can be determined right away

$$\frac{d^2T}{dt^2} = v^2 \lambda T \equiv -\omega^2 T$$

If $\lambda < 0$, $T(t)$ is a SHM with angular frequency $\omega = v\sqrt{-\lambda}$.

$$T(t) = ce^{i\omega t} = a_m e^{i\phi} e^{i\omega t}$$

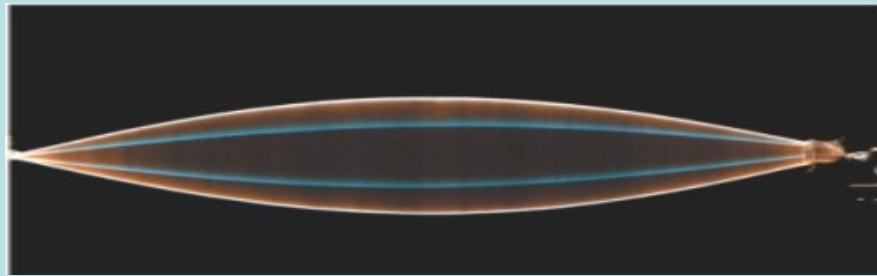
or

$$T(t) = \text{Re } a_m e^{i\phi} e^{i\omega t} = a_m \cos(\omega t + \phi) = f \cos(\omega t) + g \sin(\omega t)$$

Hence a separable solutions of wave equation is also an oscillation mode solution.

$X(x)$ satisfies a standard 2nd order ODE.

$$\frac{d^2 X}{dx^2} = \lambda X$$



The medium with finite size usually comes with Boundary Condition 邊界條件,

$$\phi(0, t) = \phi(L, t) = 0$$

Fixed End Boundary Condition

$$\frac{d^2 X}{dx^2} = \lambda X$$

$$k \equiv \sqrt{-\lambda}$$

If $\lambda < 0$ $X(x) = \text{Re}(A_m e^{i\phi'} e^{\pm ikx}) = A_m \cos(kx + \phi') = A \cos kx + B \sin kx$

Choosing B equal to zero, X automatically satisfy one of boundary conditions:

$$\phi(0, t) = 0 \rightarrow X(0) = 0 \quad X(x) = A \sin kx$$

The other boundary condition will restrict k we can choose:

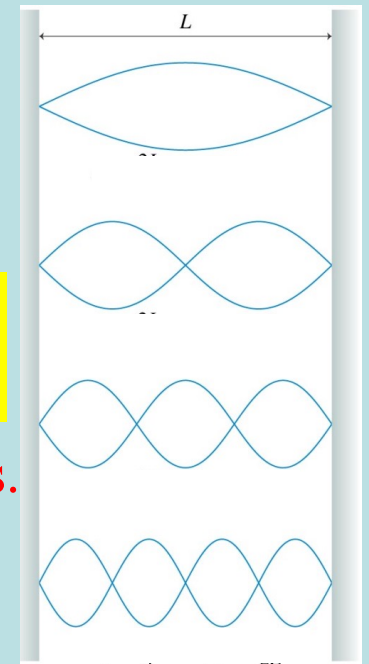
$$\phi(L, t) = 0 \rightarrow$$

$$X(L) \sim \sin(kL) = 0 \rightarrow k_n = \frac{n\pi}{L} = \sqrt{-\lambda_n} \quad \text{Wavelength} = \frac{2L}{n}$$

These $\lambda_n = -\frac{n^2\pi^2}{L^2}$ that allow a solution for X are called eigenvalues.

The solutions $X_n(x)$ for X are called eigenfunctions.

$$X_n(x) = A \sin\left(\frac{n\pi}{L} x\right) \quad n = 1, 2, \dots, \infty$$



These names remind us X_n are analogs of eigenvectors of matrix in coupled springs.

對照表

$$\frac{d^2 \mathbf{X}}{dt^2} = \mathbf{A} \cdot \mathbf{X}$$



$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

$\mathbf{A}$

$\frac{d^2}{dx^2}$ operator

$$\mathbf{A} \cdot \mathbf{a}^{(n)} = \lambda_n \mathbf{a}^{(n)}$$

eigenvalue equation of matrix $\mathbf{A}$

$$\frac{d^2 X_n(x)}{dx^2} = \lambda_n X_n(x)$$

eigenvalue equation of operator $\frac{d^2}{dx^2}$

For some function, second order derivative acting on it gives proportionally identical function.

