

Gamma Function $\Gamma(x)$

This is one of the Special Functions.

Bessel function, Legendre function etc. would be a focus this semester.

Most of them are solutions of specific ordinary differential equations.

Gamma Function is defined as

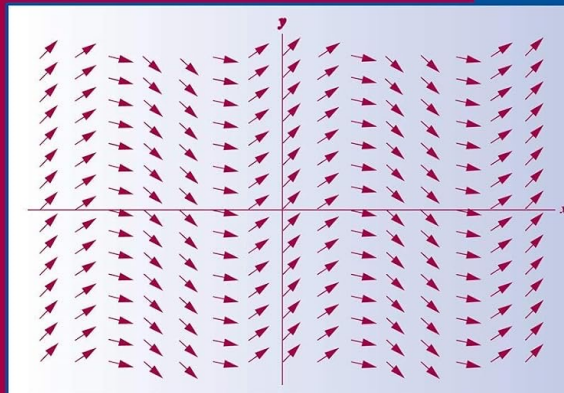
$$\Gamma(p) = \int_0^{\infty} dx \cdot x^{p-1} e^{-x}$$

It appears very often in physics

In Statistical Mechanics, we use it to compute averages in Boltzmann Distribution.

THIRD EDITION

Mathematical Methods in The Physical Sciences



Mary L. Boas

Baos Ch. 11

$$\Gamma(p) = \int_0^{\infty} dx \cdot x^{p-1} e^{-x}$$

Let's start with this simple calculation of $\Gamma(p)$ for integer p :

$$\int_0^{\infty} dx \cdot e^{-\alpha x} = -\frac{1}{\alpha} e^{-\alpha x} \Big|_0^{\infty} = 0 + \frac{1}{\alpha} = \frac{1}{\alpha}$$

Now differentiate the integral with respect to $-\alpha$ once:

$$-\frac{d}{d\alpha} \int_0^{\infty} dx \cdot e^{-\alpha x} = \int_0^{\infty} dx \cdot x e^{-\alpha x} = \frac{1}{\alpha^2}$$

Differentiate the original integral with respect to $-\alpha$ n times:

$$(-1)^n \frac{d^n}{d\alpha^n} \int_0^{\infty} dx \cdot e^{-\alpha x} = \int_0^{\infty} dx \cdot x^n e^{-\alpha x} = \frac{n!}{\alpha^n}$$

$$\alpha = 1$$



$$\Gamma(n-1) = \int_0^{\infty} dx \cdot x^n e^{-x} = n!$$

Gamma Function of positive integral n is n factorial! What a surprise!

Indeed, we can prove from definition for any p (not just integer n):

$$\Gamma(p + 1) = p\Gamma(p) \quad \text{The key equation of Gamma function.}$$

Using integration by parts to compute the left-hand side:

$$\int_a^b dx \cdot u(x)v'(x) = u(x)v(x)\Big|_a^b - \int_a^b dx \cdot u'(x)v(x)$$

$$\Gamma(p + 1) = \int_0^{\infty} dx \cdot x^p e^{-x} = -x^p e^{-x}\Big|_0^{\infty} + \int_0^{\infty} dx \cdot px^{p-1} e^{-x} = p\Gamma(p)$$

Note that if we choose p as a positive integer:

$$\Gamma(1) = \int_0^{\infty} dx \cdot e^{-x} = 1$$

Now apply the above formula, we can easily show:

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2!$$

$$\Gamma(n - 1) = n!$$

Gamma Function is n factorial in disguise.

Gamma function is not always well-defined.

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

$$\Gamma(0) = \frac{\Gamma(1)}{p \rightarrow 0} \rightarrow \infty$$

Gamma Function is infinite and divergent at $p \rightarrow 0$.

$$\Gamma(-1) = \frac{\Gamma(0)}{-1}$$

Gamma Function is divergent at negative or zero integer $p \leq 0$.

