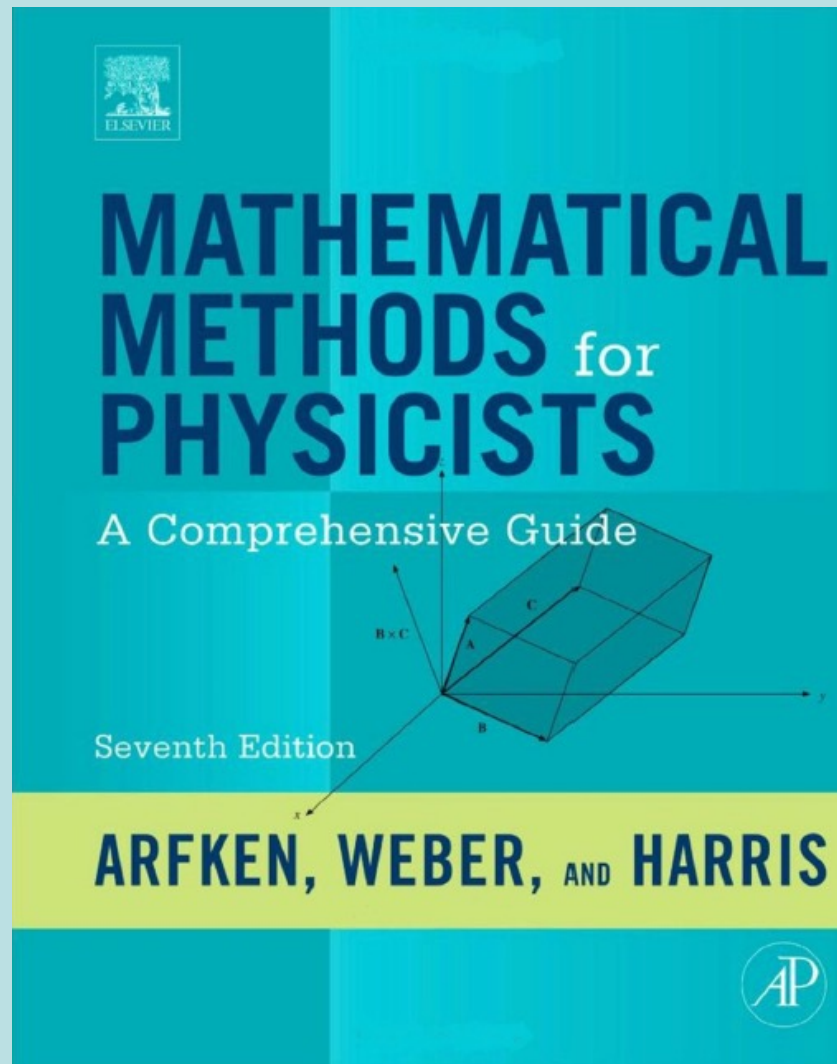


## Vectorial Calculus



Section 3.5-3.9

THE  
*Feynman*  
LECTURES ON PHYSICS

*The* NEW MILLENNIUM *Edition*

VOLUME II: MAINLY ELECTROMAGNETISM AND MATTER

Feynman - Leighton - Sands

## **2**

### ***Differential Calculus of Vector Fields***

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*Review:* Chapter 11, Vol. I, *Vectors*

#### **2-1 Understanding physics**

The physicist needs a facility in looking at problems from several points of view. The exact analysis of real physical problems is usually quite complicated, and any particular physical situation may be too complicated to analyze directly by solving the differential equation. But one can still get a very good idea of

## **3**

### ***Vector Integral Calculus***

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#### **3-1 Vector integrals; the line integral of $\nabla\psi$**

We found in Chapter 2 that there were various ways of taking derivatives of fields. Some gave vector fields; some gave scalar fields. Although we developed many different formulas, everything in Chapter 2 could be summarized in one rule: the operators  $\partial/\partial x$ ,  $\partial/\partial y$ , and  $\partial/\partial z$  are the three components of a vector operator  $\nabla$ . We would now like to get some understanding of the significance of the derivatives of fields. We will then have a better feeling for what a vector field equation means.

# Vector Calculus

University of Cambridge Part IA Mathematical Tripos

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## 3 Grad, Div and Curl

### 3.1 The Gradient

### 3.2 Div and Curl

#### 3.2.1 Some Basic Properties

#### 3.2.2 Conservative is Irrotational



## Goal: Complete Maxwell Equations in integral form

積分形式

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

微分形式

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

電磁場不單純由電荷電流產生，電（磁）場變化時也會感應產生磁（電）場。

Convert Maxwell Equations from integral formulation to differential formulation

Using differential Maxwell Eq, EM wave equation is easy to derive.

Maxwell Equations in vacuum

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Spatial change of electric field is interrelated to temporal change of magnetic field.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

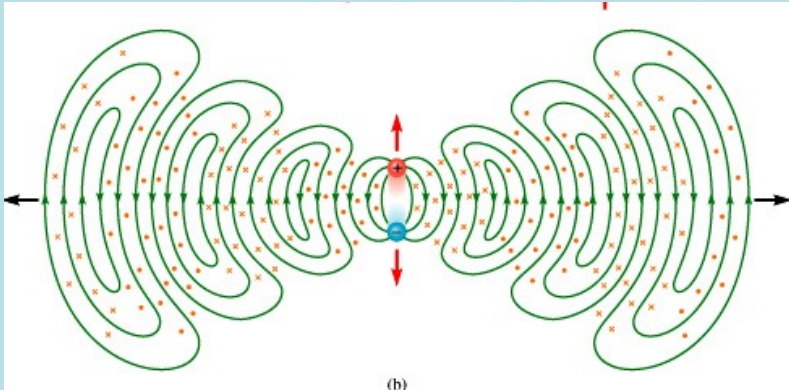
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$

Spatial change of magnetic field is interrelated to temporal change of electric field.

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Wave Equation for Electric Field



Vectors in Flat Space  $R^{2,3}$  平坦二維或三維空間中的向量

Vectorial Calculus 向量微積分

這是處理 $R^{2,3}$ 空間中的位置與時間的函數！這樣的函數一般稱為場。

This is to deal with functions of space and time, called **fields**.

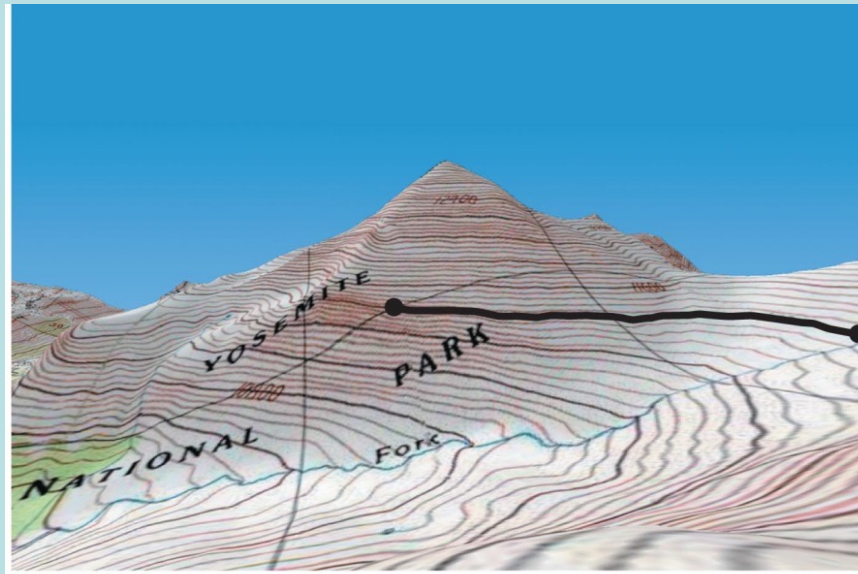
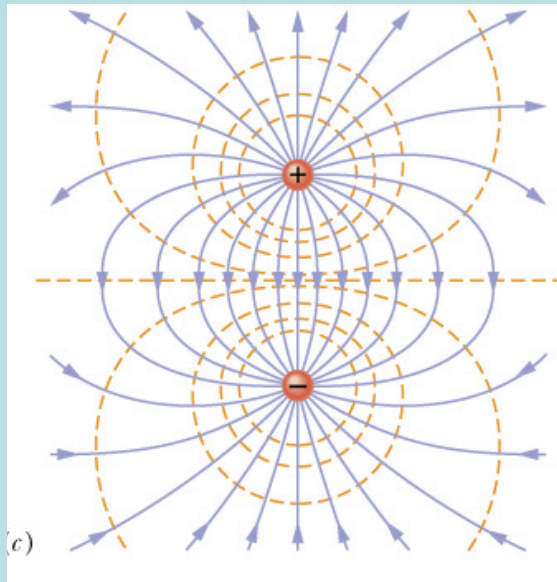
在空間中可以定義函數值為純量的函數！純量場 Scalar Field。

Scalar functions (just real number) of space and time are called scalar field.

$$V: R^3 \rightarrow R$$



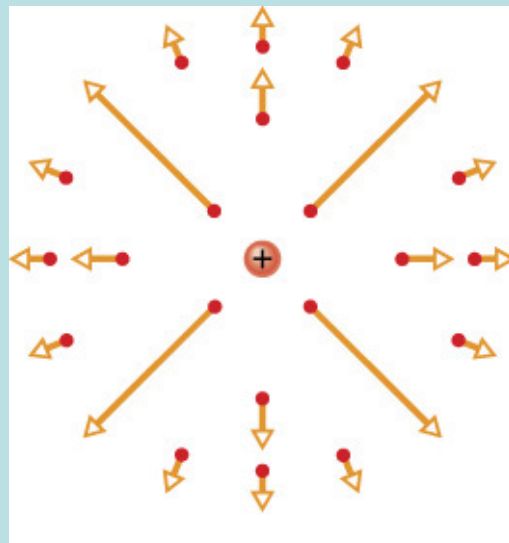
電位 $V(\vec{r})$ 是空間的純量函數。Electric Potential  $V(\vec{r})$  is a scalar function in space.  
相對的，電場 $\vec{E}(\vec{r})$ 是一個向量函數。On the contrary, electric field is a vector field.



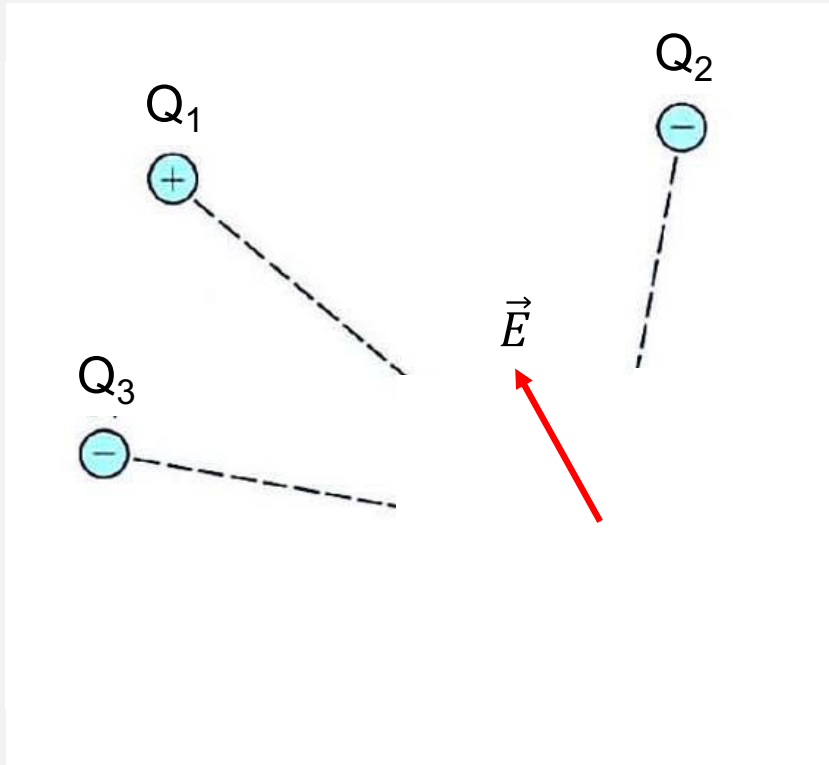
在空間中可以定義函數值為向量的函數！向量場 Vector Field。

When the function values are vectors, the fields are called vector fields.

$$\vec{E}: R^3 \rightarrow R^3$$



電場的抽象定義：



$$\vec{F} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i q}{r_i^2} \hat{r}_i$$

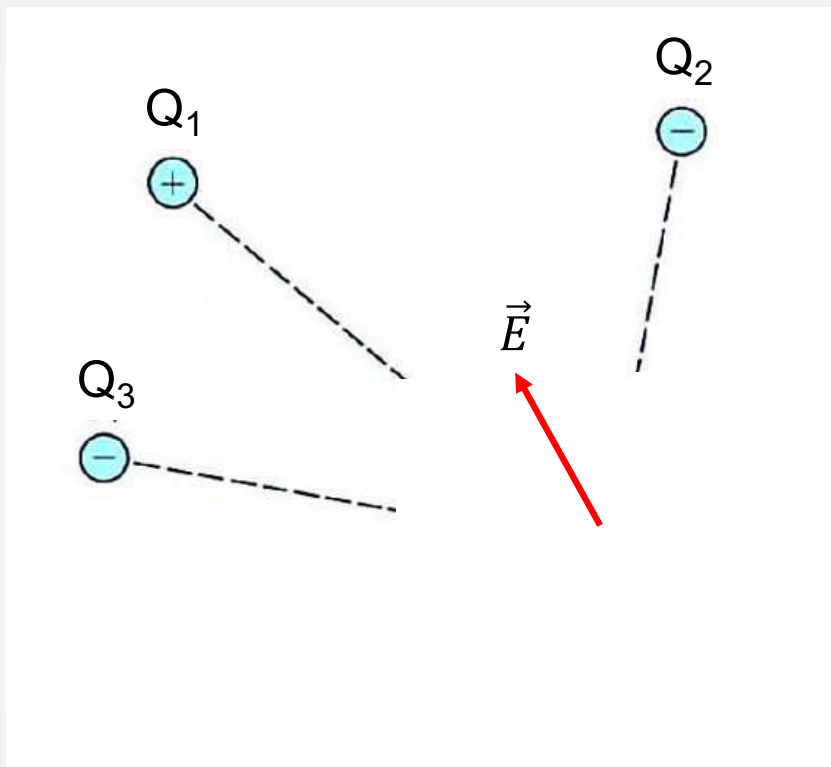
電荷量 $q$ 可以從複雜的式子中提出來：

$$= \left( \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{r_i^2} \hat{r}_i \right) \cdot q = \vec{E} \cdot q$$

電場 Electric Field

電場 $\vec{E}$ 只與 $q$ 的位置相關，與 $q$ 的電荷量無關

想像 $q$ 漸漸趨近於零 $q \rightarrow 0$ ....電場 $\vec{E}$ 應該依舊存在吧！



$$\vec{E} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{r_i^2} \hat{r}_i$$

電場與 $q$ 的電荷量無關。

由固定電荷 $Q_i$ 及 $q$ 的位置決定。

電場 $\vec{E}(\vec{r})$ 是空間的性質。

每一點都有 $\vec{E}(\vec{r})$ ，電場遍佈整個空間。

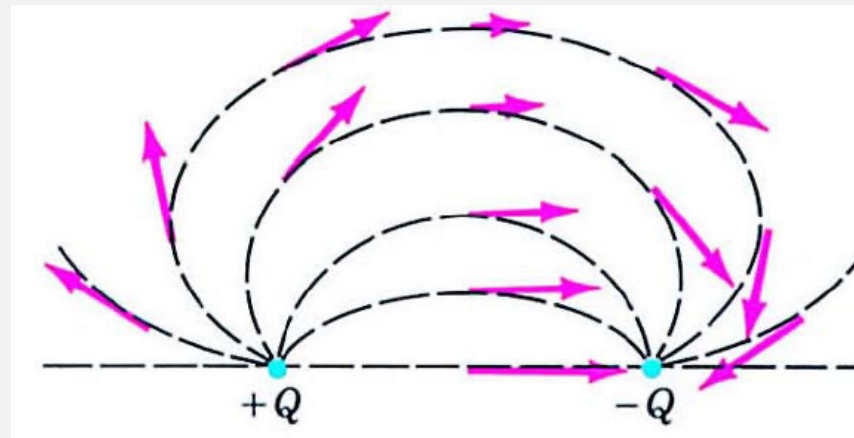
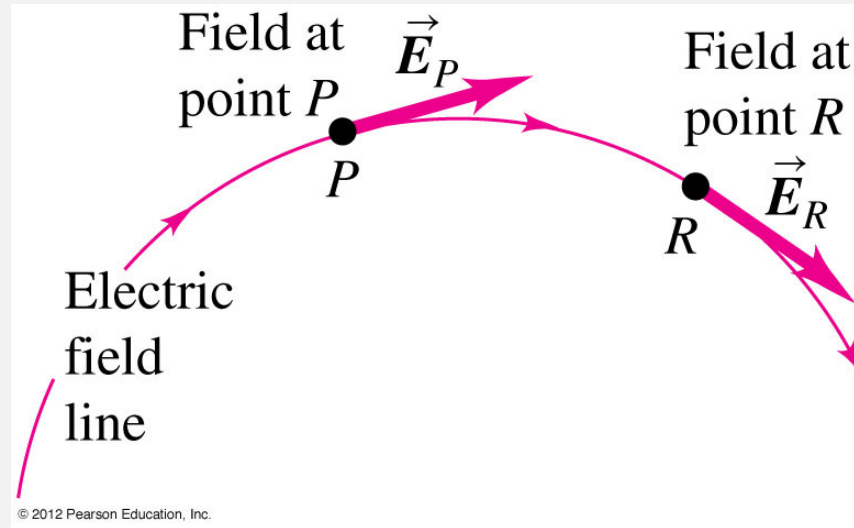
電場可以為零，但永遠存在。

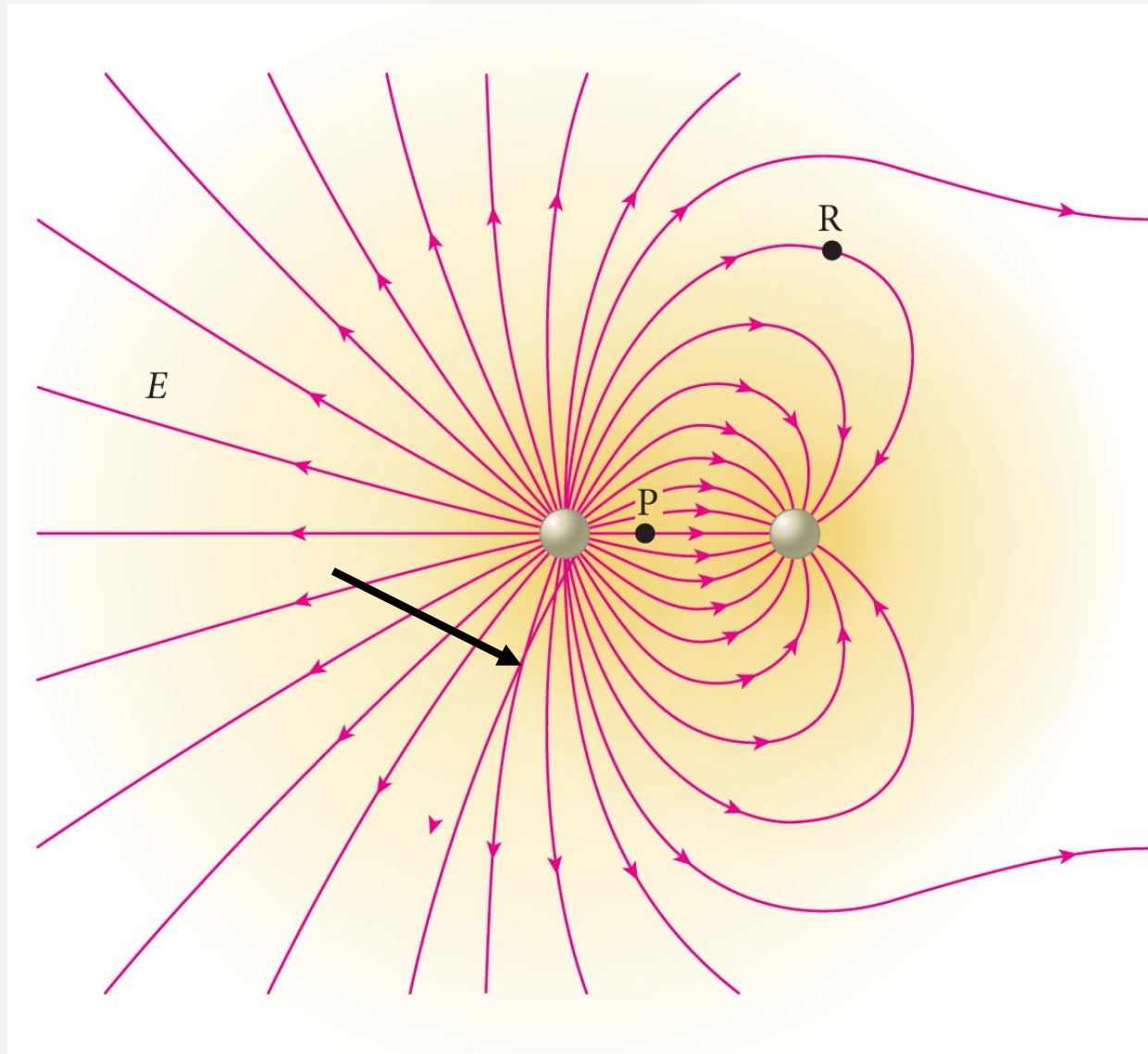
電場就是會讓位於當地的電荷 $q$ 得到靜電力 $q \cdot \vec{E}$ 的**空間的物理性質**。

電場是由固定電荷 $Q_i$ 在空間各處產生。

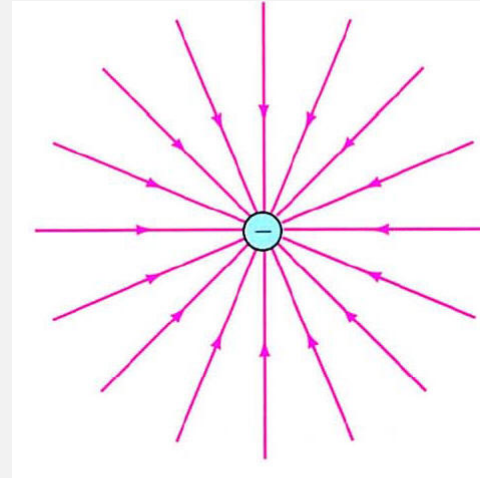
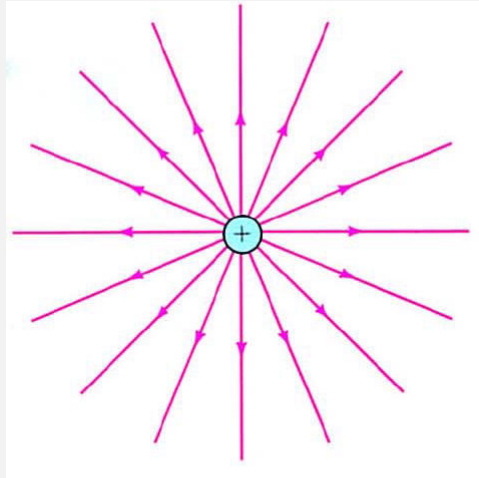
法拉第透過無數實驗觀察，歸納出場線的兩個性質：

1. 電場的方向即為通過當地之電力線的切線方向。





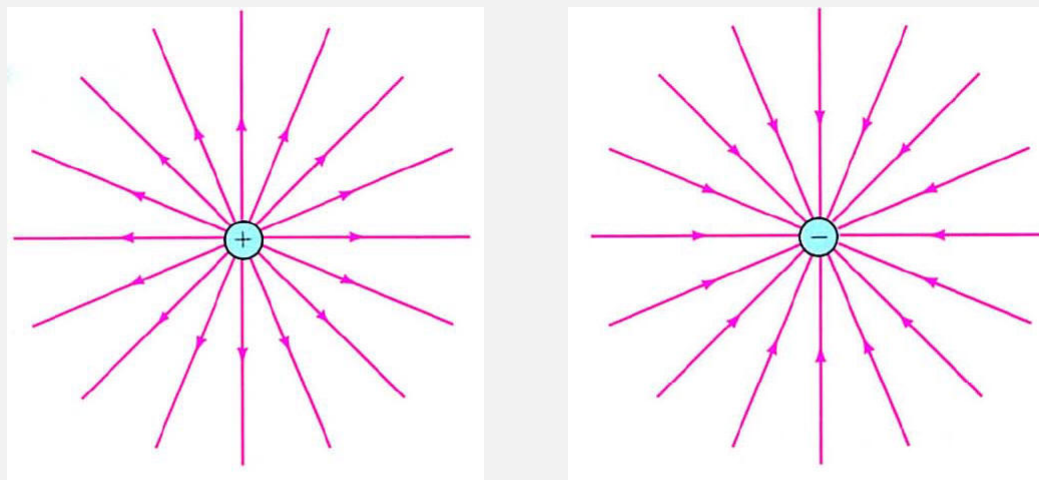
因此，電力線不會交叉。否則交叉點的電場會有兩個方向。



而且電力線不會中斷，否則中斷處電場方向將不明。

只有兩個例外，電力線由正電荷發出，由負電荷吸收！

那麼：電力線在空間中數目不變而守恆！

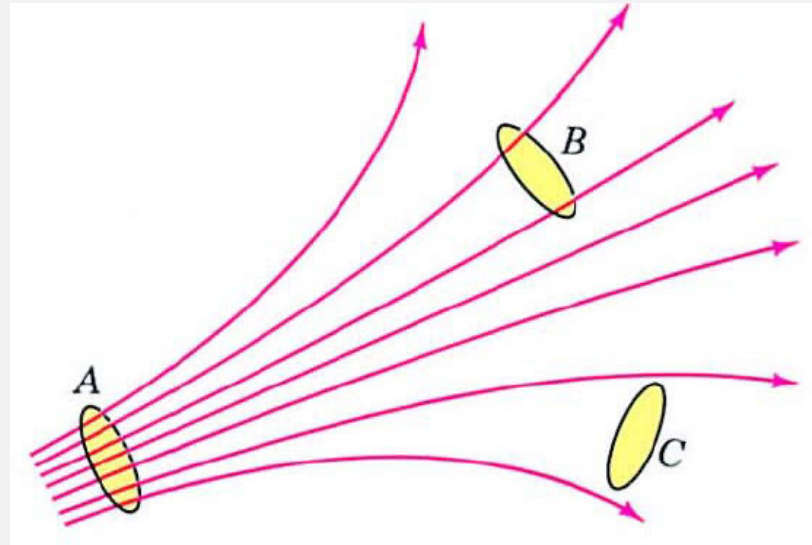


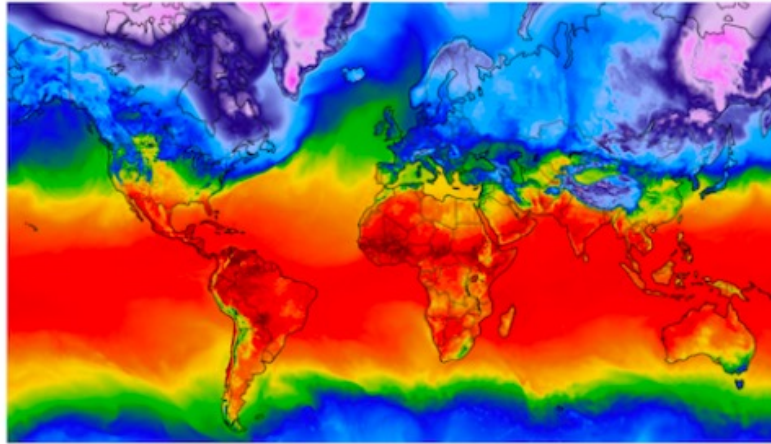
第二個特性是：電力線愈密，電場愈大！

對於點電荷，同樣不變的數目的電力線，到越遠處就分配到越大的球面，球面積正比於距離平方，所以電力線密度正好是距離平方反比，這與電場大小一樣，因此大膽猜想電力線密度正好是電場大小！

電力線密度定義為通過當地一垂直於電場的單位面積平面的電力線數目。

2. 電場大小與當地的電力線密度成正比！





**Figure 1.** On the left, the temperature on the surface of the Earth is an example of a map from  $\mathbb{R}^2 \rightarrow \mathbb{R}$ , also known as a scalar field. On the right, the wind on the surface of the Earth blows more or less horizontally and so can be viewed as a map from  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ , also known as a vector field. (To avoid being co-opted by the flat Earth movement, I should mention that, strictly speaking, each of these is a map from  $S^2$  rather than  $\mathbb{R}^2$ .)

溫度分佈就是純量場 $T(\vec{r})$

Temperature distribution is a scalar field.

風速分佈就是向量場 $\vec{v}(\vec{r})$

Wind speed distribution is a vector field.

梯度，散度，旋度

馬克斯威爾方程式的微分形式

**Differential Formulation of Maxwell Equation from Integral Formulation**

## 空間中的微分

我們可以對一空間的純量函數 $V(x, y, z)$ 作空間座標偏微分，會有三個偏微分。

We can define three partial derivatives of a scalar field  $V(x, y, z)$ .

$$\frac{\partial V}{\partial x}(x, y, z)$$

$$\frac{\partial V}{\partial y}(x, y, z)$$

$$\frac{\partial V}{\partial z}(x, y, z)$$

若 $V$ 是電位，上式類似於電場公式的三維空間推廣。

$$E = -\frac{dV}{dx}$$

電位是電位的積分，電場是電位的微分。

In one dimension, the derivative of electric potential is electric field.

這三個微分很像一個向量的三個分量。但是不是真是一個3D向量？

The three partial derivatives look like components of a vector. Are they?

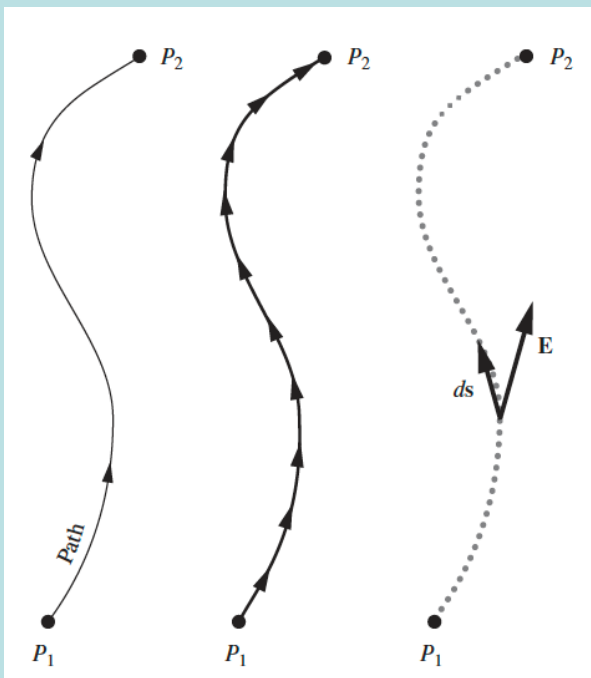
若是向量，電位做偏微分所得到的向量，就是電場向量嗎？

If so, is the vector the electric field vector  $\vec{E}$ ?

Let's start from the definition:

電位差 $\Delta V$ 是電場 $\vec{E}$ 的線積分！

Electric potential difference equals the line integral of the electric field.



$$\Delta V = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

路徑可以任選。

考慮圖中很靠近的 $P_1$ 與 $P_2$ ： If  $P_1$  and  $P_2$  are very close:

兩點的位移： Displacement:  $\Delta\vec{R} = (\Delta x, \Delta y, \Delta z)$

因為兩點很靠近，電場變化不大，可視為常數：

Since  $P_1, P_2$  are very close,  $\vec{E}$  does not change much.

兩點間的電位差：

Electric potential difference between two points equals

$$\Delta V = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} \quad \sim - \vec{E} \cdot \Delta\vec{R} = -E_x \Delta x - E_y \Delta y - E_z \Delta z$$

但 $V(x, y, z)$ 是一個多變數函數，函數變化可以以偏微分表示：

Multivariable function difference can be written in terms of partial derivatives.

$$\Delta V = \frac{\partial V}{\partial x} \Delta x + \frac{\partial V}{\partial y} \Delta y + \frac{\partial V}{\partial z} \Delta z$$

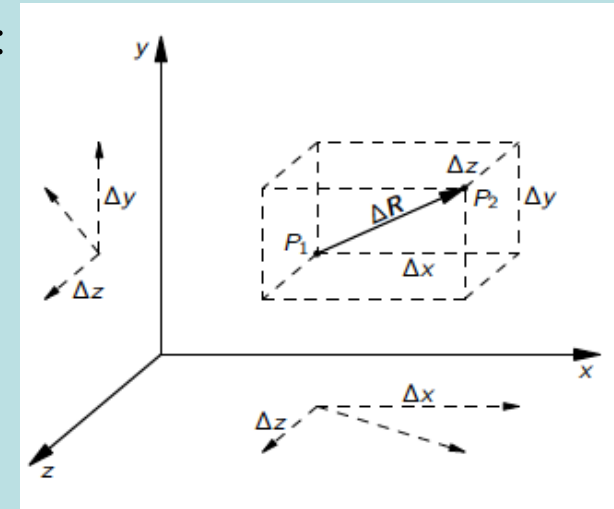
比較兩式得到電位與電場的關係：

Comparing the two formula, we find:

這顯示這三個微分真是向量的三個分量：

$$\vec{E} = \left( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$$

The three partial derivatives are indeed components of a vector.

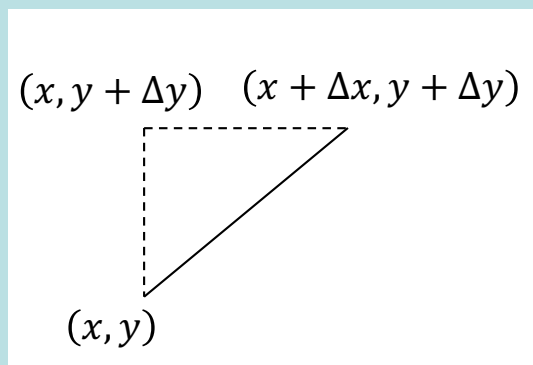


Multivariable function difference can be written in terms of partial derivatives.

中值定理的推廣：
$$f(x + \Delta x) - f(x) = \frac{df}{dx}(x_0) \cdot \Delta x$$

但 $V(x, y, z)$ 是一個多變數函數，函數變化可以以偏微分表示：

$$\Delta V = \frac{\partial V}{\partial x} \Delta x + \frac{\partial V}{\partial y} \Delta y + \frac{\partial V}{\partial z} \Delta z$$



$V(x, y)$ 函數變化：

$$\Delta V \equiv V(x + \Delta x, y + \Delta y) - V(x, y)$$

$$= V(x + \Delta x, y + \Delta y) - V(x, y + \Delta y) + V(x, y + \Delta y) - V(x, y)$$

$$= \frac{\partial V}{\partial x}(x, y + \Delta y) \Delta x + \frac{\partial V}{\partial y}(x, y) \Delta y \approx \frac{\partial V}{\partial x}(x, y) \Delta x + \frac{\partial V}{\partial y}(x, y) \Delta y$$

$$\Delta V = \frac{\partial V}{\partial x} \Delta x + \frac{\partial V}{\partial y} \Delta y$$

## 梯度 Gradient

$$\vec{A} = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi \equiv -\vec{\nabla}\phi$$

對任意 $\phi$ ， $\vec{\nabla}\phi$ 都是一個向量！ For any scalar field  $\phi$ ， $\vec{\nabla}\phi(x, y, z)$  is a vector field!

這個事實適用於任何純量場 $\phi$ ，我們根本可以把 $\vec{\nabla}$ 這個運算本身，就視為向量！

We simply treat the operation  $\vec{\nabla}$  as a vector as the notation indicate.

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

但梯度是一個運算，最終還是要作用於一個函數上。

But  $\vec{\nabla}$  is an operation after all and it must act on a field.

以上的推導並不依賴電位與電場的性質，對任一向量場 $\vec{A}$ 都對。

This derivation does not rely on  $\vec{E}$  being the electric field and is true for any vector field  $\vec{A}$ .

