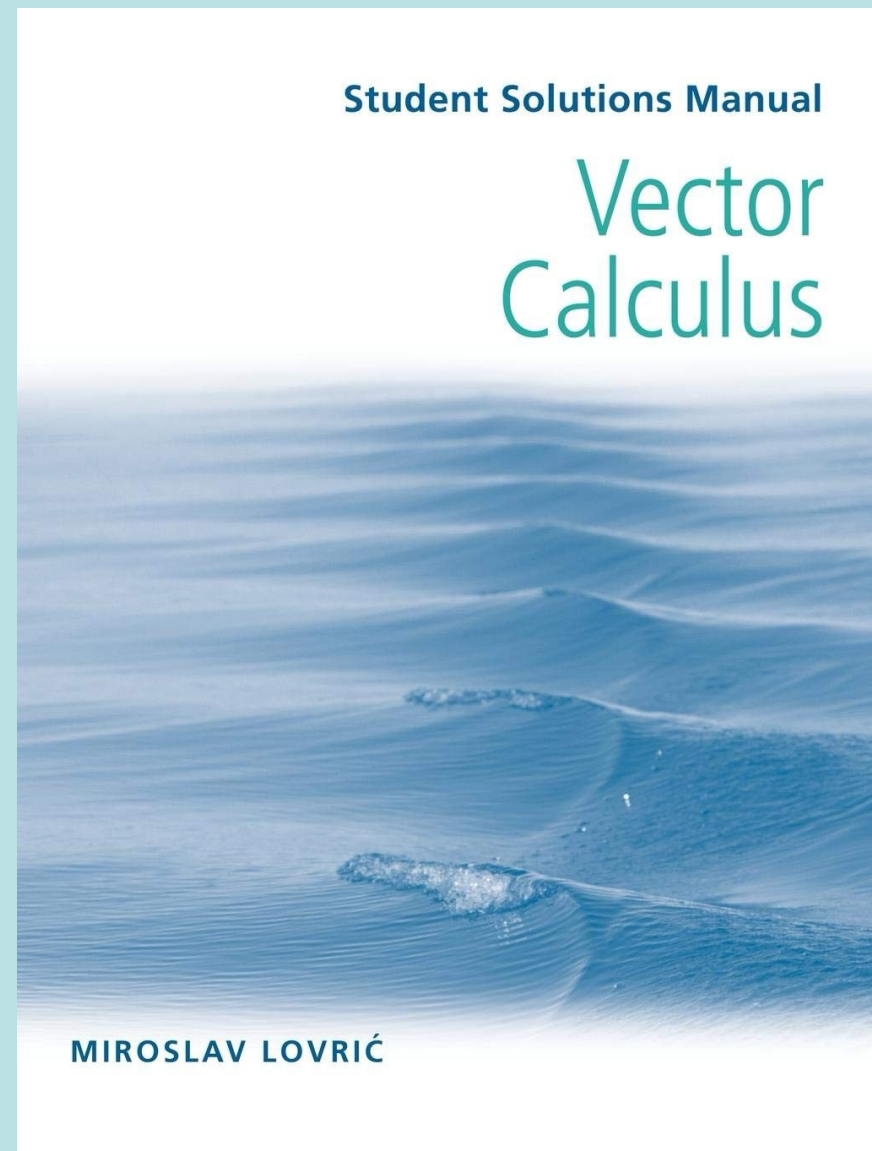
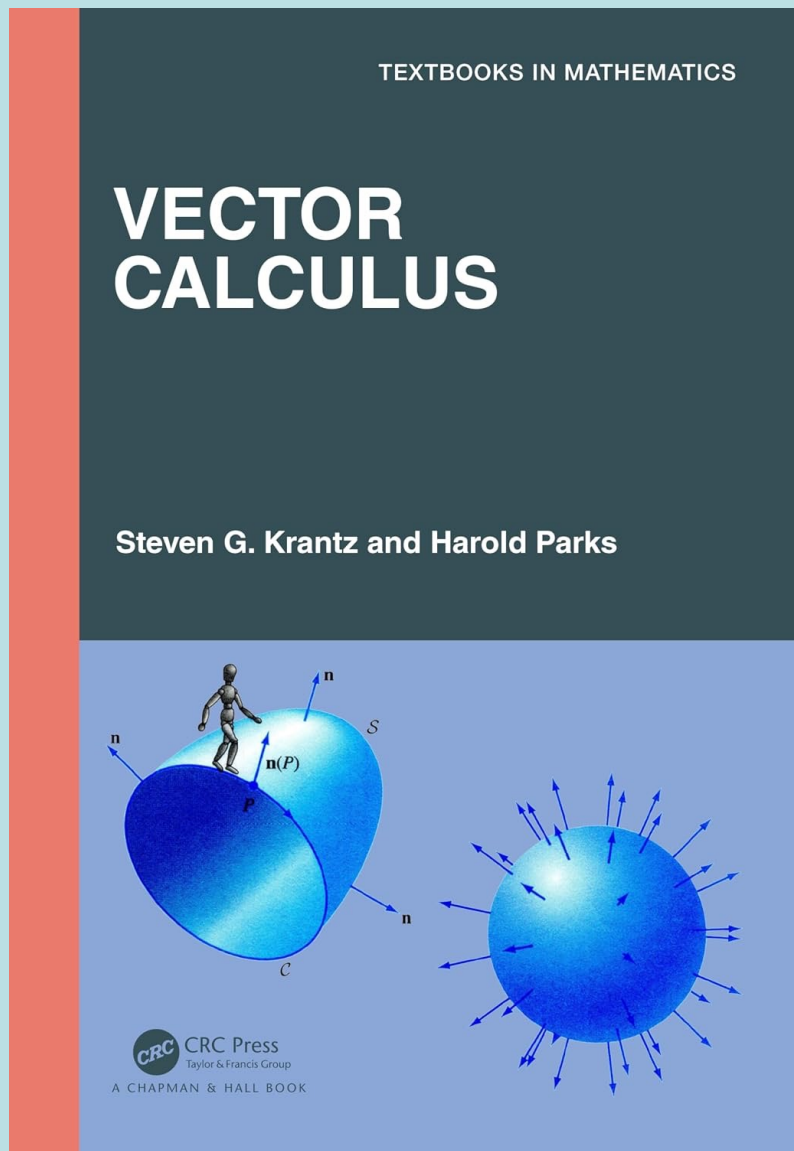
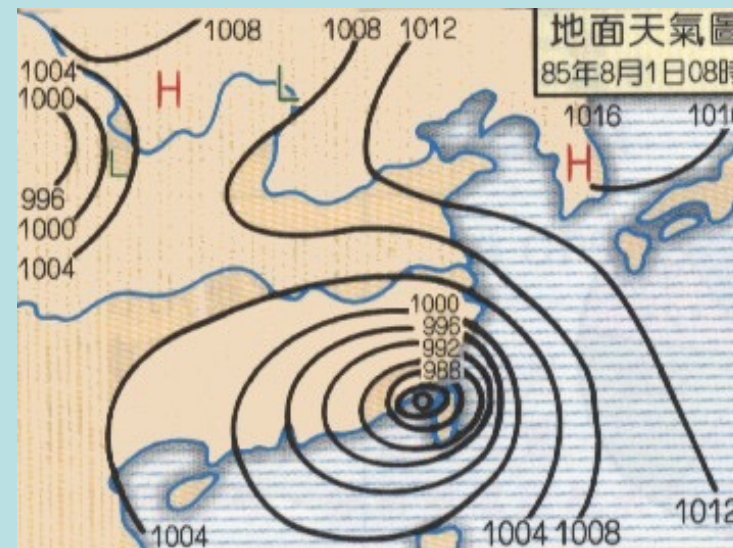
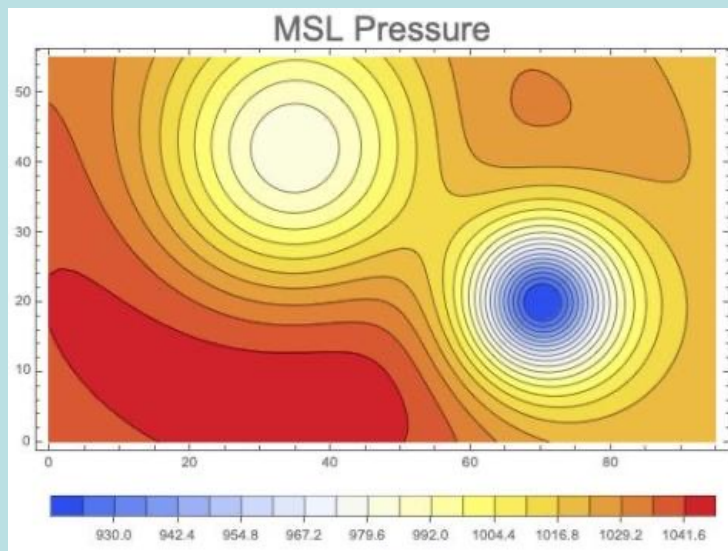
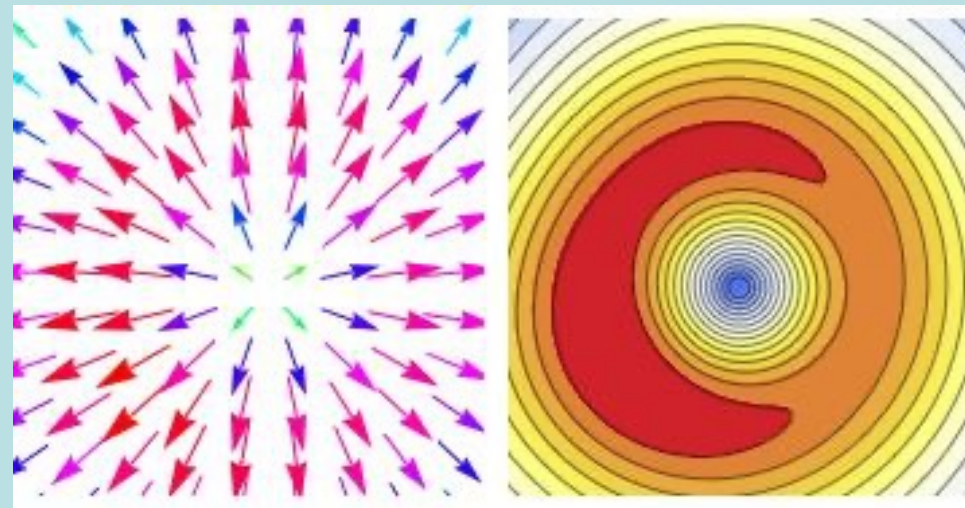