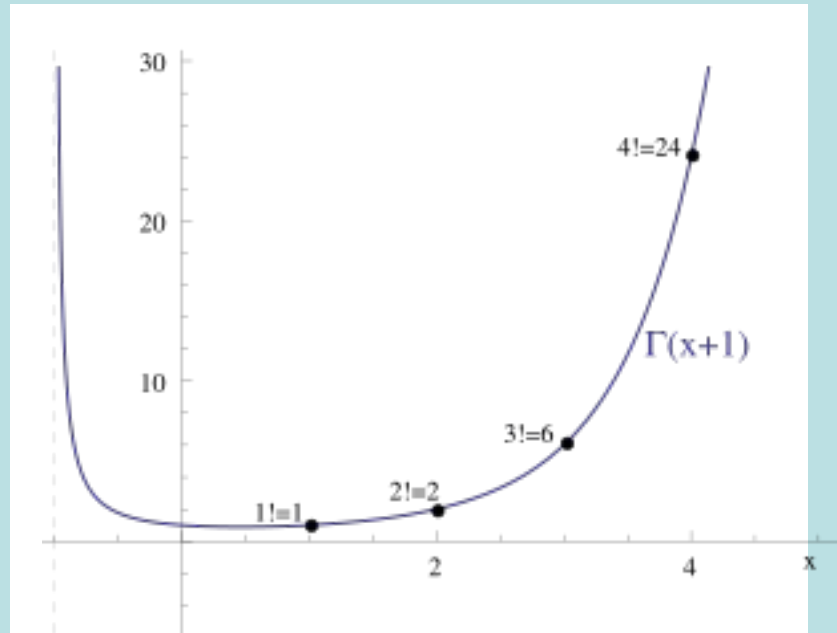
Could we can define Gamma Function for rational number (non-integer) p :

$$\Gamma(p) = \int_0^{\infty} dx \cdot x^{p-1} e^{-x}$$

The answer is yes. This integral is finite for positive and real p .

Its value is smoothly extended to $n!$ at $p = n - 1$

$$\Gamma(n - 1) = n!$$



The gamma function is a smooth curve $y = f(x)$ that connects the points of the factorial sequence: $(x, y) = (n - 1, n!)$ for all positive integer values of n .