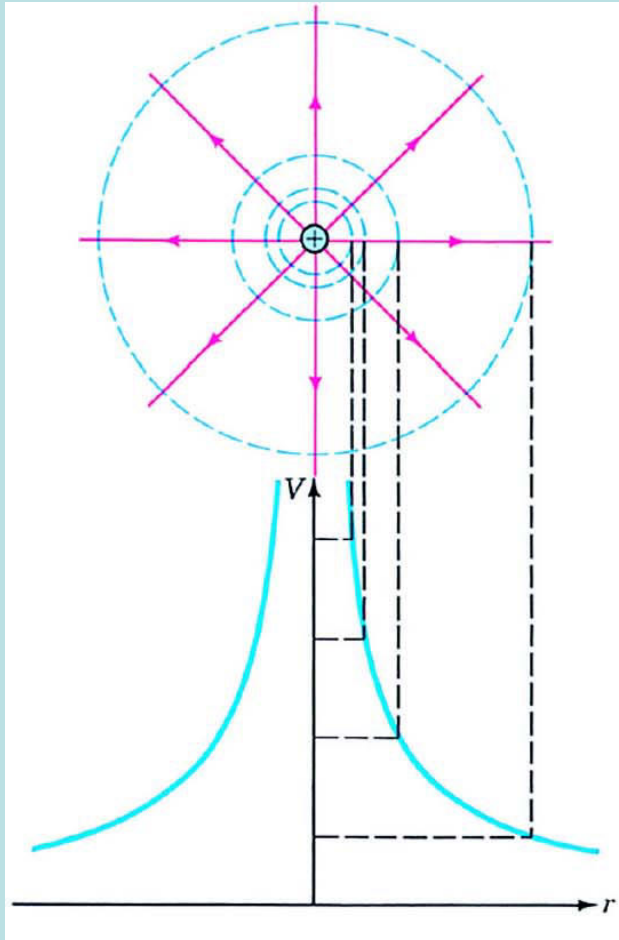
定理：任一純量場  $\phi$  它的差  $\Delta\phi$  若可寫成向量場  $\vec{A}$  的線積分：

Theorem: If the line integral of a vector field  $\vec{A}$  equals the difference of a scalar field  $\phi$ ,

$$\Delta\phi = - \int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{s}$$

$\vec{A}$  就一定可以寫成  $\phi$  的梯度： $\vec{A}$  equals the gradient of  $\phi$ :

$$\vec{A} = -\vec{\nabla}\phi$$



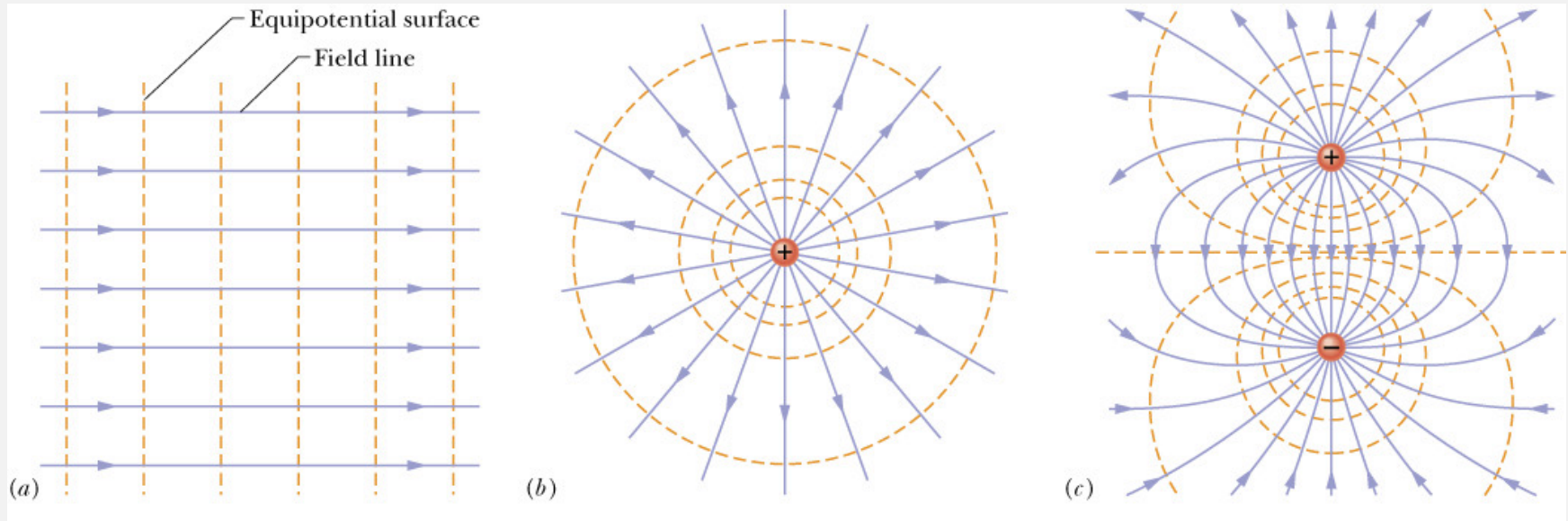
一置於原點的固定電荷 $Q$ 旁的電位：

Electric Potential of a point charge:

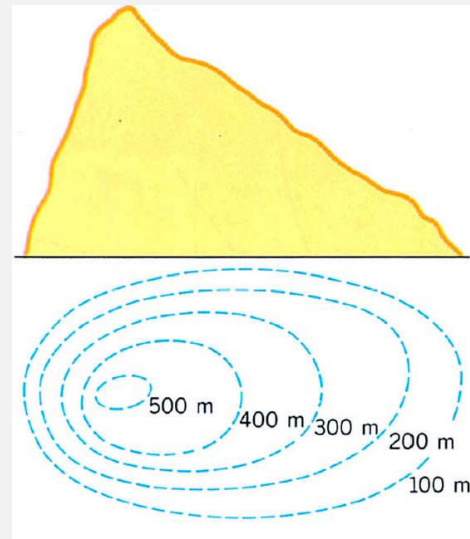
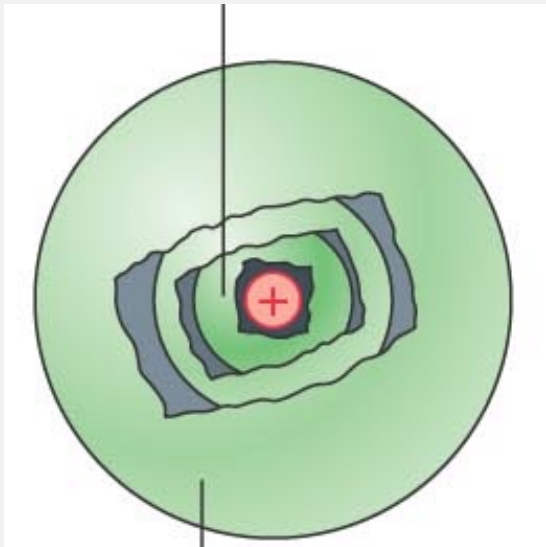
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

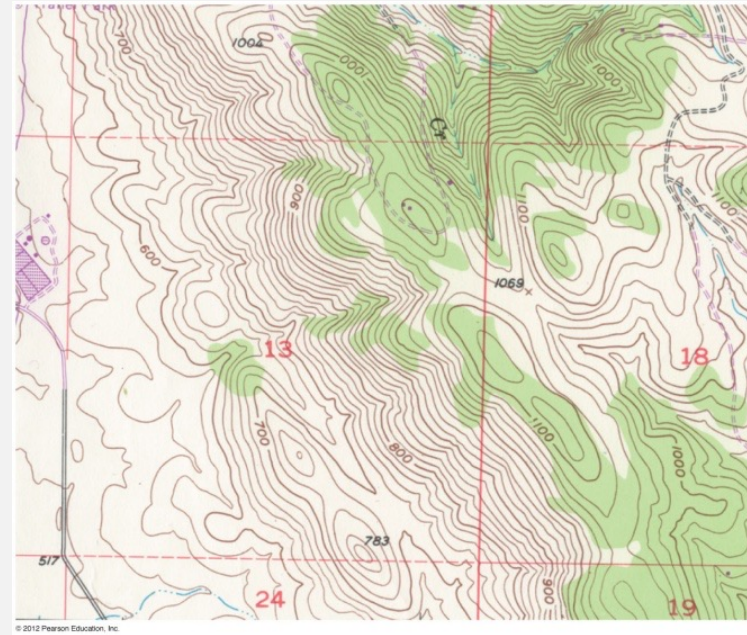
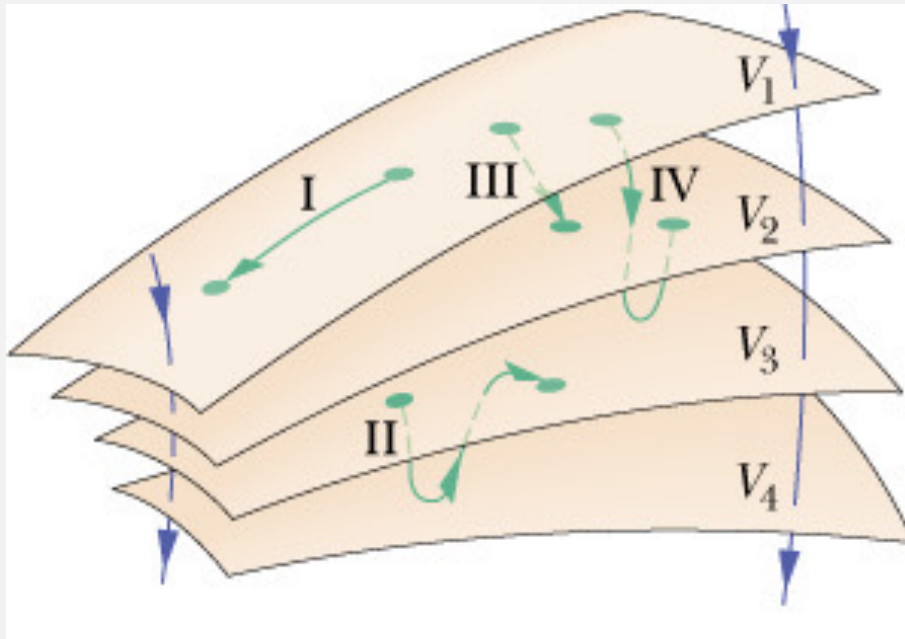
$$\vec{E} = -\vec{\nabla}V = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

要使電位 $V(\vec{r})$ 具象化最有用的就是等位面



等電位面非常像地圖上的等高線：



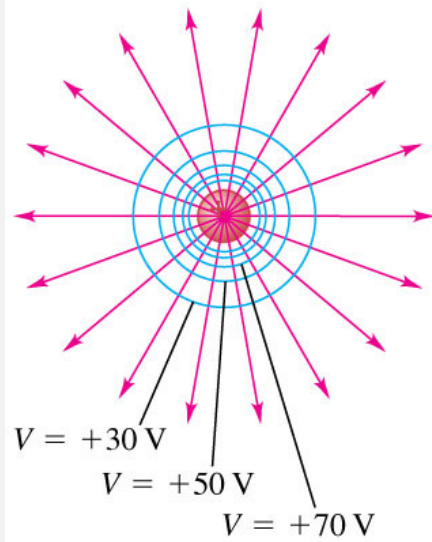


沿等位面方向的電場為零，故電位梯度電場垂直於等位面。

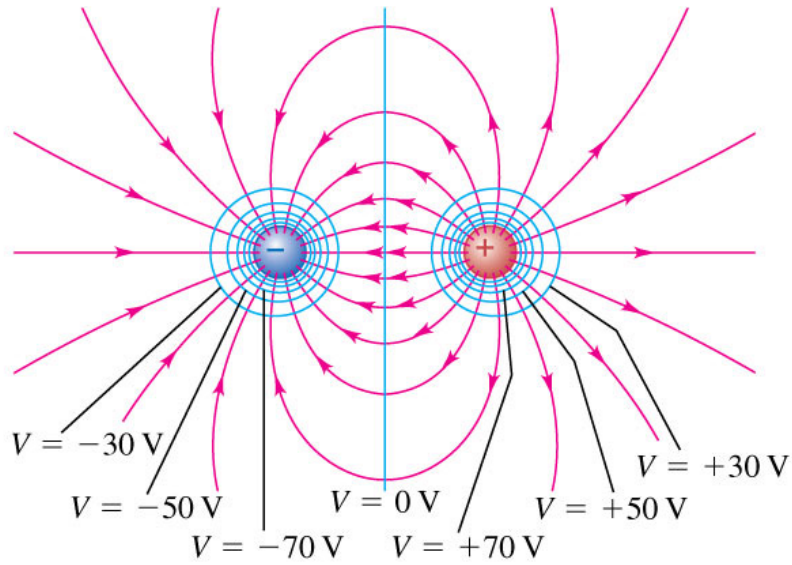
等位面越密，電場越強： $\Delta V = -Ed$

兩點之間的電位差帶動電場向量。

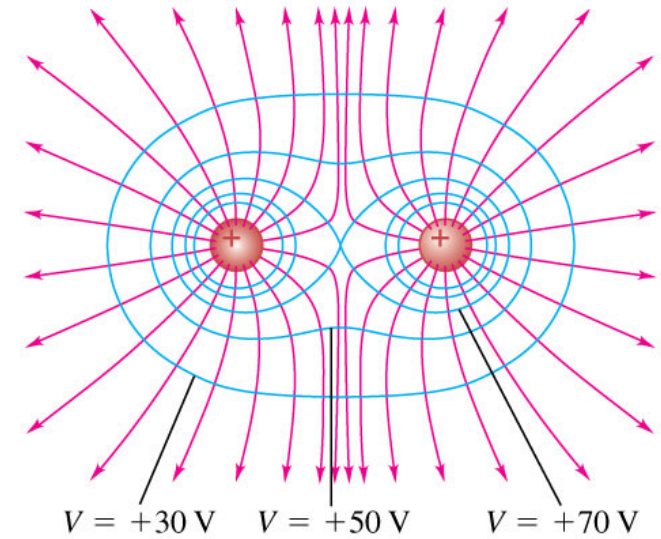
(a) A single positive charge



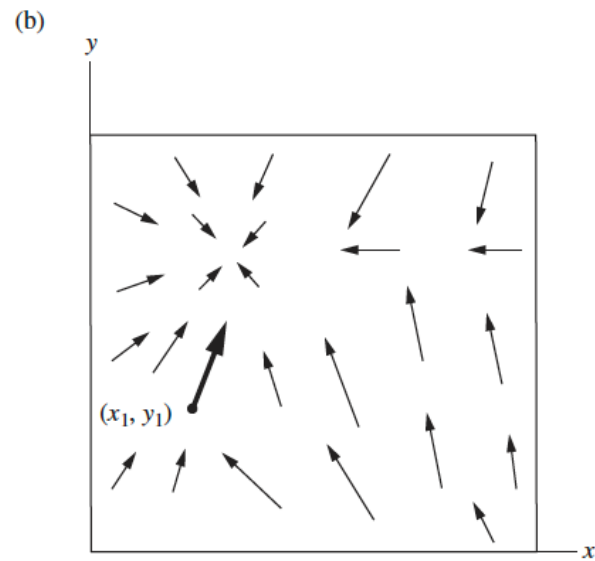
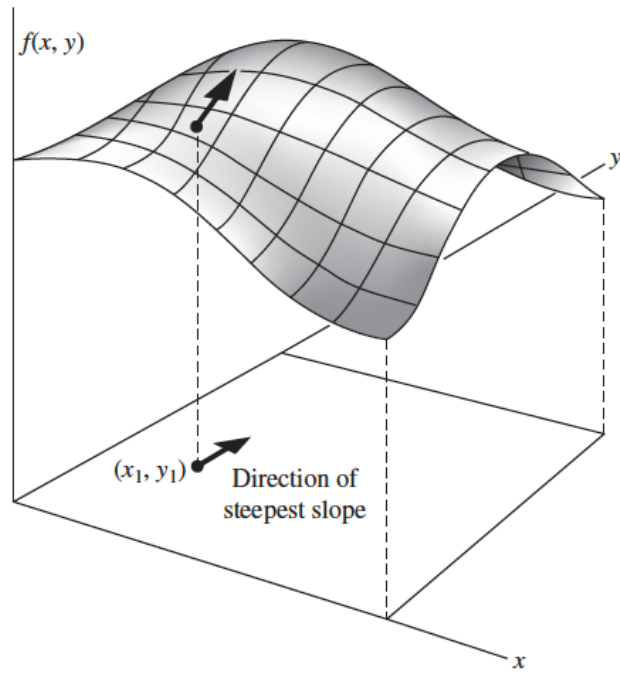
(b) An electric dipole



(c) Two equal positive charges



→ Electric field lines    — Cross sections of equipotential surfaces



**Figure 2.5.** The scalar function  $f(x, y)$  is represented by the surface in (a). The arrows in (b) represent the vector function,  $\text{grad } f$ .

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

到此梯度都運算在純量函數，得到向量場！

So far, gradients acting on scalar fields gives vector fields.

但微分運算的對象也可以是一個空間中的向量函數，例如電場： $\vec{E}$ 。

Partial derivatives in a gradient can act on vector fields too.

只是兩個向量各自有三個分量，要放在一起，這時就必須指定分量要如何組合。

There are 3 partial derivatives in gradient and vector fields. Indices need be arranged.

如同兩個向量的乘積，這裡也只有兩種選擇：內積與外積！

對此空間函數作空間微分，同時作內積組合，定義：

We can arrange the indices like an inner product while taking derivatives.

$$\vec{\nabla} \cdot \vec{E}(x, y, z) \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

可以很容易猜出這是一個純量！

The results are a scalar fields.

這個運算組合稱為**散度**，**Divergence**。This is called divergence.

對此函數作空間微分時，同時作外積組合，可以定義：

Or we can arrange the indices like a cross product. The result is a vector field.

$$\vec{\nabla} \times \vec{E}(x, y, z) \equiv \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$(\vec{A} \times \vec{B})_x = A_y B_z - A_z B_y$$

**旋度**，**Curl**

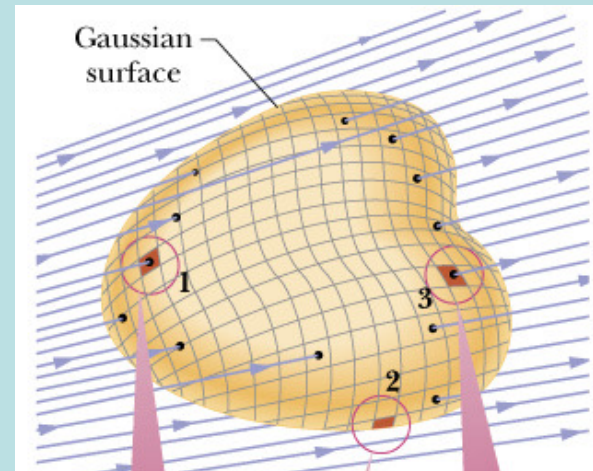
可以很容易猜出這是一個向量！

## Integration Theorems

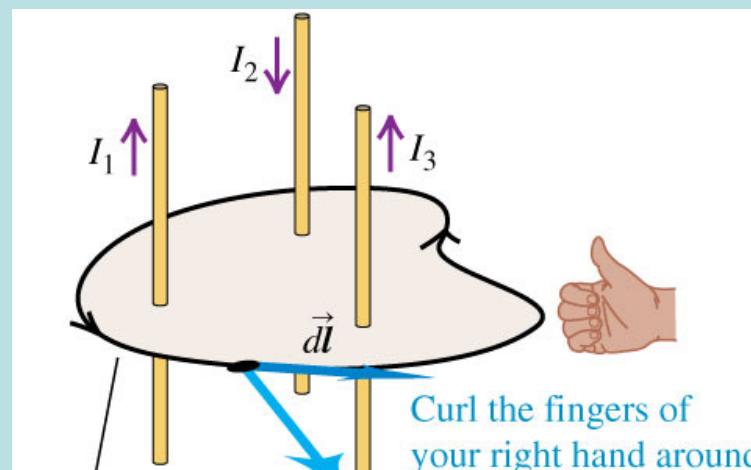
Differential operations are related to the integration of vector field in space.

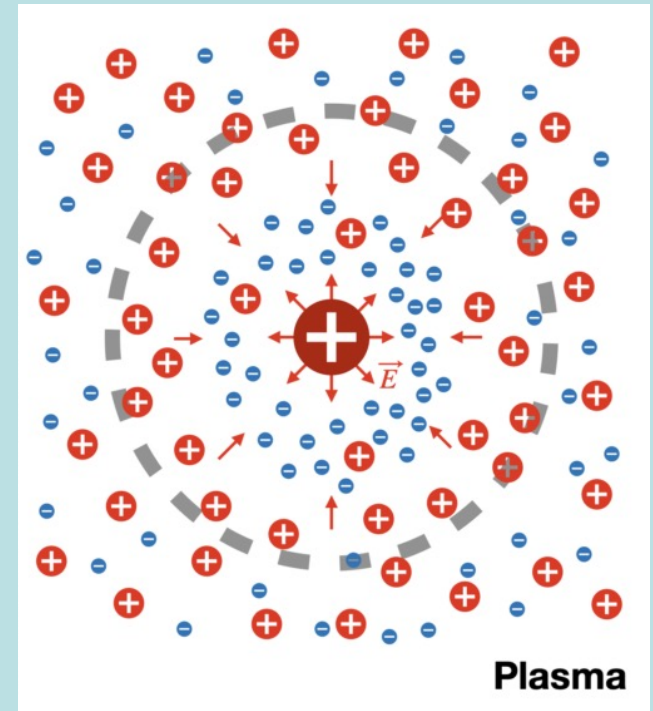
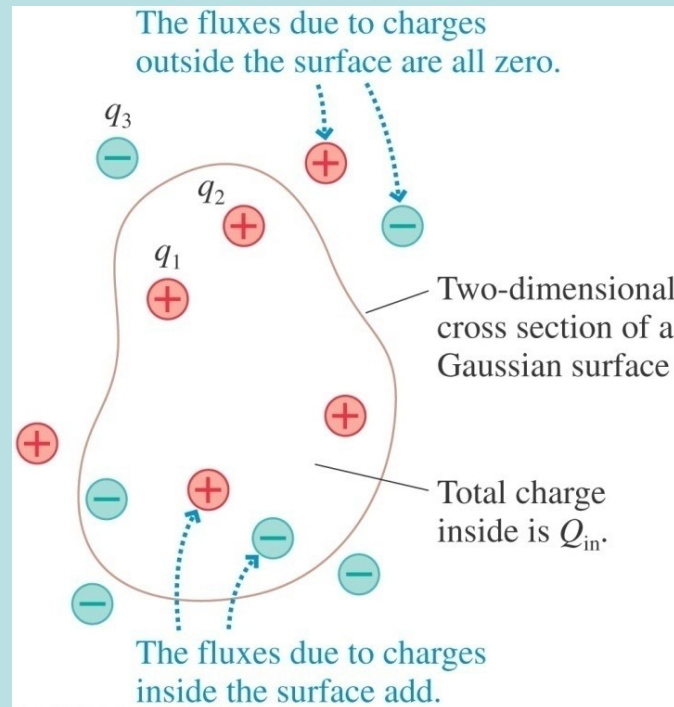
Integration of vector field in space is over curved surface or a path (line).

面積分 Surface integral



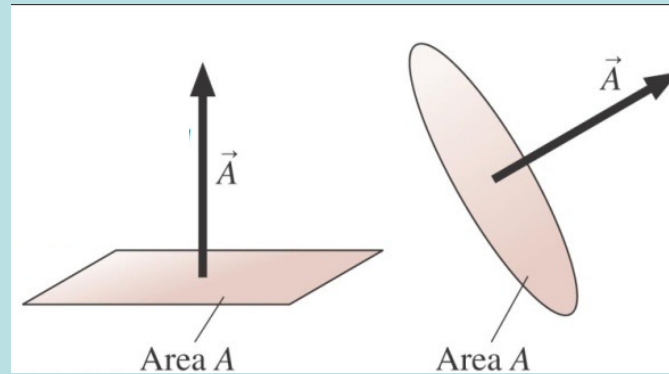
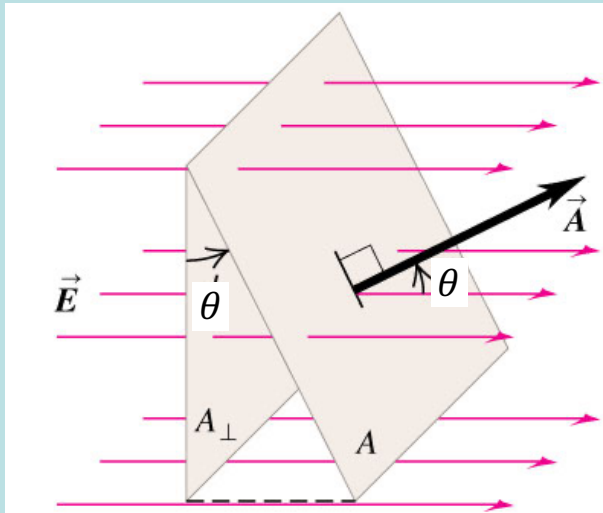
線積分 Line integral





$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

高斯定律



定義面積向量 $\vec{A}$ ：大小就是此平面的面積，方向是垂直於平面的方向。

如此定義後，角度 $\theta$ 就是向量 $\vec{A}$ 與電場 $\vec{E}$ 之間的夾角。

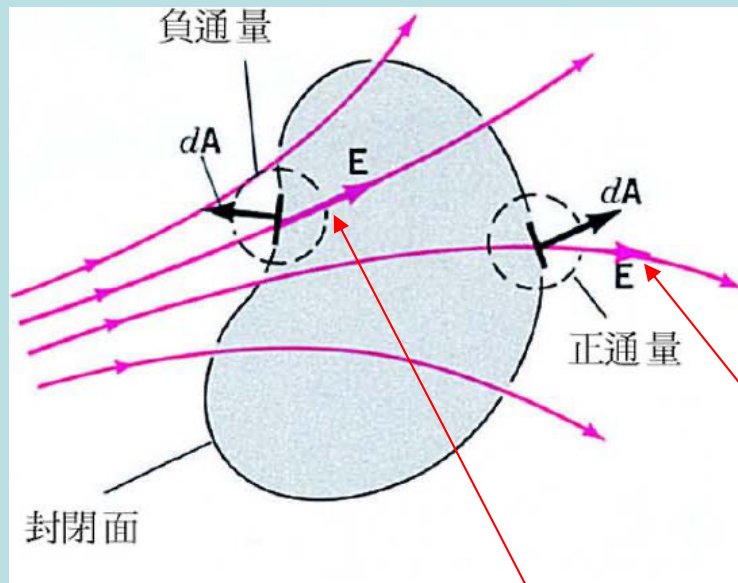
$$N = E \cdot A \cos \theta = \vec{E} \cdot \vec{A} \equiv \Phi_E$$

$\Phi_E$  電通量 Electric Flux，順著 $\vec{A}$ 的方向通過平面的電力線數目。

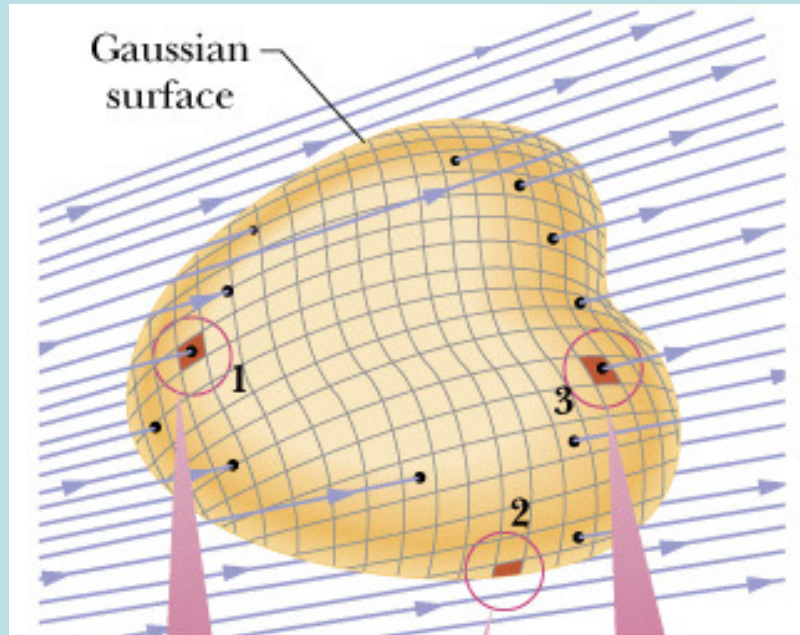
如果曲面是封閉曲面（這是高斯定律的主角，就稱高斯面）：

$$\oint \vec{E} \cdot d\vec{A} = \lim_{\substack{\delta A \rightarrow 0 \\ N \rightarrow \infty}} \sum_i \vec{E}_i \cdot \delta \vec{A}_i$$

如果是封閉曲面，空間會被分成內外兩部分，  
可以定義 $d\vec{A}$ 的方向一律是**由內指向外**：



$\vec{E}_i \cdot \delta \vec{A}_i$  為正時，電力線為離開曲面  
為負時，電力線為進入曲面

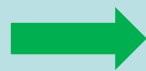


通過一封閉曲面的總電場通量就有一個很簡單的物理意義；

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \text{離開曲面的電力線數目} - \text{進入曲面的電力線數目}$$

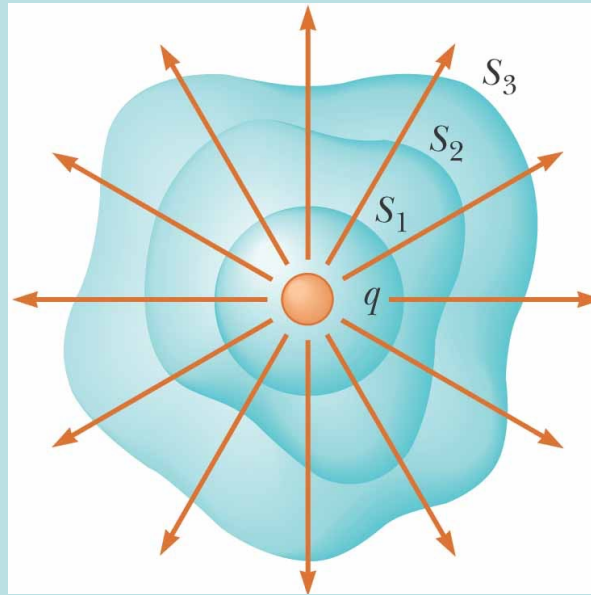
進入此封閉曲面的電場線數目必等於離開的電場線數目！

電力線數目守恆



$$\oint \vec{E} \cdot d\vec{A} = 0$$

若曲面內有一點電荷，電力線由正電荷發出（由負電荷吸收）：



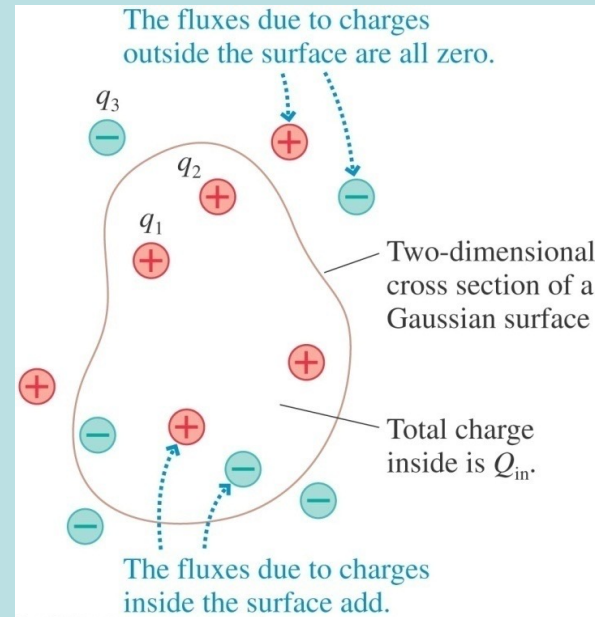
$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \text{離開曲面的電力線數目} - \text{進入曲面的電力線數目} \neq 0$$

包圍一個電荷的高斯面的電通量，等於正(負)電荷所產生(消滅)的電力線數目。

因電力線產生後不中途消失，

我們立刻可以得到：包圍點電荷 $q$ 的高斯面 $S_{1,2,3}$ ，電通量相等！

通過包圍點電荷的高斯面的電力線數與其形狀、位置、大小都無關！



$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

高斯定律 Gauss's Law

一個封閉曲面的電場面積分：通量，正比於曲面所包圍的總淨電荷量。

But Gauss's Law in this integration formulation is hard to use.

We will now calculate flux **on surface** using divergence **inside surface**.

The law will then be converted it into a differential formulation.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

First a beautiful mathematical theorem. The key observation:

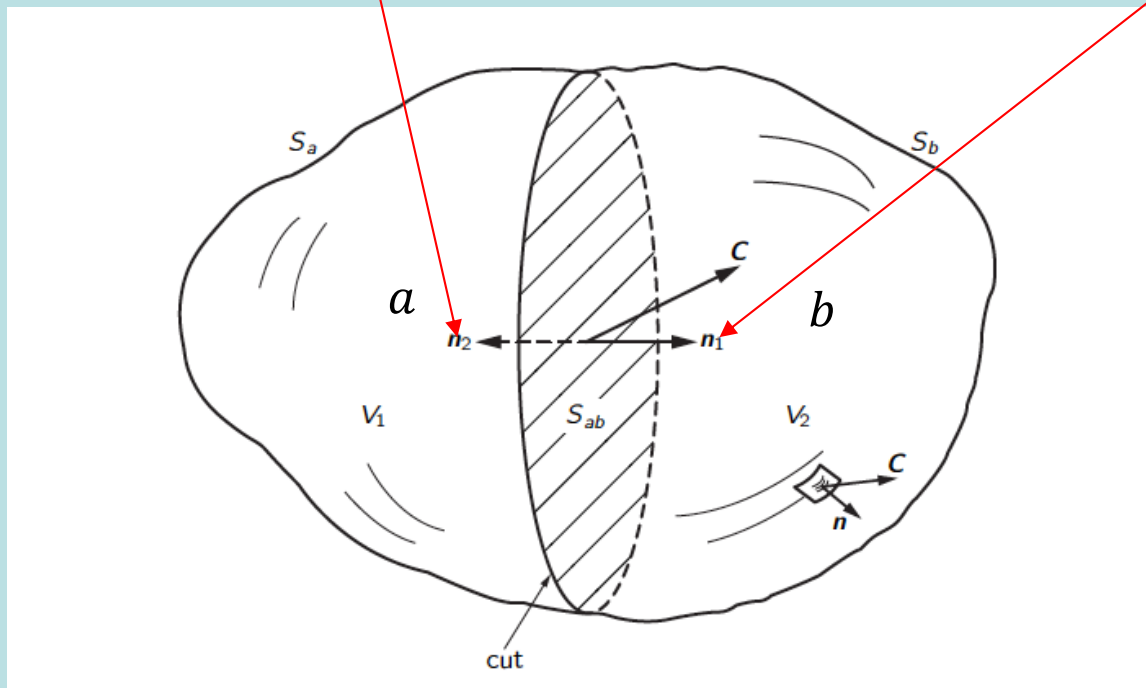
When a volume is divided into two parts:  $a + b$

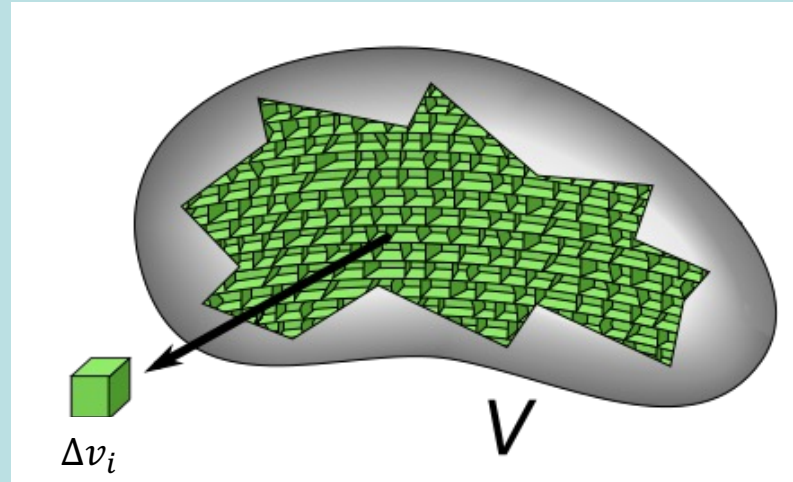
the electric flux through  $a + b$  equals the sum of flux through the two parts  $a, b$ .

$$\Phi_{a+b} = \Phi_a + \Phi_b$$

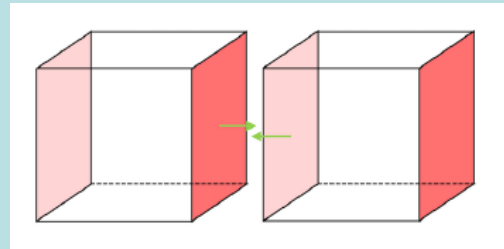
This is because when adding up, the contributions from interior faces  $S_{ab}$  cancel out.