eigenvector

$\mathbf{a}^{(n)}$

$X_n(x)$ eigenfunction

eigenvalue

λ_n

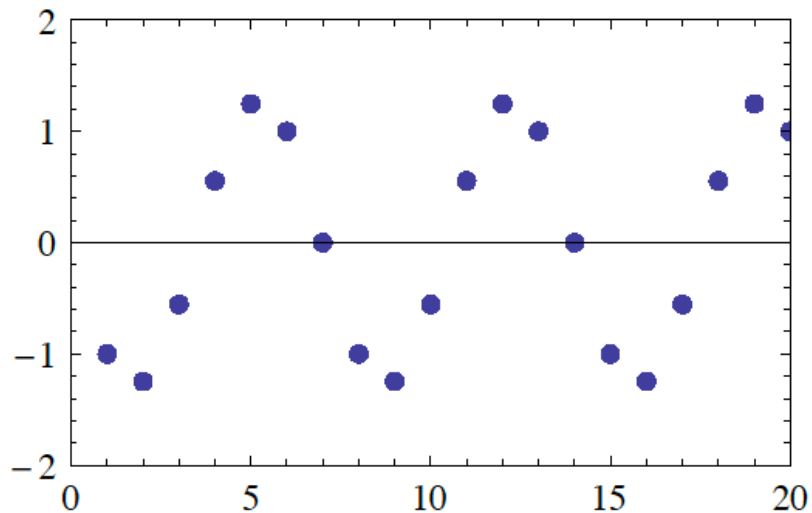
λ_n eigenvalue

$$a_j^{(p)} = \phi(dj, t = 0) \sim \sin jp \quad J\text{th component of the } p\text{th eigenvector}$$

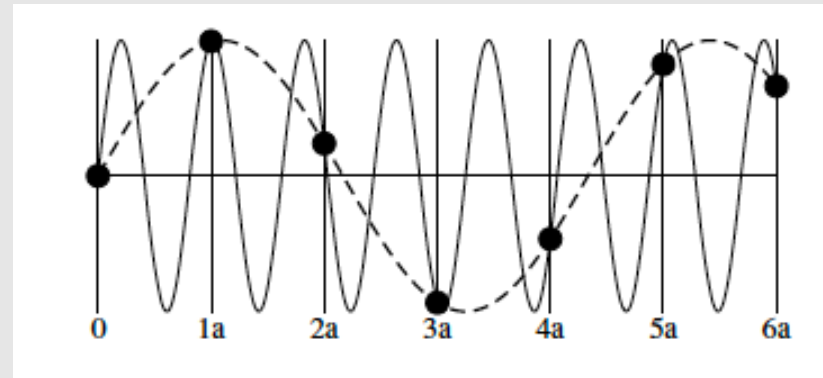
In continuous limit $N \rightarrow \infty, d \rightarrow 0$:

$$= \sin \frac{p}{d} \cdot jd \rightarrow \sin kx \quad \text{for } x = jd, j = 1 \cdots N$$

$$\frac{p}{d} \equiv k = \frac{m\pi}{(N+1)d} \rightarrow \frac{m\pi}{L}$$



$$\phi(x, t = 0) \rightarrow \sin kx \quad \text{for } x = jd, j = 1 \cdots N$$



Modes $a_j^{(p)}$ for finite N already look like a sine function.

They become $X_n = \sin(k_n x)$ exactly a sine function for infinite $N \rightarrow \infty$.

$$\frac{d^2 X_n}{dx^2} = \lambda_n X_n$$

With λ_n, k_n , the angular frequency of T corresponds to ω_n :

$$\frac{d^2 T}{dt^2} = v^2 \lambda_n T \equiv -\omega_n^2 T$$

$T(t)$ is a SHM with angular frequency $\omega_n = v\sqrt{-\lambda_n} = vk_n$.

$$T(t) = f_n \cos(\omega_n t) + g_n \sin(\omega_n t)$$

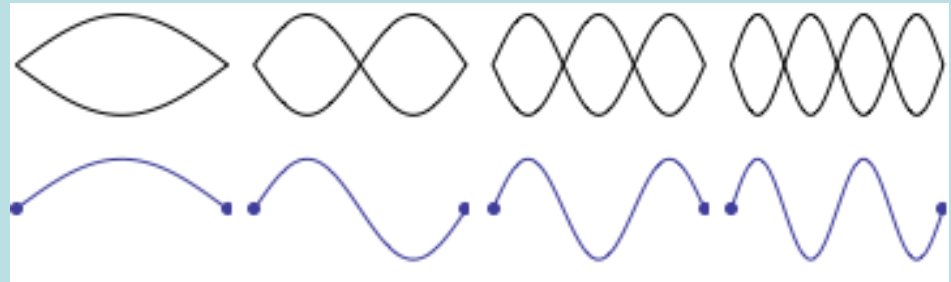
$$\omega_n = v \frac{n\pi}{L}$$

These separable solutions we will call $u^{(n)}$:

$$u^{(n)}(x, t) = X_n(x) \cdot T(t) = \sin(k_n x) \cdot (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$k_n = \frac{n\pi}{L}$$

$$\omega_n = vk_n = \frac{n\pi}{L} v$$



The time factor $T(t)$ is an oscillating function of angular frequency ω_n , and the space factor $X(x)$ is a periodic function of angular wave number k_n .

These are typical standing waves.