Calculus in 2D and 3D space

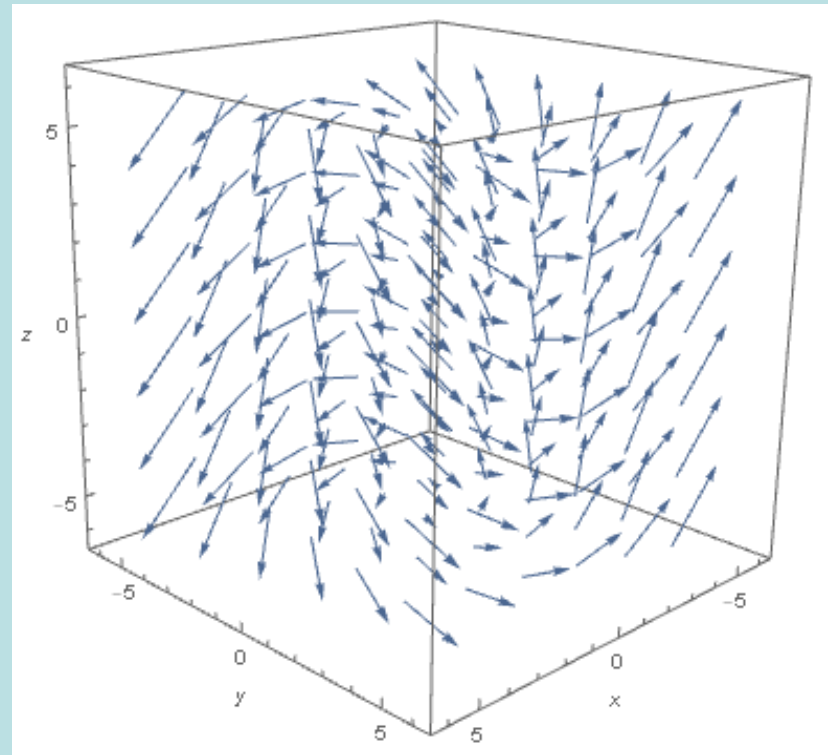
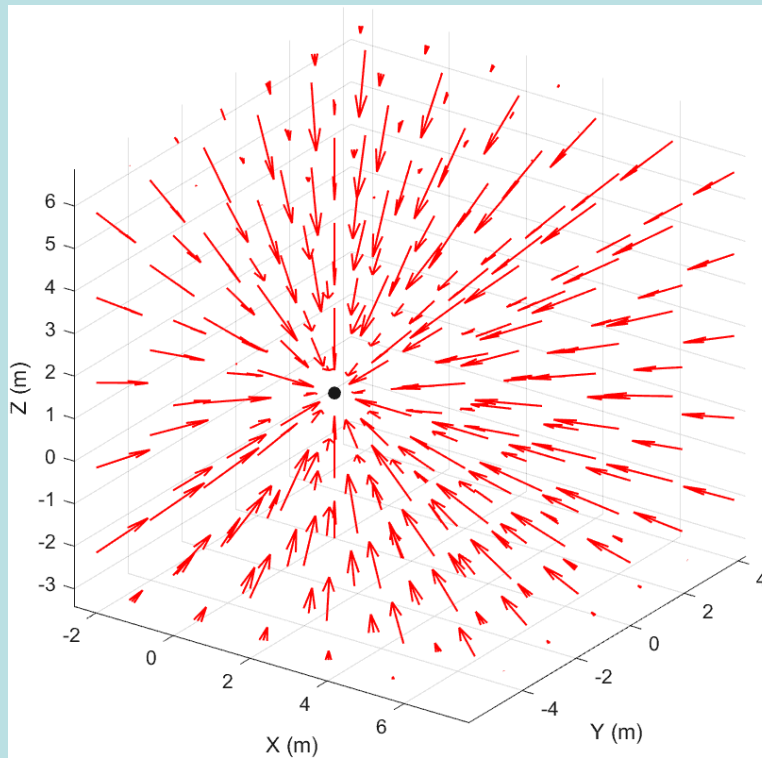


Air pressure on earth surface is a function of position



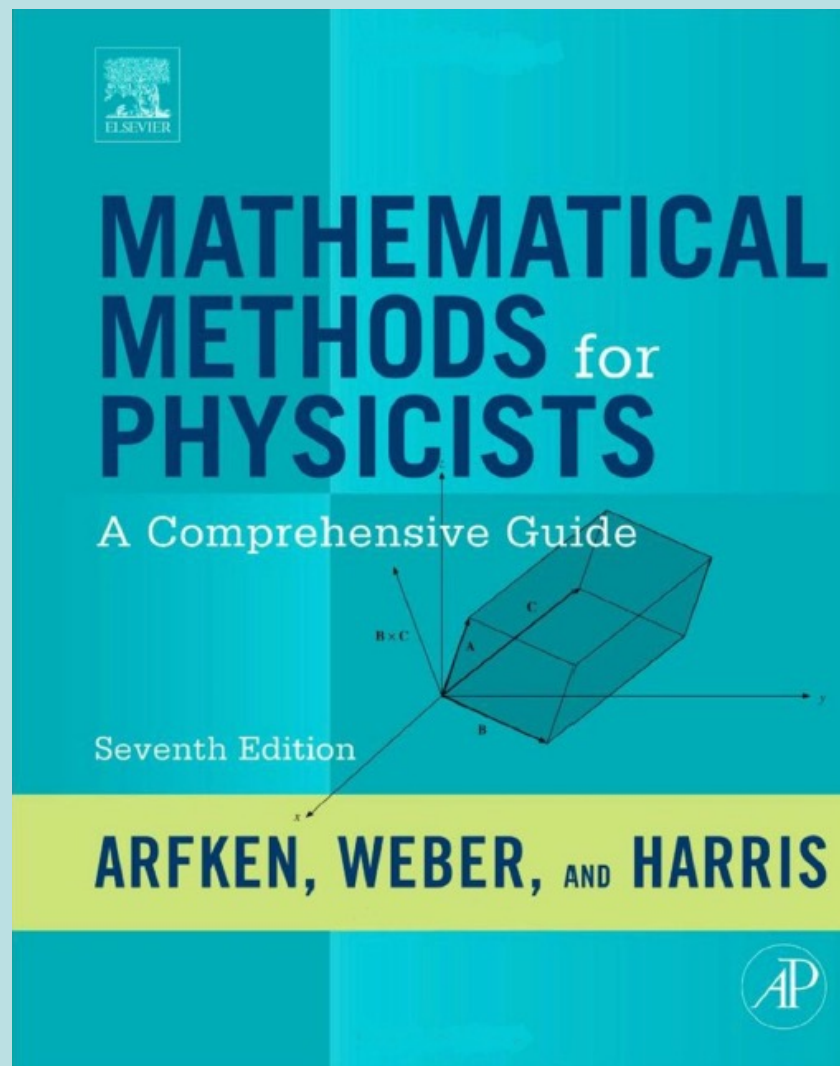
Pressure difference exert on air everywhere directional: gradient of pressure.

We note that the wind does not blow down-gradient; the pressure force is balanced by the Coriolis force arising from the Earth's rotation, so that the wind blows more-or-less along the isobars, perpendicular to the pressure gradient. This is called the geostrophic wind.



Pressure differences are a vector at every location : this is a Field.

Vectorial Calculus



Section 3.5-3.9

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VOLUME II: MAINLY ELECTROMAGNETISM AND MATTER

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2

Differential Calculus of Vector Fields

Review: Chapter 11, Vol. I, *Vectors*

2-1 Understanding physics

The physicist needs a facility in looking at problems from several points of view. The exact analysis of real physical problems is usually quite complicated, and any particular physical situation may be too complicated to analyze directly by solving the differential equation. But one can still get a very good idea of

3

Vector Integral Calculus

3-1 Vector integrals; the line integral of $\nabla\psi$

We found in Chapter 2 that there were various ways of taking derivatives of fields. Some gave vector fields; some gave scalar fields. Although we developed many different formulas, everything in Chapter 2 could be summarized in one rule: the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ are the three components of a vector operator ∇ . We would now like to get some understanding of the significance of the derivatives of fields. We will then have a better feeling for what a vector field equation means.

Vector Calculus

University of Cambridge Part IA Mathematical Tripos

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3 Grad, Div and Curl

3.1 The Gradient

3.2 Div and Curl

3.2.1 Some Basic Properties

3.2.2 Conservative is Irrotational



Vectors in Flat Space $R^{2,3}$ 平坦二維或三維空間中的向量

Vectorial Calculus 向量微積分

這是處理 $R^{2,3}$ 空間中的位置與時間的函數！這樣的函數一般稱為場。

This is to deal with functions of space and time, called **fields**.

在空間中可以定義函數值為純量的函數！純量場 Scalar Field。

Scalar functions (just real number) of space and time are called scalar field.

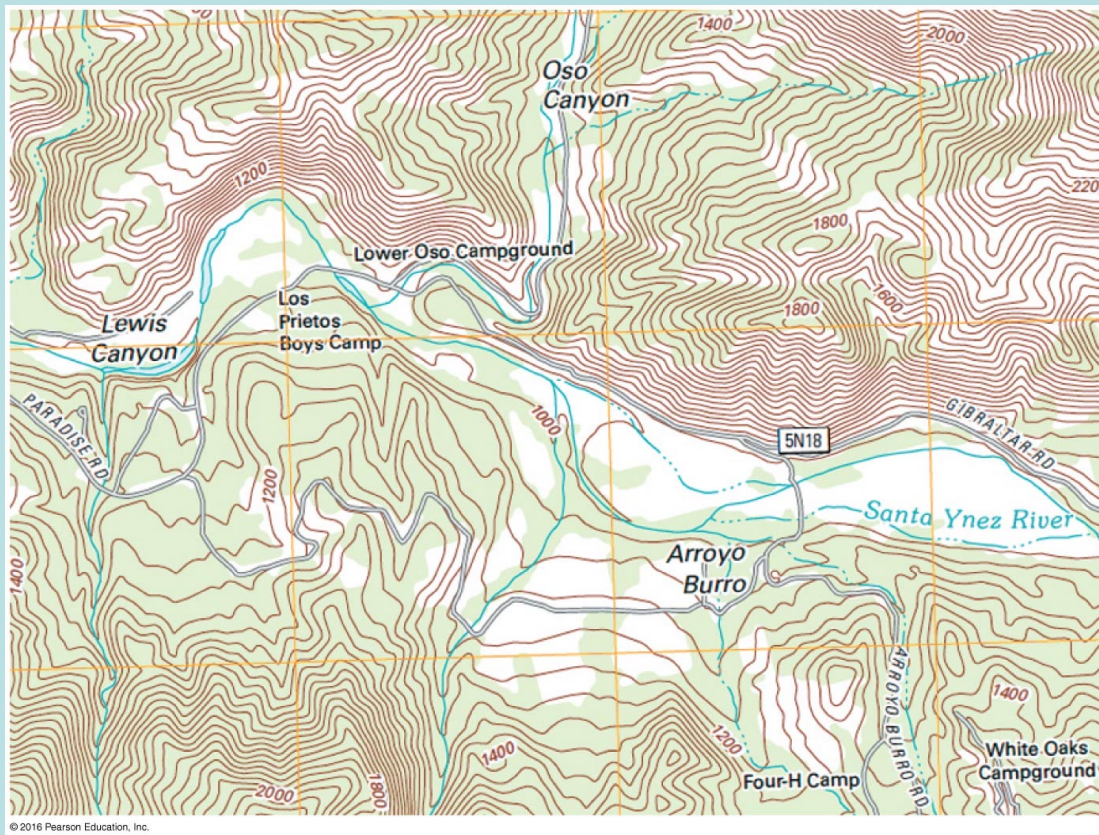
$$V: R^3 \rightarrow R$$

$U(\vec{r})$ Potential energy is determined by positions and it is a scalar.