The outward normal direction of  $S_{ab}$  from part  $b$  is opposite the direction from  $a$ .





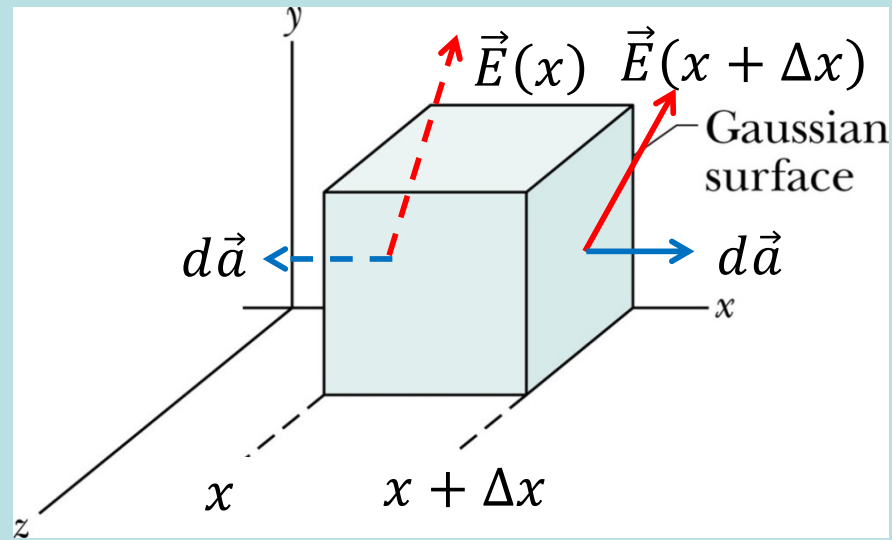
If we divide the inner space  $V$  of a Gaussian Surface into small cubes  $dV$ ,  
The sum of all the flux through the cubes equals the flux through  $V$ ,  
since the contributions from interior faces cancel out.



$$\Phi_E = \sum_i \Delta\Phi_{iE}$$

Let's calculate the flux through a small cube:

Flux through a small cube:



The flux on the two surfaces (right, left) perpendicular to  $x$  axis:  $\vec{E} \cdot d\vec{a} \sim E_x \cdot da$

$$E_x(x + \Delta x) \cdot \Delta y \Delta z - E_x(x) \cdot \Delta y \Delta z = \left( \frac{\partial E_x}{\partial x} \right) \Delta x \cdot \Delta y \Delta z$$

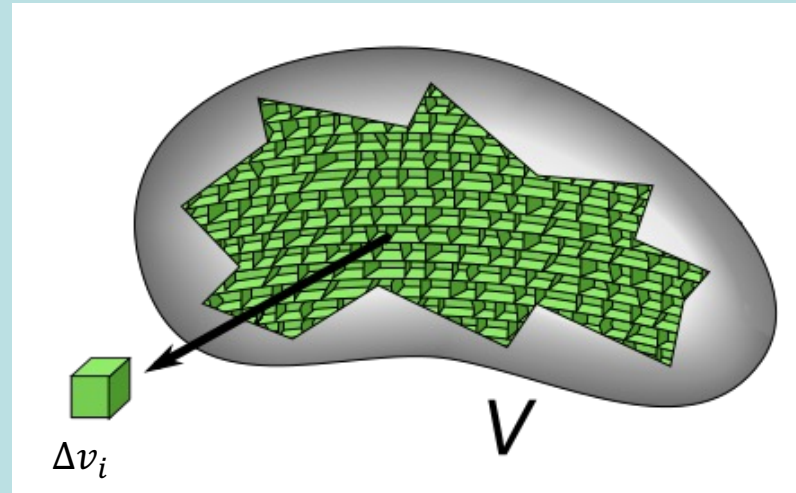
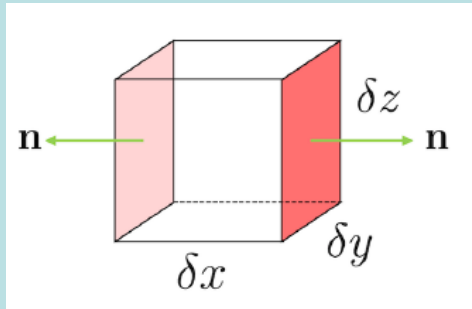
The change in  $E_x$  as only  $x$  change by  $\Delta x$  equals partial derivative  $\frac{\partial E_x}{\partial x} \Delta x$ :

Flux on two surfaces (top, bottom) perpendicular to the  $y$  axis:  $\left( \frac{\partial E_y}{\partial y} \right) \cdot \Delta x \Delta y \Delta z$

Total flux:  $\Phi_E = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \cdot \Delta x \Delta y \Delta z = (\vec{\nabla} \cdot \vec{E}) \cdot \Delta v$

$$\vec{\nabla} \cdot \vec{E} = \frac{\Phi_E}{\Delta v}$$

The physical meaning of divergence is the flux of the vector field per unit volume!



Flux through a small cube equals the divergence at the cube times its volume.

$$\Delta\Phi_E = (\vec{\nabla} \cdot \vec{E}) \cdot \Delta v$$

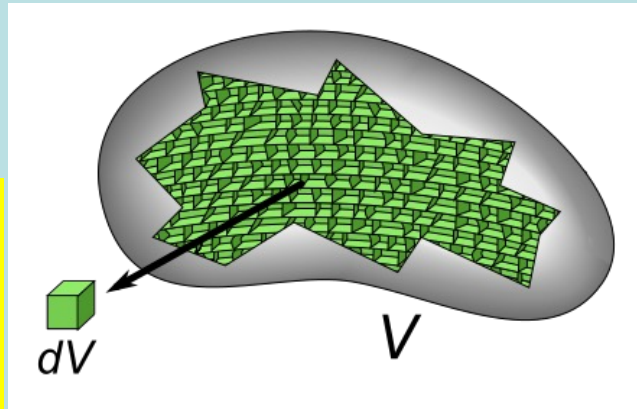
The flux through  $V$  equals the sum of all the flux through the cubes.

$$\Phi_E = \sum_i \Delta\Phi_{iE} = \sum_i (\vec{\nabla} \cdot \vec{E}_i) \cdot \Delta v_i \rightarrow \int (\vec{\nabla} \cdot \vec{E}) dv$$

Flux through a surface equals volume integral of divergence inside the surface.

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E}) dv$$

$$\oint_S \vec{A} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{A}) dv$$



It is true for any vector field.

## Divergence Theorem or Mathematical Gauss's Theorem

Integration in a volume is related to quantities on the surface.

This is intrinsically generalization of fundamental theorem of calculus.

$$\int_a^b dx \frac{df}{dx} = f(b) - f(a)$$

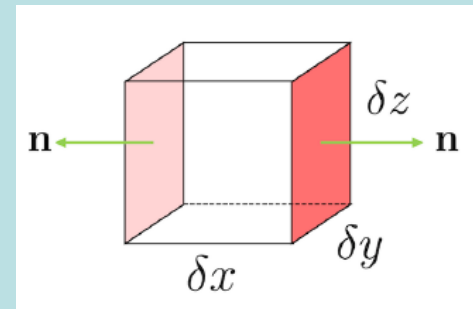
Now Physics:

Convert Gauss's Law from integral formulation to differential formulation

Apply the integral formulation of Gauss's Law to this small cube:

Electric flux equals the enclosed charge.

$$\oint \vec{E} \cdot d\vec{a} = (\vec{\nabla} \cdot \vec{E}) \cdot \Delta v = \frac{q}{\epsilon_0} = \frac{\rho \cdot \Delta v}{\epsilon_0}$$



$\rho$  is the charge density inside the cube, approximately a constant for small cube.

This is the **differential formulation of Gauss's Law**.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

This is an equation true for every point in space.

**Electric line accumulation of in unit volume is proportional to charge density.**

In reverse, take the differential formulation of Gauss's law.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Plug it into the divergence Theorem:

$$\oint \vec{E} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{E}) dv$$

We get the integral formulation of Gauss's law.

$$\oint \vec{E} \cdot d\vec{a} = \int \left( \frac{\rho}{\epsilon_0} \right) dv = \frac{q}{\epsilon_0}$$

Integral formulation of Gauss's law is equivalent to differential formulation.

Magnetic Gauss's law can also be converted into differential formulation.

$$\oint \vec{B} \cdot d\vec{a} = 0$$

Divergence Theorem

$$\oint \vec{B} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{B}) dv$$

Differential formulation: Divergence of the magnetic field is always zero.

$$\vec{\nabla} \cdot \vec{B} = 0$$

## Maxwell Equations

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$



$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

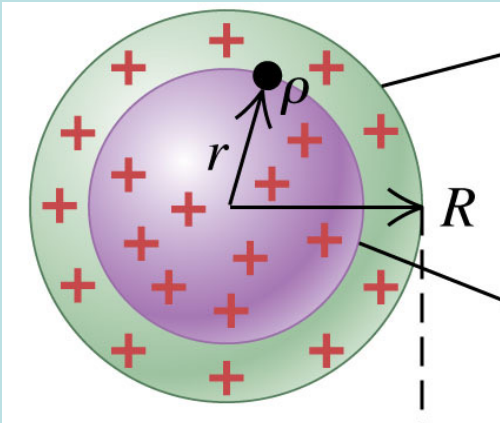
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

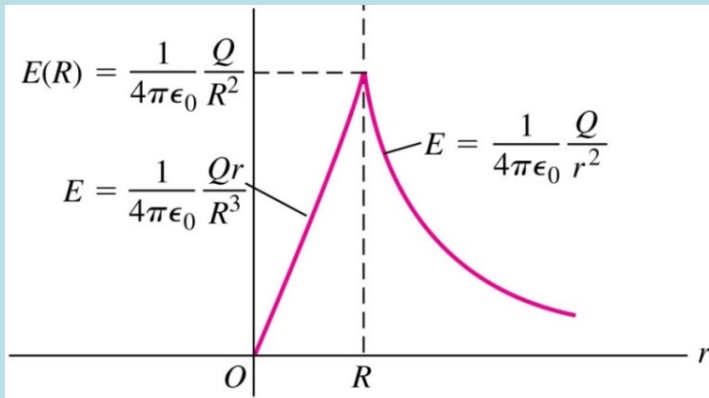


Within a uniform charge distribution :  $r < R$

Charge density:  $\rho$

$$E \cdot 4\pi r^2 = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

$$E = \frac{\rho}{3\epsilon_0} r$$



Electric flux inside the ball:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{3\epsilon_0} \vec{\nabla} \cdot \vec{r} = \frac{\rho}{3\epsilon_0} \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) = \frac{\rho}{\epsilon_0}$$

## Electric Field of a point charge

3. The electric field  $\vec{E}$  of a point charge  $Q$  fixed at the origin can be written as:

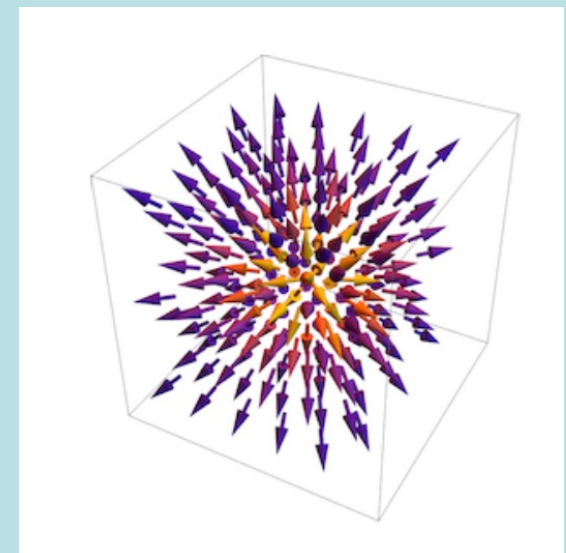
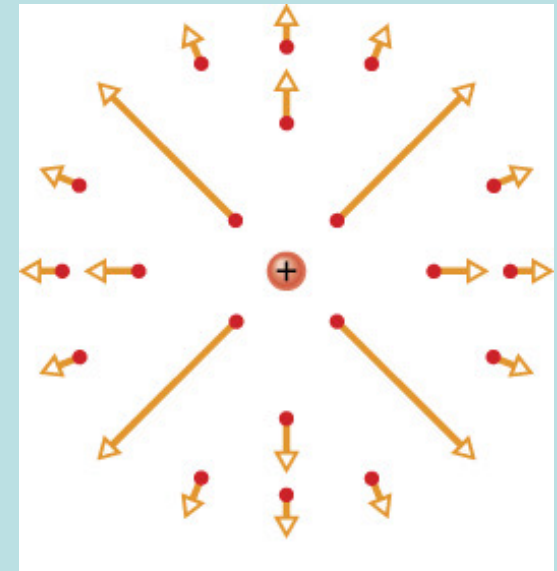
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

A. Calculate  $\vec{\nabla} \times \vec{E}$ .

B. Calculate  $\vec{\nabla} \cdot \vec{E}$  and show it is equal to zero except at the origin  $r = 0$ .

$$\vec{\nabla} \cdot \vec{E} = \frac{Q}{4\pi\epsilon_0} \vec{\nabla} \cdot \left( \frac{1}{r^2} \hat{r} \right) = 0$$

except for  $r = 0$ .



更嚴格的證明：

# Vector Calculus

University of Cambridge Part IA Mathematical Tripos

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## 3 Grad, Div and Curl

### 3.1 The Gradient

### 3.2 Div and Curl

#### 3.2.1 Some Basic Properties

#### 3.2.2 Conservative is Irrotational



3.2.3	Solenoidal Fields	58
3.2.4	The Laplacian	60
3.2.5	Some Vector Calculus Equations in Physics	60
3.3	Orthogonal Curvilinear Coordinates	61
3.3.1	Grad	64
3.3.2	Div and Curl	64
3.3.3	The Laplacian	66
<b>4</b>	<b>The Integral Theorems</b>	<b>67</b>
4.1	The Divergence Theorem	67
4.1.1	A Proof of the Divergence Theorem	69
4.1.2	Carl Friedrich Gauss (1777-1855)	74
4.2	An Application: Conservation Laws	74
4.2.1	Conservation and Diffusion	76
4.2.2	Another Application: Predator-Prey Systems	77
4.3	Green's Theorem in the Plane	80
4.3.1	George Green (1793-1841)	81
4.4	Stokes' Theorem	82
4.4.1	A Proof of Stokes' Theorem	86
4.4.2	George Gabriel Stokes (1819-1903)	88
4.4.3	An Application: Magnetic Fields	89
4.4.4	Changing Coordinates Revisited	90

### 3 Grad, Div and Curl

In this section we're going to further develop the ways in which we can differentiate. We'll be particularly interested in how we can differentiate scalar and vector fields. Our definitions will be straightforward but, at least for the time being, we won't be able to offer the full intuition behind these ideas. Perhaps ironically, the full meaning of how to differentiate will become clear only in Section 4 where we also learn the corresponding different ways to integrate.

**The 2d Divergence Theorem:** Let  $\mathbf{F}$  be a vector field in  $\mathbb{R}^2$ . Then

$$\int_D \nabla \cdot \mathbf{F} dA = \int_C \mathbf{F} \cdot \mathbf{n} ds \quad (4.6)$$

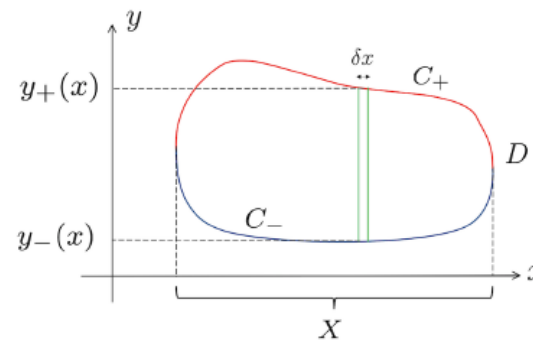
where  $D$  is a region in  $\mathbb{R}^2$ , bounded by the closed curve  $C$  and  $\mathbf{n}$  is the outward normal to  $C$ .

**Proof of the 2d Divergence Theorem:** For simplicity, we'll assume that  $\mathbf{F} = F(x, y) \hat{y}$ . The proof that we're about to give also works if  $\mathbf{F}$  points solely in the  $\hat{x}$  direction, but a general  $\mathbf{F}$  is just a linear sum of the two.

We then have

$$\int_D \nabla \cdot \mathbf{F} dA = \int_X dx \int_{y_-(x)}^{y_+(x)} dy \frac{\partial F}{\partial y}$$

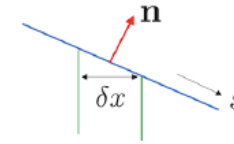
where, as the notation shows, we've chosen to do the area integral by first integrating over  $y$ , and then over  $x$ . We'll assume, for now, that the region  $D$  is convex, as shown in the figure, so that each  $\int dy$  is over just a single interval with limits  $y_{\pm}(x)$ . These limits trace out an upper curve  $C_+$ , shown



in red in the figure, and a lower curve  $C_-$  shown in blue. We then have

$$\int_D \nabla \cdot \mathbf{F} dA = \int_X dx \left( F(x, y_+(x)) - F(x, y_-(x)) \right)$$

We've succeeded in converting the area integral into an ordinary integral, but it's not quite of the line integral form that we need. The next part of the proof is to massage the integral over  $\int dx$  into a line integral over  $\int ds$ . This is easily achieved if we look at the zoomed-in figure to the right. Along the upper curve  $C_+$ , the normal  $\mathbf{n}$  points upwards and makes an angle  $\cos \theta = \hat{\mathbf{y}} \cdot \mathbf{n}$  with the vertical. Moving a small distance  $\delta s$  along the curve is equivalent to moving



$$\delta x = \cos \theta \delta s = \hat{\mathbf{y}} \cdot \mathbf{n} \delta s \quad \text{along } C_+$$

Along the lower curve,  $C_-$ , the normal  $\mathbf{n}$  points downwards and so  $\hat{\mathbf{y}} \cdot \mathbf{n}$  is negative. We then have

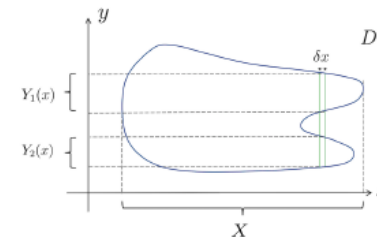
$$\delta x = -\hat{\mathbf{y}} \cdot \mathbf{n} \delta s \quad \text{along } C_-$$

The upshot is that we can write the area integral as

$$\begin{aligned} \int_D \nabla \cdot \mathbf{F} dA &= \int_X ds \left( \mathbf{n} \cdot \mathbf{F}(x, y_+(x)) + \mathbf{n} \cdot \mathbf{F}(x, y_-(x)) \right) \\ &= \int_{C_+} \mathbf{F} \cdot \mathbf{n} ds + \int_{C_-} \mathbf{F} \cdot \mathbf{n} ds \\ &= \int_C \mathbf{F} \cdot \mathbf{n} ds \end{aligned}$$

with  $C = C_+ + C_- = \partial D$  the boundary of the region.

We're left with one small loophole to close: if the region  $D$  is not convex, then the range of the integral  $\int dy$  may be over two or more disconnected intervals, as shown in the figure. In this case, the boundary curve decomposes into more pieces, but the basic strategy still holds.  $\square$



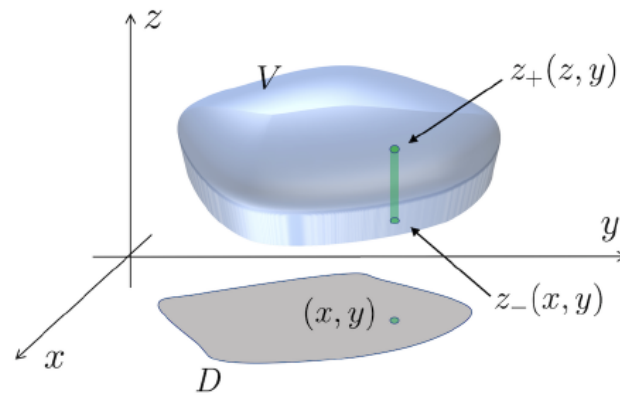


Figure 13. Performing the  $\int dz$  integral for the proof of the 3d divergence theorem.

### Proof of the 3d Divergence Theorem

The proof of the 3d (or, indeed, higher dimensional) divergence theorem follows using the same strategy. If we focus on  $\mathbf{F} = F(x, y, z) \hat{\mathbf{z}}$  we have

$$\begin{aligned} \int_V \nabla \cdot \mathbf{F} dV &= \int_D dA \int_{z_-(x,y)}^{z_+(x,y)} dz \frac{\partial F}{\partial z} \\ &= \int_D dA \left( F(x, y, z_+(x, y)) - F(x, y, z_-(x, y)) \right) \end{aligned}$$

where the limits of the integral  $z_{\pm}(x, y)$  are the upper and lower surfaces of the volume  $V$ . The area integral over  $D$  is an integral in the  $(x, y)$  plane, while to prove Gauss' theorem we need to convert this into a surface integral over  $S = \partial V$ . This step of the argument is the same as before: at any given point, the different between  $dA = dx dy$  and  $dS$  is the angle  $\cos \theta = \mathbf{n} \cdot \hat{\mathbf{z}}$  (up to a sign). This then gives the promised result (4.1).  $\square$

## Maxwell Equations

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$



$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

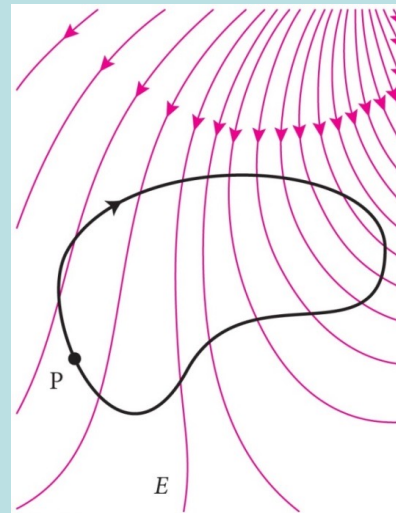
Could we write the line integral in a differential formulation?

記得電位差等於電場的線積分：

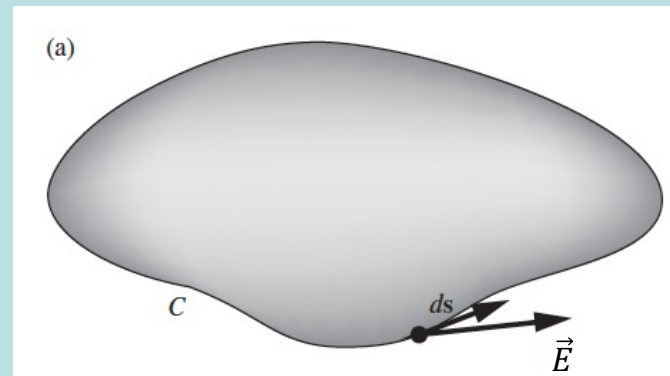
$$\Delta V = - \int_A^B \vec{E} \cdot d\vec{s}$$

若繞一個封閉曲線計算電場的線積分，因為起點與終點相同，電位差永遠等於零！

$$\oint \vec{E} \cdot d\vec{s} = 0$$



沿一個封閉曲線，電場的線積分必為零！



If a curved surface in space is divided into two regions  $a + b$ ,

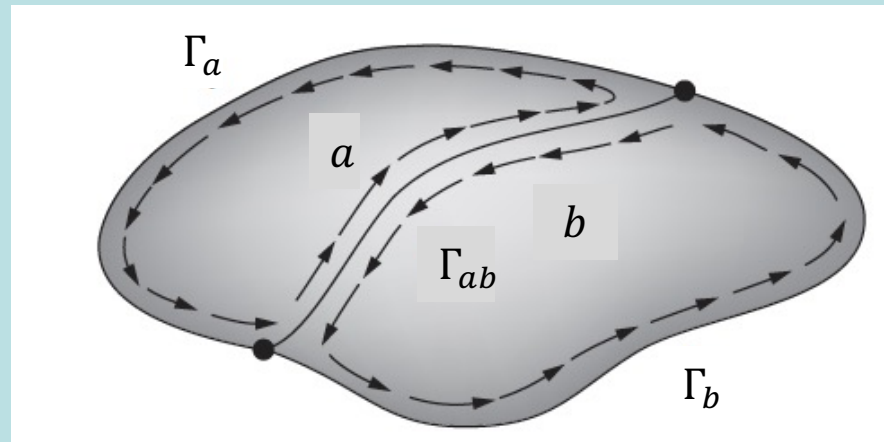
the boundary path  $\Gamma$  is decomposed into two sub-paths:  $\Gamma = \Gamma_a + \Gamma_b$

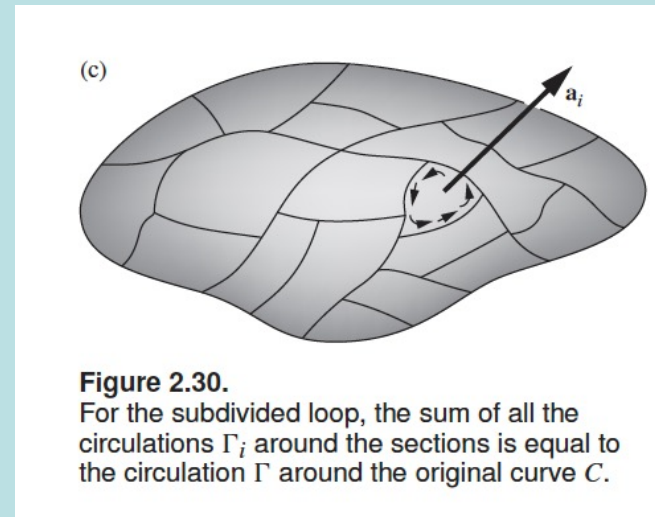
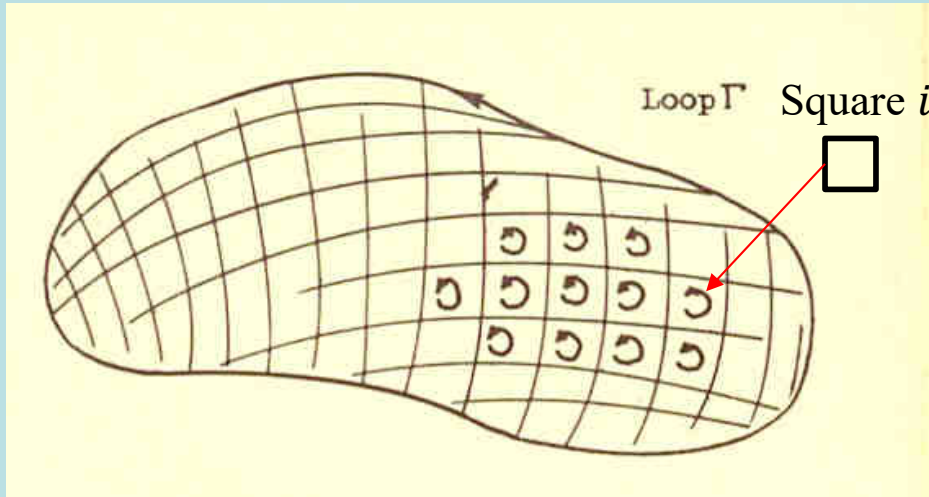
Add the line integrals along  $\Gamma_a$  and  $\Gamma_b$  and contributions on interface  $\Gamma_{ab}$  cancel.

That is because on  $\Gamma_{ab}$ , path  $\Gamma_a$  and path  $\Gamma_b$  move in opposite directions.

Line integral along  $\Gamma$  equals the sum of line integrals along  $\Gamma_a$  and  $\Gamma_b$ .

$$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = \oint_{\Gamma_a} \vec{E} \cdot d\vec{s} + \oint_{\Gamma_b} \vec{E} \cdot d\vec{s}$$



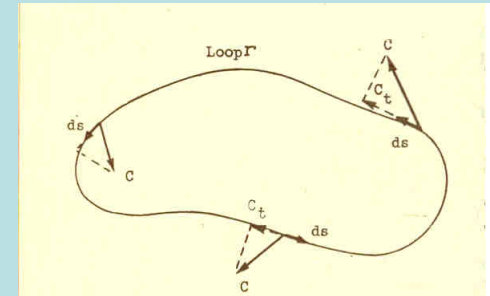
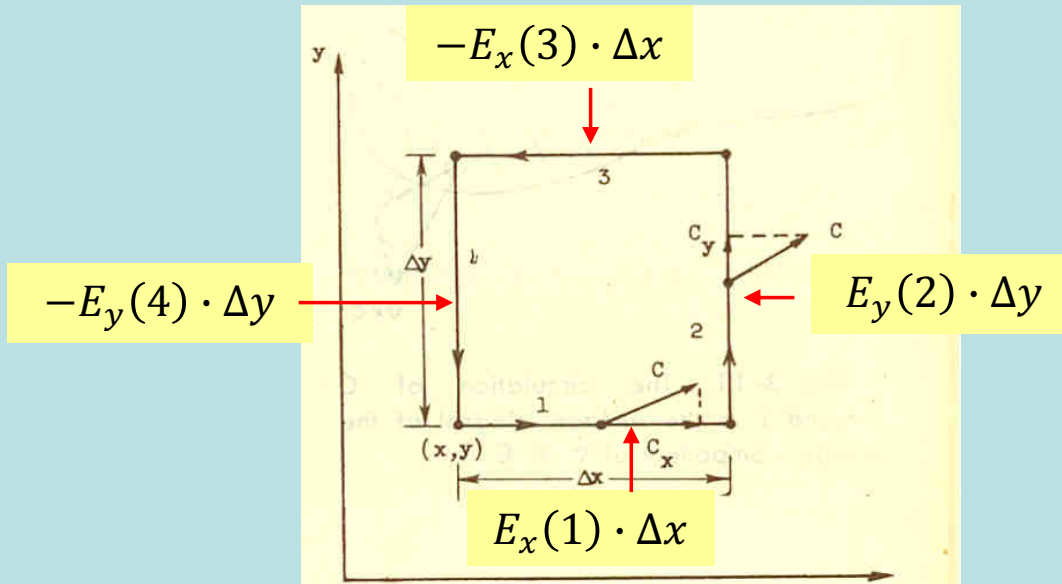


**Figure 2.30.**  
For the subdivided loop, the sum of all the circulations  $\Gamma_i$  around the sections is equal to the circulation  $\Gamma$  around the original curve  $C$ .

Any path can be decomposed into the combination of a series of small squares.  
Line integral along  $\Gamma$  equals the sum of line integrals along all the small squares.

$$\oint \vec{E} \cdot d\vec{s} = \sum_i \oint \vec{E}_i \cdot d\vec{s}_i$$

Calculate line integral along a small square on the  $x - y$  plane.



$$\oint \vec{E} \cdot d\vec{s} = [E_y(2)\Delta y - E_y(4)\Delta y] - [E_x(3)\Delta x + E_x(1)\Delta x]$$

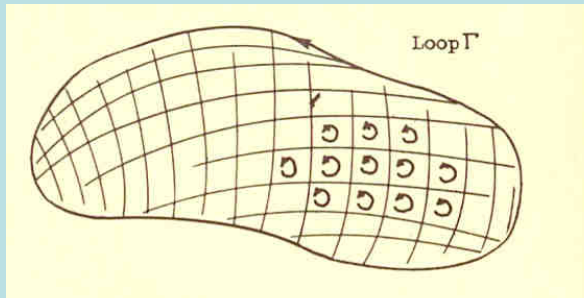
注意： $E_y(2, x + \Delta x, y) - E_y(4, x, y) \sim \frac{\partial E_y}{\partial x} \Delta x$

$$= \frac{\partial E_y}{\partial x} \Delta x \Delta y - \frac{\partial E_x}{\partial y} \Delta y \Delta x = \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \cdot \Delta x \Delta y = (\vec{\nabla} \times \vec{E})_z \cdot \Delta a = (\vec{\nabla} \times \vec{E}) \cdot \Delta \vec{a}$$

$$\oint \vec{E} \cdot d\vec{s} = (\vec{\nabla} \times \vec{E}) \cdot \Delta \vec{a}$$

$$(\vec{\nabla} \times \vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

The line integral of  $\vec{E}$  along a small square equals its curl times square area.



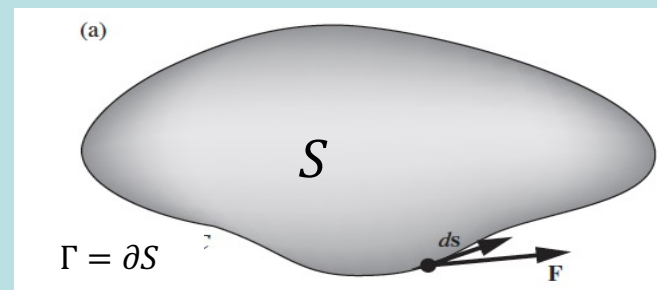
The line integral of  $\vec{E}$  along a small square equals its curl times square area.

$$(\vec{\nabla} \times \vec{E}) \cdot \Delta \vec{a}$$

Line integral along  $\Gamma$  equals the sum of line integrals along squares composing  $\Gamma$ .

$$\oint \vec{E} \cdot d\vec{s} = \sum_i \oint \vec{E}_i \cdot d\vec{s}_i = \sum_i (\vec{\nabla} \times \vec{E}_i) \cdot \Delta \vec{a}_i \rightarrow \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$

$$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$



Curve  $\Gamma = \partial S$  is defined as the boundary of  $S$ .

Line integral of a vector field along  $\Gamma = \partial S$ , equals surface integral of curl on  $S$ .

Stokes' Theorem

### 4.3 Green's Theorem in the Plane

Let  $P(x, y)$  and  $Q(x, y)$  be smooth functions on  $\mathbb{R}^2$ . Then

$$\int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P dx + Q dy \quad (4.12)$$

where  $A$  is a bounded region in the plane and  $C = \partial A$  is a piecewise smooth, non-intersecting closed curve which is traversed anti-clockwise.

**Proof:** Green's theorem is equivalent to the 2d divergence theorem (4.6). Let  $\mathbf{F} = (Q, -P)$  be a vector field in  $\mathbb{R}^2$ . We then have

$$\int_A \nabla \cdot \mathbf{F} dA = \int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (4.13)$$

If  $\mathbf{x}(s) = (x(s), y(s))$  is the parameterised curve  $C$ , then the tangent vector is  $\mathbf{t}(s) = (x'(s), y'(s))$  and the normal vector  $\mathbf{n} = (y'(s), -x'(s))$  obeys  $\mathbf{n} \cdot \mathbf{t} = 0$ .

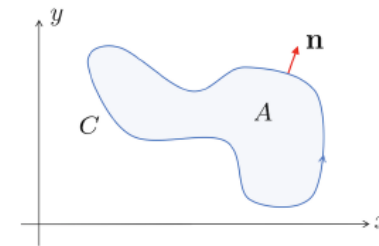
You'll need to do a little sketch to convince yourself that, as shown on the right,  $\mathbf{n}$  is the outward pointing normal provided that the arc length  $s$  increases in the anti-clockwise direction. We then have

$$\mathbf{F} \cdot \mathbf{n} = Q \frac{dy}{ds} + P \frac{dx}{ds}$$

and so the integral around  $C$  is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C P dx + Q dy \quad (4.14)$$

The 2d divergence theorem is the statement that the left-hand sides of (4.13) and (4.14) are equal; Green's theorem in the plane is the statement that the right-hand sides are equal.  $\square$



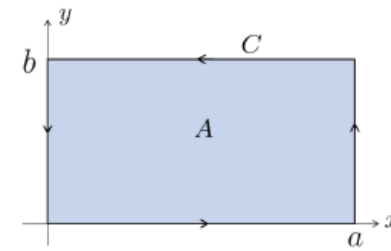
Applied to a rectangular region, Green's theorem in the plane reduces to the fundamental theorem of calculus. We take the rectangular region to be  $0 \leq x \leq a$  and  $0 \leq y \leq b$ . Then

$$\begin{aligned} \int_A -\frac{\partial P}{\partial y} dA &= -\int_0^a dx \int_0^b dy \frac{\partial P}{\partial y} \\ &= \int_0^a dx \left( -P(x, b) + P(x, 0) \right) = \int_C P dx \end{aligned}$$

where only the horizontal segments contribute, and the minus signs are such that  $C$  is traversed anti-clockwise. Meanwhile, we also have

$$\begin{aligned} \int_A \frac{\partial Q}{\partial x} dA &= \int_0^b dy \int_0^a dx \frac{\partial Q}{\partial x} \\ &= \int_0^b dy \left( Q(x, a) - Q(x, 0) \right) = \int_C Q dy \end{aligned}$$

where, this time, only the vertical segments contribute, and the minus signs are such that  $C$  is traversed anti-clockwise.