All particles oscillate as SHM by identical ω_n and phase, amplitudes $\propto \sin\left(\frac{n\pi}{L} x\right)$.

$$u^{(n)}(x, t) = \sin\left(\frac{n\pi}{L} x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

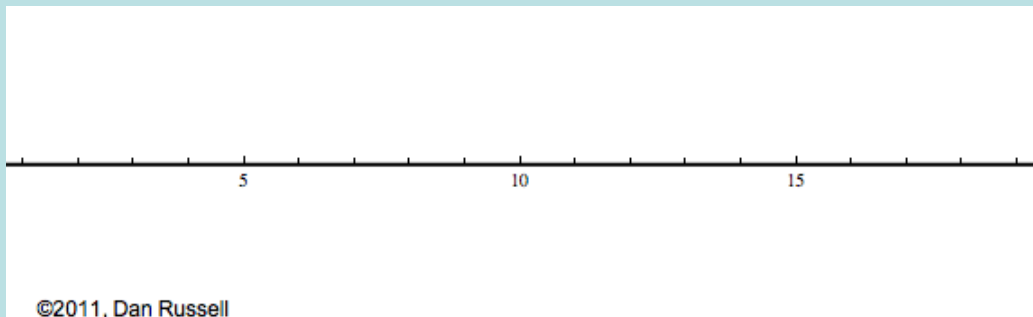
It could also be seen as a Sine function of x with time varying amplitude.

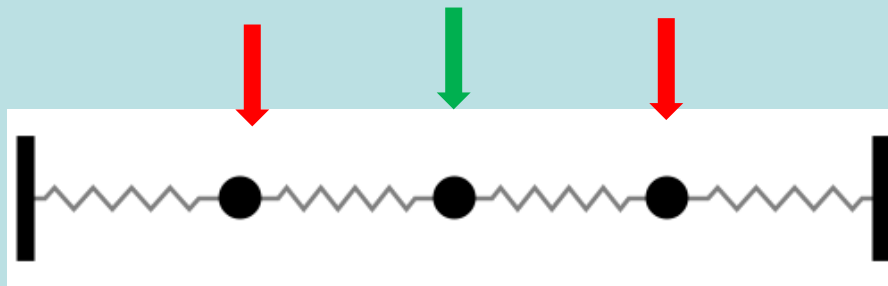
Oscillation modes in coupled springs system hold same proportion all the time:

$$x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = 1:-\sqrt{2}:1$$

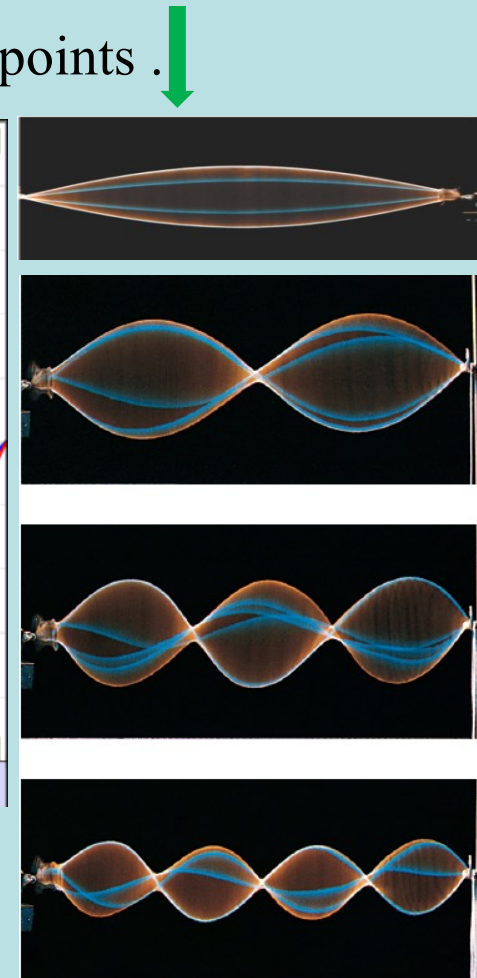
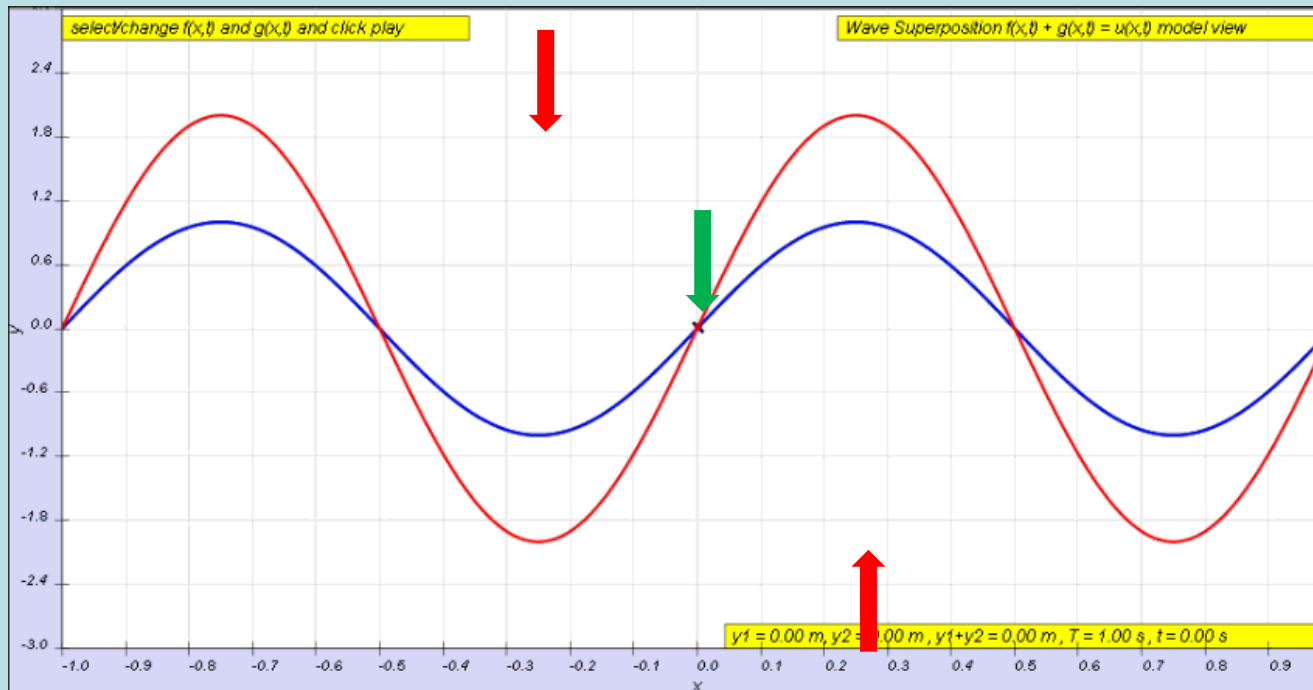
Here wavefunction at different locations are proportional to $\sin\left(\frac{n\pi}{L} x\right)$ all the time!