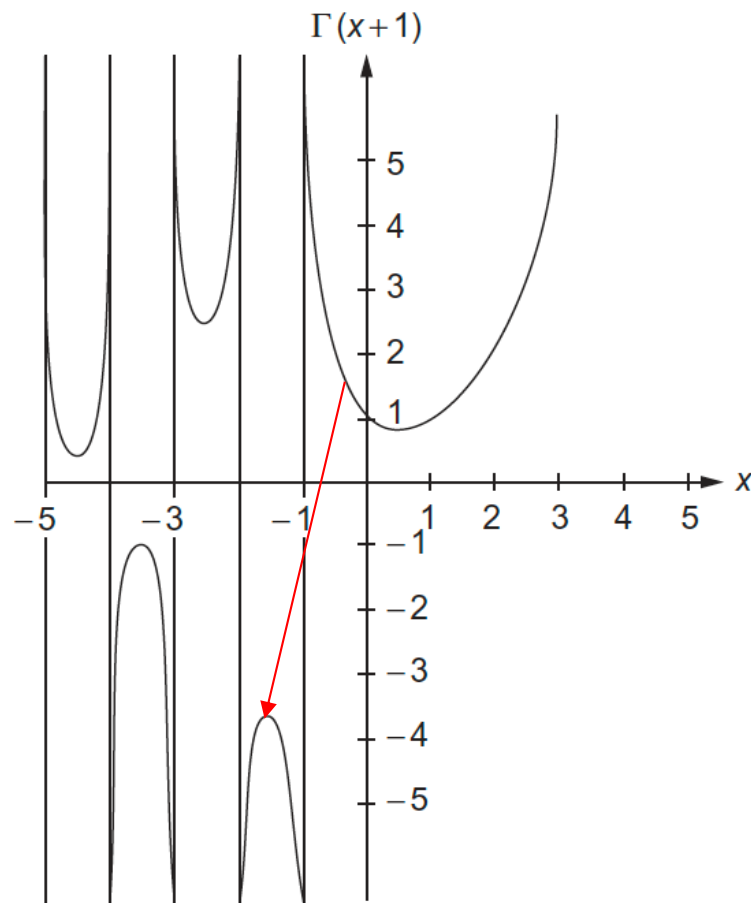


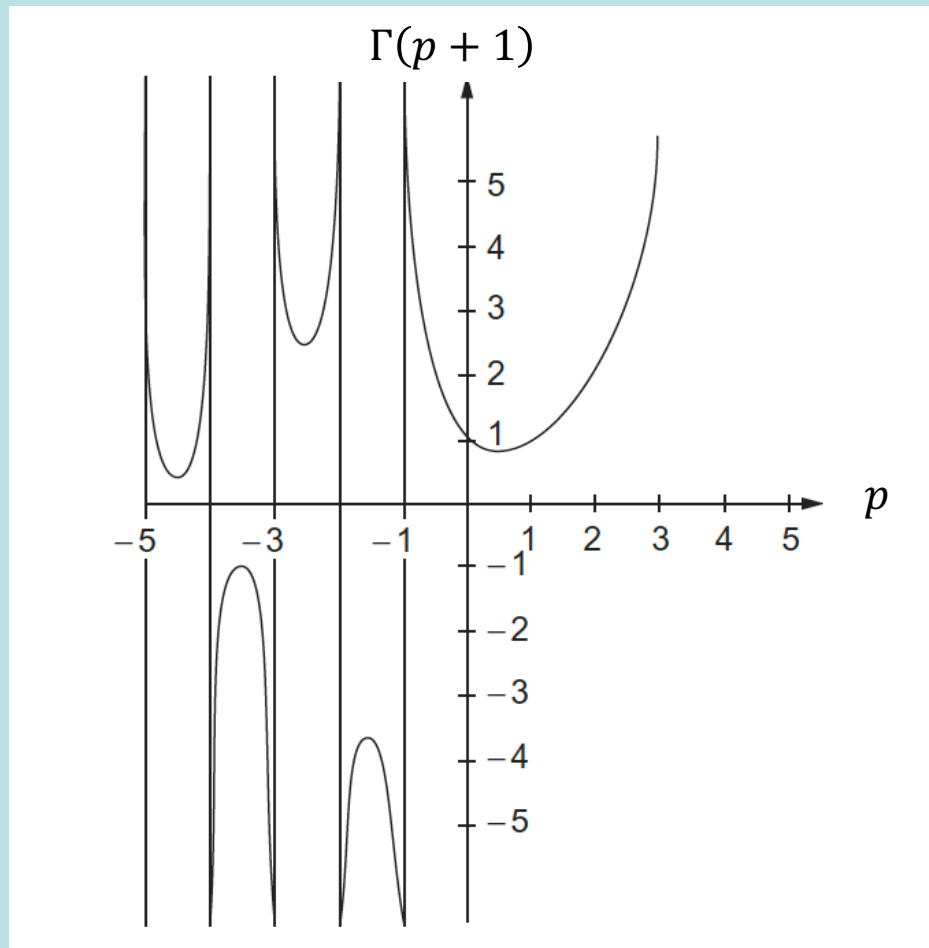
FIGURE 13.1 Gamma function $\Gamma(x + 1)$ for real x .

Remember $\Gamma(p)$ is divergent at negative integer $p < 0$.

But $\Gamma(p)$ is defined at $-1 < p < 0$ using the values at $0 < p < 1$ and

$$\Gamma(p) = \frac{\Gamma(p + 1)}{p}$$

and other intervals $-1 - m < p < -m$.



$$\Gamma(p) = \int_0^{\infty} dx \cdot x^{p-1} e^{-x}$$

$$\Gamma(n-1) = n!$$

$$\Gamma(p+1) = p\Gamma(p)$$

There are two calculable expression of Gamma function:

Euler:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z$$

We can check:

$$\Gamma(z+1) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(z+1)(z+2)(z+3) \cdots (z+n+1)} n^{z+1} =$$

$$\begin{aligned} &= z \lim_{n \rightarrow \infty} \frac{n}{(z+n+1)} \cdot \left[\frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2)(z+3) \cdots (z+n+1)} n^z \right] \\ &= z \cdot \Gamma(z) \end{aligned}$$

We are sure this formula gives:

$$\Gamma(n-1) = n!$$

You can find a rigorous proof on P601 of Arfken.

To show the equivalence of these two definitions, Eqs. (13.1) and (13.5), consider the function of two variables

$$F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt, \quad \Re(z) > 0, \quad (13.9)$$

with n a positive integer. This form was chosen because the exponential has the definition

$$\lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n \equiv e^{-t}. \quad (13.10)$$

Inserting Eq. (13.10) into Eq. (13.9), we see that the infinite- n limit of $F(z, n)$ corresponds to $\Gamma(z)$ as given by Eq. (13.5):

$$\lim_{n \rightarrow \infty} F(z, n) = F(z, \infty) = \int_0^\infty e^{-t} t^{z-1} dt \equiv \Gamma(z). \quad (13.11)$$

Our remaining task is to identify this limit also with Eq. (13.1).

Returning to $F(z, n)$, we evaluate it by carrying out successive integrations by parts. For convenience we make the substitution $u = t/n$. Then

$$F(z, n) = n^z \int_0^1 (1-u)^n u^{z-1} du. \quad (13.12)$$

The first integration by parts yields

$$\frac{F(z, n)}{n^z} = (1-u)^n \frac{u^z}{z} \Big|_0^1 + \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z du; \quad (13.13)$$

note that (because $z \neq 0$) the integrated part vanishes at both endpoints. Repeating this n times, with the integrated part vanishing at both endpoints each time, we finally get

$$\begin{aligned} F(z, n) &= n^z \frac{n(n-1) \cdots 1}{z(z+1) \cdots (z+n-1)} \int_0^1 u^{z+n-1} du \\ &= \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z. \end{aligned} \quad (13.14)$$

This is identical with the expression on the right side of Eq. (13.1). Hence

$$\lim_{n \rightarrow \infty} F(z, n) = F(z, \infty) \equiv \Gamma(z),$$

where $\Gamma(z)$ is in the form given by Eq. (13.1), thereby completing the proof.