**George Green** (14 July 1793 – 31 May 1841) was a British [mathematical physicist](#) who wrote *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in 1828.<sup>[2][3]</sup> The essay introduced several important concepts, among them a theorem similar to the modern [Green's theorem](#), the idea of [potential functions](#) as currently used in [physics](#), and the concept of what are now called [Green's functions](#). Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as [James Clerk Maxwell](#), [William Thomson](#), and others. His work on [potential theory](#) ran parallel to that of [Carl Friedrich Gauss](#).

Green's life story is remarkable in that he was almost entirely self-taught. He received only about one year of formal schooling as a child, between the ages of 8 and 9.

### Early life [\[ edit \]](#)

Green was born and lived for most of his life in the English town of [Sneinton](#), Nottinghamshire, now part of the city of [Nottingham](#). His father, also named George, was a baker who had built and owned a brick [windmill](#) used to grind grain.<sup>[1]</sup>

In his youth, Green was described as having a frail constitution and a dislike for doing work in his father's bakery. He had no choice in the matter, however, and as was common for the time he likely began working daily to earn his living at the age of five.

### Robert Goodacre's Academy [\[ edit \]](#)

During this era it was common for only 25–50% of children in Nottingham to receive any schooling.<sup>[citation needed]</sup> The majority of schools were Sunday schools, run by the

<b>George Green</b>	
<b>Born</b>	14 July 1793 Sneinton, Nottinghamshire, England
<b>Died</b>	31 May 1841 (aged 47) Nottingham, Nottinghamshire, England
<b>Alma mater</b>	Gonville and Caius College, Cambridge (BA, 1838)
<b>Known for</b>	Green measure Green's deformation tensor Green's function Green's identities Green's law Green's matrix Green's theorem Liouville–Green method
<b>Fields</b>	Mathematics
<b>Institutions</b>	Gonville and Caius College, Cambridge <sup>[1]</sup>



Green's Mill in Sneinton, the mill

以上的證明路徑與路徑內的面必須在一平面上，但此結果也適用於曲面。

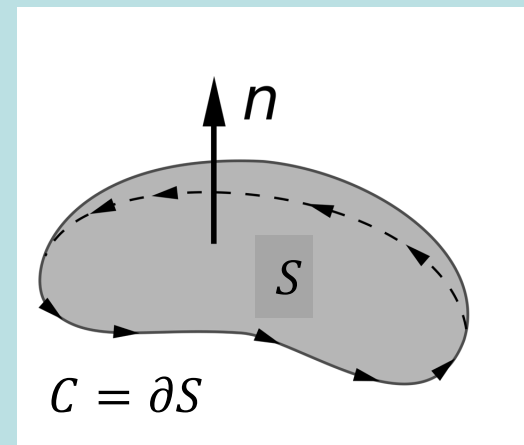
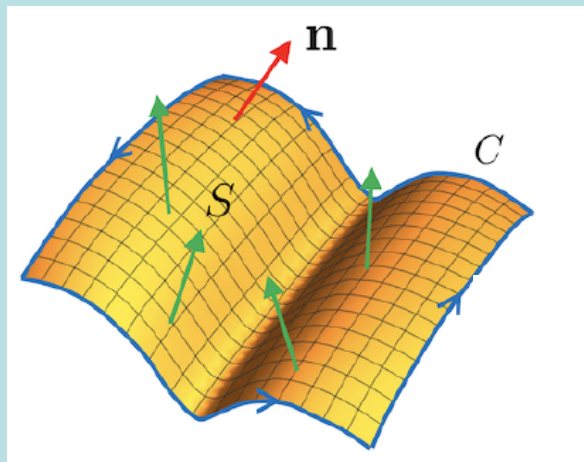
## 曲面上的史塔克定律 Stokes' Theorem

### 4.4 Stokes' Theorem

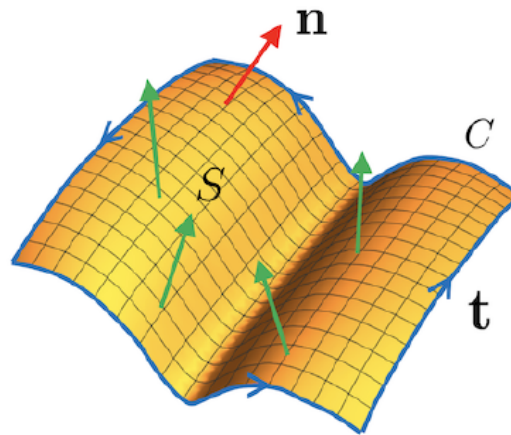
Stokes' theorem is an extension of Green's theorem, but where the surface is no longer restricted to lie in a plane.

Let  $S$  be a smooth surface in  $\mathbb{R}^3$  with boundary  $C = \partial S$  a piecewise smooth curve. *Stokes' theorem* states that, for any smooth vector field  $\mathbf{F}(\mathbf{x})$ , we have

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \oint_C \vec{F} \cdot d\vec{s}$$



沿任一曲線  $C = \partial S$  的向量場線積分，等於曲線內曲面  $S$ ，該向量場旋度的面積分  
一個曲面內的積分，竟然與邊界的積分可以連鎖。



**Figure 15.** The surface  $S$  and bounding curve  $C$  for Stokes' theorem. The normal to the surface is shown (at one point) by the red arrow. The theorem invites us to compute the flux of a vector field  $\mathbf{F}$ , shown by the green arrows, through the surface, and compare it to the line integral around the boundary.

The orientations of  $S$  and  $C$  should be *compatible*. The former is determined by the choice of normal vector  $\mathbf{n}$  to  $S$ ; the latter by the choice of tangent vector  $\mathbf{t}$  to  $C$ . The two are said to be compatible if  $\mathbf{t} \times \mathbf{n}$  points out of  $S$ . In practice, this means that if you orient the open surface so that  $\mathbf{n}$  points towards you, then the orientation of  $C$  is anti-clockwise. The general set-up is shown in Figure 15.

Note that there will typically be many surfaces  $S$  that share the same boundary  $C$ . By Stokes' theorem, the integral of  $\nabla \times \mathbf{F}$  over  $S$  must give the same answer for all such surfaces. The theorem also holds if the boundary  $\partial S$  consists of a number of disconnected components, again with their orientation determined by that of  $S$ .

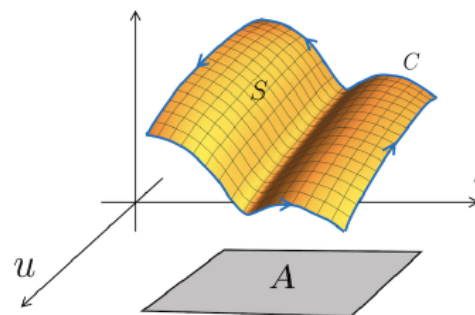
#### 4.4.1 A Proof of Stokes' Theorem

It's clear that Stokes' theorem is a version of Green's theorem in the plane, but viewed through 3d glasses. Indeed, it's trivial to show that the latter follows from the former. Consider the vector field  $\mathbf{F} = (P, Q, 0)$  in  $\mathbb{R}^3$  and a surface  $S$  that lies flat in the  $z = 0$  plane. The normal to this surface is  $\mathbf{n} = \hat{\mathbf{z}}$ , and we have

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$$

But Stokes' theorem then tells us that this can also be written as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C P dx + Q dy$$



However, with a little more work we can also show that the converse is true. In other words, we can lift Green's theorem out of the plane to find Stokes' theorem.

Consider a parameterised surface  $S$  defined by  $\mathbf{x}(u, v)$  and denote the associated area in the  $(u, v)$  plane as  $A$ . We parameterise the boundary  $C = \partial S$  as  $\mathbf{x}(u(t), v(t))$  and the corresponding boundary  $\partial A$  as  $(u(t), v(t))$ . The key idea is to use Green's theorem in the  $(u, v)$  plane for the area  $A$  and then uplift this to prove Stokes theorem for the surface  $S$ .

We start by looking at the integral around the boundary. It is

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot \left( \frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv \right) = \int_{\partial A} F_u du + F_v dv$$

where  $F_u = \mathbf{F} \cdot \partial \mathbf{x} / \partial u$  and  $F_v = \mathbf{F} \cdot \partial \mathbf{x} / \partial v$ . Now we're in a position to invoke Green's theorem, in the form

$$\int_{\partial A} F_u du + F_v dv = \int_A \left( \frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) dA$$

Now our task is clear. We should look at the partial derivatives on the right hand side. We just need to be careful about what thing depends on what thing:

$$\frac{\partial F_v}{\partial u} = \frac{\partial}{\partial u} \left( \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial v} \right) = \frac{\partial}{\partial u} \left( F_i \frac{\partial x^i}{\partial v} \right) = \left( \frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial u} \right) \frac{\partial x^i}{\partial v} + F_i \frac{\partial^2 x^i}{\partial u \partial v}$$

Meanwhile, we have

$$\frac{\partial F_u}{\partial v} = \frac{\partial}{\partial v} \left( \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial u} \right) = \frac{\partial}{\partial v} \left( F_i \frac{\partial x^i}{\partial u} \right) = \left( \frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial v} \right) \frac{\partial x^i}{\partial u} + F_i \frac{\partial^2 x^i}{\partial v \partial u}$$

Subtracting the second expression from the first, the second derivative terms cancel, leaving us with

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} \left( \frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i} \right) = (\delta_{jk} \delta_{il} - \delta_{jl} \delta_{ik}) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j}$$

At this point we wield everyone's favourite index notation identity

$$\epsilon_{jip} \epsilon_{pkl} = \delta_{jk} \delta_{il} - \delta_{jl} \delta_{ik}$$

We then have

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \epsilon_{jip} \epsilon_{pkl} \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j} = (\nabla \times \mathbf{F}) \cdot \left( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right)$$

Now we're done. Following through the chain of identities above, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{x} &= \int_A \left( \frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) dudv \\ &= \int_A (\nabla \times \mathbf{F}) \cdot \left( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right) dudv \\ &= \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\end{aligned}$$

This is Stokes' theorem. □

8. If  $X, Y, Z$  be functions of the rectangular co-ordinates  $x, y, z$ ,  $dS$  an element of any limited surface,  $l, m, n$  the cosines of the inclinations of the normal at  $dS$  to the axes,  $ds$  an element of the bounding line, shew that

$$\begin{aligned}\iint \left\{ l \left( \frac{dZ}{dy} - \frac{dY}{dx} \right) + m \left( \frac{dX}{dz} - \frac{dZ}{dx} \right) + n \left( \frac{dY}{dx} - \frac{dX}{dy} \right) \right\} dS \\ = \int \left( X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds,\end{aligned}$$

the differential coefficients of  $X, Y, Z$  being partial, and the single integral being taken all round the perimeter of the surface.

**Figure 16.** You may now turn the page... the original version of Stokes' theorem, set as an exam question.



George Stokes 1819-1903

Physics

$$\oint \vec{E} \cdot d\vec{s} = 0$$

Integral Formulation

Apply it to a small square and let the size approaches zero:



$$\oint \vec{E} \cdot d\vec{s} = (\vec{\nabla} \times \vec{E}) \cdot \Delta \vec{a}$$

$$\vec{\nabla} \times \vec{E} = 0$$

Differential Formulation

$$\vec{\nabla} \times \vec{E} = 0$$

Differential Formulation

Plug into the Mathematical Stoke's Law:

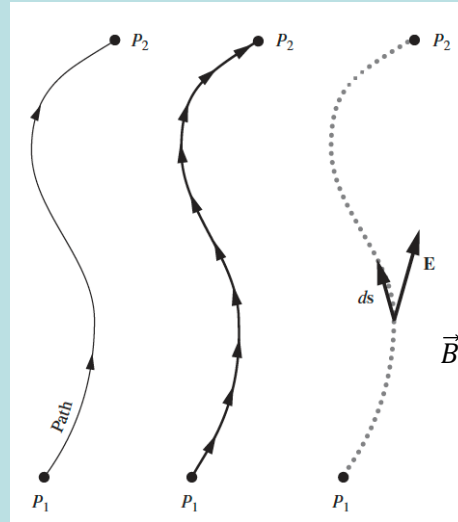


$$\oint \vec{E} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$

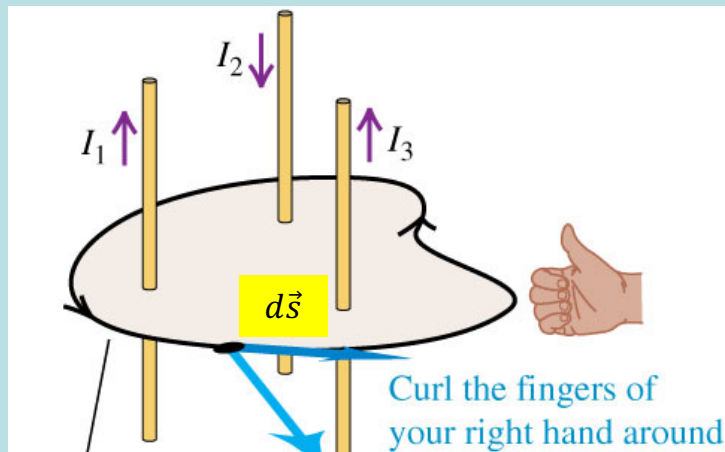
$$\oint \vec{E} \cdot d\vec{s} = 0$$

Integral Formulation

磁場 $\vec{B}$ 的線積分！



$$- \int_{P_1}^{P_2} \vec{B} \cdot d\vec{s}$$

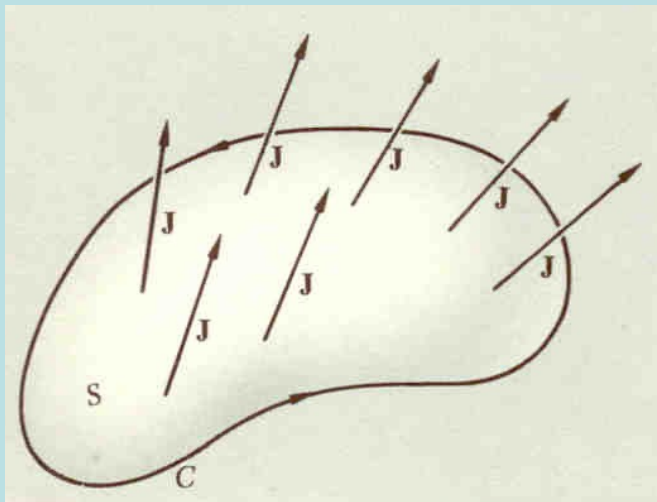
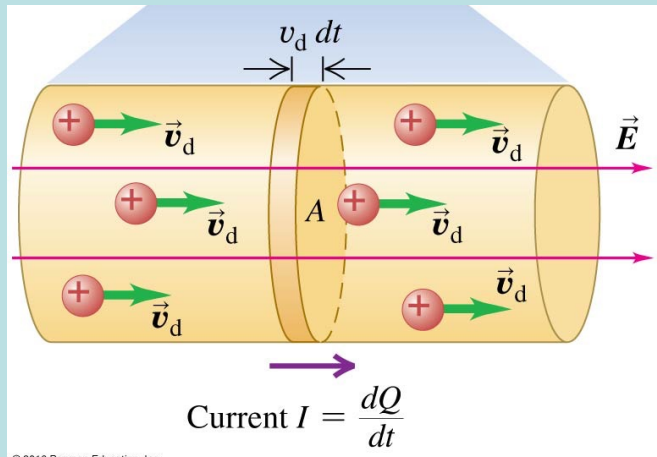


對任一封閉曲線，磁場的線積分正比於該曲線所包圍的總電流（電流疊加）。

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i \quad \text{安培定律 Ampere's Law}$$

Curl of a vector field allows us calculate its line integral.

電流為單位時間流過某一截面積 $A$ 的電荷量！  $i = \frac{q}{\Delta t}$



電流大是不是載體電荷流動快？

電流一般會與垂直截面積成正比。

將電流除以面積，才代表當地電荷的流動快慢。

$j = \frac{i}{A}$  稱為電流密度：單位面積的電流。

電流密度代表當地電荷流動，可以給予一方向！  
取電荷流動的方向為電流密度的方向

$j \rightarrow \vec{j}$

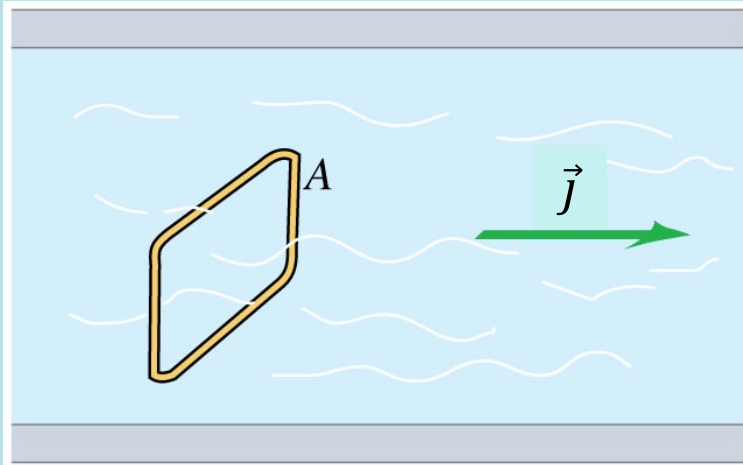
導體各處都可以有一個當地的電流密度向量。

$\vec{j} \rightarrow \vec{j}(\vec{r})$  電流密度也是向量場。

可以證明：

$\vec{j} = \rho \vec{v}$

With current density  $j$ , we can in turn calculate the current  $i$  through a flat plane.



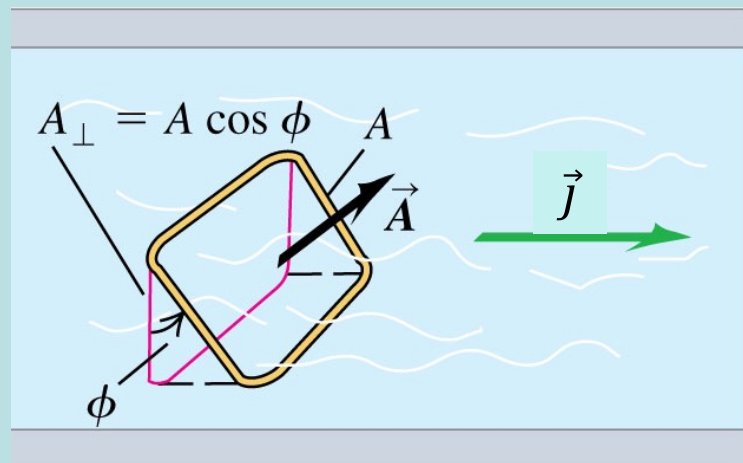
If the plane is perpendicular to  $j$ ,

$$j = \frac{i}{A}$$

$$i = jA$$



$$\phi = EA$$



If not, we can project it into perpendicular one:  $A_{\perp}$ .

$$i = jA_{\perp} = j \cdot A \cos \theta = \vec{j} \cdot \vec{A}$$

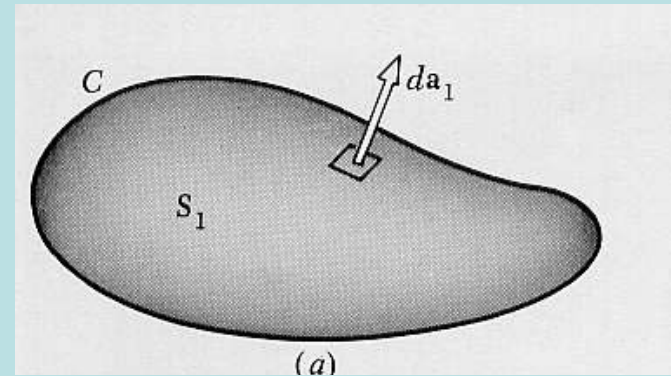
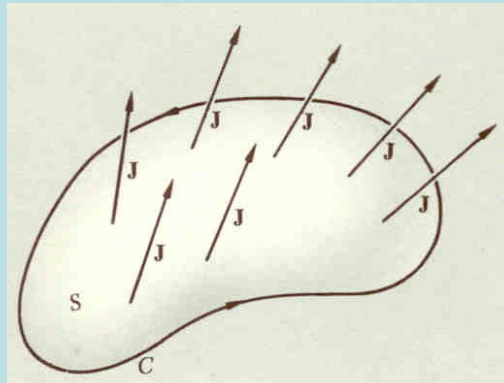
Current  $i$  equals the inner product of  $\vec{j}$  and  $\vec{A}$ , ie. flux.

$$i = \vec{j} \cdot \vec{A}$$



$$\Phi_E = \vec{E} \cdot \vec{A}$$

The above formula can also be used to calculate the current through a curved surface.



You just need cut the surface into pieces of small flat plane.

Through any one small flat plane, current equals  $\vec{j} \cdot d\vec{a}$  !

Sum up and take the limit. **Total current equals the surface integral of current density.**

$$i = \int \vec{j} \cdot d\vec{a}$$

Ampere's Law can also be written in differential formulation

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i = \mu_0 \int \vec{j} \cdot d\vec{a}$$



$$\oint \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$$

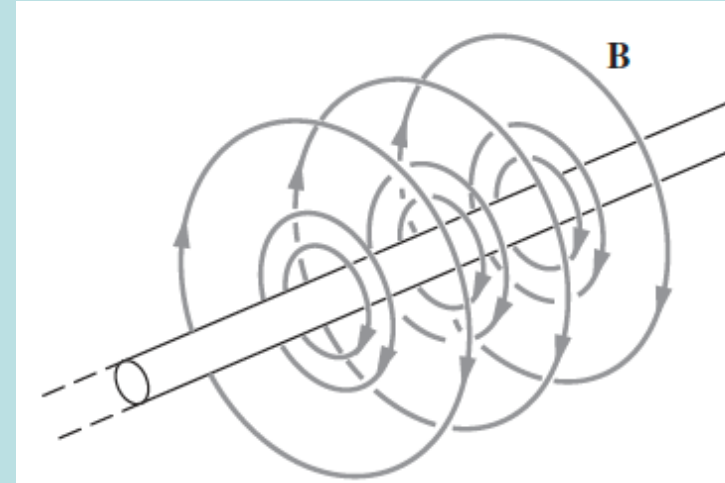
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

differential formulation of Ampere's Law

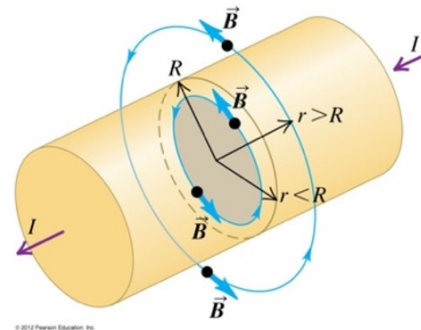
長直導線外的磁場

$$\vec{\nabla} \times \vec{B} = 0$$

except for  $r = 0$ .



5. The magnetic field inside a cylindrical current along the  $z$  axis with radius  $R$  and constant current density  $j$  can be written as ( $r < R$ ):



$$\vec{B} = \frac{\mu_0 j}{2} (y, -x, 0)$$

Calculate  $\vec{\nabla} \cdot \vec{B}$  and  $\vec{\nabla} \times \vec{B}$  and check it is consistent with Maxwell Equations.

## Maxwell Equations For Static Fields

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$

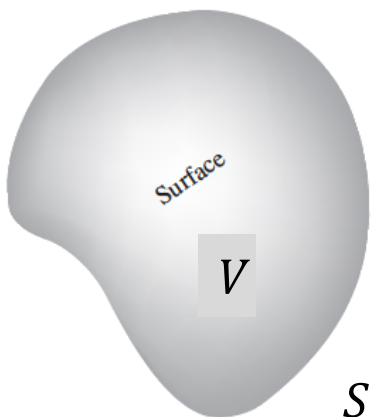
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$



散度透過高斯定律對應到面積分！旋度透過史塔克定律對應到線積分！

# Summary

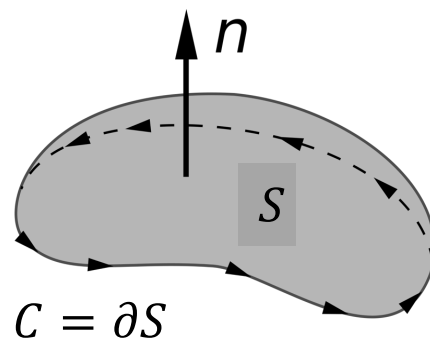
GAUSS



$$\oint_S \vec{F} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{F}) dv$$

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \nabla \cdot \mathbf{F} \end{aligned}$$

STOKES

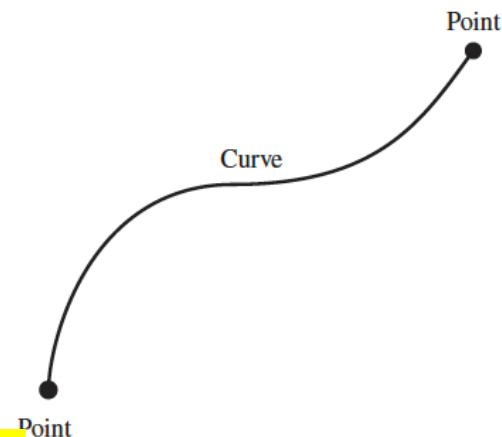


$$\oint_C \vec{A} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

In Cartesian coordinates, with  $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ :

$$\begin{aligned} \text{curl } \mathbf{A} &= \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \\ &\quad + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ &\quad + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= \nabla \times \mathbf{A} \end{aligned}$$

GRAD



Points enclose curve

$$\phi_2 - \phi_1 = \int_{\text{curve}} \text{grad } \phi \cdot ds$$

$$\begin{aligned} \text{grad } \phi &= \hat{x} \frac{\partial \phi}{\partial x} + \hat{y} \frac{\partial \phi}{\partial y} + \hat{z} \frac{\partial \phi}{\partial z} \\ &= \nabla \phi \end{aligned}$$

散度就是一個小體積的單位體積電場總通量！

旋度等於沿無限小的正方形的場線積分乘上正方形面積！

Time changing magnetic field will generate electric field by induction.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

Faraday's Law



Mathematical Stoke's Law

$$\oint \vec{E} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law in differential formulation

## Complete Maxwell Equations

積分形式

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

微分形式

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

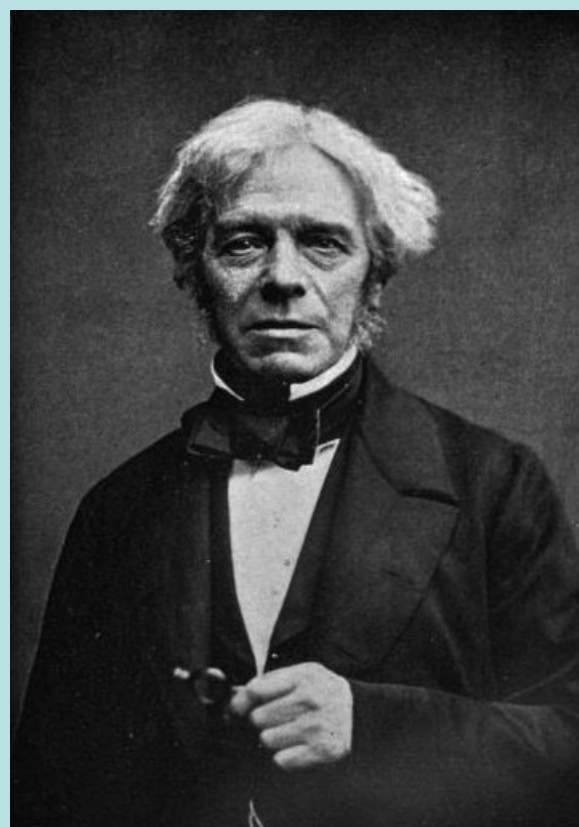
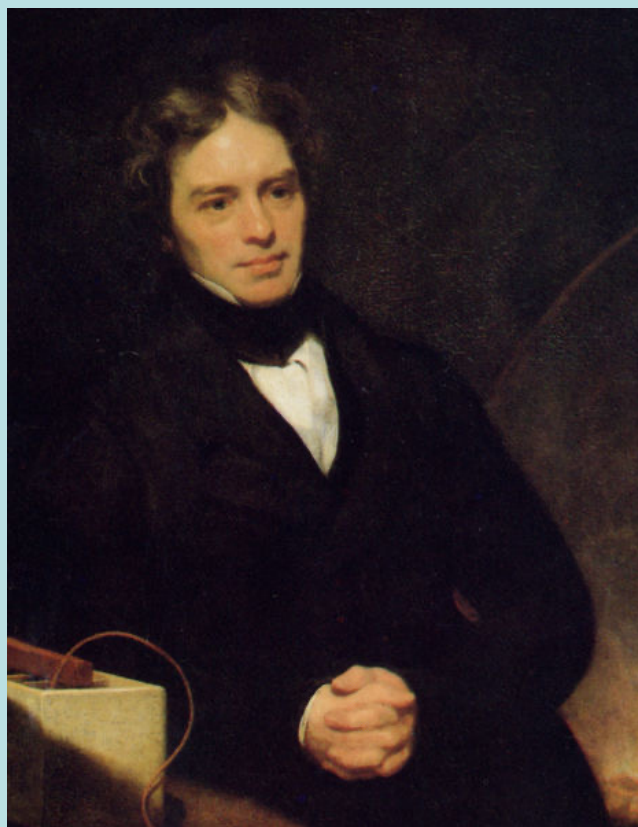
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

電磁場不單純由電荷電流產生，電（磁）場變化時也會感應產生磁（電）場。

## 電場的發明者 Faraday 1791-1861



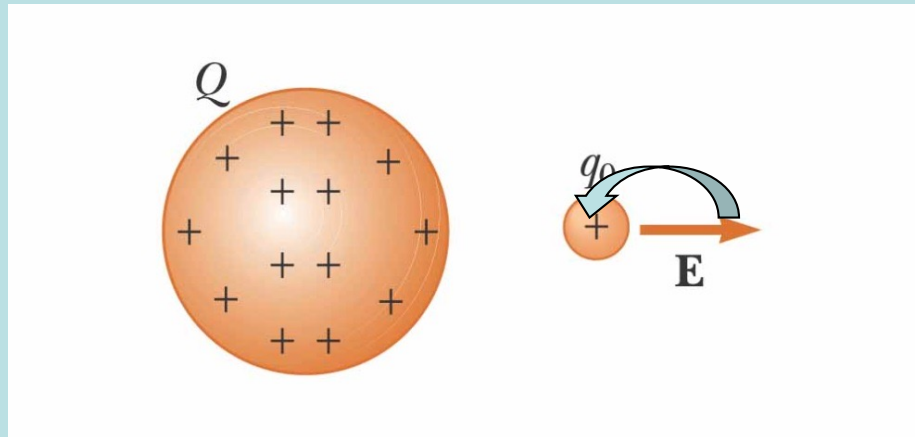
在對岸英國的法拉第憑直覺，反對靜電力是超距力，因而引進電場的概念。



Your obly. & faithful Servant  
M<sup>r</sup> Faraday

超距力並不是正確的概念！

電場的引進使得電力有機會可以不再是超距力



**Locality**在地主義：一個地方發生的事物，在當時只會影響它附近的事物。

積分形式

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

微分形式

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

左式的路徑內磁場的變化似乎與整條路徑上的電場有關。

Locality在地主義並不確定。但這只是表象。

微分形式的法拉第定律只關乎一個點上的場，

一個點上磁場的變化等於此點上電場旋度。完全在地！

馬克思威爾方程式原來完全遵守法拉第期待的Locality在地主義。

Shapes of the field lines

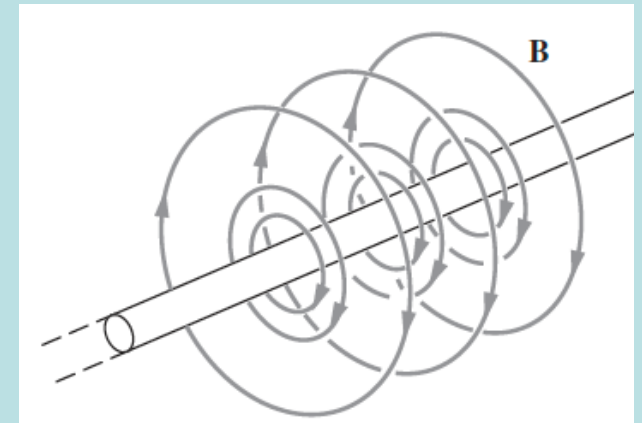
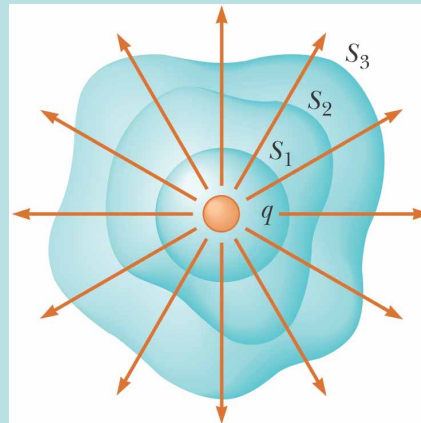
Electrostatic field and Magnetostatics field are quite different.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$



Curl of electrostatic field is zero and divergence of magnetostatics field is zero.

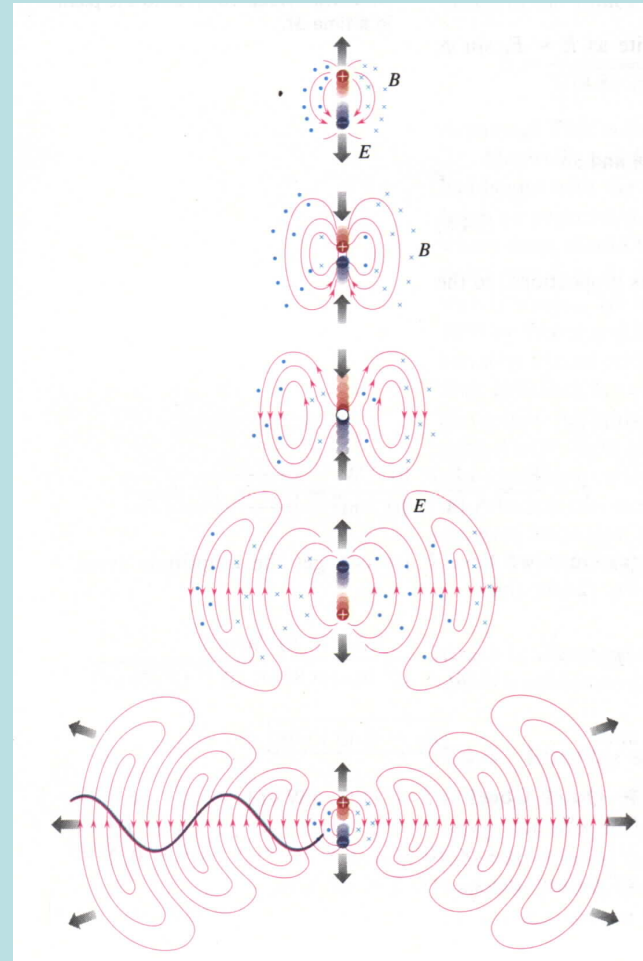
In vacuum (no charge and current),  
electric field and magnetic can only come from induction.