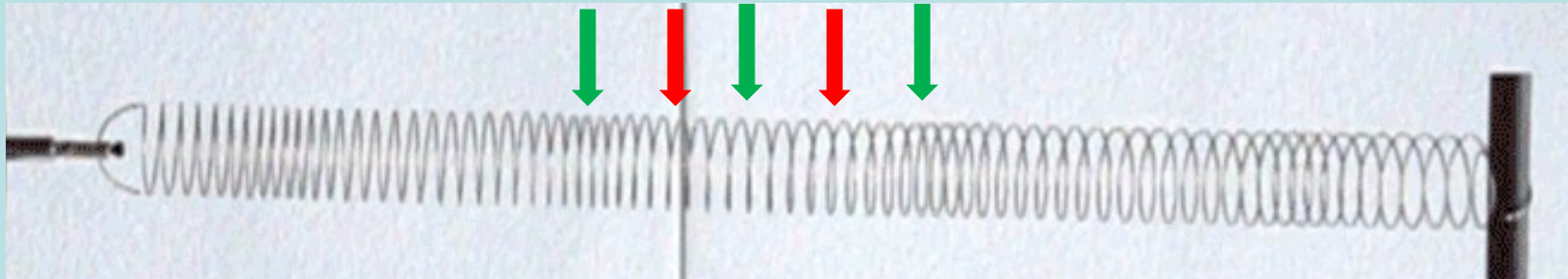
$$\frac{u^{(n)}(x_1, t)}{u^{(n)}(x_2, t)} = \frac{X_n(x_1)}{X_n(x_2)} = \frac{\sin\left(\frac{n\pi}{L} x_1\right)}{\sin\left(\frac{n\pi}{L} x_2\right)}$$





Note that there are points with no oscillation, called node points .





8.2 THE METHOD OF SEPARATION OF VARIABLES

Since the additional conditions imposed on $u(x, t)$ in our string problem fall into two groups, (a) those involving x (boundary conditions) and (b) those involving t (initial conditions), it may be reasonable to seek solutions of the PDE in the form

$$u(x, t) = X(x)T(t),$$

where X is a function of x only and T is a function of t only. If $X(x)$ is chosen to satisfy the conditions

$$X(0) = 0, \quad X(L) = 0,$$

then the function $u(x, t)$ will satisfy the same conditions. Then $T(t)$ may, perhaps, be chosen to satisfy the initial conditions.

We now require that $u(x, t)$ satisfy the PDE. We have

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{d^2 X(x)}{dx^2} T(t), \quad \frac{\partial^2 u(x, t)}{\partial t^2} = X(x) \frac{d^2 T(t)}{dt^2}.$$

Therefore

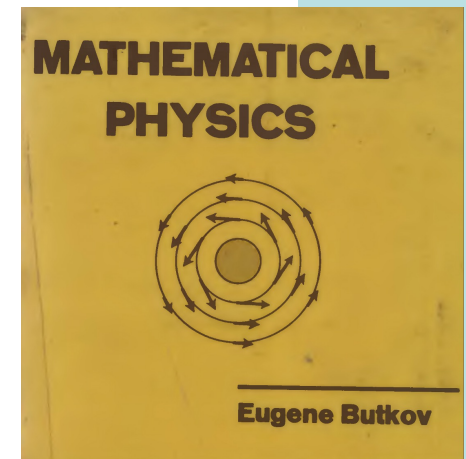
$$\frac{d^2 X}{dx^2} T = \frac{1}{c^2} X \frac{d^2 T}{dt^2}.$$