位能由位置決定，每一個座標對應一個位能值。

Almost like the heights on a map.

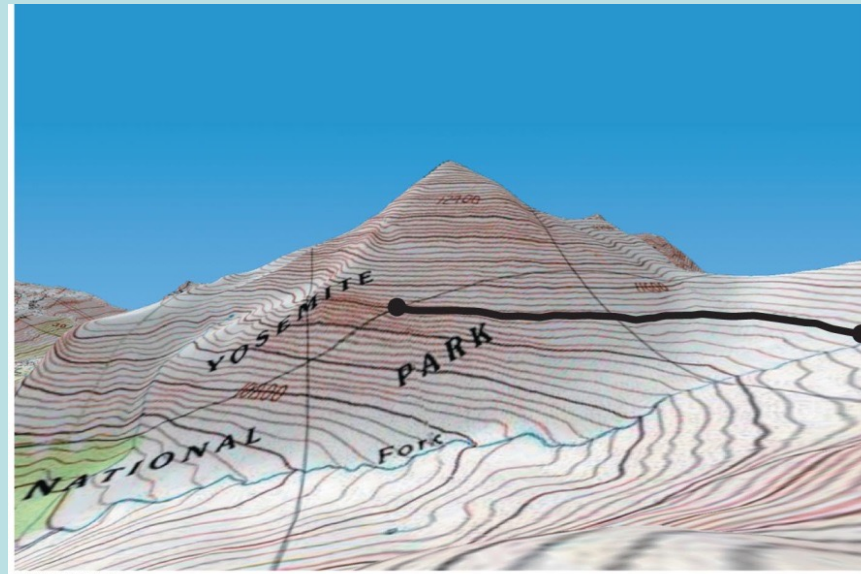
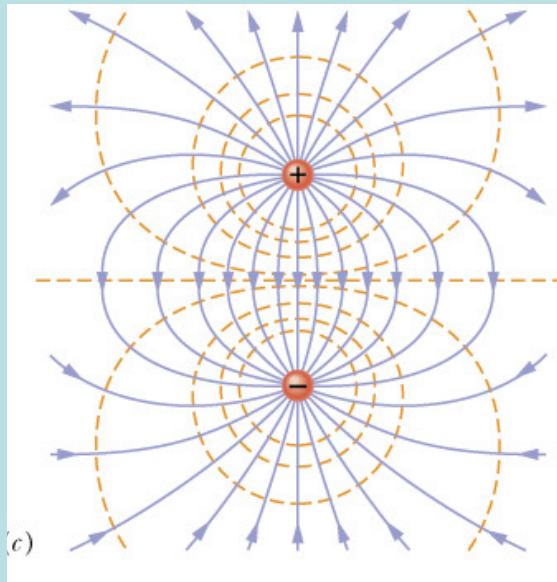
如同地圖上的高度標示一樣！



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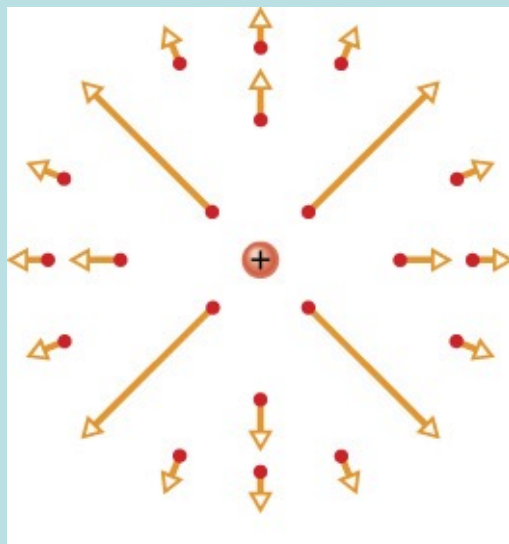
電位 $V(\vec{r})$ 是空間的純量函數。Electric Potential $V(\vec{r})$ is a scalar function in space.
相對的，電場 $\vec{E}(\vec{r})$ 是一個向量函數。On the contrary, electric field is a vector field.



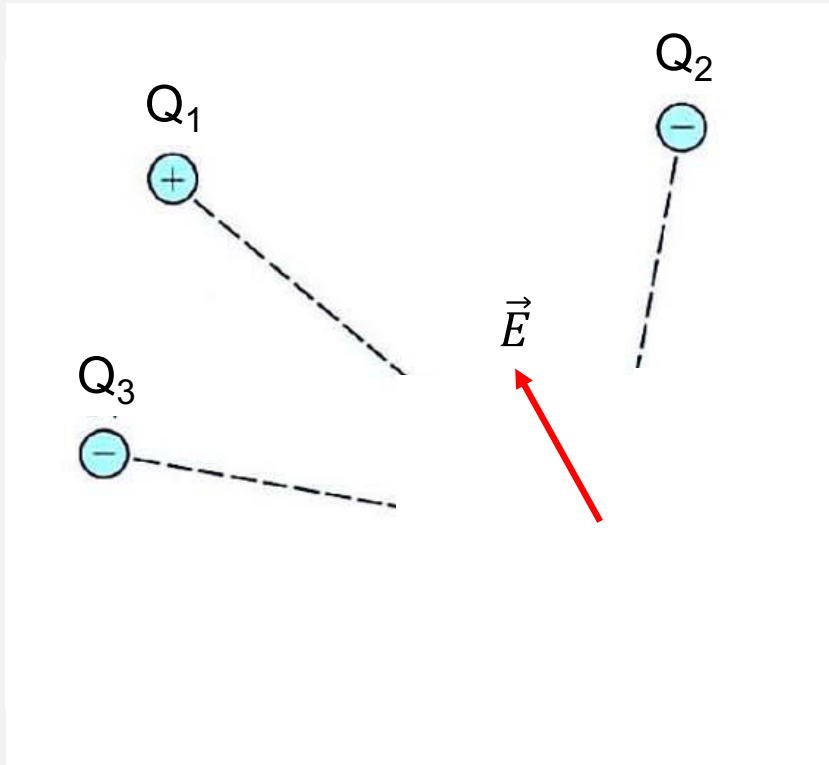
在空間中可以定義函數值為向量的函數！向量場 Vector Field。

When the function values are vectors, the fields are called vector fields.

$$\vec{E}: R^3 \rightarrow R^3$$



電場的抽象定義：



$$\vec{F} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i q}{r_i^2} \hat{r}_i$$

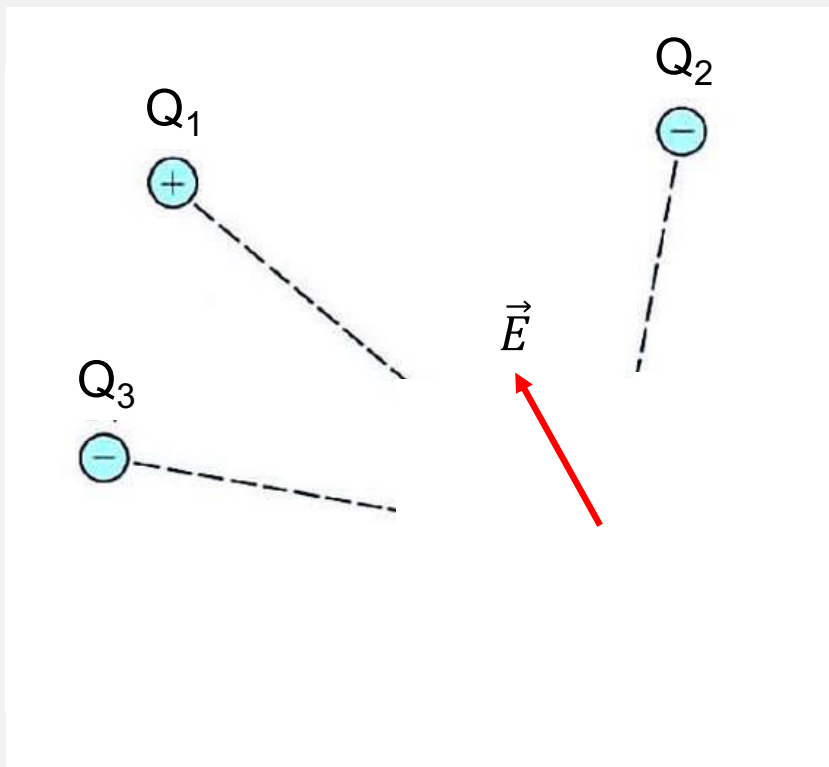
電荷量 q 可以從複雜的式子中提出來：

$$= \left(\sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{r_i^2} \hat{r}_i \right) \cdot q = \vec{E} \cdot q$$

電場 Electric Field

電場 $\vec{E}$ 只與 q 的位置相關，與 q 的電荷量無關

想像 q 漸漸趨近於零 $q \rightarrow 0$ 電場 $\vec{E}$ 應該依舊存在吧！



$$\vec{E} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{r_i^2} \hat{r}_i$$