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

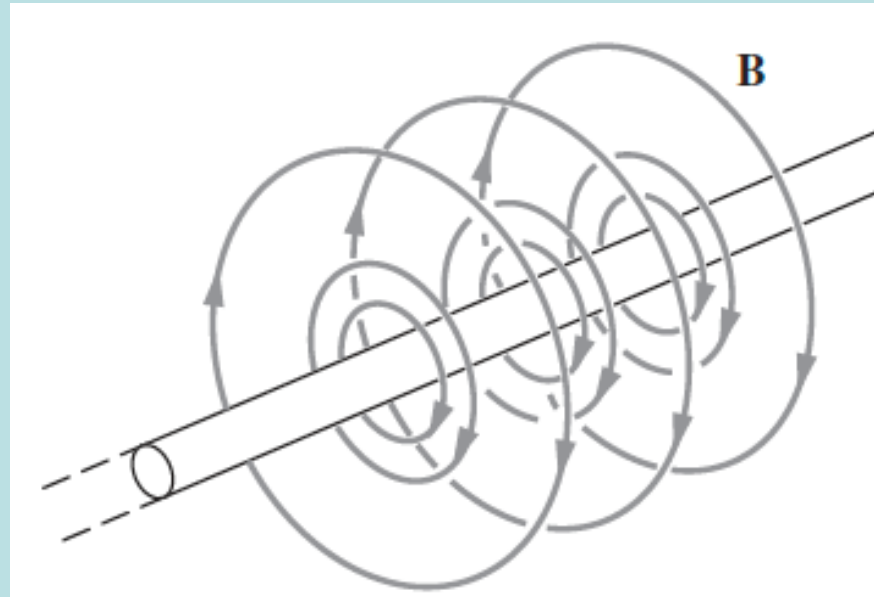
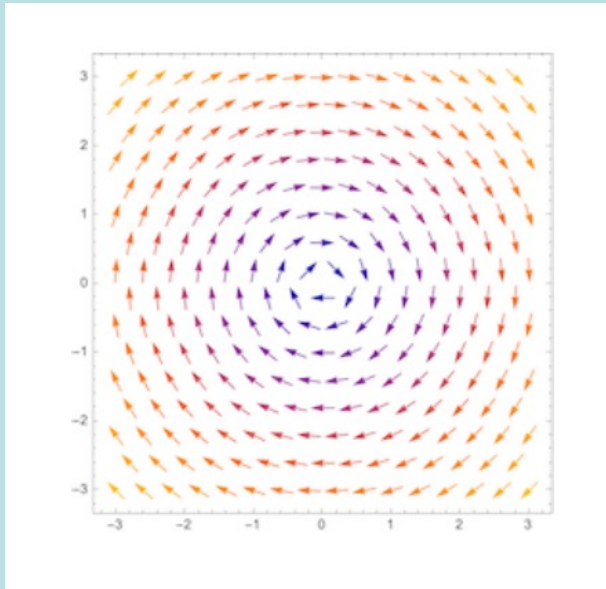
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$



Divergence of electric field and magnetic fields are both zero.

以靜磁場與靜電場的場線來理解是最方便的！



靜磁場是零散度，靜磁場線是一條一條的封閉曲線（不一定是圓）。

沒有起點與終點，也就沒有磁荷了！

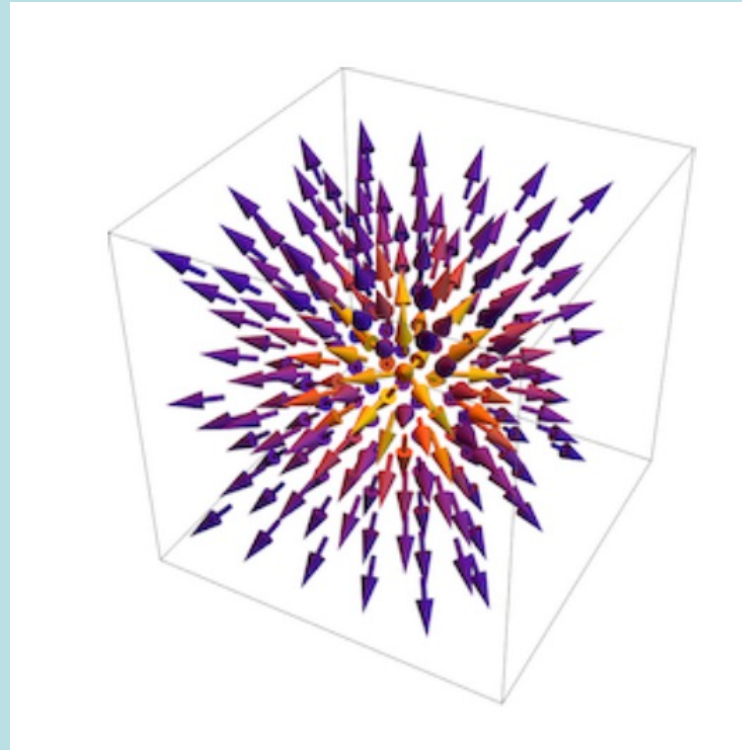
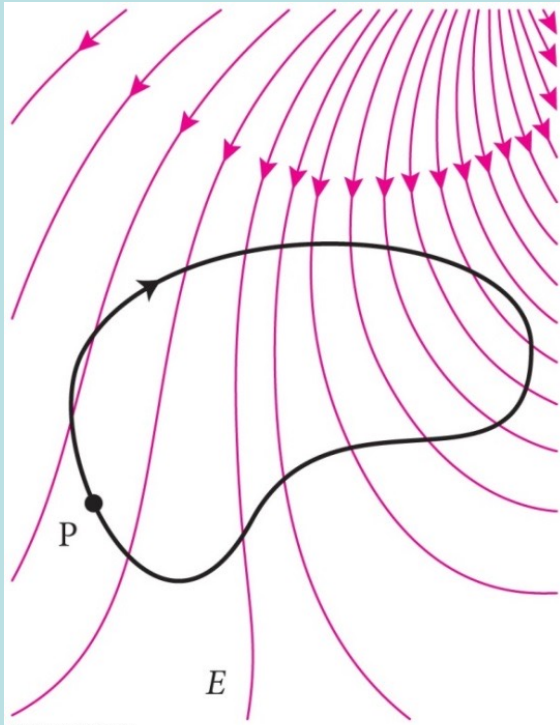
我們可以把它們稱為**漩渦狀**的場線。

**漩渦狀**場線沒有起點或末點，因此對任一高斯面，進入的場線一定離開。

**散度永遠為零**！也稱為無散度場**Divergence Free Field**.

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{B}_{\text{漩渦}} \cdot d\vec{A} = 0$$



庫倫電場是由點狀源頭發出的場線，可以稱為放射狀，  
它是保守力場，**放射狀**場線對線積分貢獻永遠為零！

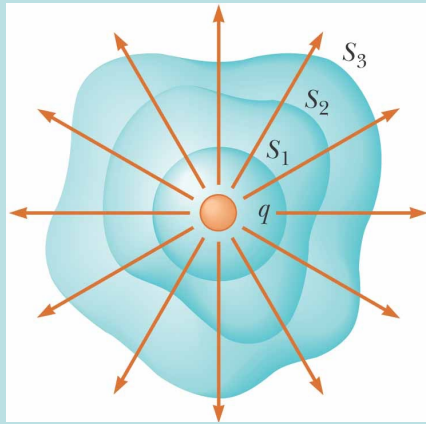
對任一封閉曲線

$$\oint \vec{E}_{\text{放射}} \cdot d\vec{s} = 0$$

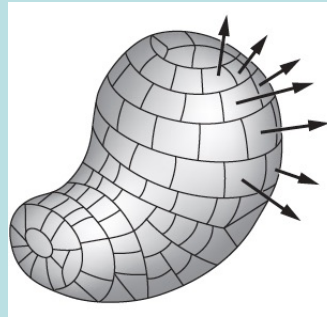
因此

$$\vec{\nabla} \times \vec{E} = 0$$

又稱為無旋度場 **Irrotational Field**.

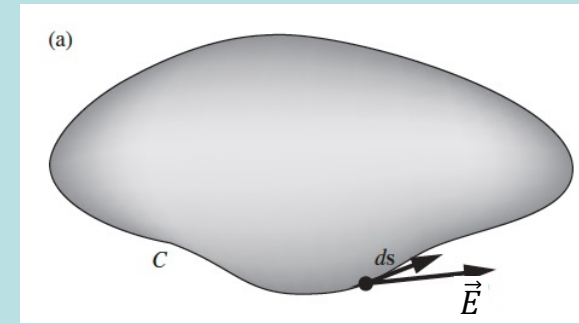


放射狀的場線



面積分

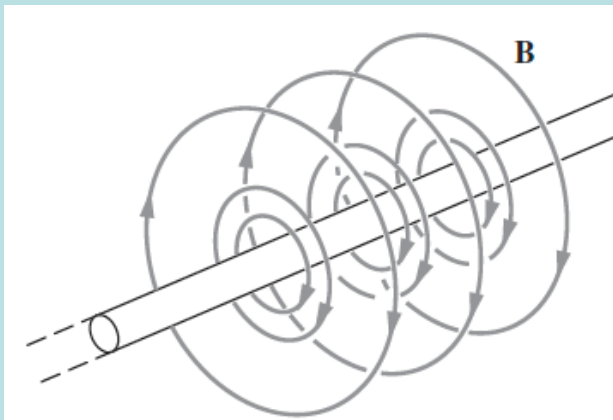
$$\oint \vec{E}_{\text{放射}} \cdot d\vec{A}$$



線積分

$$\oint \vec{E}_{\text{放射}} \cdot d\vec{s} = 0$$

面積分會補抓放射狀的場線！

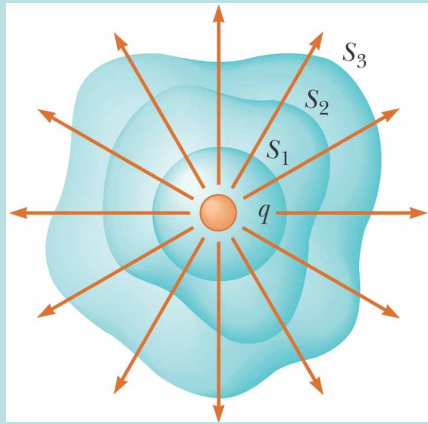


漩渦狀的場線

$$\oint \vec{E}_{\text{漩渦}} \cdot d\vec{A} = 0$$

$$\oint E_{\text{漩渦}} \cdot ds$$

線積分會補抓漩渦狀的場線！



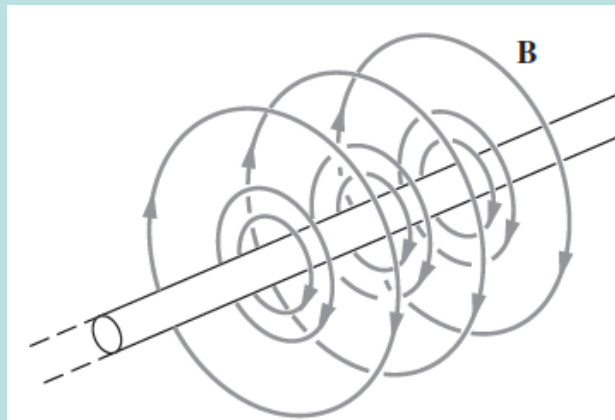
精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

圖像：

放射狀的場線，

稱為無旋度場 Irrotational Field



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線，

稱為無散度場 Divergence Free Field.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

電荷是靜電場的來源，產生的靜電場是放射狀，**無旋度場 Irrotational Field**。

電流是靜磁場的來源，產生的靜磁場是漩渦狀，**稱為無散度場 Divergence Free Field**。

這兩種場分別各有一個定理！這兩種場都可以寫成另一個場的特定微分！

## Potentials

$\vec{\nabla} \times \vec{E} = 0$  電場線積分為零，所以有電位存在，是電位的梯度： $\vec{E} = \vec{\nabla} V$

定理：任一無散度向量場 $\vec{F}$ ， $\vec{\nabla} \times \vec{F} = 0$

等價於：此一向量場 $\vec{F}$ 可以寫成一個純量場 $\phi$ 的梯度： $\vec{F} = \vec{\nabla} \phi$

倒過來的陳述的證明：

純量場 $\phi$ 的梯度，必無散度：

$$\vec{\nabla} \times (\vec{\nabla} \phi) = 0$$

向量外積的規則顯示梯度的旋度必為零！ $\vec{\nabla} \times \vec{\nabla} \sim 0$ ，電場滿足：

$$\vec{\nabla} \times \vec{F} = 0$$

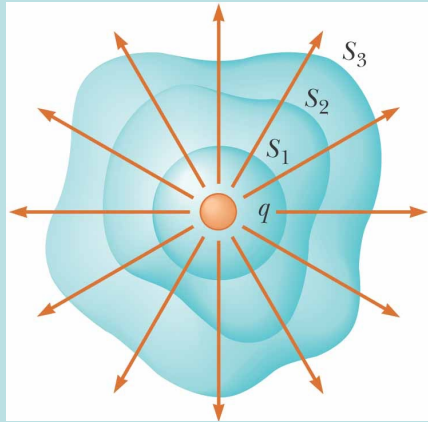
正向的證明：根據 Stokes Theorem

$$\vec{\nabla} \times \vec{F} = 0 \quad \longrightarrow \quad \oint \vec{F} \cdot d\vec{s} = 0$$

是一個保守力場，因此可以寫下位能 $\phi$ ：

$$\Delta \phi = \int \vec{F} \cdot d\vec{s}$$

如同電場為電位的梯度， $\vec{F}$ 也是 $\phi$ 的梯度： $\vec{F} = \vec{\nabla} \phi$  得證！



精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

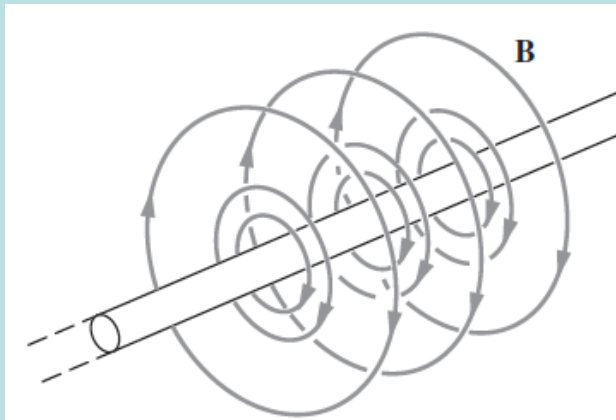
圖像：

放射狀的場線

另一個場的微分：

$$\vec{E} = \vec{\nabla} \phi$$

稱為無旋度場 Irrotational Field



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線

?

稱為無散度場 Divergence Free Field.

定理：任一無散度向量場 $\vec{B}$ ，  $\vec{\nabla} \cdot \vec{B} = 0$

等價於此一向量場 $\vec{B}$ 可以寫成一個向量場 $\vec{A}$ 的旋度： $\vec{B} = \vec{\nabla} \times \vec{A}$

倒過來的陳述的證明：

向量外積的規則顯示旋度的散度必為零！向量場 $\vec{A}$ 的旋度的散度為零。

$$\vec{\nabla} \cdot (\vec{\nabla} \times)$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

正向的證明：我們可以直接定義向量場 $\vec{A}$ ，見下一頁。

This time, we will prove the converse statement by explicitly exhibiting a vector potential  $\mathbf{A}$  such that  $\mathbf{F} = \nabla \times \mathbf{A}$ . We pick some arbitrary point  $\mathbf{x}_0 = (x_0, y_0, z_0)$  and then construct the following vector field

$$\mathbf{A}(\mathbf{x}) = \left( \int_{z_0}^z F_y(x, y, z') dz' , \int_{x_0}^x F_z(x', y, z_0) dx' - \int_{z_0}^z F_x(x, y, z') dz' , 0 \right) \quad (3.6)$$

Since  $A_z = 0$ , the definition of the curl (3.4) becomes

$$\nabla \times \mathbf{A} = \left( -\frac{\partial A_y}{\partial z} , \frac{\partial A_x}{\partial z} , \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Using the ansatz (3.6), we find that the first two components of  $\nabla \times \mathbf{A}$  immediately give what we want

$$(\nabla \times \mathbf{A})_x = F_x(x, y, z) \quad \text{and} \quad (\nabla \times \mathbf{A})_y = F_y(x, y, z)$$

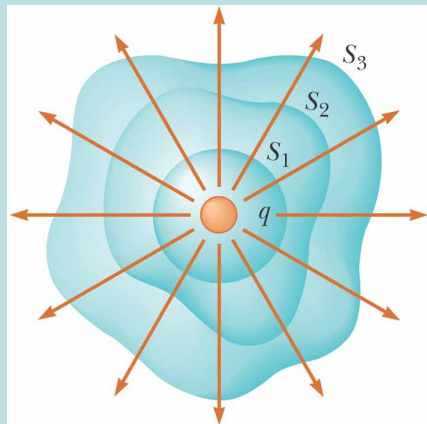
both of which follow from the fundamental theorem of calculus. Meanwhile, we still have a little work ahead of us for the final component

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) - \int_{z_0}^z \frac{\partial F_x}{\partial x}(x, y, z') dz' - \int_{z_0}^z \frac{\partial F_y}{\partial y}(x, y, z') dz'$$

At this point we use the fact that  $\mathbf{F}$  is solenoidal, so  $\nabla \cdot \mathbf{F} = 0$  and so  $\partial F_z / \partial z' = -(\partial F_x / \partial x + \partial F_y / \partial y)$ . We then have

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) + \int_{z_0}^z \frac{\partial F_z}{\partial z'}(x, y, z') dz' = F_z(x, y, z)$$

This is the result we want. □



精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

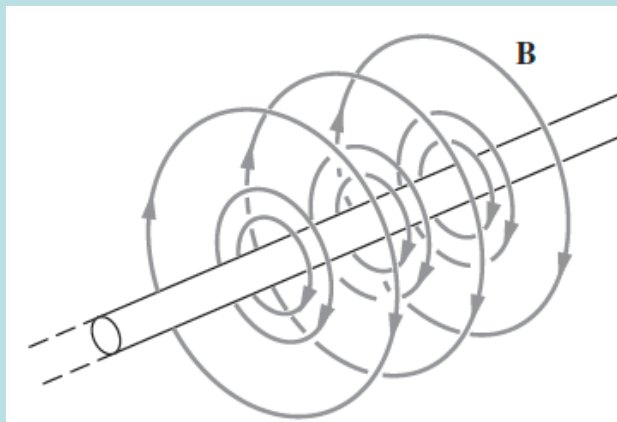
圖像：

放射狀的場線

另一個場的微分：

$$\vec{E} = \vec{\nabla} \phi$$

稱為無旋度場 **Irrotational Field**



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

稱為無散度場 **Divergence Free Field**.

所以  $\phi, \vec{A}$  可以代替  $\vec{E}, \vec{B}$ ，作為討論電磁現象的物理場。

在量子力學中  $\phi, \vec{A}$  甚至比  $\vec{E}, \vec{B}$  更真實。

以上分類不只適用於電磁場。

# Applications

If  $\vec{\nabla}$  acts like a vector, combining two  $\vec{\nabla}$ 's as inner product generates a scalar.

$$\vec{\nabla} \cdot \vec{\nabla} = \left( \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \nabla^2$$

This is an operation to be acted on functions.

$$\vec{\nabla} \cdot \vec{\nabla} U = \nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

This operation is called Laplacian ◦

$$\vec{E} = \vec{\nabla} V$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Plug the first formula into the second :

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (\vec{\nabla} V) = \nabla^2 V = \frac{\rho}{\epsilon_0}$$

Pierre-Simon Laplace 1749-1827



**Electric Potential satisfies Laplacian Equation!**

$$\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\rho}{\epsilon_0}$$

Using differential Maxwell Eq, EM wave equation is easy to derive.

Maxwell Equations in vacuum

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Spatial change of electric field is interrelated to temporal change of magnetic field.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

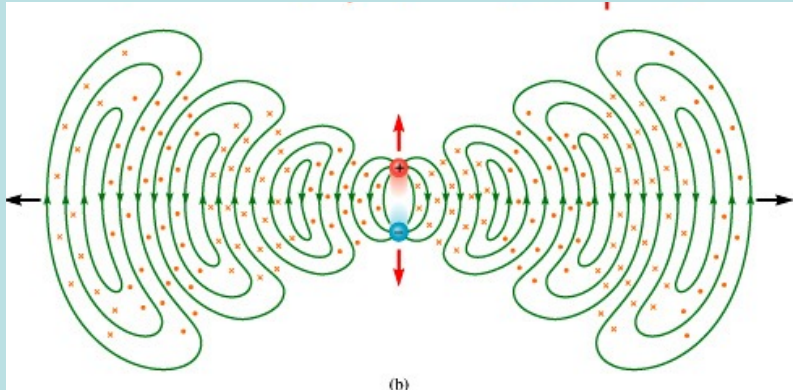
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$

Spatial change of magnetic field is interrelated to temporal change of electric field.

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Wave Equation for Electric Field



$$\vec{A} \times (\vec{B} \times \vec{C})$$

Calculate the components directly:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_z = A_x (\vec{B} \times \vec{C})_y - A_y (\vec{B} \times \vec{C})_x$$

$$= A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)$$

$$= B_z (A_x C_x + A_y C_y) - (A_x B_x + A_y B_y) C_z$$

$$= B_z (A_x C_x + A_y C_y + A_z C_z) - (A_x B_x + A_y B_y + A_z B_z) C_z$$

$$= B_z (\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B}) C_z$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_z = B_z (\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B}) C_z \quad \text{對另外兩個分量也成立！}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{F}) - (\vec{\nabla} \cdot \vec{\nabla}) \vec{F} = \vec{\nabla} (\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$

## Levi-Civita Tensor

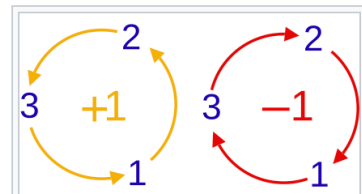
$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123, 231, 312 \\ -1 & ijk = 132, 213, 321 \\ 0 & \text{some subscripts are equal} \end{cases}$$

$$(\vec{A} \times \vec{B})_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k$$

Levi-Civita Tensor 滿足

$$\sum_{k=1}^3 \epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

左式不為零，只能發生於  $ij$  與  $lm$  是  $k$  以外的兩個不同足標，只能  $i$  與  $l$  同， $j$  與  $m$  同  $\delta_{il} \delta_{jm}$ ，或者  $i$  與  $m$  同， $j$  與  $l$  同  $-\delta_{im} \delta_{jl}$ ， $k$  求和時，只有一個足標，不同於  $ij$  者，會有貢獻，因此乘 1。得證。



For the indices  $(i, j, k)$  in  $\epsilon_{ijk}$ , the values 1, 2, 3 occurring in the ■ cyclic order (1, 2, 3) correspond to  $\epsilon = +1$ , while occurring in the ■ reverse cyclic order correspond to  $\epsilon = -1$ , otherwise  $\epsilon = 0$ .

$$\sum_{j,k,l,m=1}^3 \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \sum_{j,k,l,m=1}^3 A_j B_l C_m (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

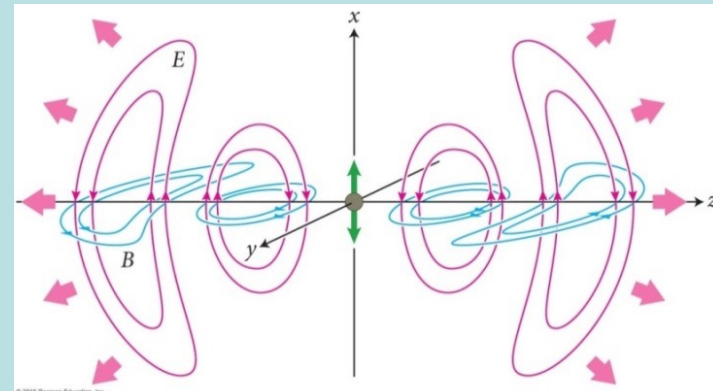
$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \nabla \times \frac{\partial \vec{E}}{\partial t}$$

$$\nabla(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial(\nabla \times \vec{E})}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

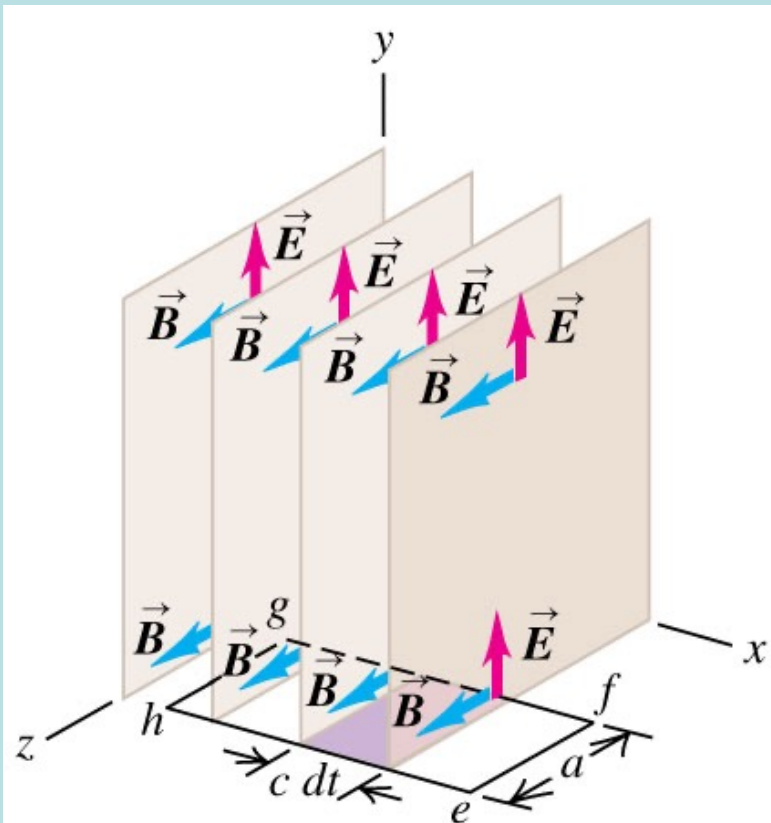
Wave Equation for Magnetic Field



Consider a special version of EM wave with  $\vec{E}$  in  $y$  and  $\vec{B}$  in  $z$  directions. This wave will move in  $x$  direction and we allow fields to be  $x$  dependent.

$$E_y(x, t), B_z(x, t)$$

The field is constant on  $yz$  plane and this wave is called Plane Wave

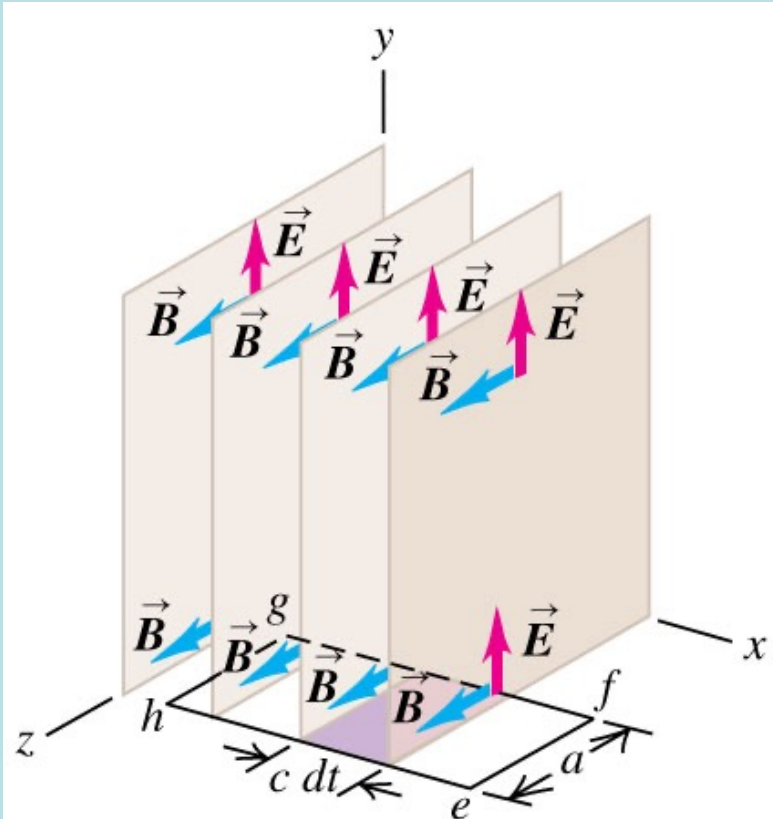


$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$



$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$E_y(x, t), B_z(x, t)$$



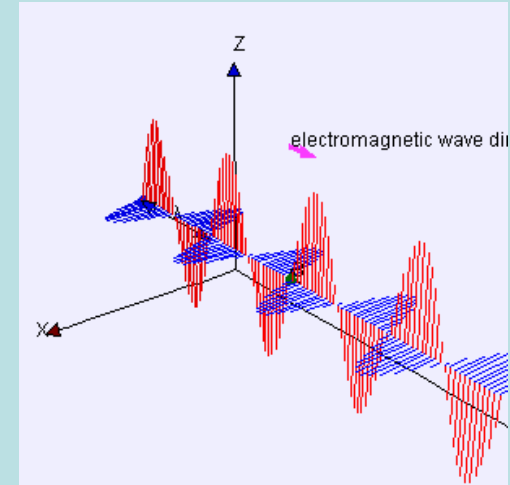
$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$



$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$



$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2}$$



$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

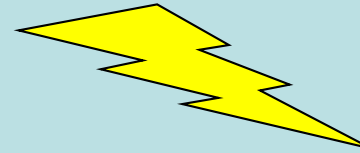
## 電磁波的速度

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= \frac{1}{\sqrt{1.26 \times 10^{-6} \cdot 8.85 \times 10^{-12}}} \sim 2.99 \times 10^8 \text{ m/s}$$

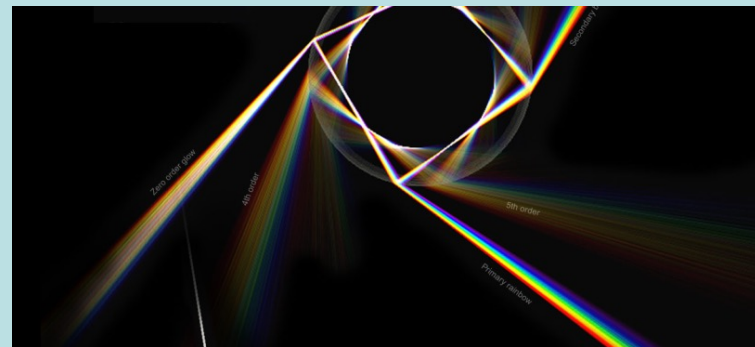
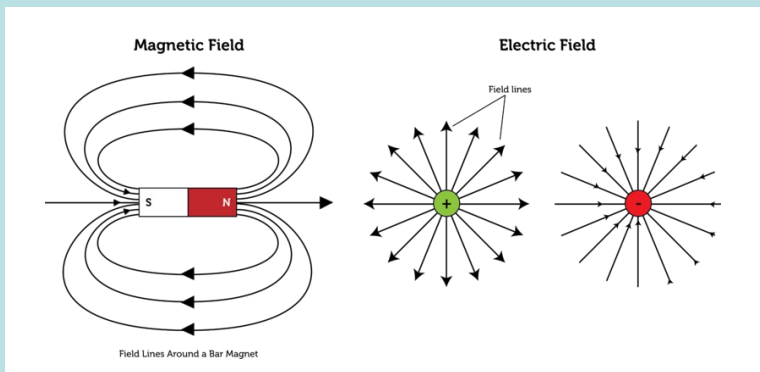
$$v = c$$

光速！！



Light is an electromagnetic wave.

馬克斯威爾統一了電磁學與光學！



## PART VI.

### ELECTROMAGNETIC THEORY OF LIGHT.

(91) At the commencement of this paper we made use of the optical hypothesis of an elastic medium through which the vibrations of light are propagated, in order to shew that we have warrantable grounds for seeking, in the same medium, the cause of other phenomena as well as those of light. We then examined electromagnetic phenomena, seeking for their explanation in the properties of the field which surrounds the electrified or magnetic bodies. In this way we arrived at certain equations expressing certain properties of the electromagnetic field. We now proceed to investigate whether these properties of that which constitutes the electromagnetic field, deduced from electromagnetic phenomena alone, are sufficient to explain the propagation of light through the same substance.

$$x = \kappa t, \quad y = \kappa t, \quad z = \kappa t, \quad \dots \quad (66)$$

These electromotive forces are due to the variations either of the electromagnetic or the electrostatic functions, as there is no motion of conductors in the field; so that the equations of electromotive force (D) are

$$\left. \begin{aligned} P &= -\frac{dF}{dt} - \frac{d\Psi}{dx}, \\ Q &= -\frac{dG}{dt} - \frac{d\Psi}{dy}, \\ R &= -\frac{dH}{dt} - \frac{d\Psi}{dz}. \end{aligned} \right\} \dots \dots \dots (67)$$

(94) Combining these equations, we obtain the following:—

$$\left. \begin{aligned} k\left(\frac{dJ}{dx} - \nabla^2 F\right) + 4\pi\mu\left(\frac{d^2 F}{dt^2} + \frac{d^2 \Psi}{dx dt}\right) &= 0, \\ k\left(\frac{dJ}{dy} - \nabla^2 G\right) + 4\pi\mu\left(\frac{d^2 G}{dt^2} + \frac{d^2 \Psi}{dy dt}\right) &= 0, \\ k\left(\frac{dJ}{dz} - \nabla^2 H\right) + 4\pi\mu\left(\frac{d^2 H}{dt^2} + \frac{d^2 \Psi}{dz dt}\right) &= 0. \end{aligned} \right\} \dots \dots \dots (68)$$

If we differentiate the third of these equations with respect to  $y$ , and the second with respect to  $z$ , and subtract,  $J$  and  $\Psi$  disappear, and by remembering the equations (B) of magnetic force, the results may be written

$$\left. \begin{aligned} k\nabla^2 \mu\alpha &= 4\pi\mu \frac{d^2}{dt^2} \mu\alpha, \\ k\nabla^2 \mu\beta &= 4\pi\mu \frac{d^2}{dt^2} \mu\beta, \\ k\nabla^2 \mu\gamma &= 4\pi\mu \frac{d^2}{dt^2} \mu\gamma. \end{aligned} \right\} \dots \dots \dots (69)$$

(95) If we assume that  $\alpha, \beta, \gamma$  are functions of  $lx + my + nz - Vt = w$ , the first equation becomes

$$k\mu \frac{d^2 \alpha}{dw^2} = 4\pi\mu^2 V^2 \frac{d^2 \alpha}{dw^2}, \quad \dots \dots \dots (70)$$

or

$$V = \pm \sqrt{\frac{k}{4\pi\mu}}. \quad \dots \dots \dots (71)$$

The other equations give the same value for  $V$ , so that the wave is propagated in either direction with a velocity  $V$ .

(95) If we assume that  $\alpha, \beta, \gamma$  are functions of  $lx + my + nz - Vt = w$ , the first equation becomes

$$k\mu \frac{d^2\alpha}{dw^2} = 4\pi\mu^2 V^2 \frac{d^2\alpha}{dw^2} \dots\dots\dots (70),$$

or

$$V = \pm \sqrt{\frac{k}{4\pi\mu}} \dots\dots\dots (71).$$

The other equations give the same value for  $V$ , so that the wave is propagated in either direction with a velocity  $V$ .

This wave consists entirely of magnetic disturbances, the direction of magnetization being in the plane of the wave. No magnetic disturbance whose direction of magnetization is not in the plane of the wave can be propagated as a plane wave at all.

Hence magnetic disturbances propagated through the electromagnetic field agree with light in this, that the disturbance at any point is transverse to the direction of propagation, and such waves may have all the properties of polarized light.