But Euler's formula is still not easy to evaluate.

Infinite Product of Weierstrass:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Note that the expression in the infinite product is getting closer to 1 as $n \rightarrow \infty$.

To a certain accuracy, we only need to include finite number of terms.

This expression is readily computable.

Take Euler's Formula:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z$$

inverse it:

$$\frac{1}{\Gamma(z)} = \lim_{n \rightarrow \infty} z \left(\frac{1+z}{1} \right) \left(\frac{2+z}{2} \right) \cdots \left(\frac{n+z}{n} \right) n^{-z} = z \lim_{n \rightarrow \infty} \prod_{m=1}^n \left(1 + \frac{z}{m} \right) e^{-z \ln n}$$

$\ln n$ can be approximated by a series of $1/m$:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right) = \lim_{n \rightarrow \infty} \left(\sum_{m=1}^n \frac{1}{m} - \ln n \right) = \gamma$$

$$\lim_{n \rightarrow \infty} \left(\sum_{m=1}^n \frac{z}{m} + z \ln n \right) = \gamma z$$

Plug this formula back, we proved Weierstrass's formula is correct:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \lim_{n \rightarrow \infty} \prod_{m=1}^n \left(1 + \frac{z}{m} \right) e^{-\frac{z}{m}} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}}$$

You can find a rigorous proof on P603 of Arfken.

This infinite-product form is useful for proving various properties of $\Gamma(z)$. It can be derived from the original definition, [Eq. \(13.1\)](#), by rewriting it as

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1) \cdots (z+n)} n^z = \lim_{n \rightarrow \infty} \frac{1}{z} \prod_{m=1}^n \left(1 + \frac{z}{m}\right)^{-1} n^z. \quad (13.17)$$

Taking the reciprocal of [Eq. \(13.17\)](#) and using

$$n^{-z} = e^{(-\ln n)z}, \quad (13.18)$$

we obtain

$$\frac{1}{\Gamma(z)} = z \lim_{n \rightarrow \infty} e^{(-\ln n)z} \prod_{m=1}^n \left(1 + \frac{z}{m}\right). \quad (13.19)$$

Multiplying and dividing the right-hand side of [Eq. \(13.19\)](#) by

$$\exp \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) z \right] = \prod_{m=1}^n e^{z/m}, \quad (13.20)$$

we get

$$\begin{aligned} \frac{1}{\Gamma(z)} = z \left\{ \lim_{n \rightarrow \infty} \exp \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) z \right] \right\} \\ \times \left[\lim_{n \rightarrow \infty} \prod_{m=1}^n \left(1 + \frac{z}{m}\right) e^{-z/m} \right]. \end{aligned} \quad (13.21)$$

Comparing with [Eq. \(1.13\)](#), we see that the parenthesized quantity in the exponent approaches as a limit the Euler-Mascheroni constant, thereby confirming [Eq. \(13.15\)](#).

Gamma Functions at half integers $p = \frac{2m+1}{2}$ are especially useful:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} dx \cdot x^{-\frac{1}{2}} e^{-x} = \int_0^{\infty} dx \cdot \frac{1}{\sqrt{x}} e^{-x}$$

Change the variable to $x = y^2$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} dx \cdot \frac{1}{\sqrt{x}} e^{-x} = 2 \int_0^{\infty} dy \cdot y \frac{1}{y} e^{-y^2} = 2 \int_0^{\infty} dy \cdot e^{-y^2} = \int_{-\infty}^{\infty} dy \cdot e^{-y^2}$$

$\Gamma\left(\frac{1}{2}\right)$ is the integral of the Gaussian distribution.

For the general case:

$$\Gamma\left(\frac{2m+1}{2}\right) = \int_0^{\infty} dx \cdot x^{\frac{2m-1}{2}} e^{-x} = \int_0^{\infty} dx \cdot \frac{x^m}{\sqrt{x}} e^{-x} = \int_{-\infty}^{\infty} dy \cdot y^{2m} e^{-y^2}$$

We'll use $\Gamma\left(\frac{3}{2}\right)$ very soon.

$$\Gamma\left(\frac{3}{2}\right) = \int_{-\infty}^{\infty} dy \cdot y^2 e^{-y^2}$$

To compute integral of Gaussian distribution, multiply two $\Gamma\left(\frac{1}{2}\right)$ together.

$$\int_{-\infty}^{\infty} dx \cdot e^{-x^2} \int_{-\infty}^{\infty} dy \cdot e^{-y^2} = \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} dx dy \cdot e^{-(x^2+y^2)}$$

It equals surface integral on $x - y$ plane, Baos P242-245, and we can change to polar coordinates.