電場與 q 的電荷量無關。

由固定電荷 Q_i 及 q 的位置決定。

電場 $\vec{E}(\vec{r})$ 是空間的性質。

每一點都有 $\vec{E}(\vec{r})$ ，電場遍佈整個空間。

電場可以為零，但永遠存在。

電場就是會讓位於當地的電荷 q 得到靜電力 $q \cdot \vec{E}$ 的空間的物理性質。

電場是由固定電荷 Q_i 在空間各處產生。

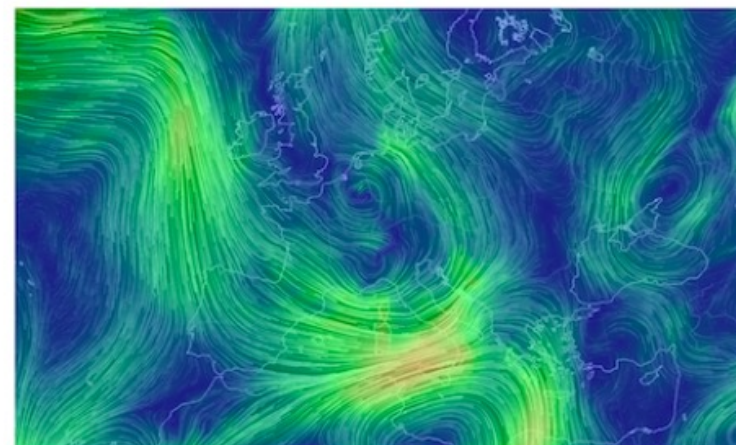
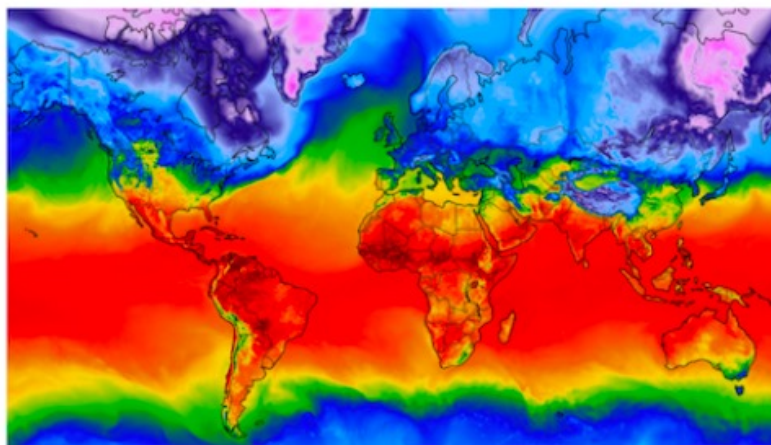


Figure 1. On the left, the temperature on the surface of the Earth is an example of a map from $\mathbb{R}^2 \rightarrow \mathbb{R}$, also known as a scalar field. On the right, the wind on the surface of the Earth blows more or less horizontally and so can be viewed as a map from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, also known as a vector field. (To avoid being co-opted by the flat Earth movement, I should mention that, strictly speaking, each of these is a map from S^2 rather than $\mathbb{R}^2$.)

溫度分佈就是純量場 $T(\vec{r})$

Temperature distribution is a scalar field.

風速分佈就是向量場 $\vec{v}(\vec{r})$

Wind speed distribution is a vector field.

梯度，散度，旋度

Grad, Div, Curl

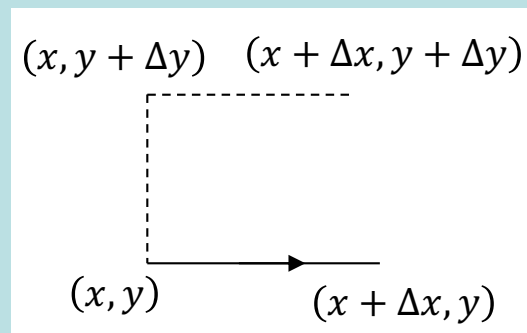
Major tools in 3D vector calculus: Partial Derivatives

$$\frac{\partial V}{\partial x}(x, y, z)$$

$$\frac{\partial V}{\partial y}(x, y, z)$$

$$\frac{\partial V}{\partial z}(x, y, z)$$

$$\frac{\partial V}{\partial x} \rightarrow \lim_{\Delta x \rightarrow 0} \frac{V(x + \Delta x, y, z) - V(x, y, z)}{\Delta x}$$

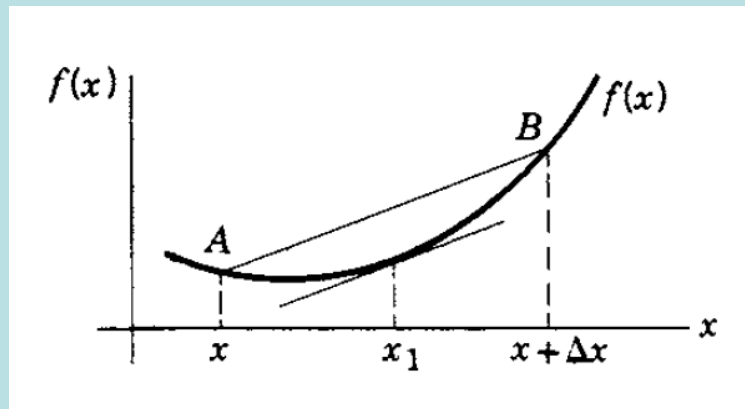


Partial derivative is the rate of change along displacement in one axis.

First, an extremely useful theorem!

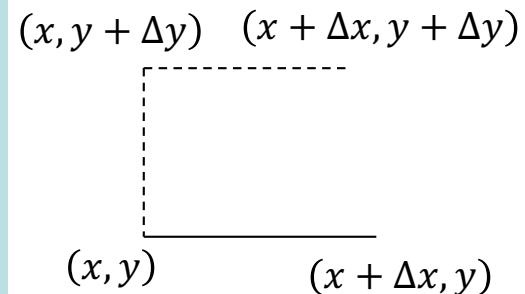
Single-variable function difference equals derivative times variable difference.

$$f(x + \Delta x) - f(x) \rightarrow \frac{df}{dx} \cdot \Delta x \quad \Delta x \rightarrow 0$$

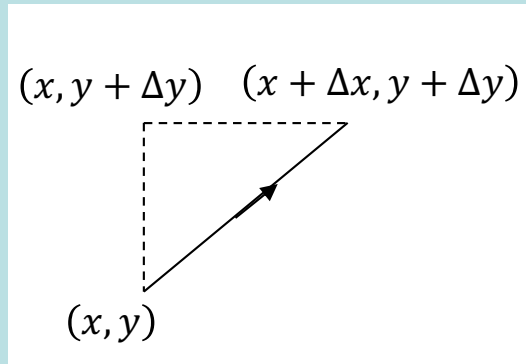


If we fix y, z , multivariable function $V(x, y, z)$ becomes single-variable function:
Derivative as $\Delta x \rightarrow 0$ is partial derivative $\frac{\partial V}{\partial x}$.

$$V(x + \Delta x, y, z) - V(x, y, z) \rightarrow \frac{\partial V}{\partial x} \Delta x \quad \Delta x \rightarrow 0$$



General displacement can be decomposed into displacements along 3 directions.



$$\Delta V \equiv V(x + \Delta x, y + \Delta y) - V(x, y)$$

Change of V can be decomposed into changes along 3 directions.

$$= [V(x + \Delta x, y + \Delta y) - V(x, y + \Delta y)] + [V(x, y + \Delta y) - V(x, y)]$$

$$= \frac{\partial V}{\partial x}(x, y + \Delta y)\Delta x + \frac{\partial V}{\partial y}(x, y)\Delta y \approx \frac{\partial V}{\partial x}(x, y)\Delta x + \frac{\partial V}{\partial y}(x, y)\Delta y$$

$$\Delta V = \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y$$

$$\Delta V = \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z$$

Differentiation in 3D space

We can define three partial derivatives of a scalar field $V(x, y, z)$.

$$\frac{\partial V}{\partial x}(x, y, z)$$

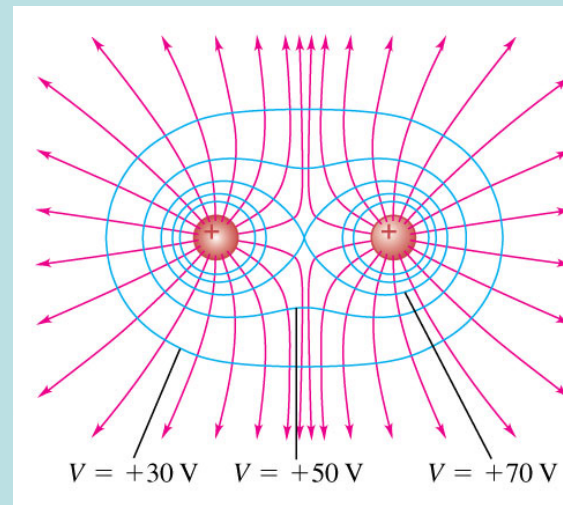
$$\frac{\partial V}{\partial y}(x, y, z)$$

$$\frac{\partial V}{\partial z}(x, y, z)$$

The three partial derivatives look like components of a vector. Are they?

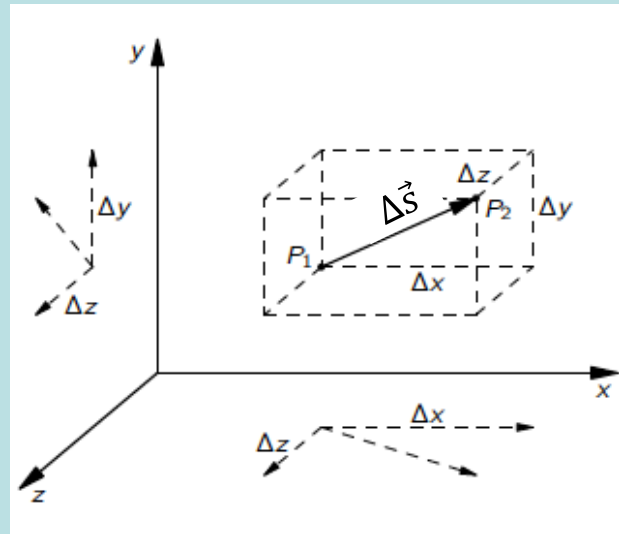
The answer is yes.