(96) The only medium in which experiments have been made to determine the value of  $k$  is air, in which  $\mu = 1$ , and therefore, by equation (46),

$$V = v \dots\dots\dots (72).$$

By the electromagnetic experiments of MM. Weber and Kohlrausch\*,

$$v = 310,740,000 \text{ metres per second}$$

The velocity of light in air, by M. Fizeau's\* experiments, is

$$V = 314,858,000 ;$$

according to the more accurate experiments of M. Foucault †,

$$V = 298,000,000.$$

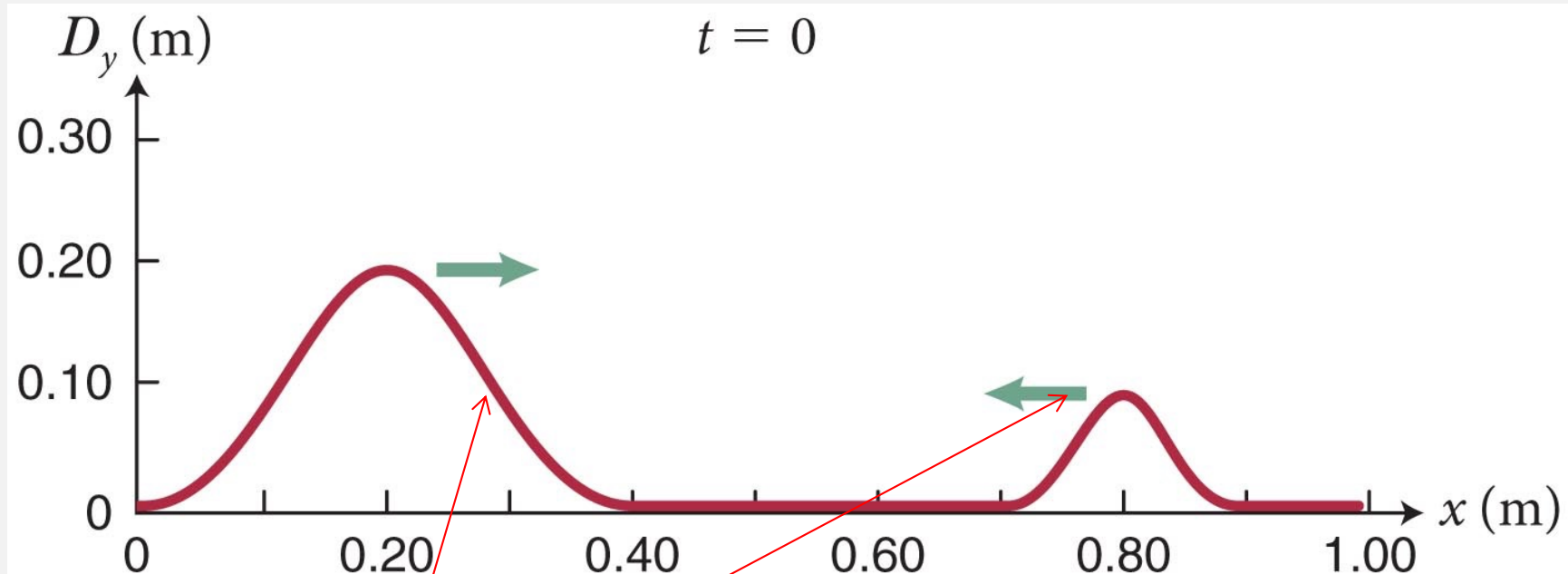
The velocity of light in the space surrounding the earth, deduced from the coefficient of aberration and the received value of the radius of the earth's orbit, is

$$V = 308,000,000.$$

(97) Hence the velocity of light deduced from experiment agrees sufficiently well with the value of  $v$  deduced from the only set of experiments we as yet possess. The value of  $v$  was determined by measuring the electromotive force with which a condenser of known capacity was charged, and then discharging the condenser through a galvanometer, so as to measure the quantity of electricity in it in electromagnetic measure. The only use made of light in the experiment was to see the instruments. The value of  $V$  found by M. Foucault was obtained by determining the angle through which a revolving mirror turned, while the light reflected from it went and returned along a measured course. No use whatever was made of electricity or magnetism.

The agreement of the results seems to shew that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.

弦波的波函數的最普遍解： $y(x, t) = f(x - vt) + g(x + vt)$



而  $f(x)$  ( $g(x)$ ) 即為向右 (左) 傳播的波的瞬間波型

波型以定速傳播

波型在傳播過程中不變形

以上結果適用於任何滿足波方程式的波動現象！

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

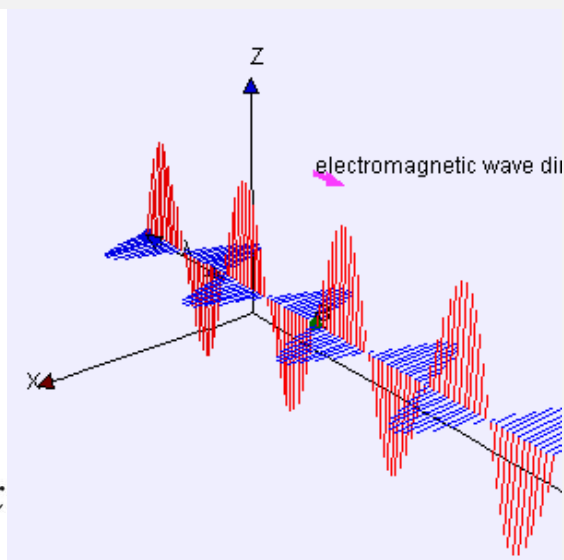
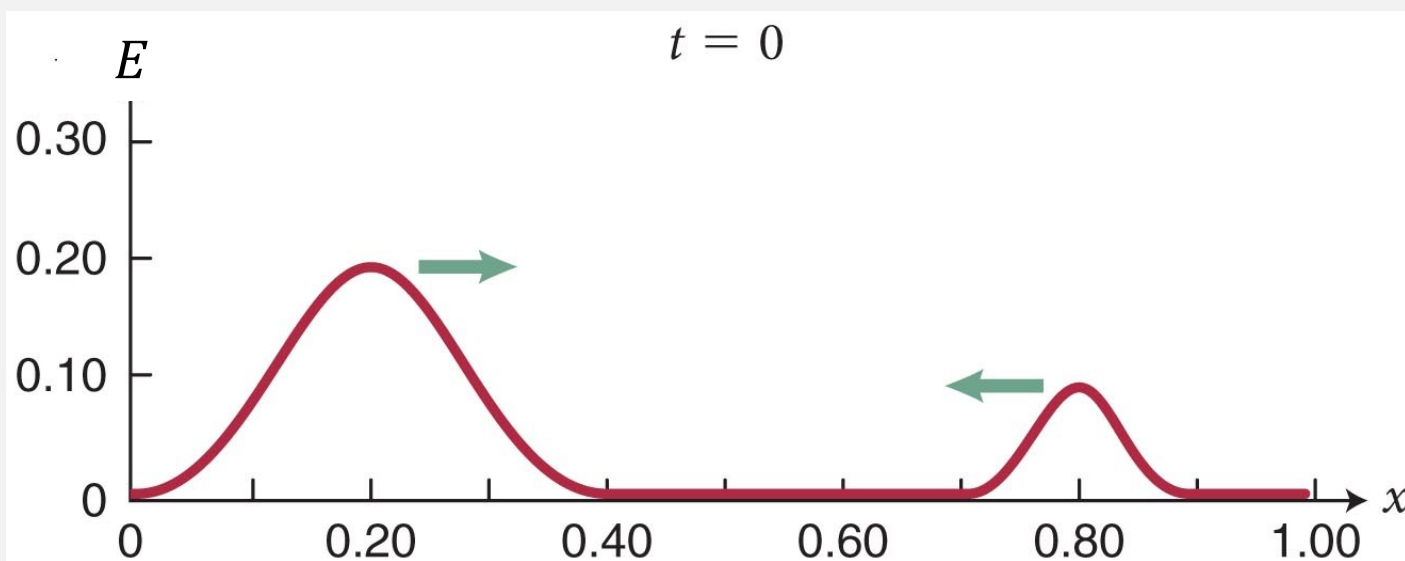
波方程式的解為：

$$y(x, t) = f(x - vt) + g(x + vt)$$

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

可見電磁波的電場也是如此：

$$E(x, t) = E_1(x - vt) + E_2(x + vt)$$



$E_1(x), E_2(x)$  為向右及向左以定速傳播的電場波。

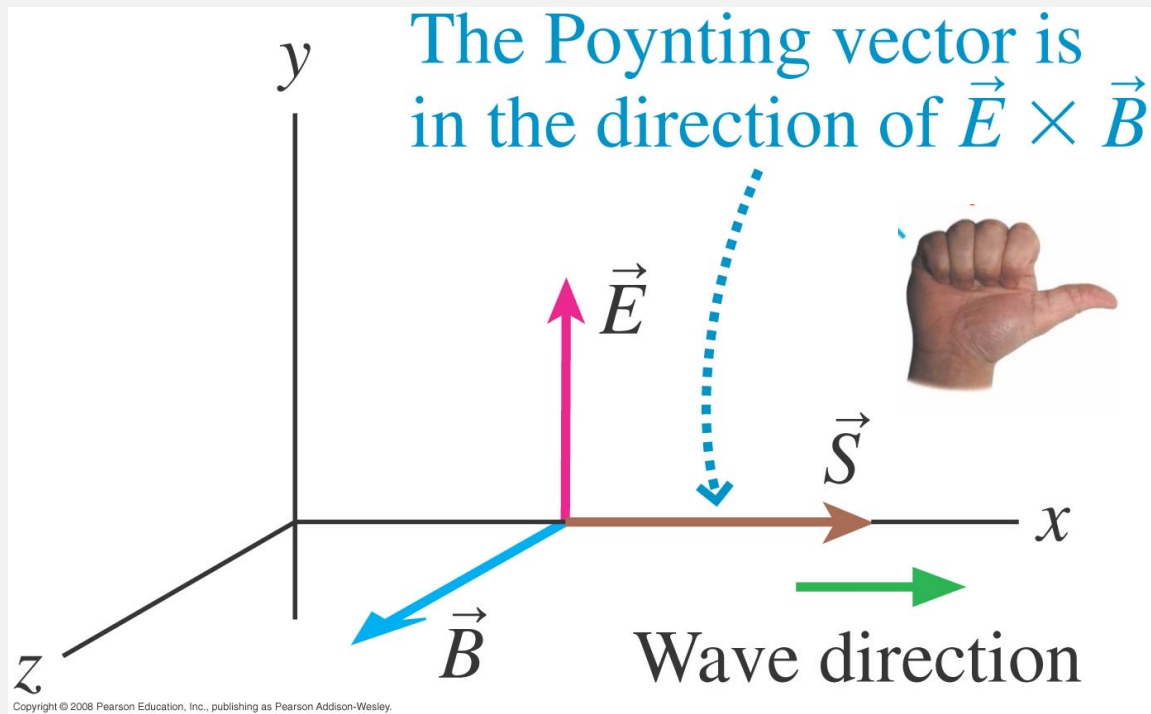
可以確認電磁場的瞬間變化是以定速  $c$  在空間中自源頭向外傳播。

電磁波是依賴電磁感應，因此其電場與磁場必須互相垂直  
波的傳播方向又與電場及磁場皆垂直。

會在空間中傳播的電磁場，方向及大小必須滿足特定關係！

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Poynting vector  $\vec{S}$  功率傳播向量，就是波傳播的方向。



右手定則

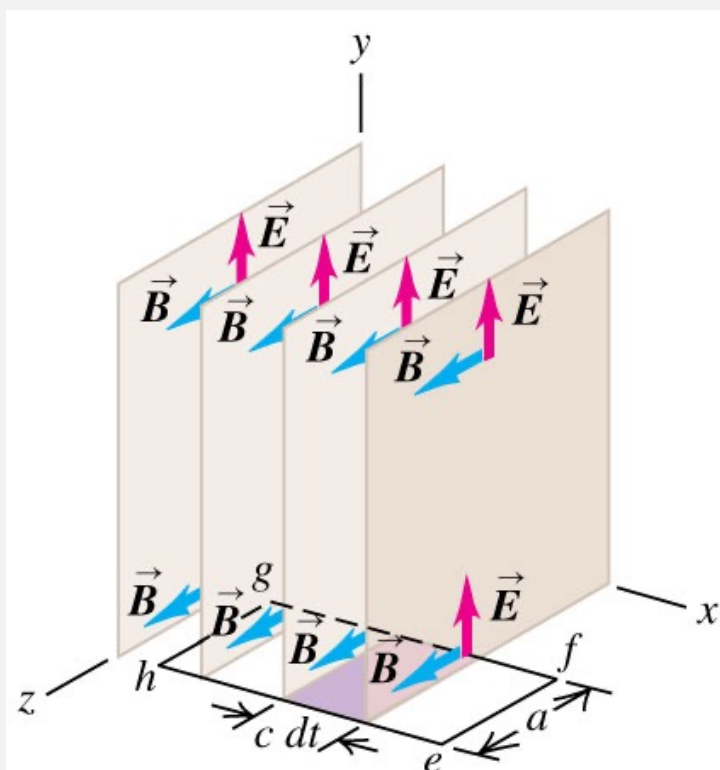
考慮沿 $y$ 方向的電場與沿 $z$ 方向的磁場。

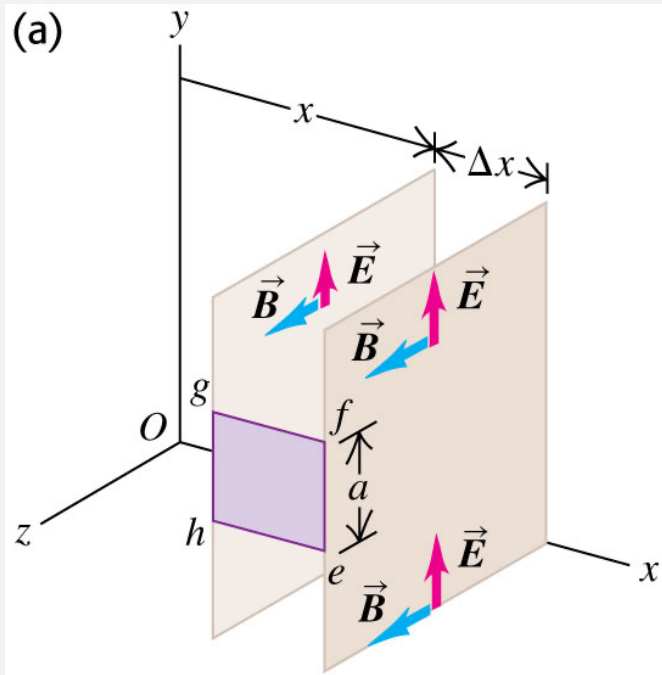
波會沿 $x$ 方向傳播，我們現在容許電磁場與座標 $x$ 有關！

$$E_y(x, t), B_z(x, t)$$

因為此波在 $yz$ 平面上是一個常數，且電磁場方向是沿 $yz$ 平面上。

這樣的波稱為平面波。海嘯方塊波是平面波的特例！





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磁場變化與垂直的電場滿足法拉第定律。

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$



選擇如左圖 *efgh* 的封閉曲線

$$E(x + \Delta x) \cdot a - E(x) \cdot a = -\frac{d(B \cdot a\Delta x)}{dt}$$

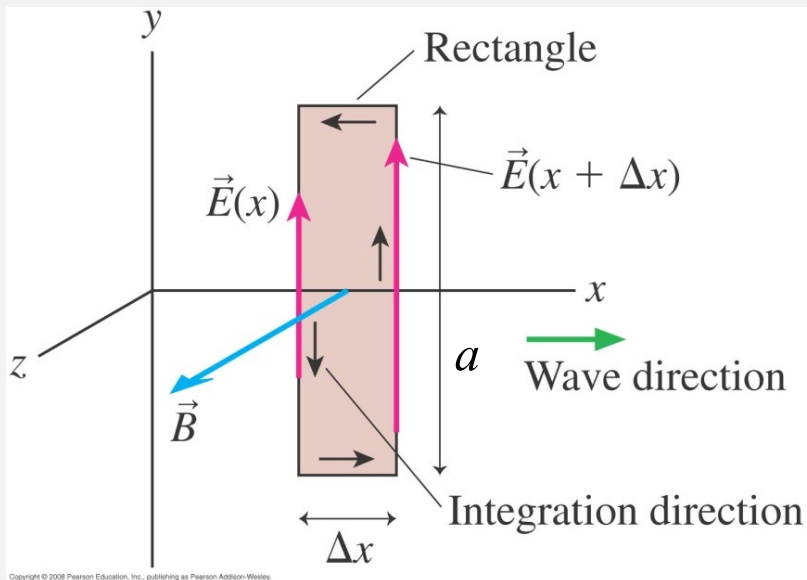


一小片空間前後的電場差，  
等於通過其截面磁通量的變化率！

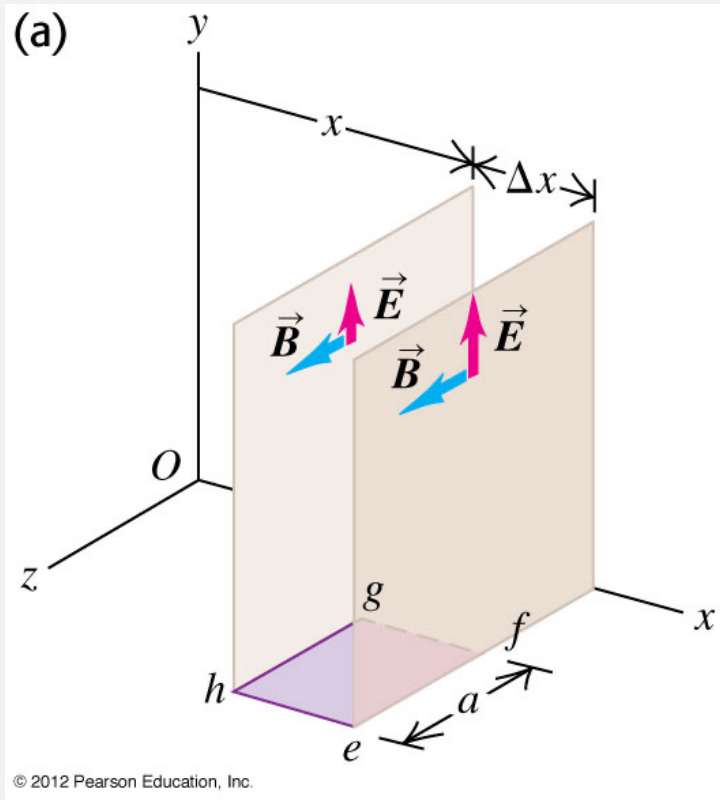
$$\frac{\partial E}{\partial x} \Delta x \cdot a = -a\Delta x \cdot \frac{\partial B}{\partial t}$$



$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$



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電場變化與垂直的磁場滿足安培定律。

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

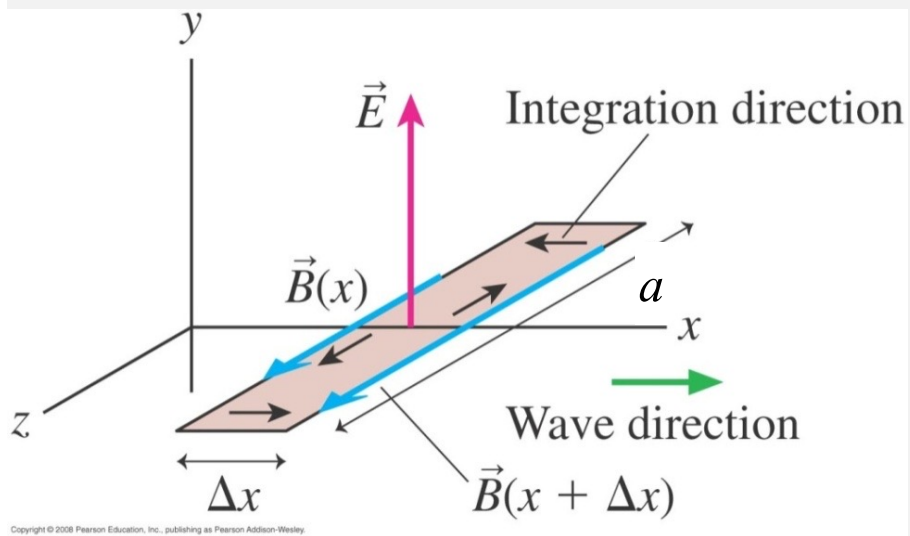


選擇如左圖的封閉曲線

$$-B(x + \Delta x) \cdot a + B(x) \cdot a = \mu_0 \epsilon_0 \frac{d(E \cdot a \Delta x)}{dt}$$



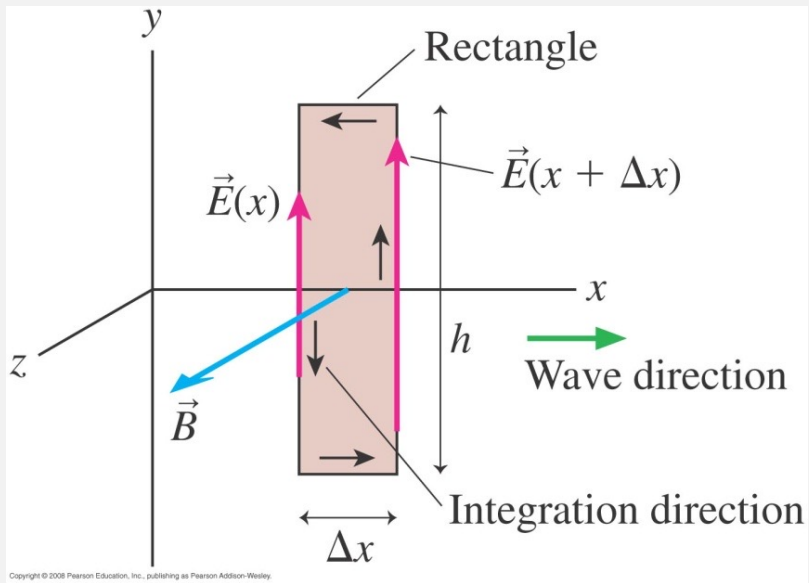
一小片空間前後的磁場差，  
等於通過其截面電通量的變化率！



$$-\frac{\partial B}{\partial x} \Delta x \cdot a = \mu_0 \epsilon_0 \cdot a \Delta x \cdot \frac{\partial E}{\partial t}$$

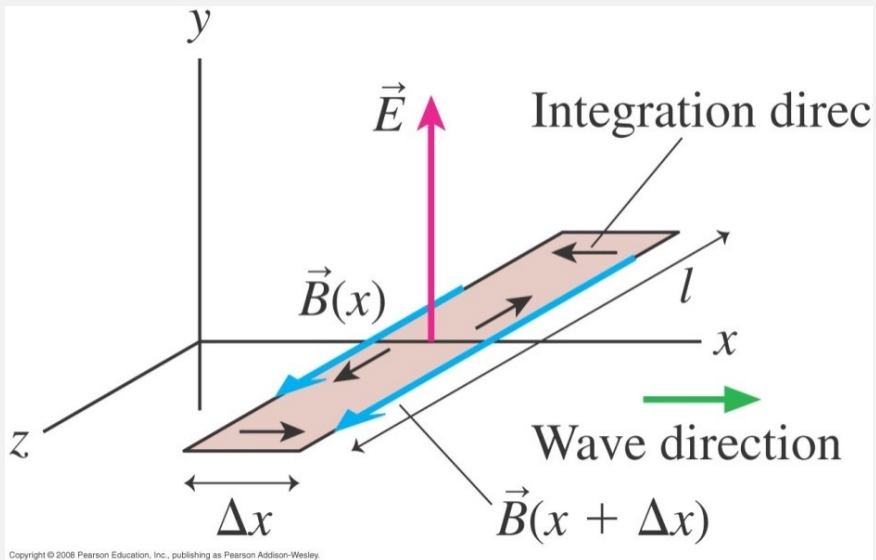
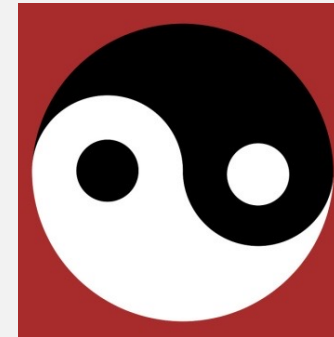


$$\frac{\partial B}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$



$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

變化的磁場與感應產生的電場必須滿足的關係



$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

變化的電場與感應產生的磁場必須滿足的關係

變化的電場感應產生的變化的磁場，感應產生變化的電場  
彼此互相感生的感應電磁場，兩個條件都必須滿足！

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

對x作偏微分

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial x \partial t}$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

對t作偏微分

$$\frac{\partial^2 B}{\partial t \partial x} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

第一式的右方等於第二式的左方



$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial x \partial t} = -\frac{\partial^2 B}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$



$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

電場滿足波方程式

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

對  $t$  作偏微分

$$\frac{\partial^2 E}{\partial t \partial x} = -\frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

對  $x$  作偏微分

$$\frac{\partial^2 B}{\partial x^2} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial x \partial t}$$

第一式的左方等於第二式的右方



$$\frac{\partial^2 B}{\partial x^2} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial x \partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$



$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

磁場也滿足波方程式

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

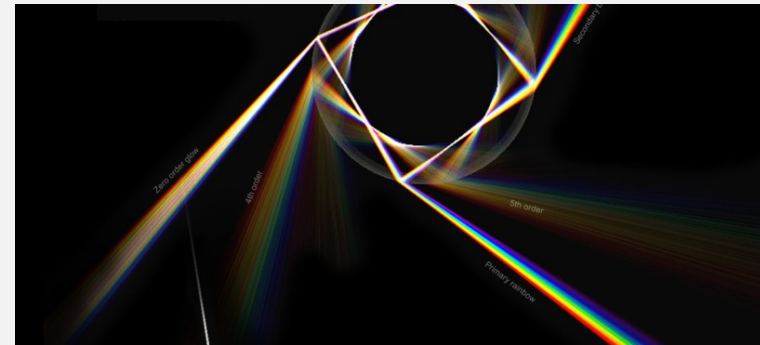
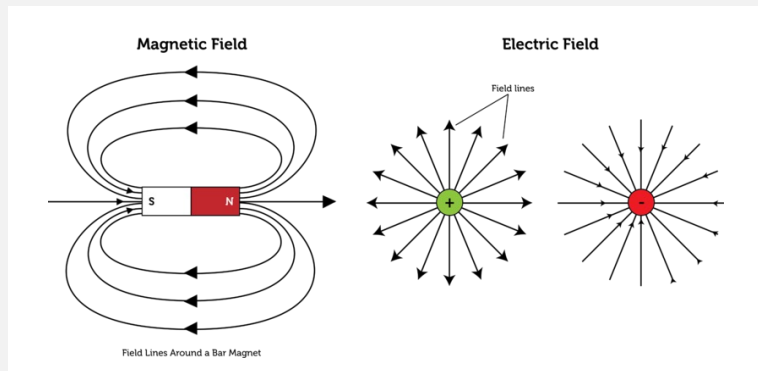
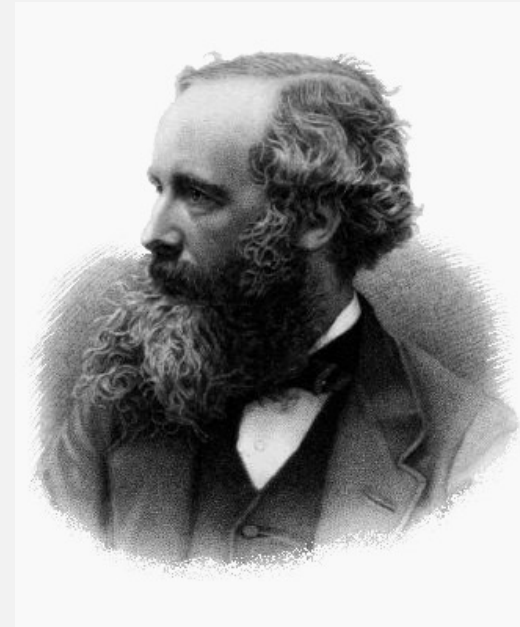
$$\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$$

電磁波 Electromagnetic Wave

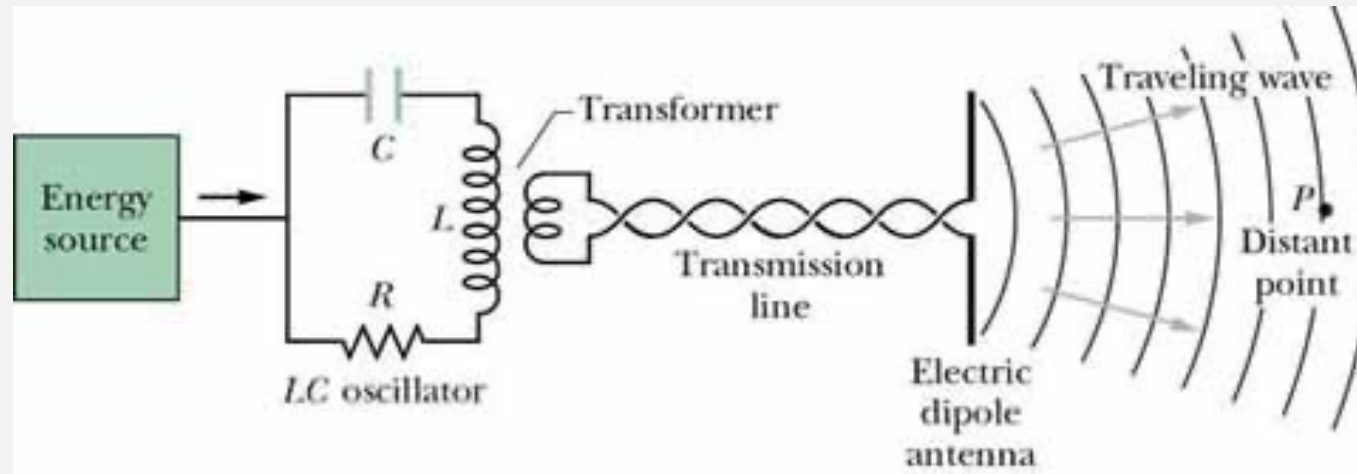
$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

光原來是一種電磁波！

馬克斯威爾統一了電磁學與光學！

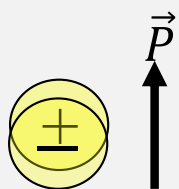


## 電偶極作為電磁波波源

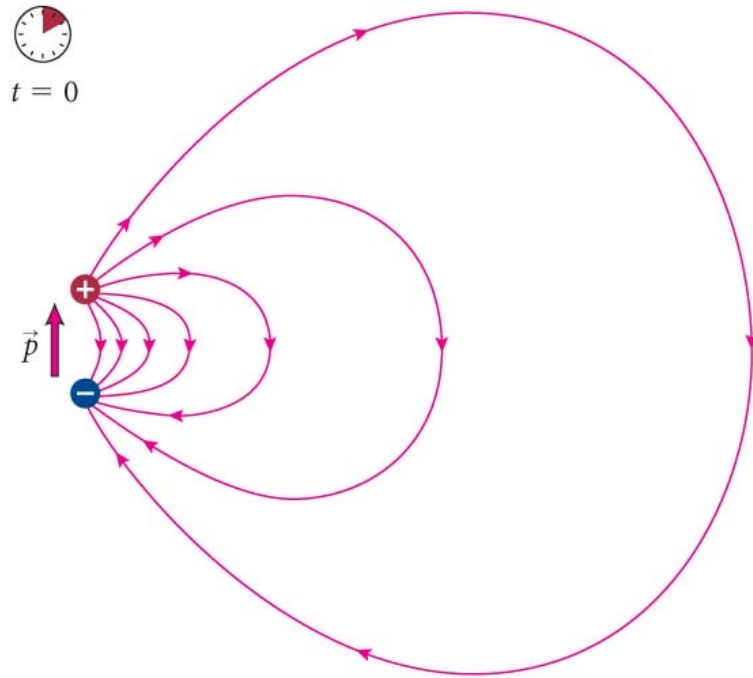


如果有一個電偶極突然形成，遠處的電磁場如何被影響？

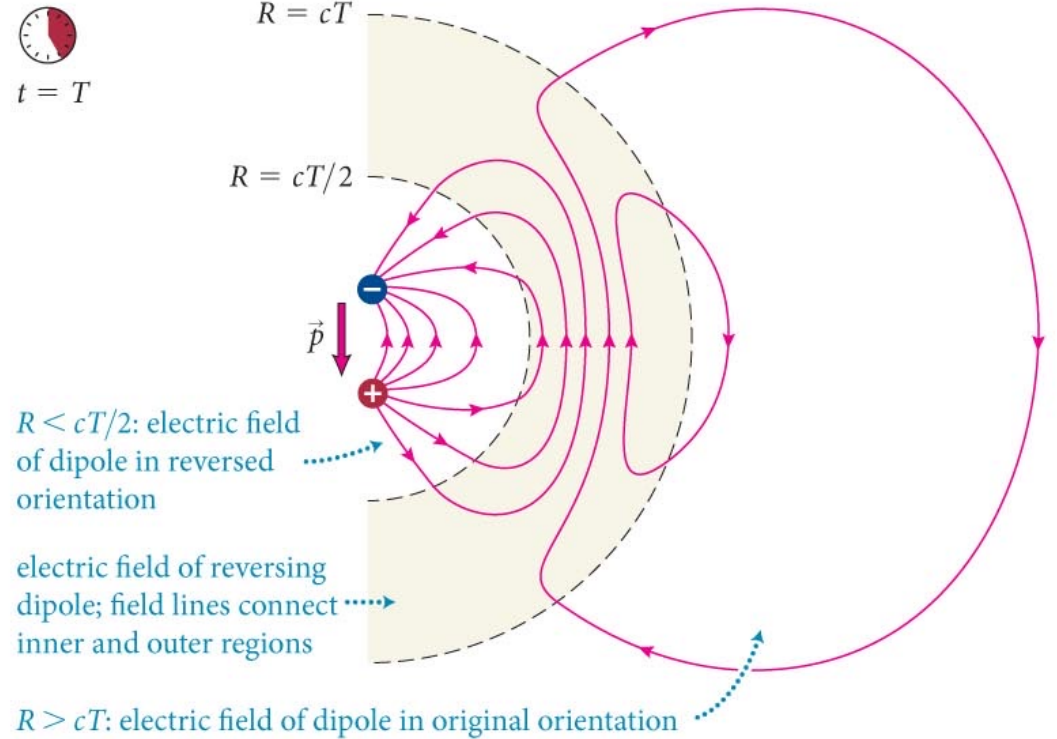
$\vec{E}, \vec{B}$  ?