Dividing both sides by $X(x)T(t)$, we obtain

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2}.$$

The left-hand side of this equation depends on x alone; the right-hand side depends on t alone. If this equality is to hold for all x and t , it is evident that either side must be a constant (same for both sides):

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -k^2, \quad \frac{1}{T} \frac{d^2 T}{dt^2} = -k^2,$$



The constant λ is known as a *separation constant*. The equation for $X(x)$ can be written as

$$\frac{d^2 X}{dx^2} = \lambda X$$

and will lead to exponential functions if $\lambda > 0$, to trigonometric functions if $\lambda < 0$, and to a linear function if $\lambda = 0$:

$$X(x) = \begin{cases} Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}} & (\lambda > 0), \\ A' \cos(x\sqrt{-\lambda}) + B' \sin(x\sqrt{-\lambda}) & (\lambda < 0), \\ A''x + B'' & (\lambda = 0). \end{cases}$$

It is not difficult to verify that the boundary conditions $X(0) = 0$, $X(L) = 0$ can be satisfied *only* if $\lambda < 0$ and, moreover, *only* if A' is set equal to zero and the values of λ satisfy the condition

$$\sqrt{-\lambda} = n\pi/L \quad (n = 1, 2, 3, \dots).$$

Exercise. Show, in detail, that it is possible to satisfy either $X(0) = 0$ or $X(L) = 0$, but not both, if $\lambda \geq 0$. Also, prove the statement made for the case $\lambda < 0$.

These “allowed” values of the separation constant λ ,

$$\lambda_n = -n^2\pi^2/L^2 \quad (n = 1, 2, 3, \dots),$$

are usually called the *eigenvalues*, or *characteristic values*, of the problem under consideration.* By this we mean the problem of finding functions satisfying the given DE *and* the given boundary conditions. In our case there is an infinity of such functions, called *eigenfunctions*, and they read

$$X_n(x) = B'_n \sin(n\pi x/L) \quad (n = 1, 2, 3, \dots),$$

where B'_n is an arbitrary (nonzero) constant which may, in general, be different for different eigenfunctions.

In our problem of the stretched string the function $T(t)$ which is multiplied by $X(x)$ must satisfy the DE with the same separation constant as $X(x)$. Therefore, to each eigenfunction $X_n(x)$ there corresponds a function $T_n(t)$ satisfying

$$\frac{1}{c^2} \frac{1}{T_n} \frac{d^2 T_n}{dt^2} = \lambda_n = -\frac{n^2 \pi^2}{L^2}.$$

This yields

$$T_n(t) = C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L},$$

where C_n and D_n are arbitrary constants.

* The word *eigenvalue* is an adaptation of the German term *eigenwert* derived from *eigen* = proper and *wert* = value.

Summarizing our results we may say that the attempt to find the solution of our PDE with given boundary conditions and initial conditions in the form

$$u(x, t) = X(x)T(t)$$

leads us, so far, to an *infinite number* of such functions which may be written as

$$u_n(x, t) = \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

where $A_n = B'_n C_n$ and $B_n = B'_n D_n$ are arbitrary constants. Each of these functions $u_n(x, t)$ satisfies the PDE and the boundary conditions. It remains for us to select from among these functions, adjusting the constants A_n and B_n , those functions that will also satisfy the desired initial conditions. Before we do this, however, note that each function $u_n(x, t)$ represents, on its own, *some* kind of possible motion of the stretched string (corresponding to some special initial conditions). These types of motion are known as the *characteristic modes* (or *normal modes*) of vibration of the string. Each one represents a harmonic motion (vibration) with the characteristic frequency (or “eigenfrequency”)

$$\omega_n = n\pi c/L \quad (n = 1, 2, 3, \dots).$$

A bit of reflection will show that it is not possible to satisfy *arbitrary* initial conditions

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = v_0(x)$$

by any *single* function $u_n(x, t)$. Indeed, any particular function $u_n(x, t)$ will satisfy initial conditions of the form

$$u_0(x) = A_n \sin \frac{n\pi x}{L}, \quad v_0(x) = B_n \frac{cn\pi}{L} \sin \frac{n\pi x}{L}$$

and *no others*. To overcome this difficulty we can lean back on the powerful *principle of superposition*. Our PDE is *linear homogeneous*, and so are our boundary conditions. It is trivial to verify that if two functions $u_n(x, t)$ and $u_m(x, t)$ satisfy the PDE and the boundary conditions, then their linear combination

$$f(x, t) = C_1 u_n(x, t) + C_2 u_m(x, t)$$

will also satisfy the PDE and the boundary conditions. By induction this will also be true for a linear combination of a *finite* number of functions $u_n(x, t)$. It is not unreasonable to conjecture that the same properties will hold for an *infinite series* formed by the functions $u_n(x, t)$:

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

provided the series converges (or if not, provided it can be treated as a distribution, in conformity with the principles stated in Chapter 6).