Take electric potential, a function of 3D coordinates, as an example:



Electric potential difference equals potential energy difference divided by charge.

$$\Delta V = V(P_2) - V(P_1) = \frac{\Delta U}{q}$$



If P_1 and P_2 are very close with infinitesimal displacement :
potential energy difference equals electric force times $\Delta \vec{R}$.

$$\Delta \vec{S} = (\Delta x, \Delta y, \Delta z)$$

$$\Delta V = \frac{\Delta U}{q} \sim \frac{\vec{F} \cdot \Delta \vec{S}}{q} = \vec{E} \cdot \Delta \vec{S} = -E_x \Delta x - E_y \Delta y - E_z \Delta z$$

Electric potential difference equals inner product of electric field and displacement.

Since $\vec{E}$ does not change much, its value can be approximately taken at P_1 .

There is another way to compute function difference in space.

Function difference equals partial derivatives times variable differences.

$$\Delta V = \frac{\partial V}{\partial x} \Delta x + \frac{\partial V}{\partial y} \Delta y + \frac{\partial V}{\partial z} \Delta z$$

The two computations must equal.

$$\Delta V \sim -\vec{E} \cdot \Delta \vec{s} = -E_x \Delta x - E_y \Delta y - E_z \Delta z$$

Comparing the two formulae, we find:

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

The three partial derivatives are indeed components of a vector.

$$\vec{E} = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$$

This derivation does not rely on $\vec{E}$ being the electric field.

It is true for any vector field $\vec{A}$ and scalar field $\Delta \phi = \vec{A} \cdot \Delta \vec{s}$.

$$\vec{A} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

梯度 Gradient

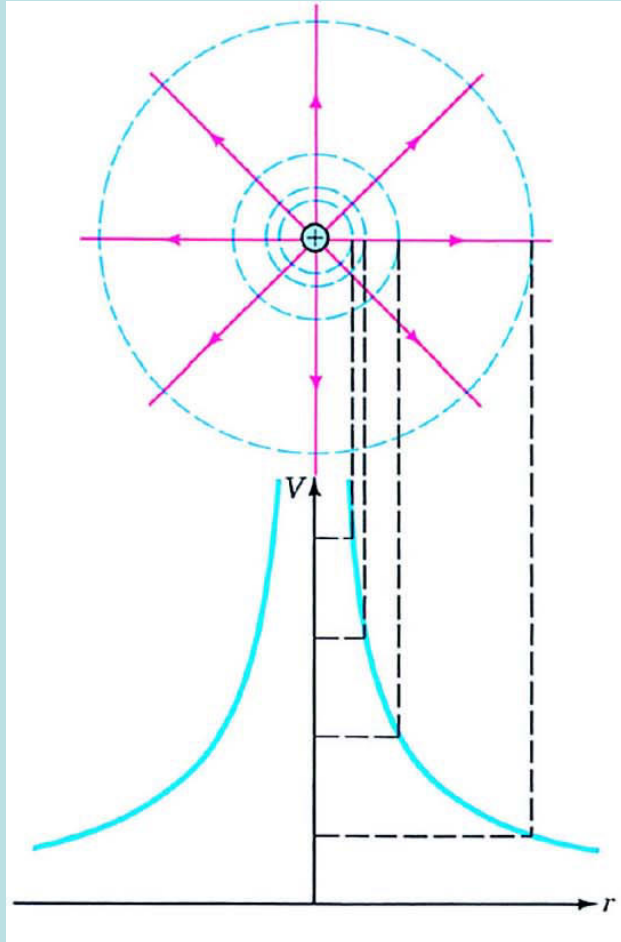
$$\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \equiv \vec{\nabla}\phi(x, y, z)$$

For any scalar field ϕ , $\vec{\nabla}\phi(x, y, z)$ is a vector field!

We simply treat the operation $\vec{\nabla}$ as a vector as the notation indicate.

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

But $\vec{\nabla}$ is an operation after all and it must act on a field.



Electric Potential of a point charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

$$\vec{E} = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$\frac{\partial V}{\partial x} = \frac{dV}{dr} \frac{\partial r}{\partial x}$$

$$\frac{dV}{dr} = -\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{x}{r} = \hat{r}_x$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \hat{r}_y$$

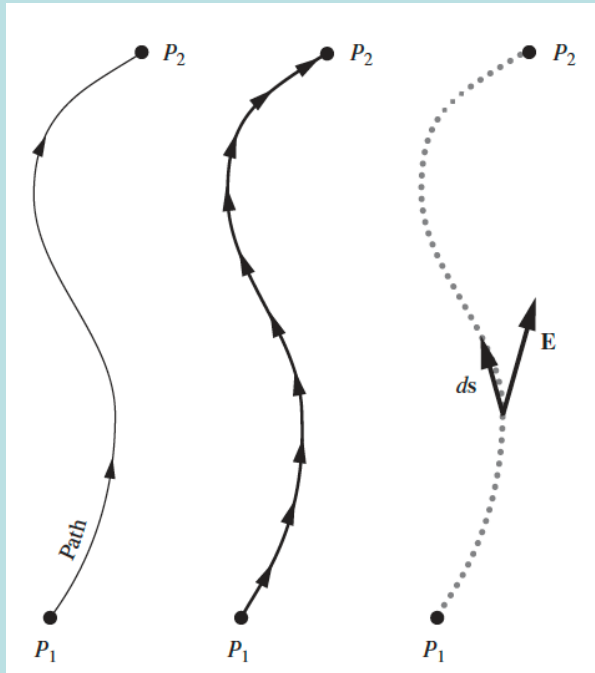
$$\frac{\partial r}{\partial z} = \frac{z}{r} = \hat{r}_z$$

$$\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = -\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} (\hat{r}_x, \hat{r}_y, \hat{r}_z)$$

$$\vec{E} = -\vec{\nabla} V = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

We can add up potential difference $\vec{E} \cdot \Delta\vec{s}$ of infinitesimal intervals.

General electric potential difference equals the line integral of the electric field.



$$\Delta V = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

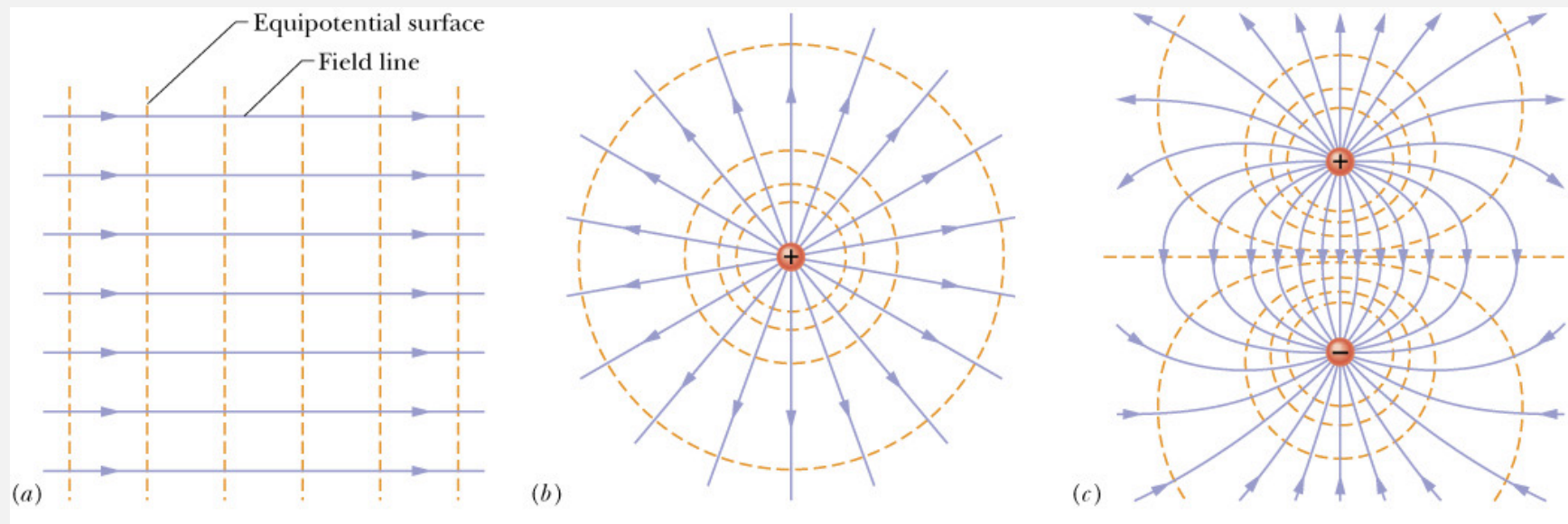
Theorem: If the line integral of a vector field $\vec{A}$ equals difference of a scalar field ϕ ,

$$\Delta\phi = - \int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{s}$$

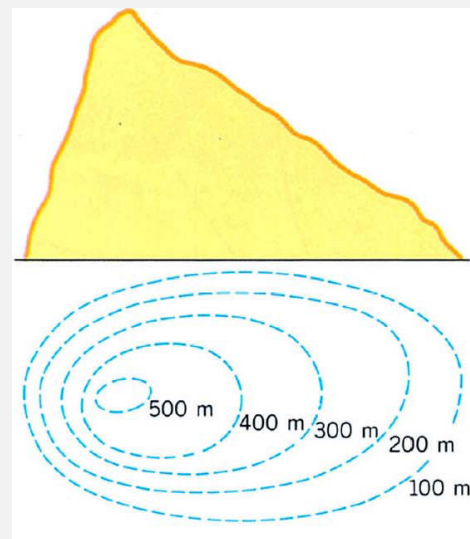
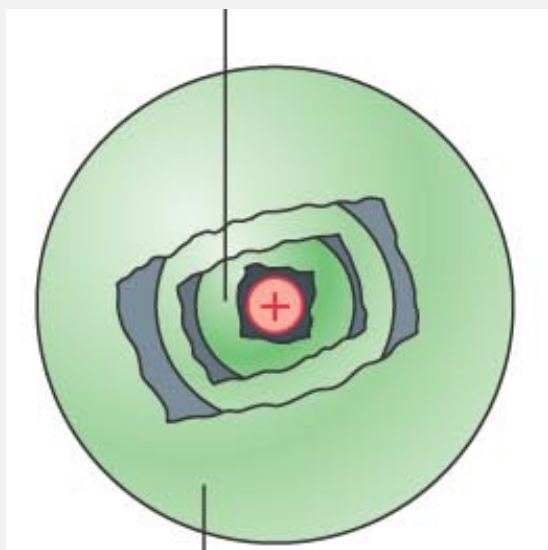
$\vec{A}$ equals the gradient of ϕ :