$$= \int_0^{\infty} dr \cdot \int_0^{2\pi} r d\theta e^{-r^2} = 2\pi \int_0^{\infty} r dr \cdot e^{-r^2}$$

$$= \pi \int_0^{\infty} d(r^2) \cdot e^{-r^2} = \pi \int_0^{\infty} dt \cdot e^{-t} = \pi$$

Baos p258

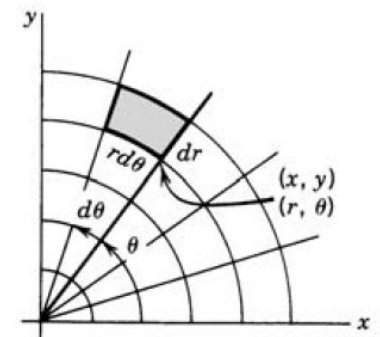
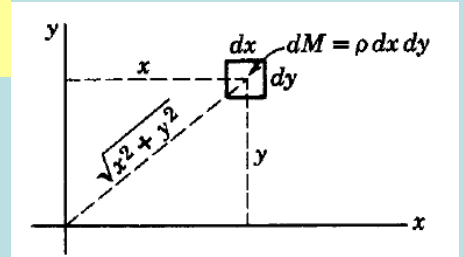


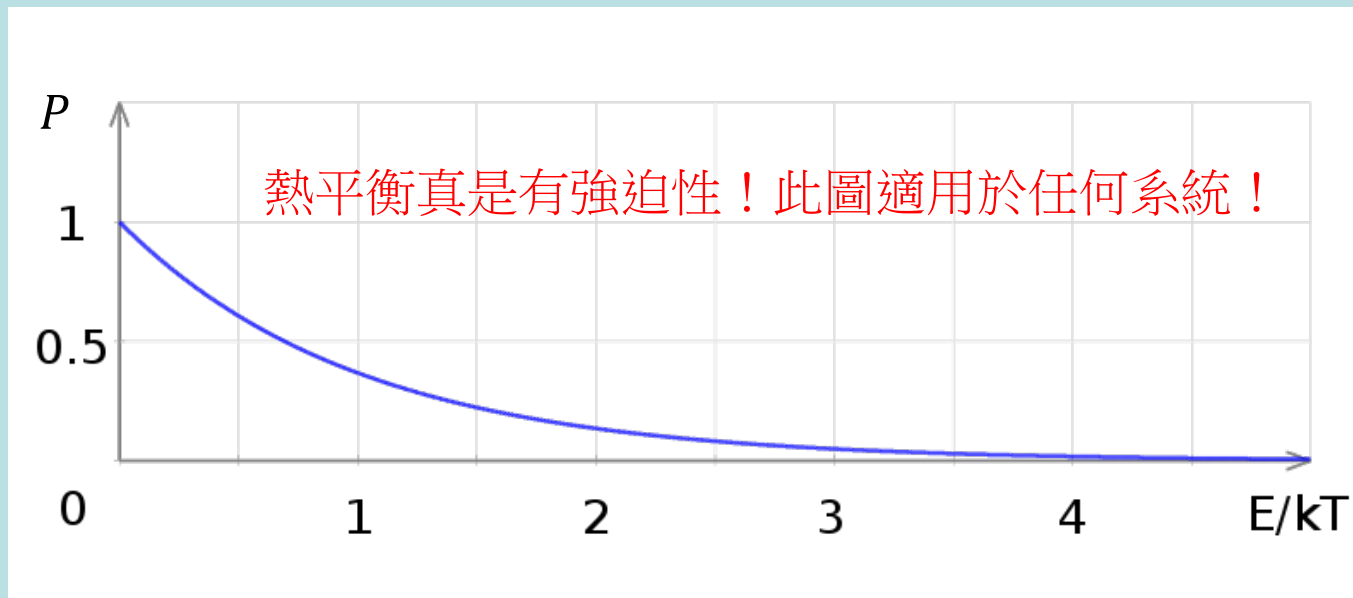
Figure 4.1

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Boltzmann Distribution分布或Canonical Distribution正則分布

$$\frac{P(s_1)}{P(s_2)} = e^{-\Delta E/kT} = \frac{e^{-E(s_1)/kT}}{e^{-E(s_2)/kT}} \quad \text{或} \quad P \propto e^{-E/kT} \quad \text{Boltzmann Factor}$$

一個系統在定溫下，出現在一特定狀態的機率正比於Boltzmann Factor。
完全由溫度與能量就可以決定！和其他細節完全無關。



$$E \gg kT, \quad P \ll 1$$

Consider a SHO in an environment of temperature T .

It means SHO interacts constantly and randomly with particles in environment.