(a) Electric field of stationary electric dipole



(b) Field shortly after dipole has reversed



近處的電場線是放射狀，遠處的電場線是迴旋狀！

這是三度空間的立體波，用電位 $\phi$ 與 $\vec{A}$ 討論比較容易，

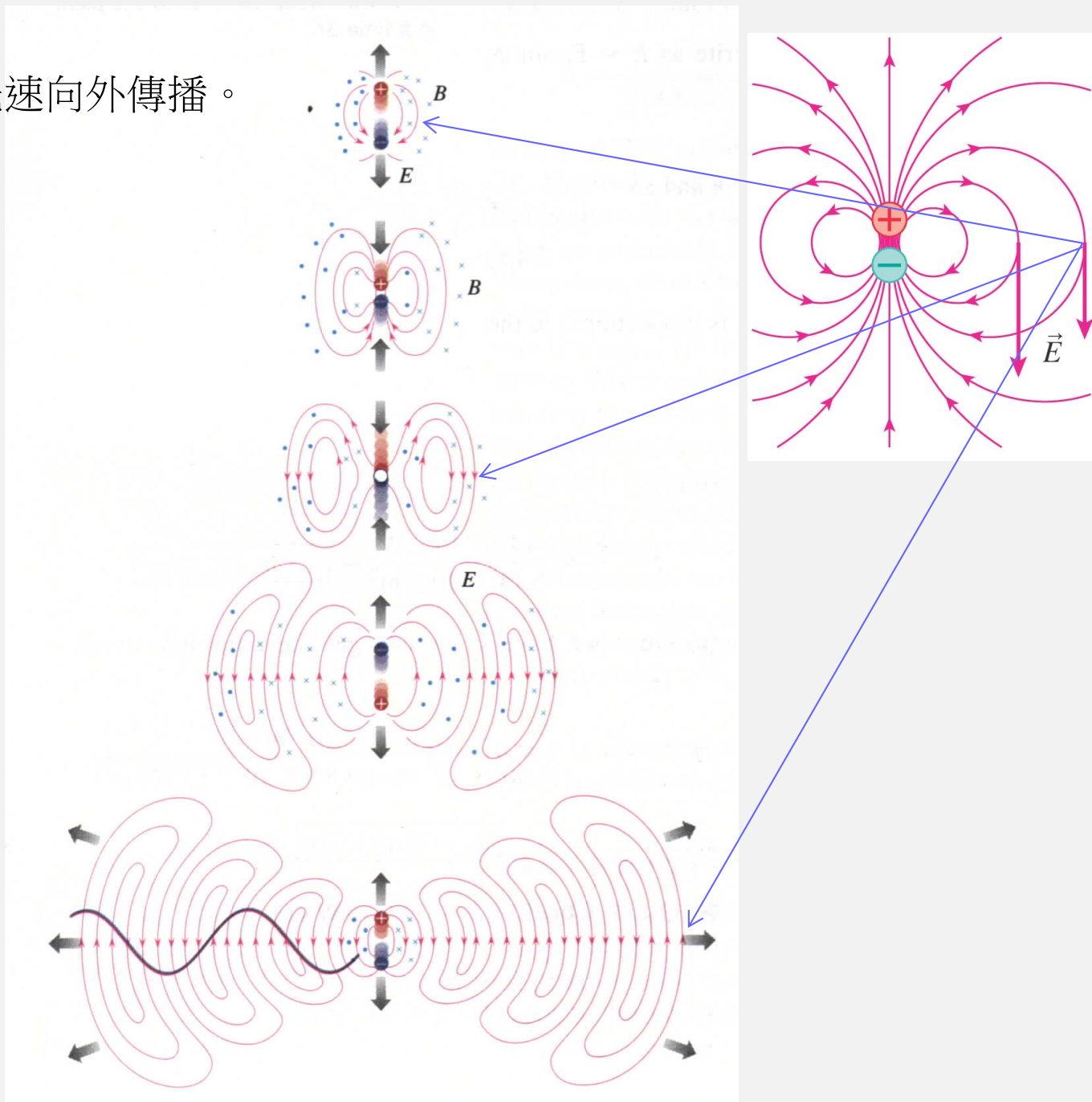
電位 $\phi$ 與 $\vec{A}$ 都滿足波方程式：

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

$$\nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$$

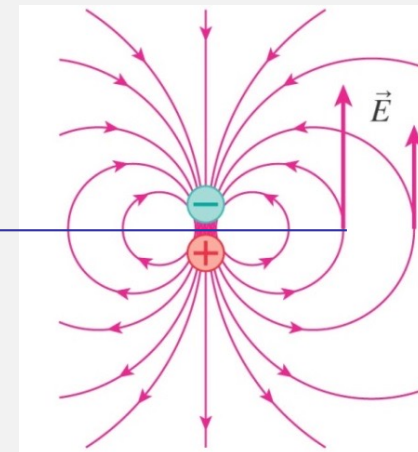
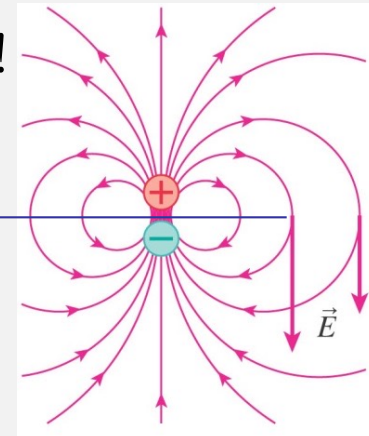
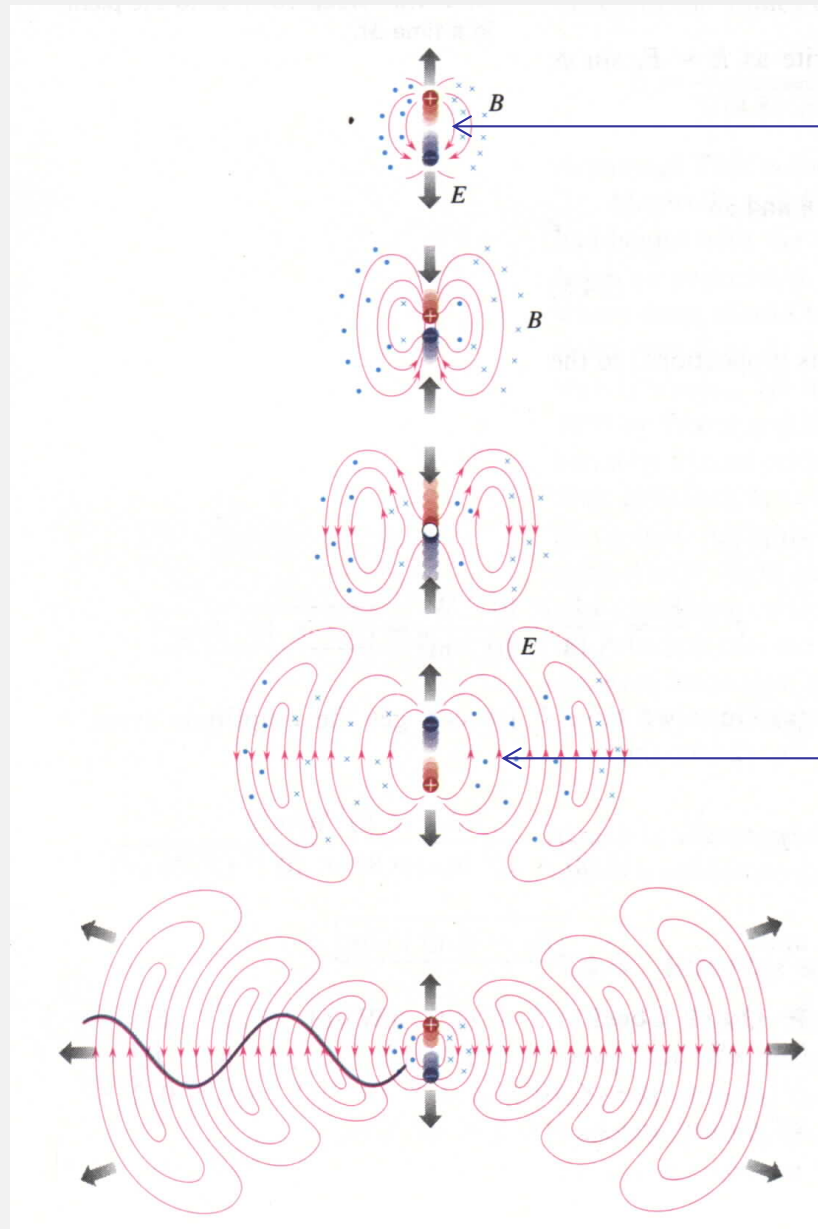
在極靠近電偶極處，靜電學得到的電場結果還是對的！

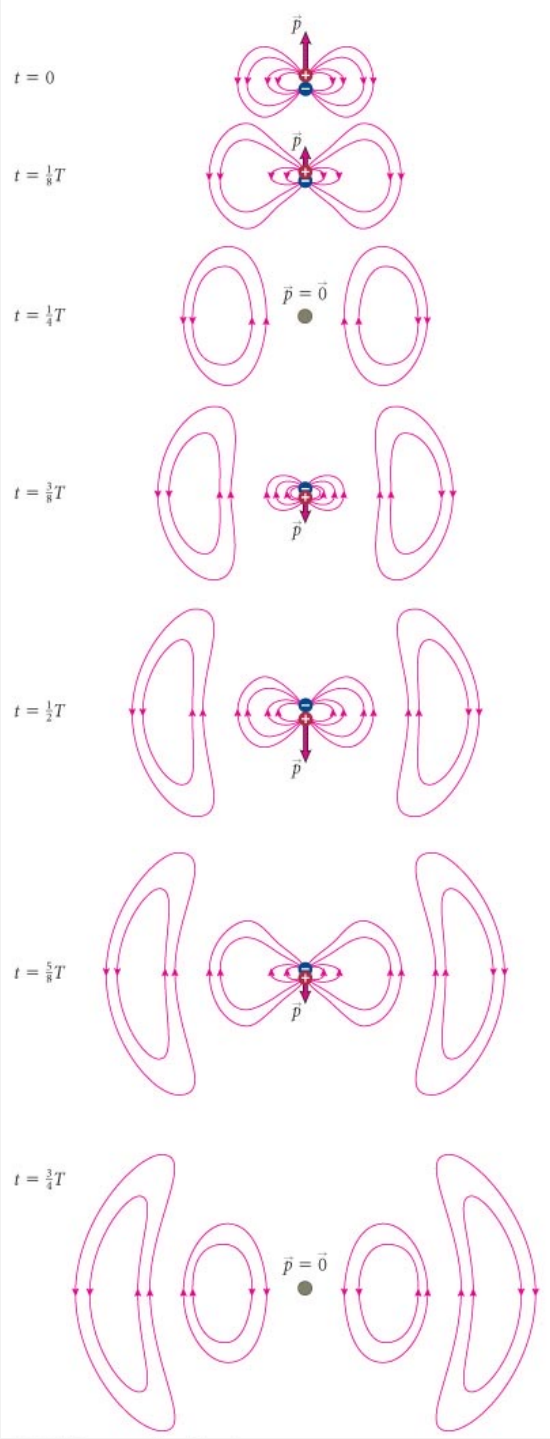
這個電磁場會以光速向外傳播。

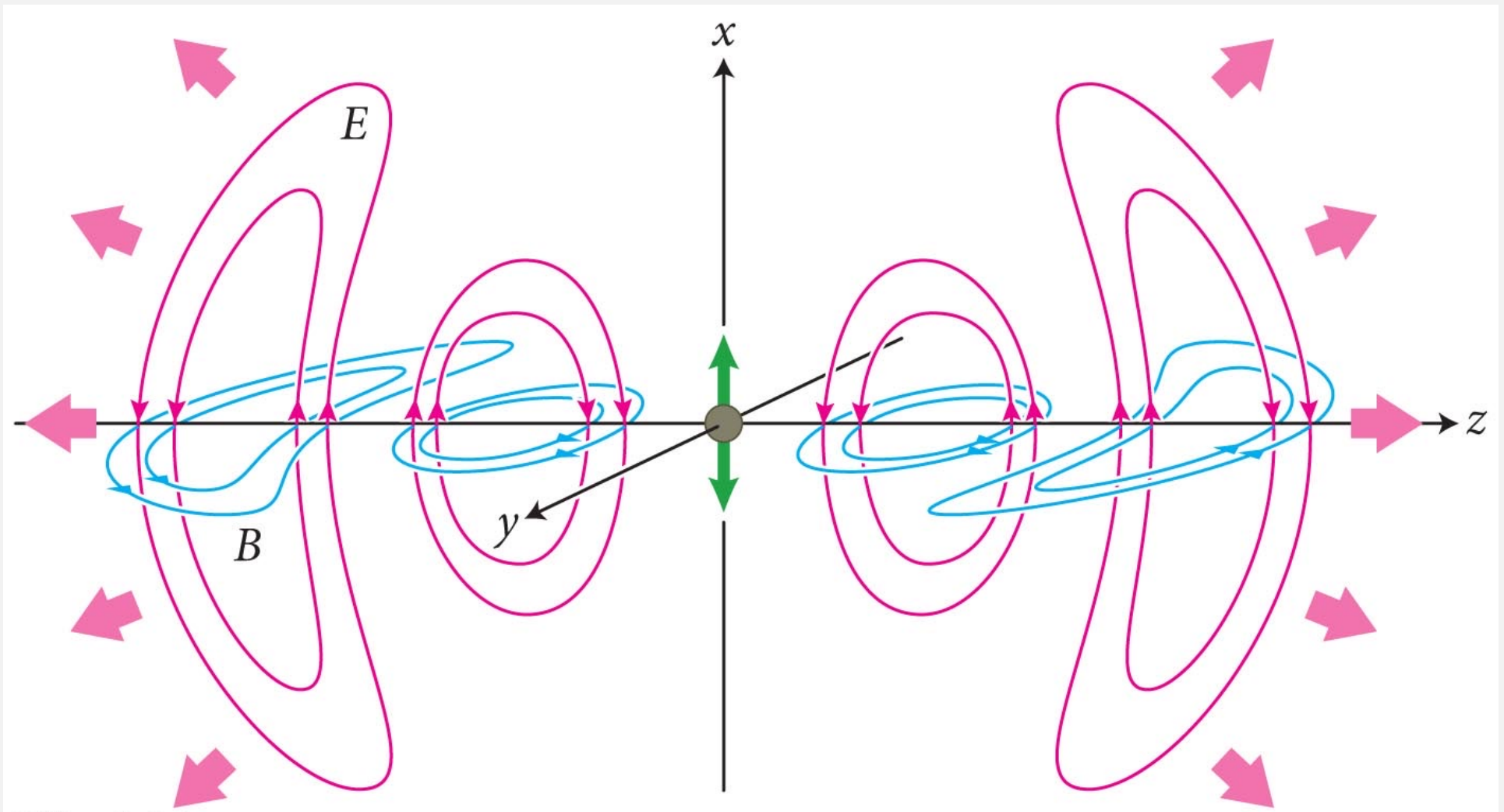


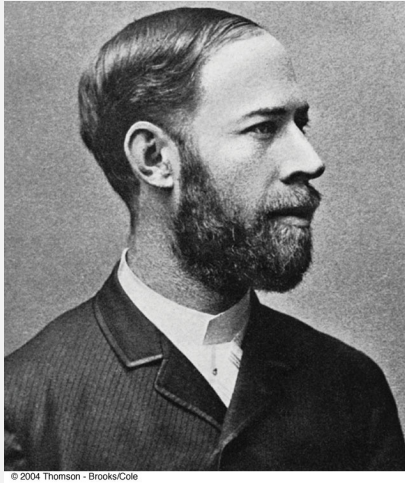
當電偶極做簡諧運動時，

在極靠近電偶極處，電磁場也會以時間的正弦函數來振盪！

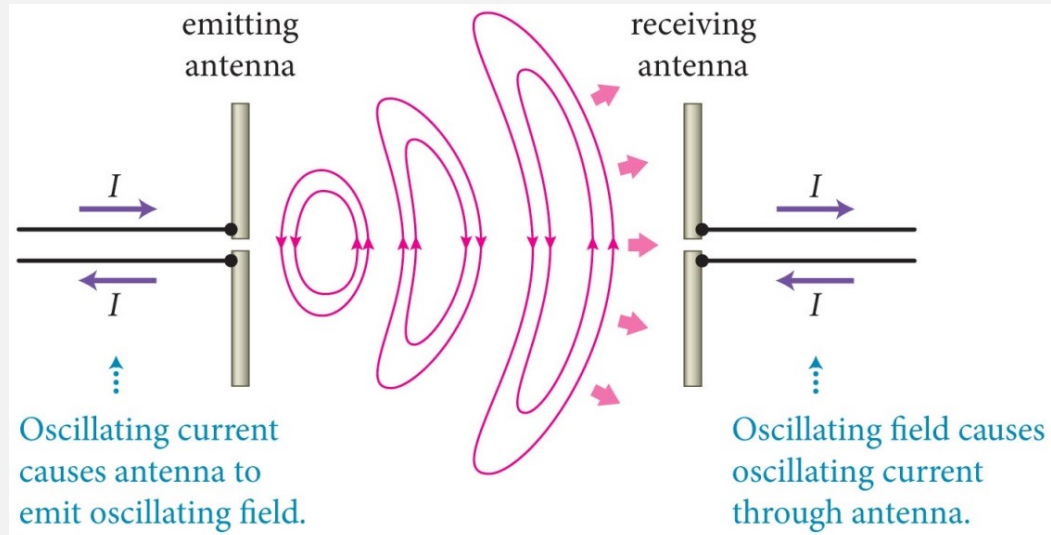








Hertz (1887)

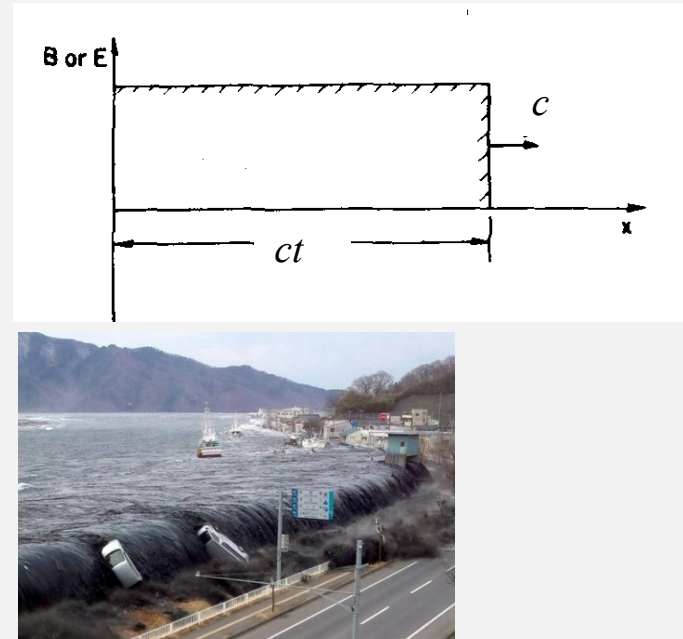
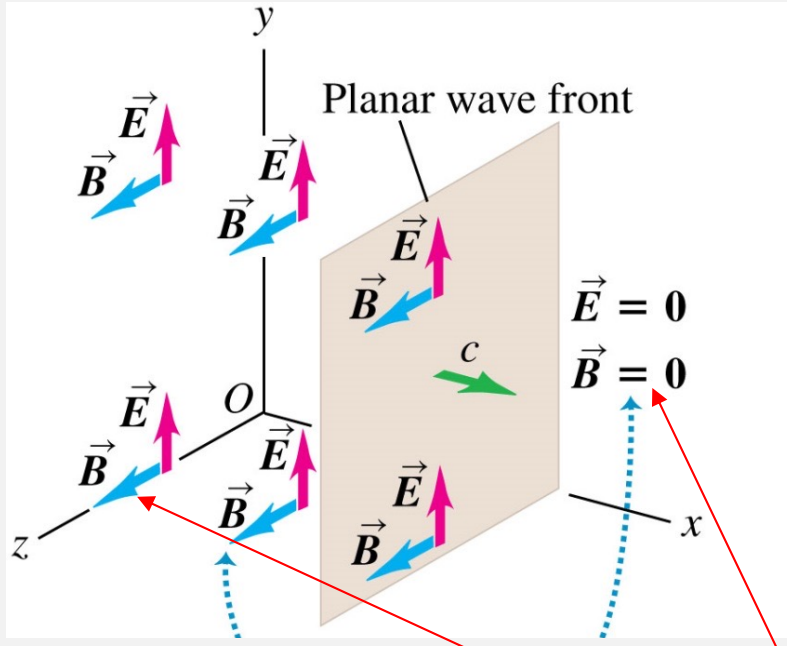


發射端與接收端都是電偶極，就是俗稱的天線。

## 海嘯電磁波：稍微嚴格地推導



如果有電偶極突然出現，近處的電磁場與遠處的電磁場（= 0）會有差異。  
最極端的情況就是如海嘯一般，差異集中在一波前平面上發生。



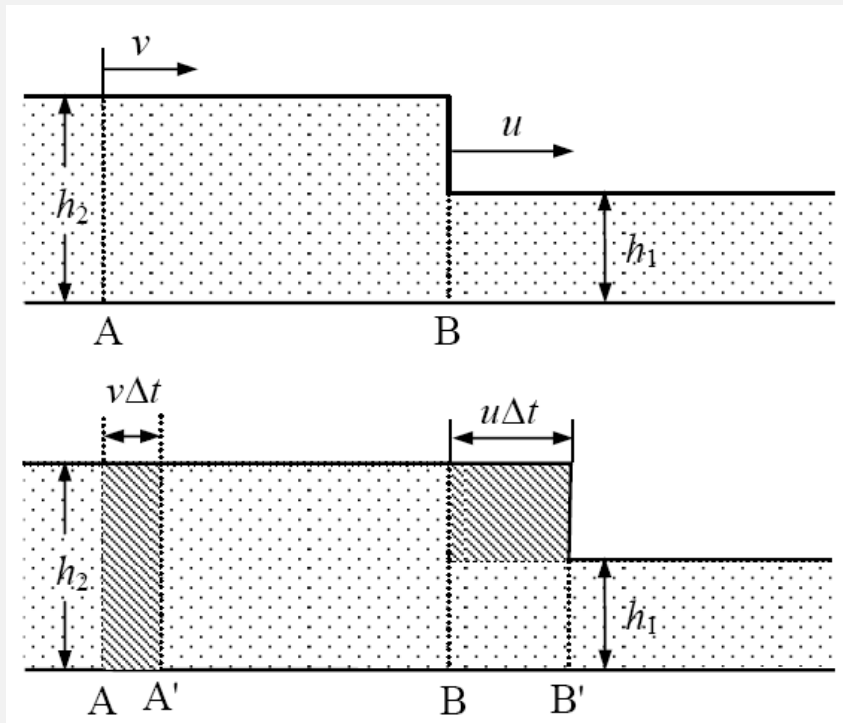
電場與磁場在一個波前平面的前面為零。

電場與磁場在一個波前平面的後面不為零，假設兩者都是常數，與位置無關。

因為電場與磁場互相感生，所以彼此需要垂直！

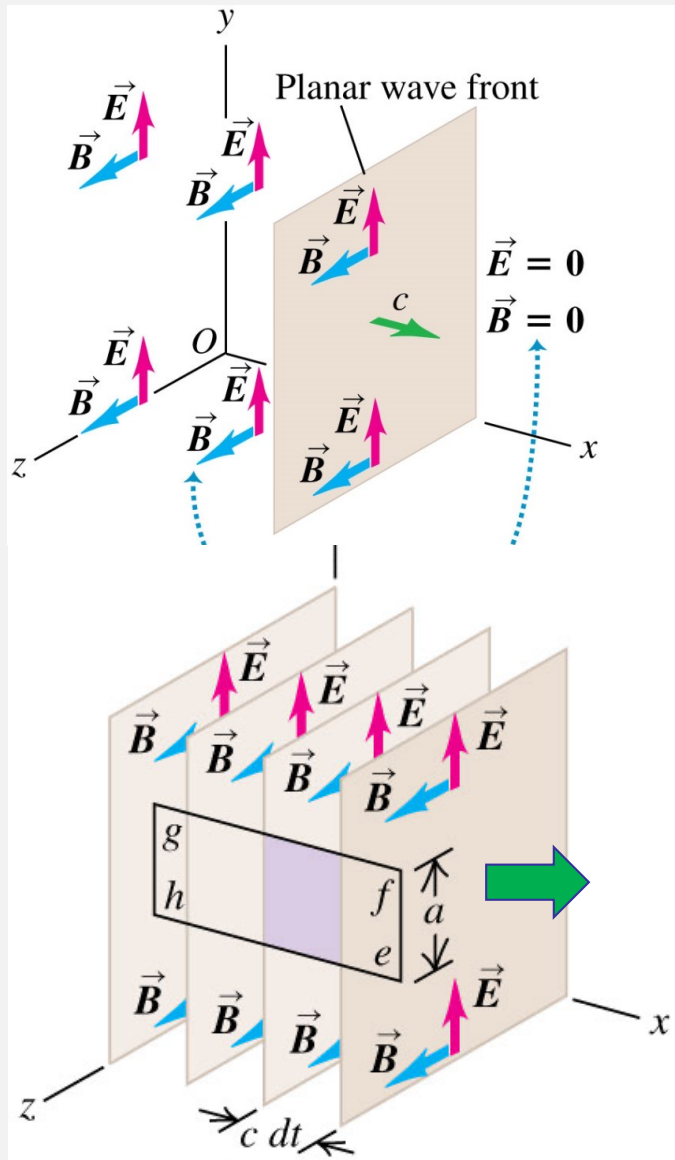
可以想像時間一長，所有地方都會出現電磁場，可見波前平面會往前移動。

現在，研究中間的歷程，也就是這樣的海嘯電磁波可否存在與如何傳播。



一般的海嘯發生時，水面也會出現一個垂直切面，切面後方水位較前方高。前後水位的高度差，會造成水的壓力差，推動切面後方的水向前移動。因此，此切面會以垂直於切面的方向向前移動。

電磁場會有內在的機制，如海嘯高度差的水壓，推動波前平面移動嗎？



波前平面前後的電磁場有差異，會帶動波前向前移嗎？

感應電磁場遵守法拉第定律：
$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

考慮一跨越波前平面的垂直安培圈： $ghcf$

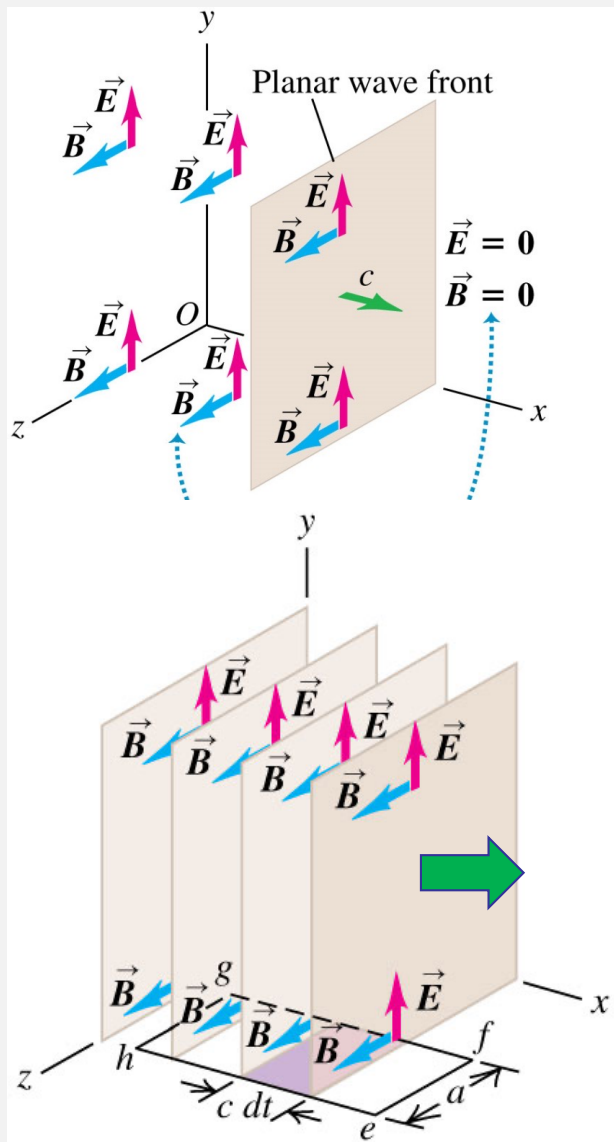
注意波前平面前後的電場有差異，電場線積分不為零，  
法拉第定律就要求安培圈內的磁通量必須增加。

不難，若波前平面前進，磁場的區域的面積A就增加。  
磁通量就增加。

電場的空間變化等於磁場的時間變化！

海嘯波戲劇化了空間的變化差距。

電場在波前平面前後的差異，根據法拉第定律，可以推動磁場波前平面的前進。



而同時，波前平面前後的磁場也不同，

感應電磁場也必須遵守安培馬克斯威爾定律：

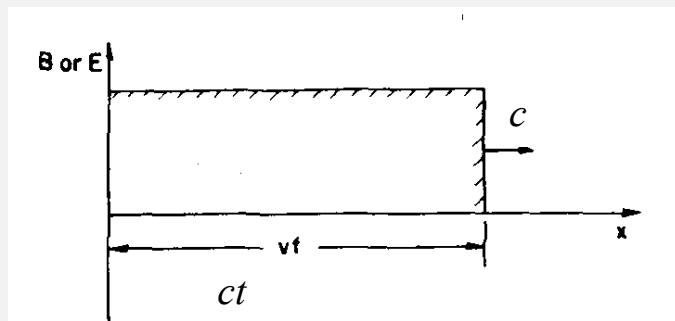
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

考慮另一跨越波前平面的水平安培圈：*ghef*

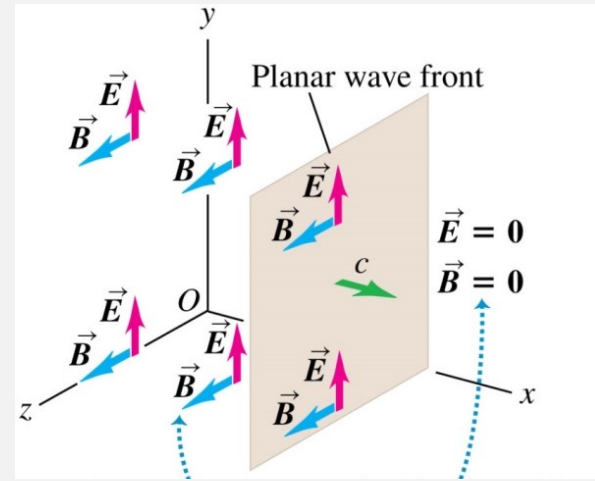
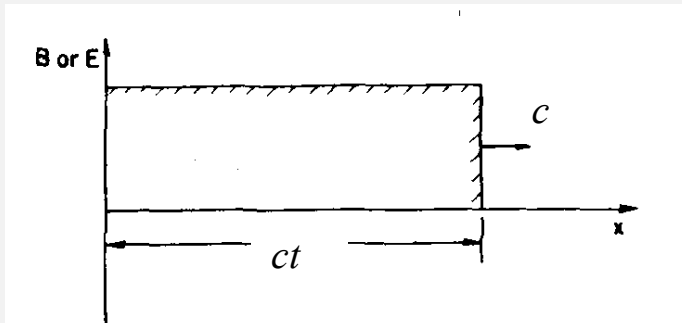
平面前後的磁場不同，則磁場線積分不為零，  
安培馬克斯威爾定律要求電通量增加。

電場波前平面若前進，便增加了電通量。

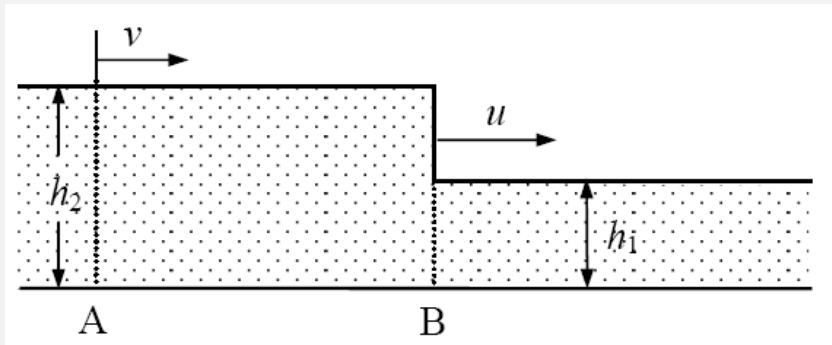
同時滿足以上討論的配置，即為下圖：



磁場在波前平面前後的差異，根據安培馬克斯威爾定律，可以推動電場波前平面的前進。

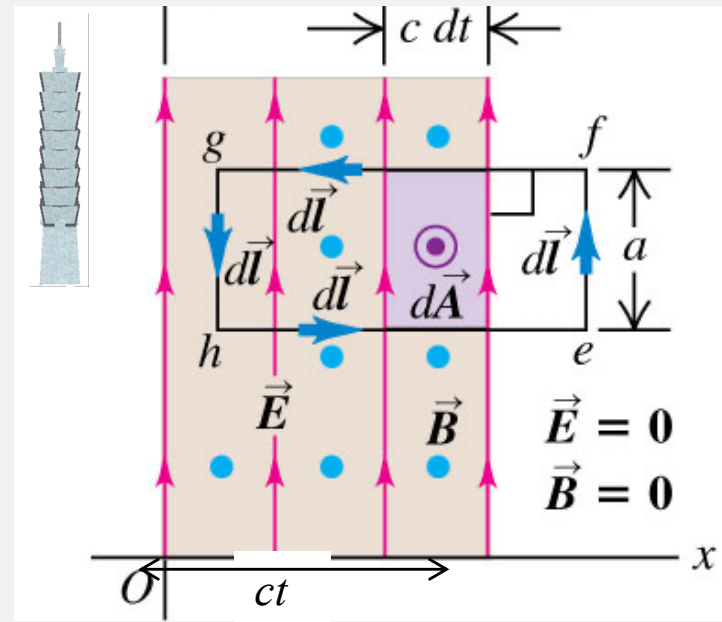
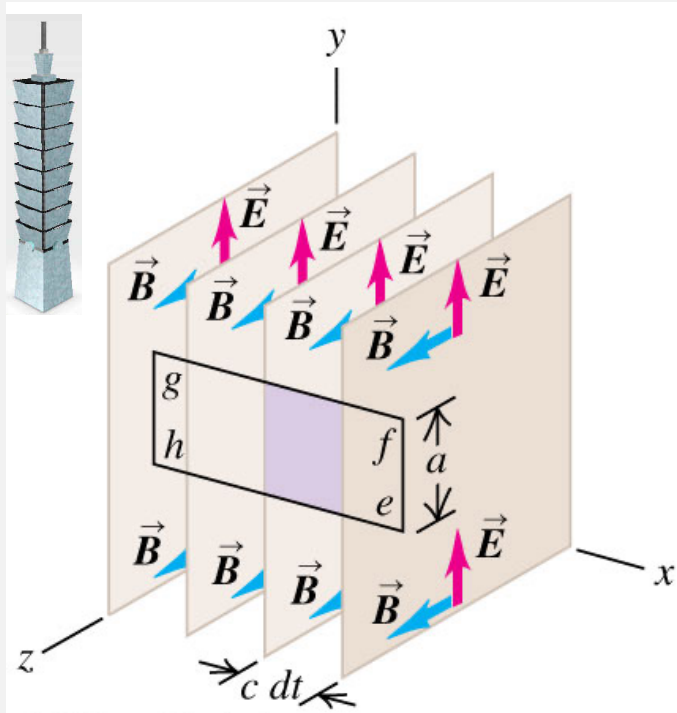


如果如前面的猜想，電場與磁場有同樣的波前平面。  
 電磁場在波前平面前後的差異，同時推動整個波前平面的前進。



這就如同海嘯波前平面前後水的高度差，造成水壓差，推動海嘯波前向前移動。

假設，此波前平面以有限速度 $c$ 移動。



考慮跨越波前平面的安培圈： $ghef$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

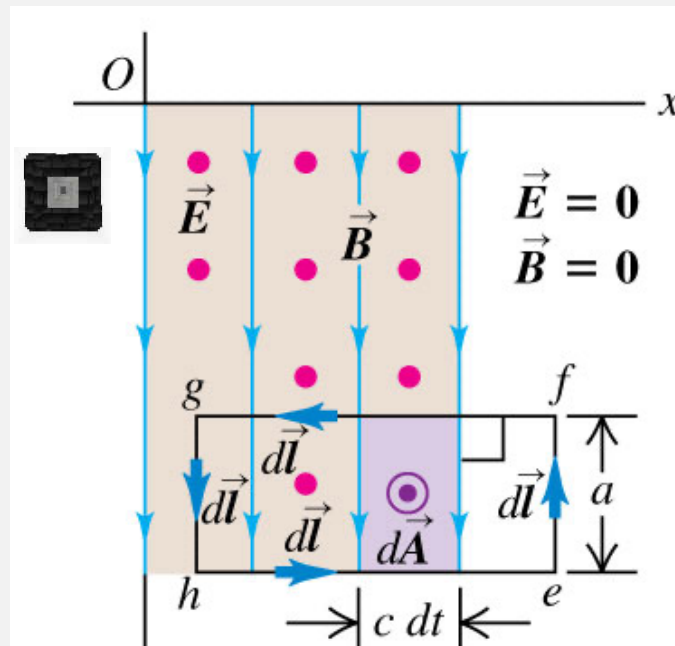
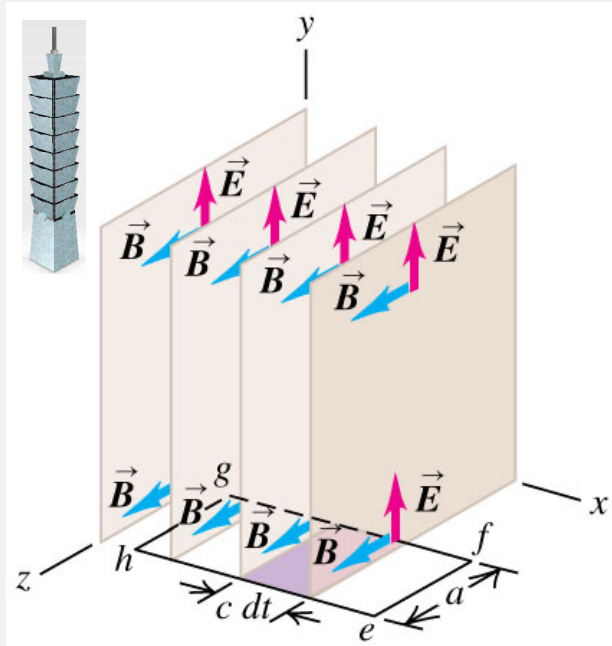
$$\oint \vec{E} \cdot d\vec{s} = 0 \cdot a - E \cdot a + 0 + 0 = -Ea = -\frac{d\Phi_B}{dt} = -B \cdot \frac{dA}{dt} = -B \cdot ac$$

$$E = B \cdot c$$

波前平面前後的電場 $E$ 差所帶動，有磁場 $B$ 區域的邊界會以有限的速度 $c$ 移動，要滿足法拉第定律，電磁場與前進速度必須滿足此條件。

但若要形成海嘯，有電場 $E$ 的區域也要跟著前進，會嗎？

波前平面的前後也有磁場差，而感應電磁場遵守安培馬克斯威爾定律：



考慮如圖水平的安培圈： $ghfe$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = -0 \cdot a + B \cdot a + 0 + 0 = Ba = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 E \cdot \frac{dA}{dt} = \mu_0 \epsilon_0 E \cdot ac$$