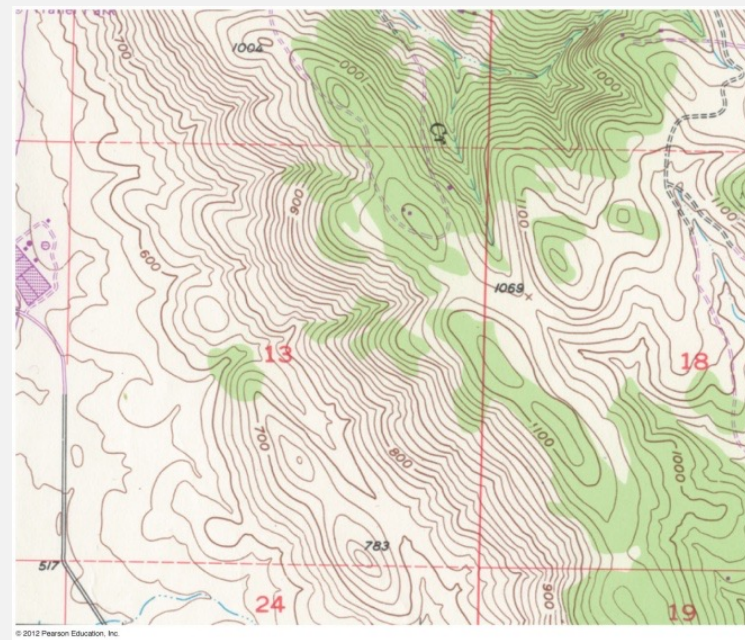
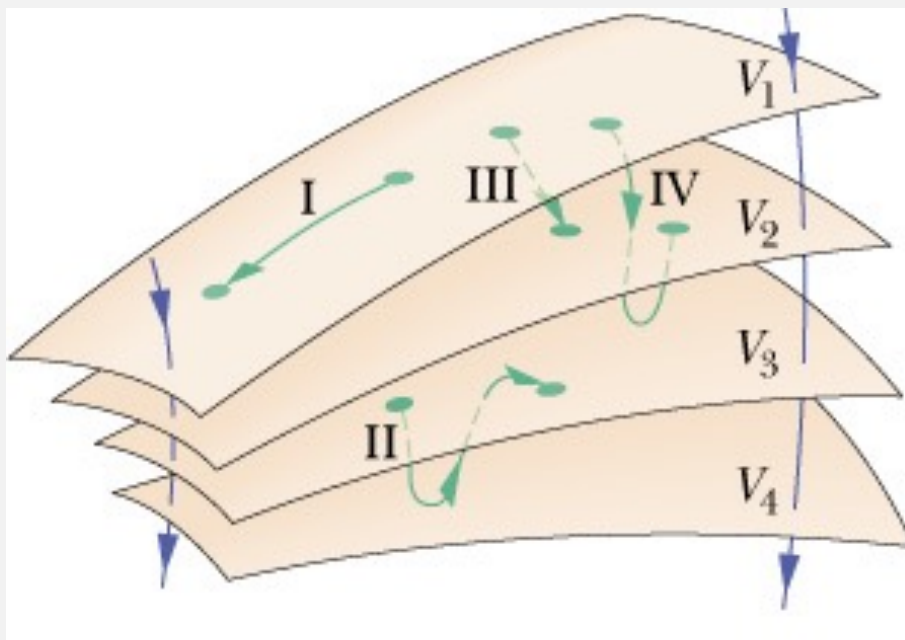
$$\vec{A} = -\vec{\nabla}\phi$$

要使電位 $V(\vec{r})$ 具象化最有用的就是等位面



等電位面非常像地圖上的等高線：



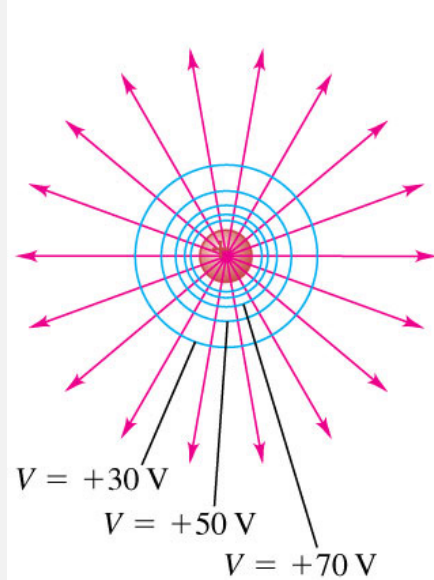


沿等位面方向的電場為零，故電位梯度電場垂直於等位面。

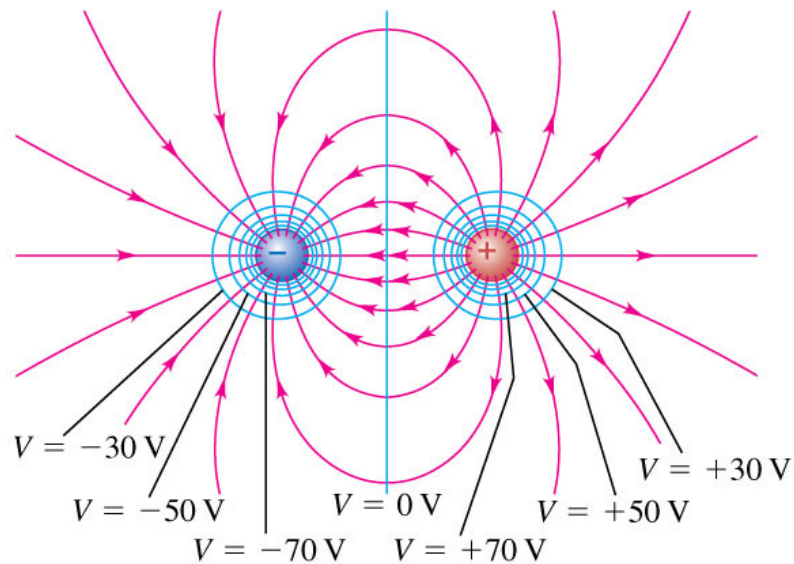
等位面越密，電場越強： $\Delta V = -Ed$

兩點之間的電位差帶動電場向量。

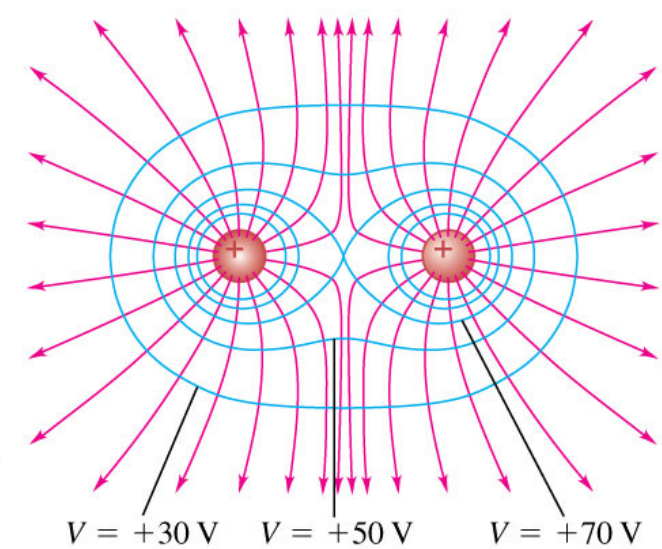
(a) A single positive charge



(b) An electric dipole



(c) Two equal positive charges



— Electric field lines — Cross sections of equipotential surfaces

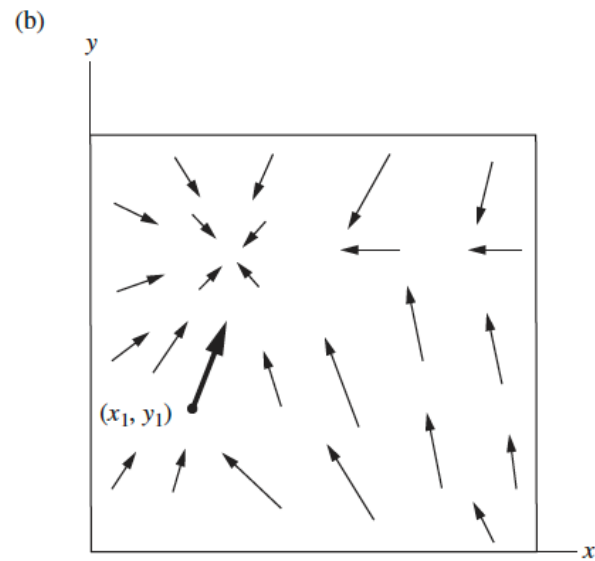
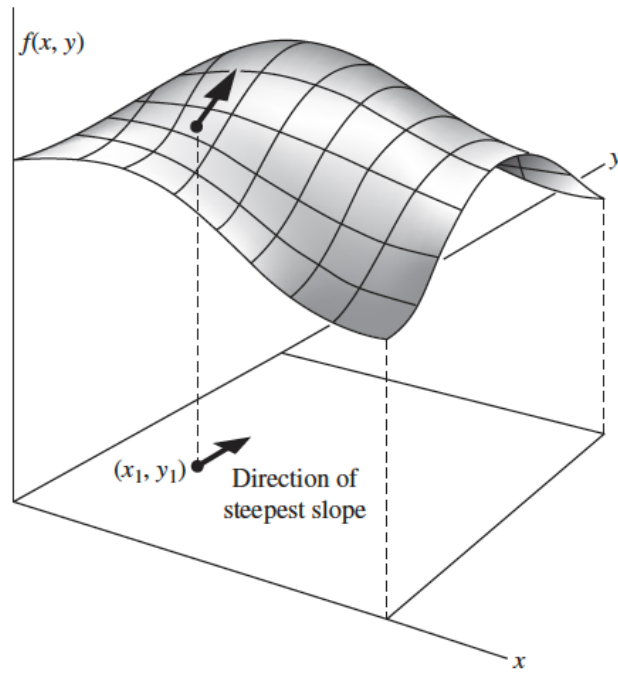


Figure 2.5.
The scalar function $f(x, y)$ is represented by the surface in (a). The arrows in (b) represent the vector function, $\text{grad } f$.



So far, gradient acts on scalar fields to give vector fields.

Partial derivatives in a gradient can act on vector fields too.

We can arrange the indices as an inner product while taking derivatives.

$$\vec{\nabla} \cdot \vec{E}(x, y, z) \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

The results are a scalar fields.

This is called 散度 , Divergence.