To be more precise, the function defined above certainly satisfies the boundary conditions

$$y(0, t) = y(L, t) = 0$$

(since each term satisfies them). It will also satisfy the PDE, *provided* it can be differentiated twice term by term with respect to both x and t . From the point of view of the classical theory of functions, this may impose considerable restrictions on A_n and B_n ; e.g., the twice differentiated series $y(x, t)$ must be uniformly convergent. On the other hand, from the point of view of the theory of distributions it is sufficient* that A_n and B_n are of the order of some power of n :

$$|A_n| < An^N, \quad |B_n| < Bn^N \quad (\text{some } N),$$

and this point of view may be adequate for physical applications.

In any case, however, the justification of the method depends on the properties of A_n and B_n , and it is usually convenient to *assume* its validity and *justify it later*, after the solution has been developed.

It is needless to emphasize that the function $y(x, t)$ is a Fourier sine series in x (it is also a Fourier series in t , but this fact is of much less importance). Setting $t = 0$ and using the first initial condition, we obtain

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

If the function $u_0(x)$ can be expanded into a Fourier sine series, the coefficients A_n can be determined. In physical problems $u_0(x)$ is invariably continuous, piecewise very smooth, and vanishes at $x = 0$ and $x = L$. Therefore, it can be represented as above.

Similarly, calculating $(\partial u / \partial t)(x, t)$ and using the second initial condition, we can obtain

$$v_0(x) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{L} \sin \frac{n\pi x}{L}.$$

In physical problems $v_0(x)$ is sometimes assumed to be discontinuous. However, it is invariably piecewise continuous and piecewise very smooth, and the coefficients B_n can be determined as well.

Consequently, we have constructed a solution to our problem in the form of a series

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

which satisfies the boundary conditions and initial conditions for all physically reasonable functions $u_0(x)$ and $v_0(x)$.

Two extensions of the separation of variables method

若是在三度空間，波函數可以分離為 $\vec{r}, t$ 個別的函數的模式解：

$$\phi(\vec{r}, t) = \psi(\vec{r}) \cdot T(t)$$

Method of Separation of Variables

代入波方程式：

$$\frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi$$

得到：

$$\frac{1}{v^2} \psi \frac{\partial^2 T}{\partial t^2} = T \nabla^2 \psi$$

一般會把 v^2 移到左邊，同時左右邊都除以 $\psi(\vec{r}) \cdot T(t)$ ：

$$\frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{\psi} \nabla^2 \psi$$

現在左邊只與 t 有關，右邊只與 $\vec{r}$ 有關，但兩者是獨立變數！相等並不可能。

唯一的例外：左右兩式都與各自的變數無關，是一常數。設此常數為 λ 。

$$\frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{\psi} \nabla^2 \psi \equiv \lambda$$