波前平面前後的磁場差，因安培定律，帶動有電場區域的邊界以速度 $c$ 移動！

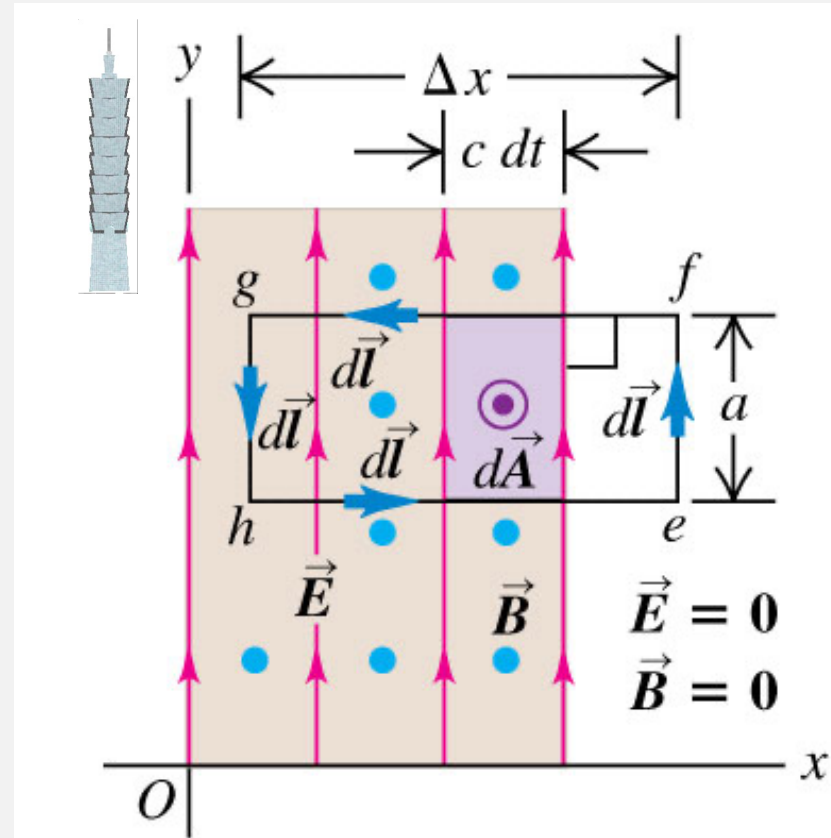
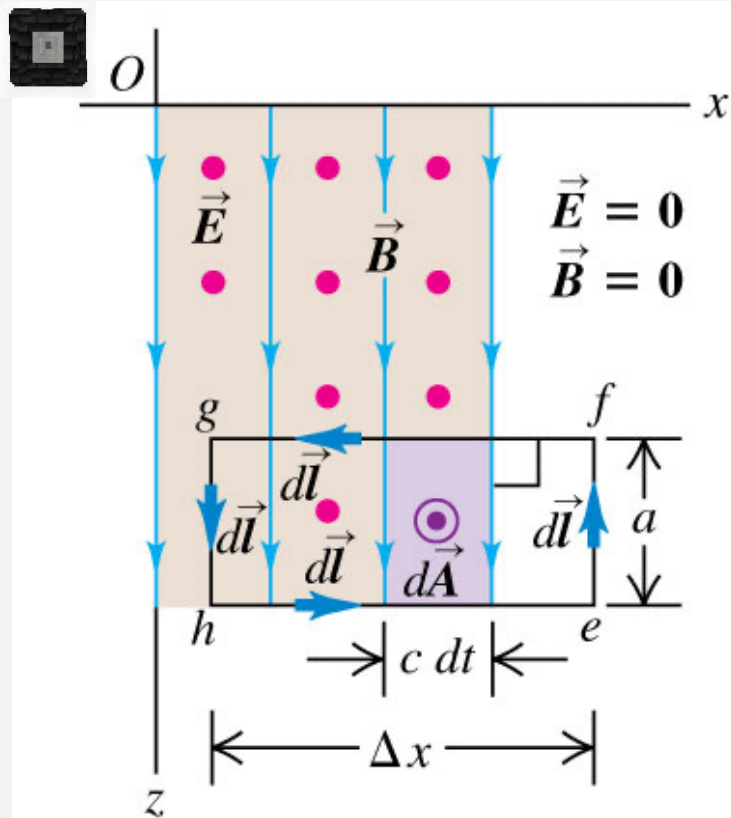
$$B = \mu_0 \epsilon_0 E \cdot c$$

安培-馬克思威爾定律

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

法拉弟定律

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$



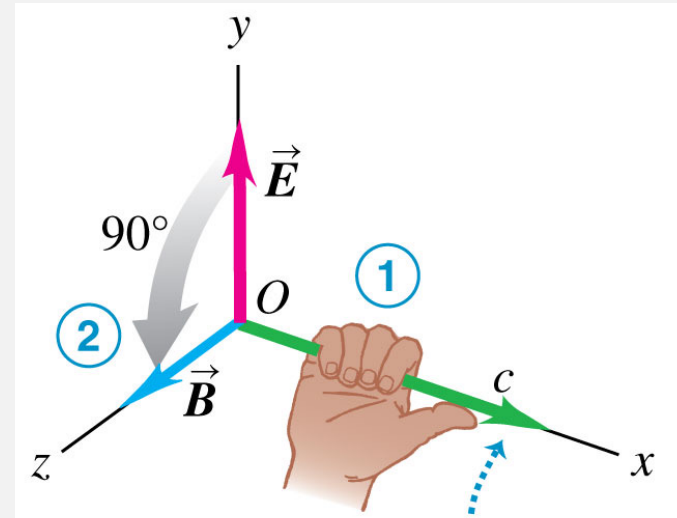
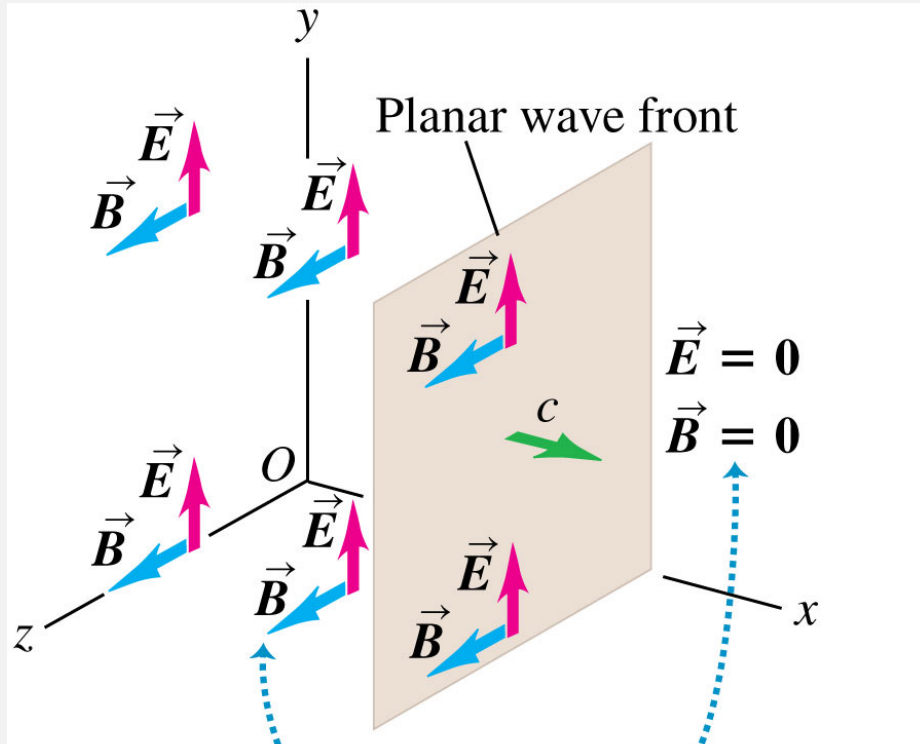
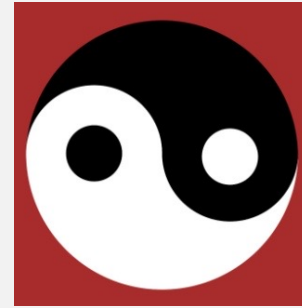
波前前後的磁場差，帶動電場區域的前進！ 波前前後的電場差，帶動磁場區域的前進！

磁場的空間變化等於電場的時間變化！ 電場的空間變化等於磁場的時間變化！

$$B = \mu_0 \epsilon_0 E \cdot c$$

海嘯波戲劇化了空間的變化差距。

$$E = B \cdot c$$



這個互相感生的機制，

可以讓感應電磁場在遠離電荷與電流的情況下，獨立地在空間中傳播！

若要彼此相生，獨立的電磁場必須滿足一定的條件：

電磁場方向必須彼此垂直，又垂直於傳播方向，

電場與磁場大小成正比： $E = B \cdot c$        $B = \mu_0 \epsilon_0 E \cdot c$

兩個條件同時成立，則有電場  $E$  的區域及有磁場  $B$  的區域就可以同步前進。

要求兩個條件同時成立，卻出現非常重要的結果。

$$B = \mu_0 \epsilon_0 E \cdot c$$

$$E = B \cdot c$$

$$c^2 \cdot \mu_0 \epsilon_0 = 1$$

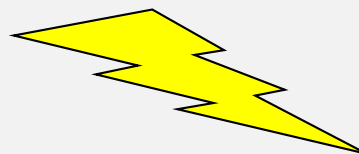
$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

波前平面傳播的速度是一個常數，完全給定！

因此推導出：電磁相生的骨牌效應的傳播速度是有限的。

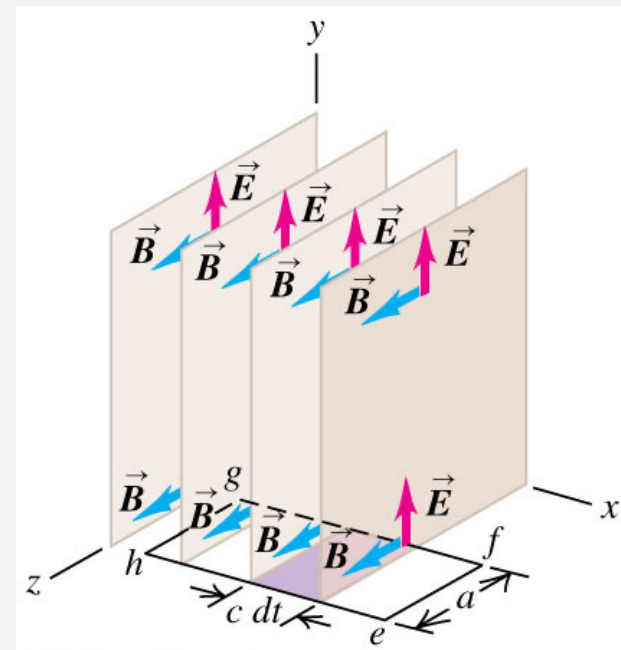
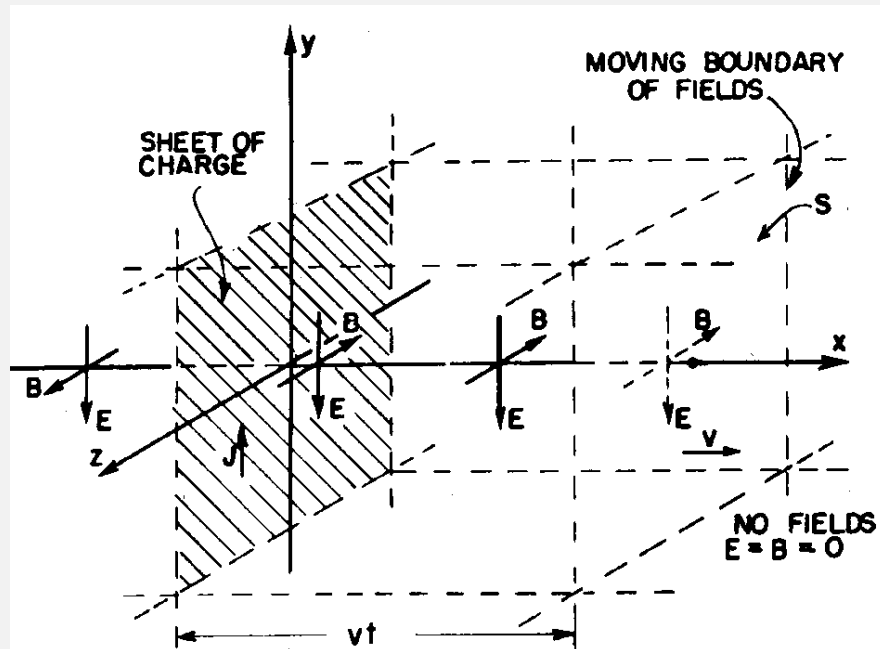
$$= \frac{1}{\sqrt{1.26 \times 10^{-6} \cdot 8.85 \times 10^{-12}}} \sim 2.99 \times 10^8 \text{ m/s}$$

光速！！

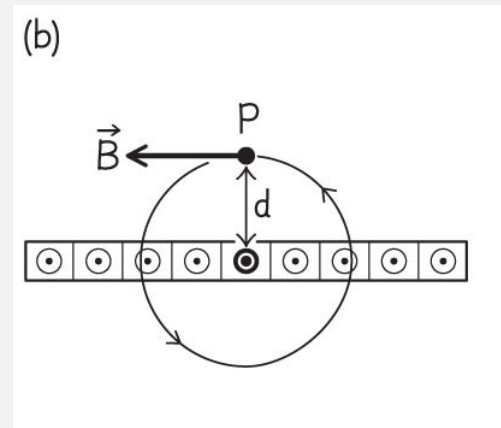
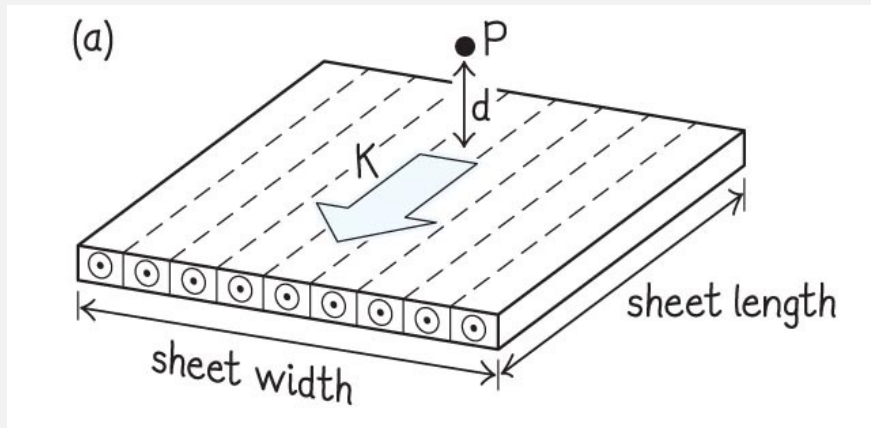


光原來是一種電磁波！

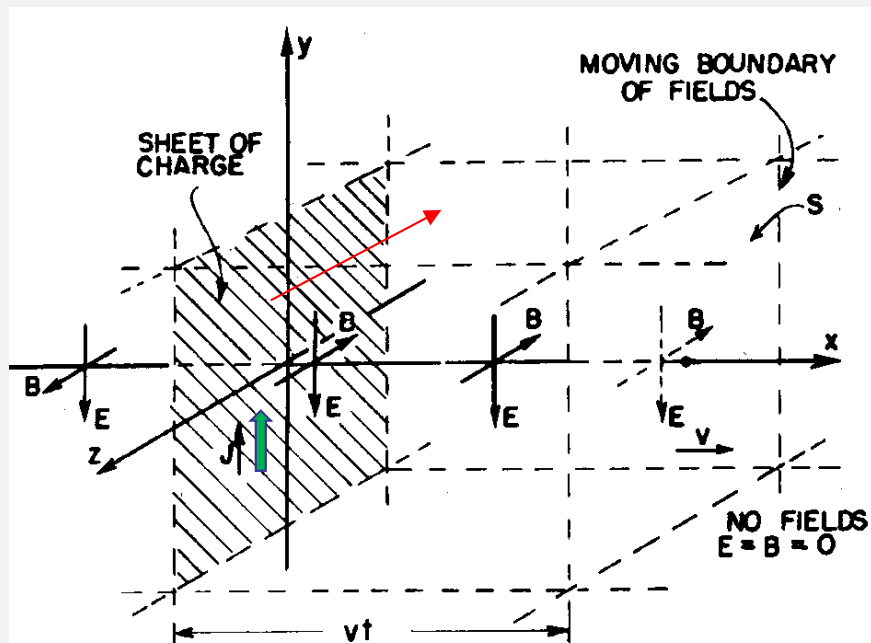
這樣一個海嘯波可以來自一個無限大的導體平板上突然出現的平面電流。



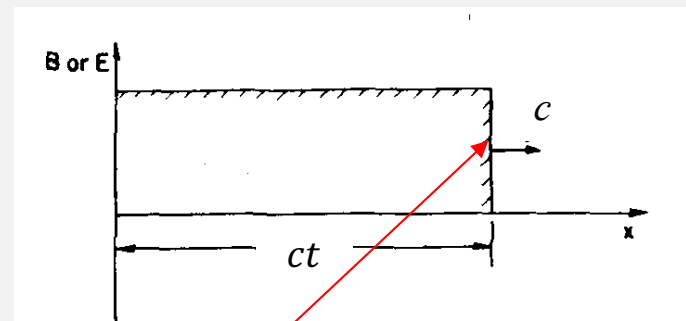
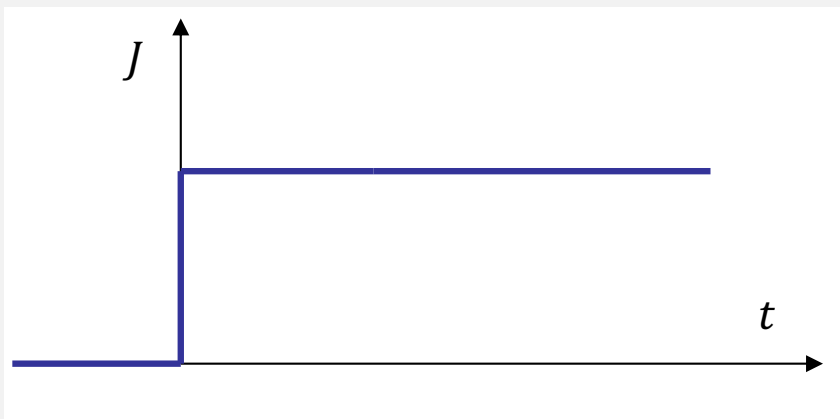
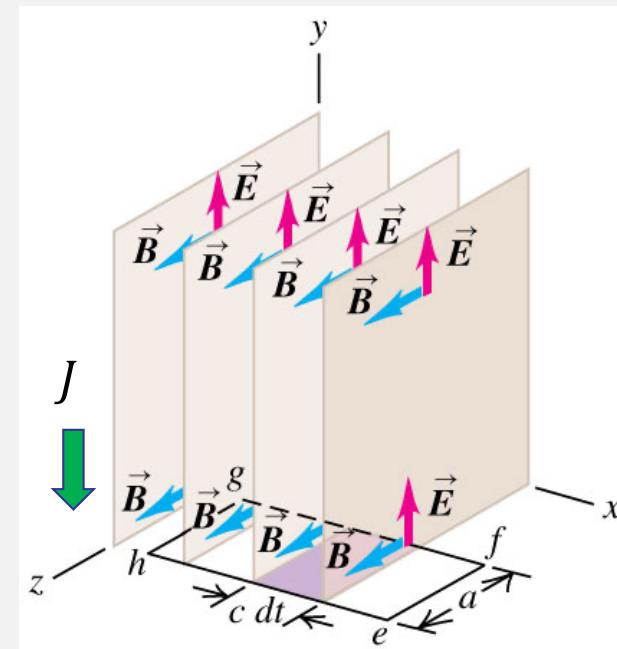
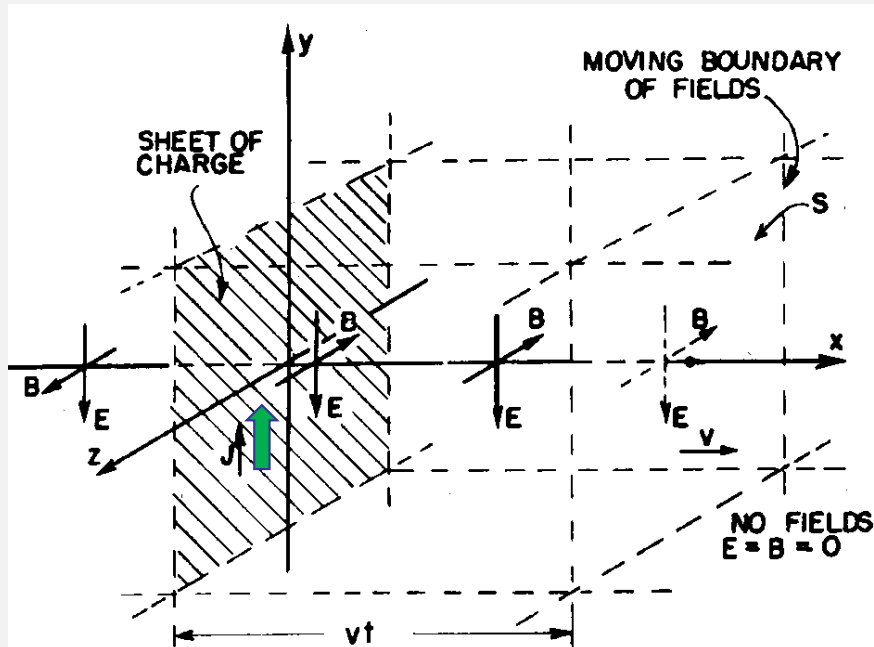
## 平面電流板周圍的磁場



磁場的方向為平行於平板，而垂直於電流方向。  
忽然出現的磁場會感生垂直方向的電場。

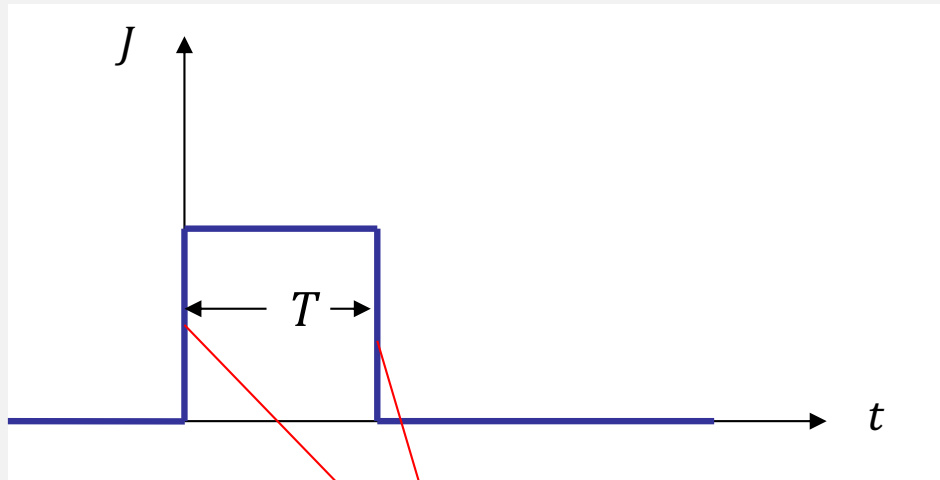


這樣一個海嘯波可以來自一個無限大的導體平板上突然出現的平面電流。  
 在  $t = 0$  時突然出現平面電流，然後保持定值。

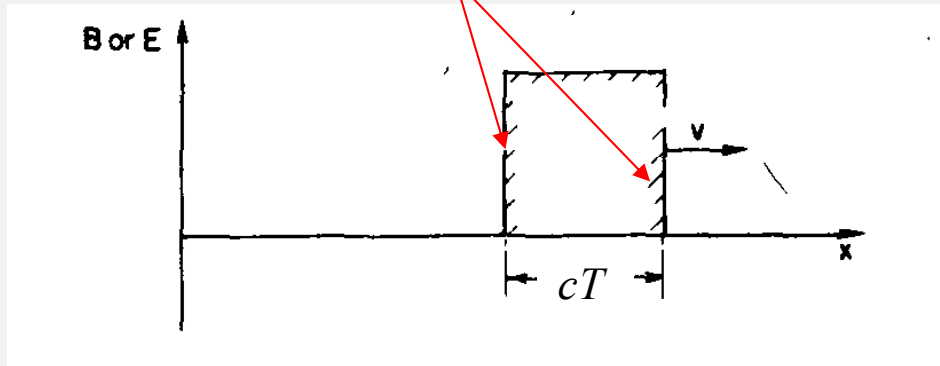


在  $t = 0$ ，只有極近處產生磁場，因此波前平面位於  $x = 0$ 。  
 波前平面以定速  $c$  在空間中向外傳播，位置為  $x = ct$ 。

若電流在產生一段時間  $T$  後， $t = T$  時突然停止：



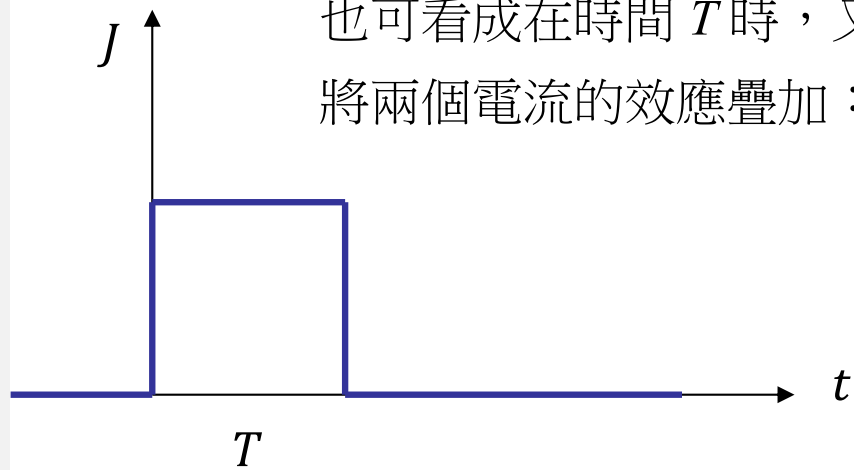
此停止也應該以定速  $c$  向外傳播，因此.....



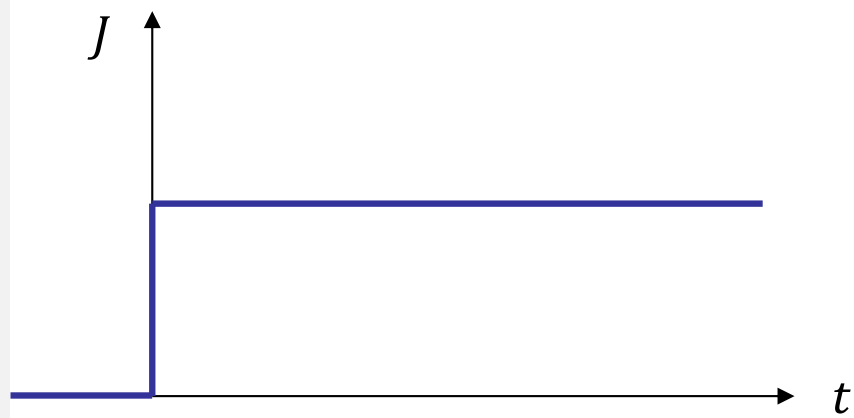
現在平板  $x = 0$  附近沒有電場也沒有磁場！

一個脈衝方塊的電磁場（前後都為零）將以定速  $c$  向前孤獨地傳播。

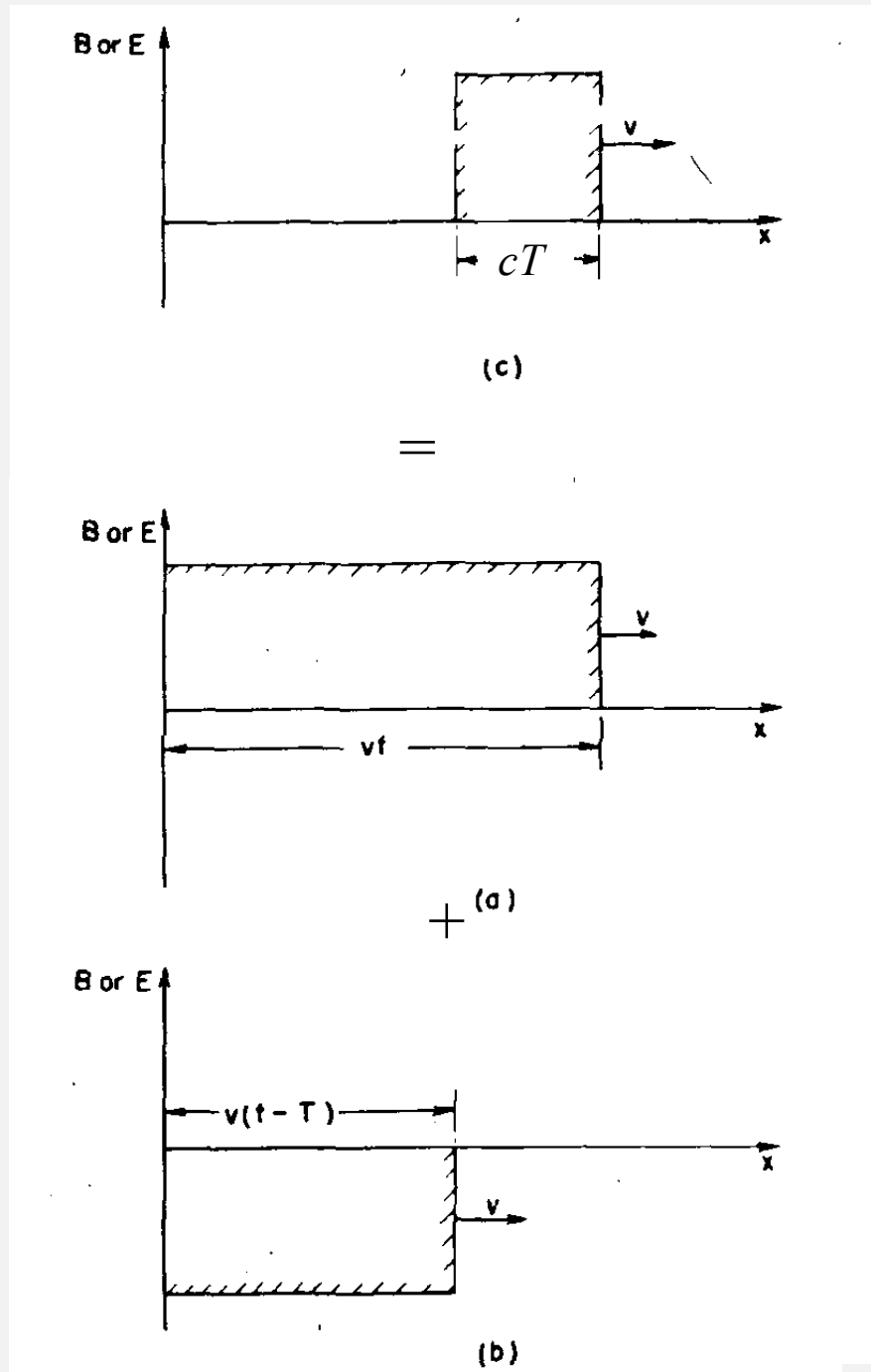
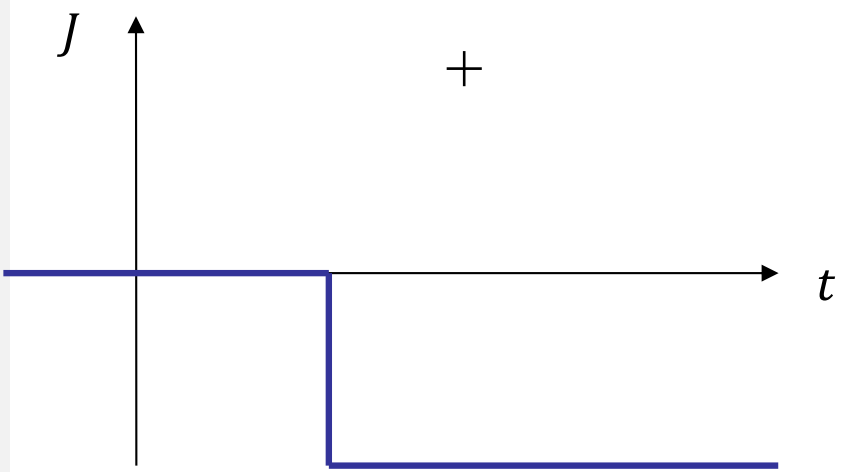
也可看成在時間  $T$  時，又產生一反向的電流，  
將兩個電流的效應疊加：



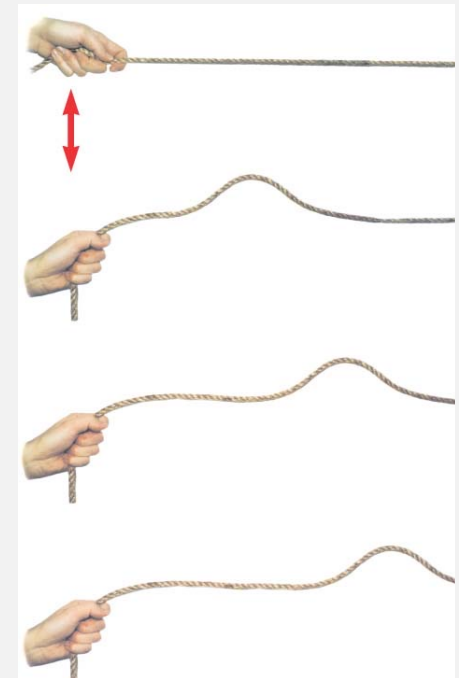
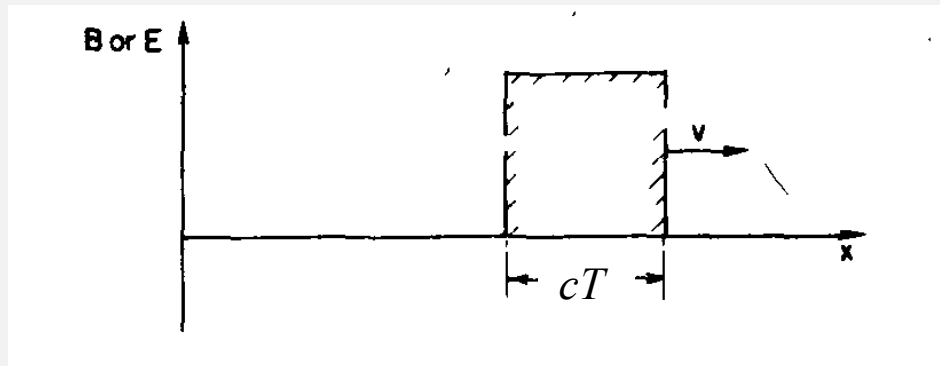
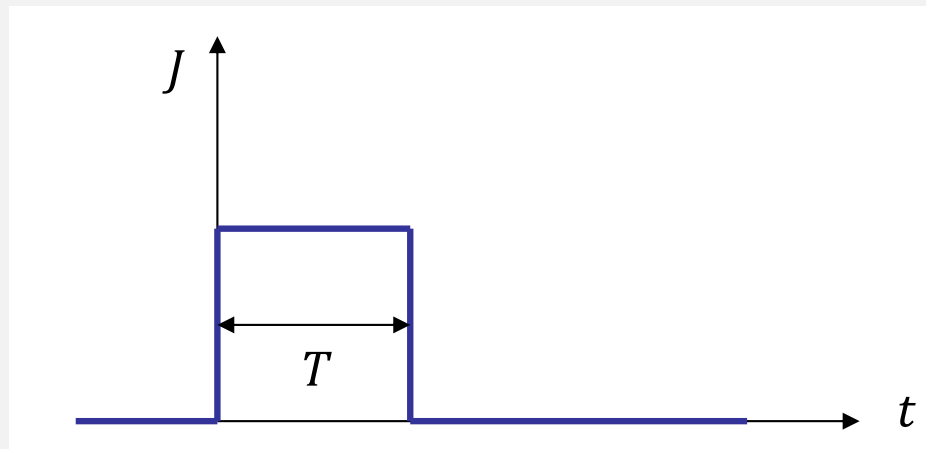
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方塊波：若電流突然產生，一段時間  $T$  後突然停止：



這段時間內的電流，產生一個電磁場的脈衝！

此脈衝在電流消失後還會在遠方繼續傳播！

此脈衝的電場與磁場是獨立於產生它們的電流而存在的！

這是波動。方塊脈衝波！