Vector Product The vector or cross product of $\mathbf{A}$ and $\mathbf{B}$ is written $\mathbf{A} \times \mathbf{B}$. By definition, $\mathbf{A} \times \mathbf{B}$ is a vector whose magnitude and direction are given as follows:

The magnitude of $\mathbf{A} \times \mathbf{B}$ is

$$(4.14) \quad |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta,$$

where θ is the positive angle ($\leq 180^\circ$) between $\mathbf{A}$ and $\mathbf{B}$. The direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and in the sense $\mathbf{C}$ of advance of a right-handed screw rotated from $\mathbf{A}$ to $\mathbf{B}$ as in Figure 4.12.

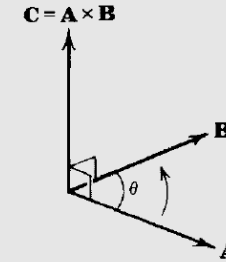


Figure 4.12

It is convenient to find the direction of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ by the following right-hand rule. Think of grasping the line $\mathbf{C}$ (or a screwdriver driving a right-handed screw in the direction $\mathbf{C}$) with the right hand. The fingers then curl in the direction of rotation of $\mathbf{A}$ into $\mathbf{B}$ (arrow in Figure 4.12) and the thumb points along $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

Perhaps the most startling result of the vector product definition is that $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ are not equal; in fact, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. In mathematical language, vector multiplication is not commutative.

We find from (4.14) that the cross product of any two parallel (or antiparallel) vectors has magnitude $|\mathbf{A} \times \mathbf{B}| = AB \sin 0^\circ = 0$ (or $AB \sin 180^\circ = 0$). Thus

$$(4.15) \quad \begin{aligned} \mathbf{A} \times \mathbf{B} &= 0 \quad \text{if } \mathbf{A} \text{ and } \mathbf{B} \text{ are parallel or antiparallel,} \\ \mathbf{A} \times \mathbf{A} &= 0 \quad \text{for any } \mathbf{A}. \end{aligned}$$

Then we have the useful results

$$(4.16) \quad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$

Also from (4.14) we find

$$|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}| |\mathbf{j}| \sin 90^\circ = 1 \cdot 1 \cdot 1 = 1,$$

and similarly for the magnitude of the cross product of any two different unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. From the right-hand rule and Figure 4.13, we see that the direction of $\mathbf{i} \times \mathbf{j}$ is $\mathbf{k}$, and since its magnitude is 1, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; however, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. Similarly evaluating the other cross products, we find

$$(4.17) \quad \begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j}. \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{array}$$

A good way to remember these is to write them cyclically (around a circle as indicated in Figure 4.14). Reading around the circle counterclockwise (positive θ direction), we get the positive products (for example, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$); reading the other way we get the negative products (for example, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$).

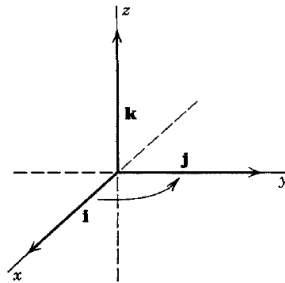


Figure 4.13

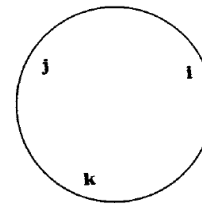


Figure 4.14

It is well to note here that the results (4.17) depend upon the way we have labeled the axes in Figure 4.13. We have arranged the (x, y, z) axes so that a rotation of the x into the y axis (through 90°) corresponds to the rotation of a right-handed screw advancing in the positive z direction. Such a coordinate system is called a *right-handed system*. If we used a left-handed system (say exchanging x and y), then all the equations in (4.17) would have their signs changed. This would be confusing; consequently, we practically always use right-handed coordinate systems, and we must be careful about this in drawing diagrams. (See Chapter 10, Section 6.)

To write $\mathbf{A} \times \mathbf{B}$ in component form we need the distributive law, namely

$$(4.18) \quad \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}.$$

(see Problem 7.18).

Then we find

$$(4.19) \quad \begin{aligned} \mathbf{A} \times \mathbf{B} &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\ &= \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \end{aligned}$$

The second line in (4.19) is obtained by multiplying out the first line (getting nine products) and using (4.16) and (4.17). The determinant in (4.19) is the most convenient way to remember the component form of the vector product. You should verify that multiplying out the determinant using the elements of the first row gives the result in the line above it.

Or we can arrange the indices as a cross product. The result will be a vector field.

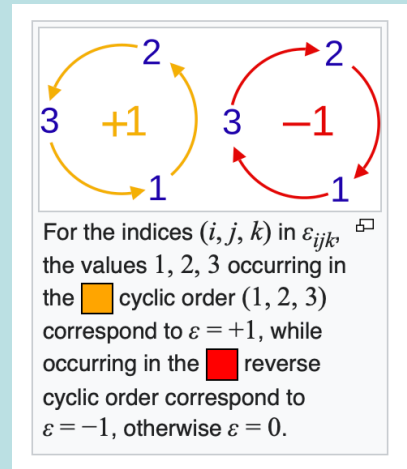
Levi-Civita Tensor



$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123, 231, 312 \\ -1 & ijk = 132, 213, 321 \\ 0 & \text{some subscripts are equal} \end{cases}$$

$$(\vec{A} \times \vec{B})_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k$$

$$(\vec{A} \times \vec{B})_x = A_y B_z - A_z B_y$$



$$\vec{\nabla} \times \vec{E}(x, y, z) \equiv \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

This is called 旋度, Curl.



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Levi-Civita symbol

 28 languages 

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Not to be confused with [Levi-Civita connection](#).

See [Ricci calculus](#), [Einstein notation](#), and [Raising and lowering indices](#) for the [index notation](#) used in the article.

In [mathematics](#), particularly in [linear algebra](#), [tensor analysis](#), and [differential geometry](#), the **Levi-Civita symbol** or **Levi-Civita epsilon** represents a collection of numbers defined from the [sign of a permutation](#) of the [natural numbers](#) 1, 2, ..., n , for some positive integer n . It is named after the Italian mathematician and physicist [Tullio Levi-Civita](#). Other names include the **permutation symbol**, **antisymmetric symbol**, or **alternating symbol**, which refer to its [antisymmetric](#) property and definition in terms of permutations.

Three dimensions [\[edit \]](#)

Index and symbol values [\[edit \]](#)

In three dimensions, when all i, j, k, m, n, p, q each take values 1, 2, and 3:^[3]

$$\varepsilon_{ijk}\varepsilon^{pqk} = \delta_i^p \delta_j^q - \delta_i^q \delta_j^p \quad (4)$$

$$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta_j^i \quad (5)$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6. \quad (6)$$

Product [\[edit \]](#)

The Levi-Civita symbol is related to the [Kronecker delta](#). In three dimensions, the relationship is given by the following equations (vertical lines denote the determinant):^[4]

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) - \delta_{im} (\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl}) + \delta_{in} (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}). \end{aligned}$$

A special case of this result occurs when one of the indices is repeated and summed over:

$$\sum_{i=1}^3 \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

In Einstein notation, the duplication of the i index implies the sum on i . The previous is then denoted $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$.

If two indices are repeated (and summed over), this further reduces to:

$$\sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}$$

Integration Theorems

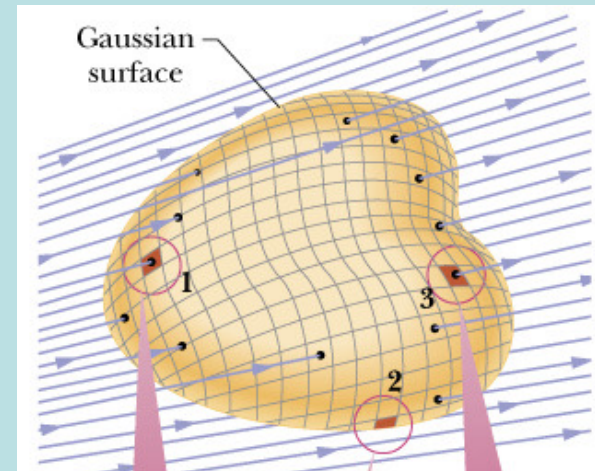


Differential operations are related to integration of vector field in space.

Integration of vector field in space is over curved surface or a path (line).

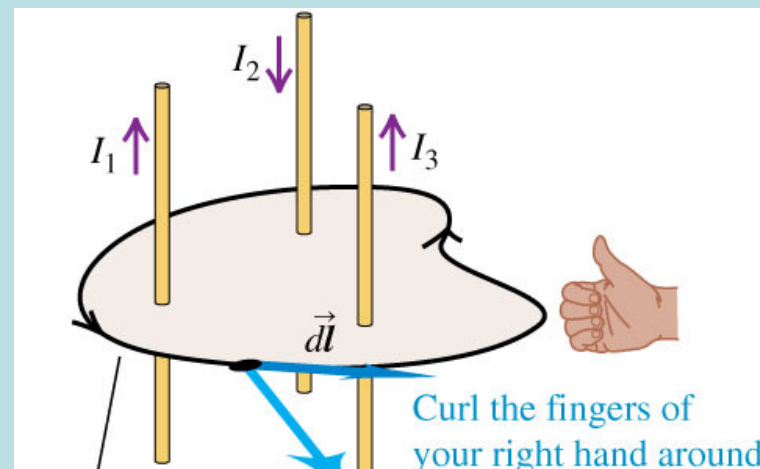
面積分 Surface integral

is related to 散度, **Divergence**.



線積分 Line integral

is related to 旋度, **Curl**.





積分形式

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

微分形式

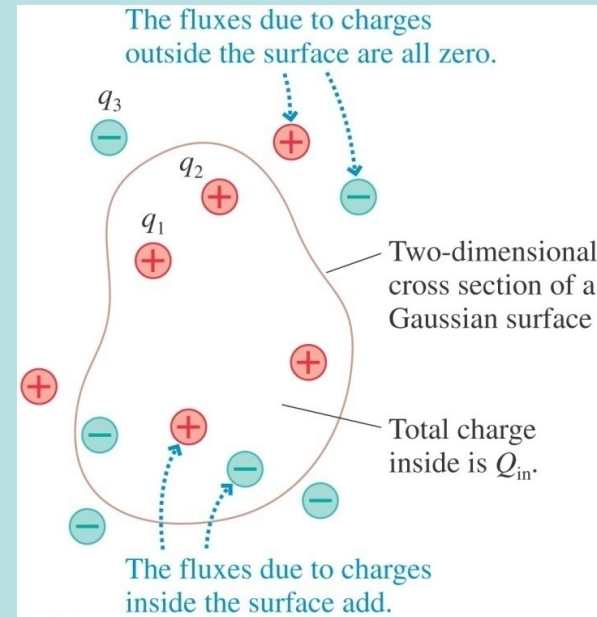
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Convert Maxwell Equations from integral formulation to differential formulation



$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \quad \text{高斯定律 Gauss's Law}$$

一個封閉曲面的電場面積分：通量，正比於曲面所包圍的總淨電荷量。

But Gauss's Law in this integration formulation is hard to use.