$$\frac{d^2 T}{dt^2} = v^2 \lambda \cdot T$$

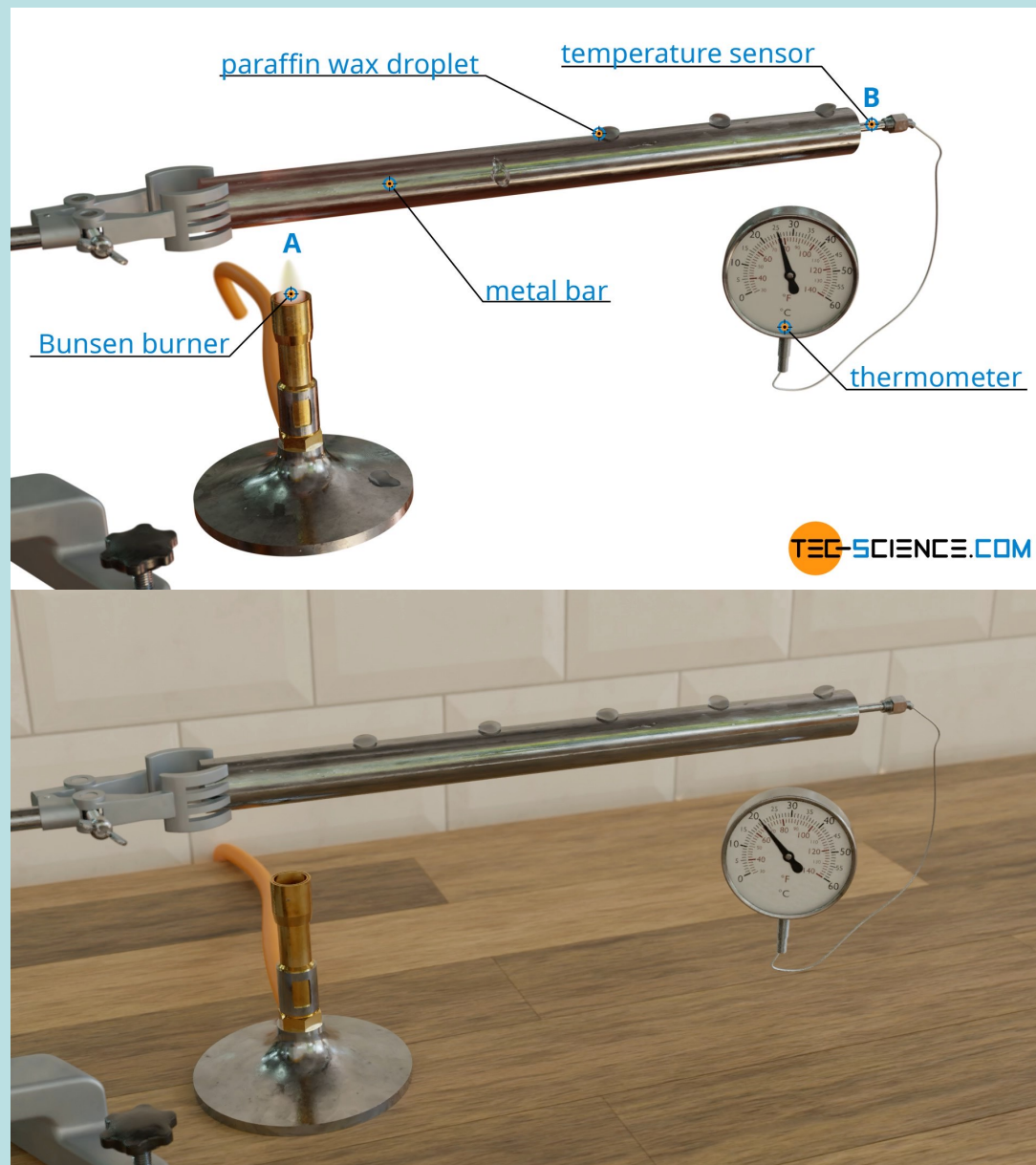
$$\nabla^2 \psi = \lambda \cdot \psi$$

Helmholtz Equation

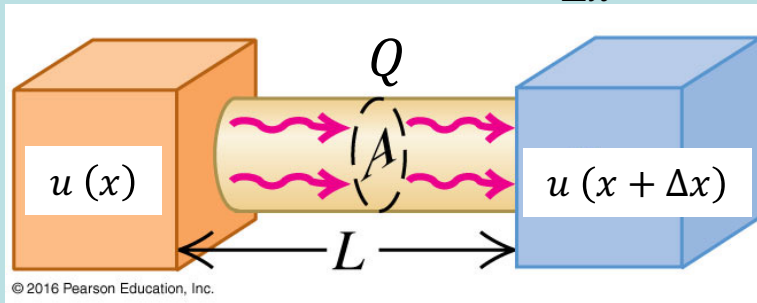


Hermann von Helmholtz 1821-1894

Temperature distribution of a heat conducting bar



Treat the bar as a series of points with temperature distribution $u(x, t)$.
 Temperature difference $\frac{\Delta u}{\Delta x}$ at neighboring point drive heat flow $\frac{dQ}{dt}$.

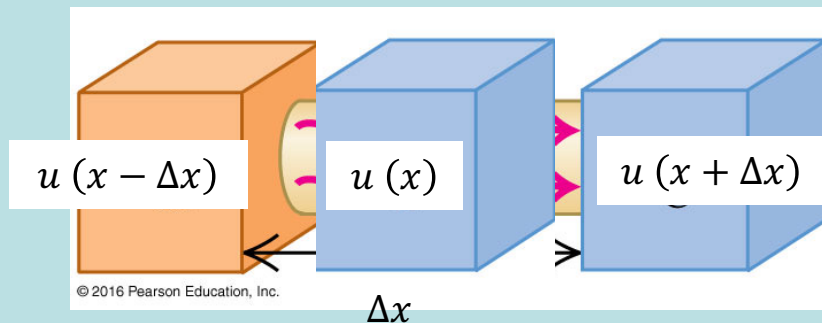


$$\frac{dQ}{dt} = h \frac{\Delta u}{\Delta x} \rightarrow h \frac{\partial u}{\partial x}$$

fixed t

$$H = \frac{dQ}{dt} = kA \frac{T_H - T_C}{L}$$

Labels in diagram:
 - Rate of heat flow: H
 - Heat current in conduction: H
 - Thermal conductivity of rod material: k
 - Cross-sectional area of rod: A
 - Length of rod: L
 - Temperatures of hot and cold ends of rod: T_H and T_C



Net heat flow into a point on the bar change its temperature $\frac{\partial u}{\partial t}$.

Net heat flow equals heat flow $\frac{dQ}{dt}$ to the right minus from the left.