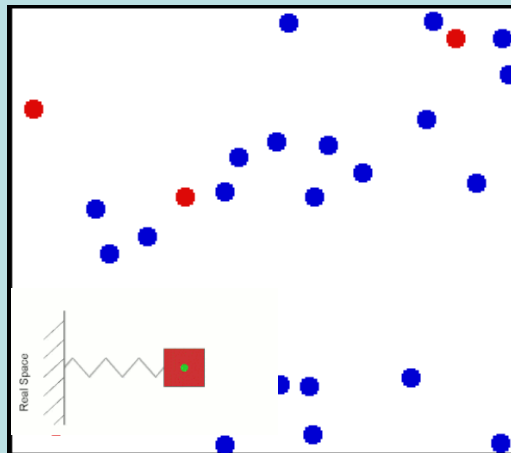
The probability $P(v)$ of finding SHO to have velocity v is $\propto$ Boltzmann Factor :

$$P(v) \propto e^{-E/kT} \propto e^{-\frac{mv^2}{2kT}}$$

The total probability $P(v)$ equals one.

Hence, we divide $e^{-\frac{mv^2}{2kT}}$ by its integration over all possible velocity to get $P(v)$.

$$P(v) = \frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2kT}}} e^{-\frac{mv^2}{2kT}}$$



$$\int_0^{\infty} dv' \cdot e^{-\frac{mv'^2}{2kT}} = \frac{1}{2} \int_{-\infty}^{\infty} dv' \cdot e^{-\frac{mv'^2}{2kT}}$$

This integration is integration of Gaussian function and hence is just $\Gamma\left(\frac{1}{2}\right)$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} dx \cdot \frac{1}{\sqrt{x}} e^{-x} = 2 \int_0^{\infty} dy \cdot y \frac{1}{y} e^{-y^2} = 2 \int_0^{\infty} dy \cdot e^{-y^2} = \int_{-\infty}^{\infty} dy \cdot e^{-y^2}$$

We just need to change variable:

$$\frac{1}{2} \int_{-\infty}^{\infty} dv' \cdot e^{-\frac{mv'^2}{2kT}} = \sqrt{\frac{kT}{2m}} \int_{-\infty}^{\infty} dy \cdot e^{-y^2}$$

$$v' = \sqrt{\frac{2kT}{m}} y$$

$$\int_0^{\infty} dv' \cdot e^{-\frac{mv'^2}{2kT}} = \sqrt{\frac{kT}{2m}} \Gamma\left(\frac{1}{2}\right)$$

We can now calculate the average of kinetic energy of SHO in T :

$$\left\langle \frac{1}{2}mv^2 \right\rangle_{\text{av}} = \int_0^{\infty} dv \cdot \frac{1}{2}mv^2 \cdot P(v) = \frac{1}{\int_0^{\infty} dv \cdot e^{-\frac{mv^2}{2kT}}} \int_0^{\infty} dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2kT}}$$

$$\int_0^{\infty} dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2kT}} = \sqrt{\frac{kT}{2m}} \int_{-\infty}^{\infty} dy \cdot kTy^2 e^{-y^2}$$

$$v = \sqrt{\frac{2kT}{m}} y$$

Compared with the formula:

$$\Gamma\left(\frac{3}{2}\right) = \int_{-\infty}^{\infty} dy \cdot y^2 e^{-y^2}$$

The above integration is just $kT \sqrt{\frac{kT}{2m}} \Gamma\left(\frac{3}{2}\right)$.

$$\left\langle \frac{1}{2}mv^2 \right\rangle_{\text{av}} = \frac{1}{\int_0^{\infty} dv \cdot e^{-\frac{mv^2}{2kT}}} \int_0^{\infty} dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2kT}} = \frac{kT \sqrt{\frac{kT}{2m}} \Gamma\left(\frac{3}{2}\right)}{\sqrt{\frac{kT}{2m}} \Gamma\left(\frac{1}{2}\right)} = \frac{1}{2} kT$$

I don't need the value of Gamma Functions. I use just the following formula.

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

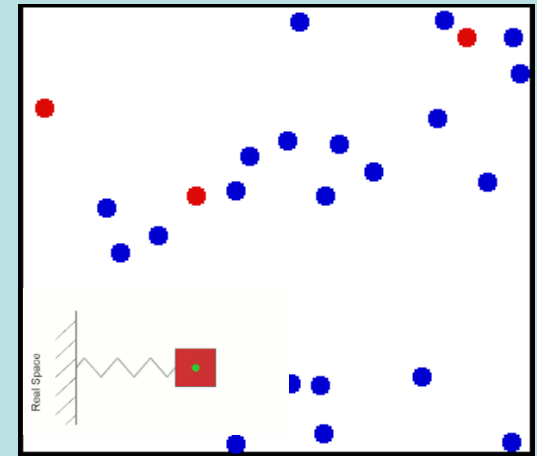
$$\left\langle \frac{1}{2} m v^2 \right\rangle_{\text{av}} = \frac{1}{\int_0^\infty dv \cdot e^{-\frac{mv^2}{2kT}}} \int_0^\infty dv \cdot \frac{1}{2} m v^2 \cdot e^{-\frac{mv^2}{2kT}} = \frac{1}{2} kT$$