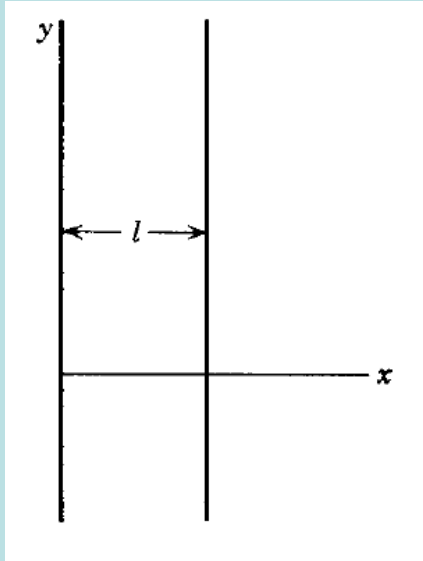
$$C(\rho A \Delta x) \frac{\partial u}{\partial t} = h \frac{\partial u}{\partial x}(x + \Delta x) - h \frac{\partial u}{\partial x}(x) = h \Delta x \frac{\partial^2 u}{\partial x^2}$$

fixed x fixed t

$$\frac{1}{\alpha^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{1}{\alpha^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Heat Flow Equation

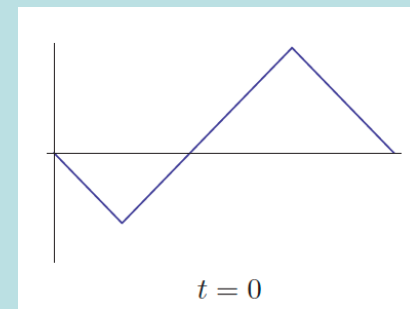


Boundary condition :

$$u(t, 0) = 273 \text{ K} \quad u(t, l) = 273 \text{ K}$$

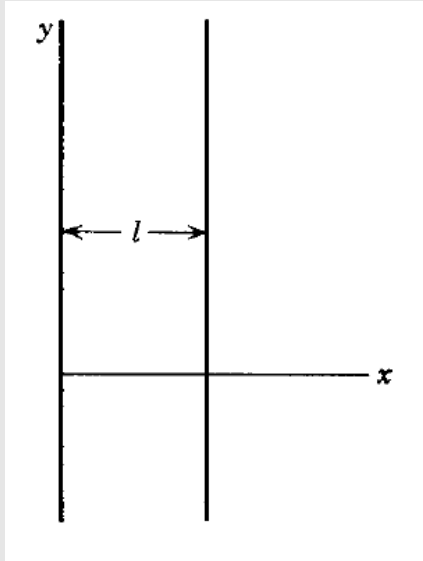
Initial condition :

$$u(0, x) = 273 \text{ K} + f(x)$$



$$\frac{1}{\alpha^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Heat Flow Equation



Boundary condition :

$$u(t, 0) = 273 \text{ K} \quad u(t, l) = 373 \text{ K}$$

Initial condition :

$$u(0, x) = 273 \text{ K}$$

$$\frac{1}{\alpha^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Heat Flow Equation

The Heated Ring

So far, we have not paid any attention to boundary conditions. As noted above, these will eliminate nonphysical eigensolutions and thereby reduce the collection to a manageable, albeit still infinite, number. In this subsection, we will discuss a particularly important case, which, following Fourier's line of reasoning, leads us directly into the heart of Fourier series.

Consider the heat equation on the interval $-\pi \leq x \leq \pi$, subject to the *periodic boundary conditions*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(t, -\pi) = u(t, \pi), \quad \frac{\partial u}{\partial x}(t, -\pi) = \frac{\partial u}{\partial x}(t, \pi). \quad (3.21)$$



HEAT CONDUCTIVITY RING - WOOD BASE

A\$19.25

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Periodic Boundary Condition

$$u(t, 0) = u(t, 2\pi)$$

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 2\pi)$$

Search for separable solutions, products of a function of x and a function of t .

$$u(x, t) = X(x) \cdot T(t) \quad \text{Method of Separation of Variables}$$

Plug in the heat flow equation :

$$\frac{1}{\alpha^2} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{1}{\alpha^2} X \frac{\partial T}{\partial t} = T \frac{\partial^2 X}{\partial x^2}$$

Devide both side by $X(x) \cdot T(t)$, and change partial d to ordinary d.

$$\frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

Now left-hand side is only t related while right-hand side only x related.

But x, t are independent variables. General equality is impossible.

The only exception: the two formulae are both constants. Call it λ .

$$\frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} \equiv \lambda$$

$X(x)$ satisfies exactly the same ODE as in wave equation.

$$\frac{d^2 X}{dx^2} = \lambda X$$

With fixed end boundary condition:

$$X_n(x) = A \sin\left(\frac{n\pi}{L} x\right) \quad n = 1, 2, \dots \infty$$

Equation for the time part is different, even simpler:

$$\frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = \lambda$$

$$\frac{dT(t)}{dt} = -\alpha^2 \lambda T(t)$$

$$T(t) = ce^{-\alpha^2 \lambda t}$$

Temperature in a conducting bar does not oscillate. It decays exponentially.

$$u_n(x, T) = c \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \quad n = 1, 2, \dots, \infty$$

Temperature difference will dissipate, decrease slowly with time.