注意此結果完全與質量 m 無關！

因此彈力位能的計算也得到完全相同結果：

$$\left\langle \frac{1}{2} K x^2 \right\rangle_{\text{av}} = \frac{1}{\int_0^\infty dx \cdot e^{-\frac{Kx^2}{2kT}}} \int_0^\infty dx \cdot \frac{1}{2} K x^2 \cdot e^{-\frac{Kx^2}{2kT}} = \frac{1}{2} kT$$

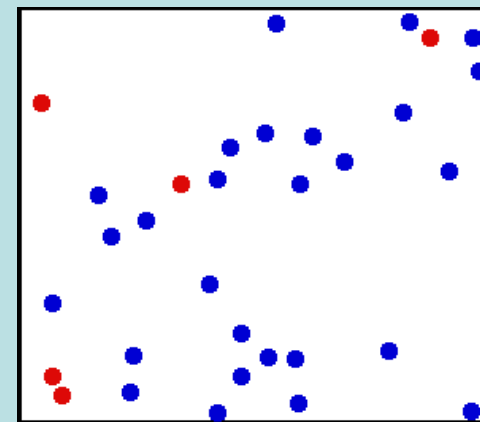
任何能量只要是動力座標的平方都會得到相同結果！



能量均分原則 Equipartition of Energy

一個系統中，**任一個**可以儲存能量的型式（稱為**自由度**），
只要是動力座標的平方，在頻繁的熱作用（混亂的能量交換）
達到**熱平衡**後，都會得到同樣的平均能量。

這個平均能量就是溫度： $\frac{1}{2}kT$



溫度的微觀意義原來是每一個自由度所分配的能量！

$$\langle E \rangle_{\text{av}} = \frac{1}{2}kT$$

理想氣體溫度為動能平均是這個更廣泛的原則的一個特例。

混亂製造了平衡態，混亂也使我們可以預測平衡態的性質！

關鍵是需要**頻繁混亂**的能量交換！這是**熱現象**的本質。

定義一個新的符號： $\beta \equiv \frac{1}{kT}$

$$\frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2}\beta}} \int_0^\infty dv \cdot \frac{1}{2} mv^2 \cdot e^{-\frac{mv^2}{2}\beta} = - \frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2}\beta}} \frac{\partial}{\partial \beta} \int_0^\infty dv \cdot e^{-\frac{mv^2}{2}\beta}$$

分子積分中的 mv^2 可以由對 β 的微分得到！

注意分子與分母中的積分是一樣的，因此可寫成對數的微分：

$$= - \frac{\partial}{\partial \beta} \ln \int_0^\infty dv \cdot e^{-\frac{mv^2}{2}\beta}$$

$$\frac{\partial}{\partial x} \ln f = \frac{1}{f} \cdot \frac{df}{dx}$$

接著變換變數：

$$x \equiv \sqrt{\frac{m\beta}{2}} v$$

$$dv \equiv \sqrt{\frac{2}{m\beta}} dx$$

如此，注意式中的 β 可以與積分分離。
積分可以不用算。

$$\begin{aligned} &= - \frac{\partial}{\partial \beta} \ln \sqrt{\frac{2}{m\beta}} \int_0^\infty dx \cdot e^{-x^2} = - \frac{\partial}{\partial \beta} \left[\ln \sqrt{\frac{1}{\beta}} + \ln \sqrt{\frac{2}{m}} \int_0^\infty dx \cdot e^{-x^2} \right] \\ &= - \frac{\partial}{\partial \beta} \ln \sqrt{\frac{1}{\beta}} = \frac{1}{2} \frac{\partial}{\partial \beta} \ln \beta = \frac{1}{2} \frac{1}{\beta} = \frac{1}{2} kT \end{aligned}$$

In Statistical Mechanics it is essential to have an approximation of $n!$ for large n .

Stirling Formula

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \quad n \rightarrow \infty$$

Write it in terms of gamma function, which has an integration definition.

$$\Gamma(p+1) \sim p^p e^{-p} \sqrt{2\pi p} \quad p \rightarrow \infty$$

$$\Gamma(p+1) = \int_0^{\infty} dx \cdot x^p e^{-x} = \int_0^{\infty} dx \cdot (e^{\ln x})^p e^{-x} = \int_0^{\infty} dx \cdot e^{p \ln x - x}$$

We know how to integrate Gaussian function.

Maybe, $p \ln x - x$ can be approximated by a quadratic function of x as $p \rightarrow \infty$.

It does have a minimum at $x = p$. $\left. \frac{d}{dx} (p \ln x - x) \right|_{x=p} = 0$

$p \ln x - x$ can be approximated by a quadratic function of $(x - p)$ as $p \rightarrow \infty$

$p \ln x - x$ can be approximated by a quadratic function of $x - p$ as $p \rightarrow \infty$

Change the variable: $x - p = y\sqrt{p}$ $dx = dy\sqrt{p}$

$$\int_0^{\infty} dx \cdot e^{p \ln x - x} = \sqrt{p} \int_{-\infty}^{\infty} dy \cdot e^{p \ln(p+y\sqrt{p}) - p - y\sqrt{p}}$$

Note that as $x \rightarrow 0$, $y \rightarrow -\sqrt{p} \rightarrow -\infty$. See Baos p553.

Expand the log in the exponential by: $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$

$$\ln(p + y\sqrt{p}) = \ln \left[p \left(1 + \frac{y}{\sqrt{p}} \right) \right] = \ln p + \ln \left(1 + \frac{y}{\sqrt{p}} \right) \sim \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \dots$$

$$p \ln(p + y\sqrt{p}) - p - y\sqrt{p} = p \ln p - p - \frac{y^2}{2}$$

Note that there is no linear term in y . It is indeed an integration of Gaussian function.

$$\Gamma(p + 1) \sim \sqrt{p} e^{p \ln p - p} \int_{-\infty}^{\infty} dy \cdot e^{-\frac{y^2}{2}} = \sqrt{p} e^{p \ln p - p} \sqrt{2\pi} \Gamma\left(\frac{1}{2}\right) = p^p e^{-p} \sqrt{2\pi p}$$

This is true for all p . When $p = n$, $n! \sim n^n e^{-n} \sqrt{2\pi n}$ $n \rightarrow \infty$

Laplace's method

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Not to be confused with [Laplace's approximation](#) or [Laplace smoothing](#).



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In [mathematics](#), **Laplace's method**, named after [Pierre-Simon Laplace](#), is a technique used to approximate [integrals](#) of the form

$$\int_a^b e^{Mf(x)} dx,$$

where f is a twice-[differentiable function](#), M is a large [number](#), and the endpoints a and b may be infinite. This technique was originally presented in the book by [Laplace \(1774\)](#).

In [Bayesian statistics](#), [Laplace's approximation](#) can refer to either approximating the [posterior normalizing constant](#) with Laplace's method or approximating the [posterior distribution](#) with a [Gaussian](#) centered at the [maximum a posteriori estimate](#).^{[1][2]} Laplace approximations are used in the [integrated nested Laplace approximations](#) method for fast approximations of [Bayesian inference](#).

Formal statement and proof [\[edit\]](#)

Suppose $f(x)$ is a twice continuously [differentiable function](#) on $[a, b]$, and there exists a unique point $x_0 \in (a, b)$ such that:

$$f(x_0) = \max_{x \in [a, b]} f(x) \quad \text{and} \quad f''(x_0) < 0.$$

Then:

$$\lim_{n \rightarrow \infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{n(-f''(x_0))}}} = 1.$$

Proof

[\[hide\]](#)

Lower bound: Let $\varepsilon > 0$. Since f'' is continuous there exists $\delta > 0$ such that if $|x_0 - c| < \delta$ then $f''(c) \geq f''(x_0) - \varepsilon$. By [Taylor's Theorem](#), for any $x \in (x_0 - \delta, x_0 + \delta)$,

$$f(x) \geq f(x_0) + \frac{1}{2}(f''(x_0) - \varepsilon)(x - x_0)^2.$$

Then we have the following lower bound:

$$\begin{aligned} \int_a^b e^{nf(x)} dx &\geq \int_{x_0 - \delta}^{x_0 + \delta} e^{nf(x)} dx \\ &\geq e^{nf(x_0)} \int_{x_0 - \delta}^{x_0 + \delta} e^{\frac{n}{2}(f''(x_0) - \varepsilon)(x - x_0)^2} dx \\ &= e^{nf(x_0)} \sqrt{\frac{1}{n(-f''(x_0) + \varepsilon)}} \int_{-\delta\sqrt{n(-f''(x_0) + \varepsilon)}}^{\delta\sqrt{n(-f''(x_0) + \varepsilon)}} e^{-\frac{1}{2}y^2} dy \end{aligned}$$

where the last equality was obtained by a change of variables

$$y = \sqrt{n(-f''(x_0) + \varepsilon)}(x - x_0).$$