We will now calculate flux **on surface** using divergence **inside surface**.

The law will then be converted it into a differential formulation.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

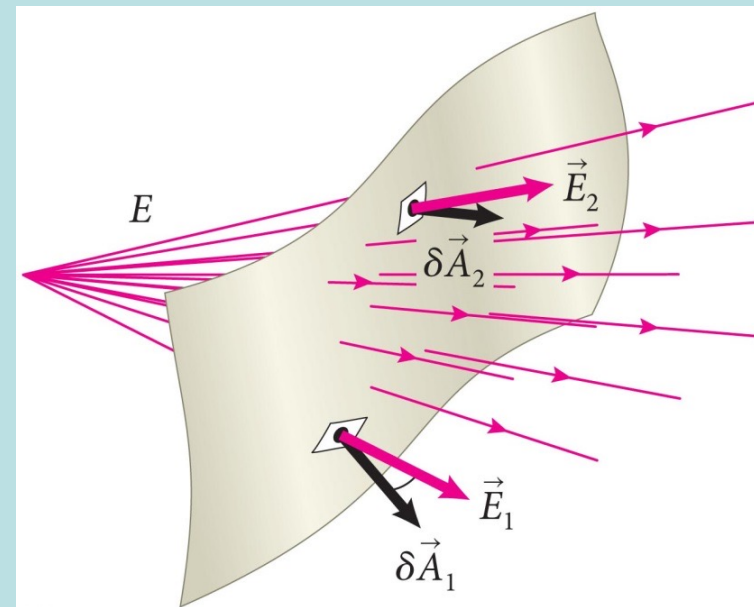
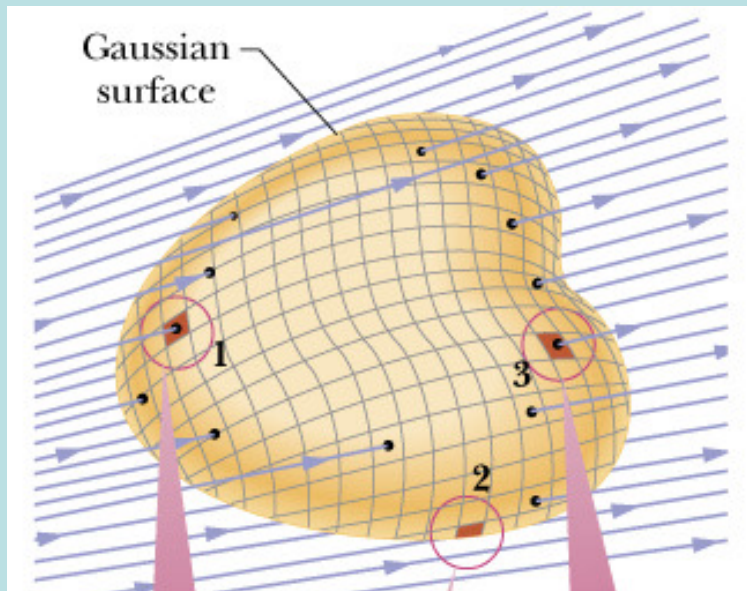
Electric Flux is defined as the inner product of $\vec{E}$ and $\vec{A}$:

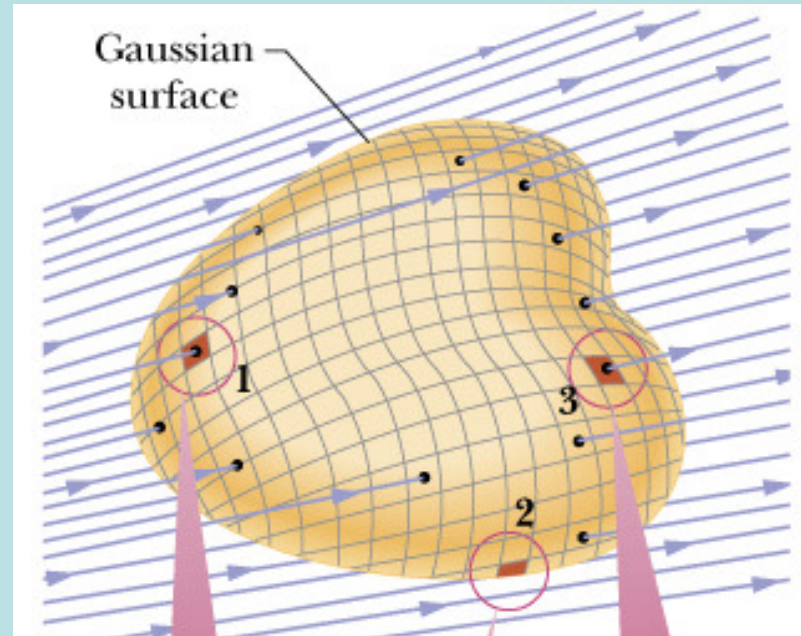


$$\Phi_E = \sum_i \vec{E}_i \cdot \delta \vec{A}_i$$

The sum becomes integration as $\delta A \rightarrow 0$, defined as a surface integral.

$$\Phi_E = \sum_i \vec{E}_i \cdot \delta \vec{A}_i \xrightarrow[N \rightarrow \infty]{\delta A \rightarrow 0} \int_{\text{surface}} \vec{E} \cdot d\vec{A}$$





The total flux through a closed surface has a simple meaning:

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \text{number of electric lines leaving} - \text{number of entering}$$

About flux, first a beautiful mathematical theorem. The key observation:

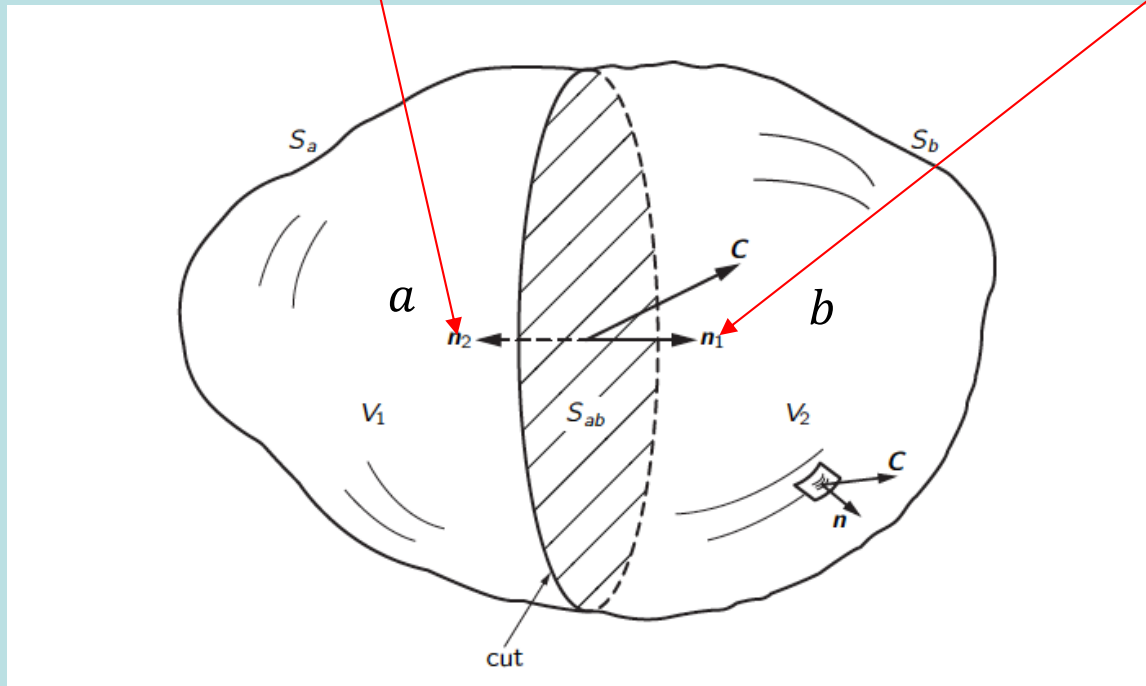
When a volume is divided into two parts: $a + b$

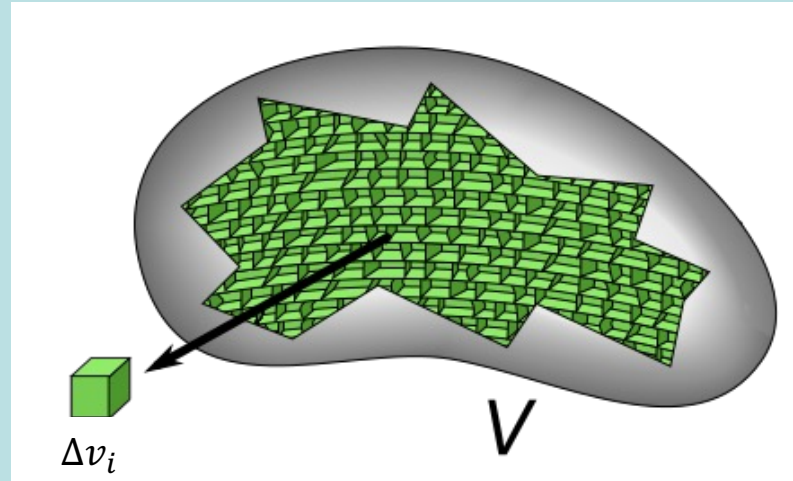
the electric flux through $a + b$ equals the sum of flux through the two parts a, b .

$$\Phi_{a+b} = \Phi_a + \Phi_b$$

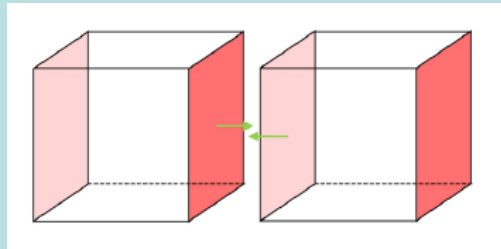
This is because when adding up, the contributions from interior faces S_{ab} cancel out.

The outward normal direction of S_{ab} from part b is opposite the direction from a .





If we divide the inner space V of a Gaussian Surface into small cubes dV ,
 The sum of all the flux through the cubes equals the flux through V ,
 since the contributions from interior faces cancel out.

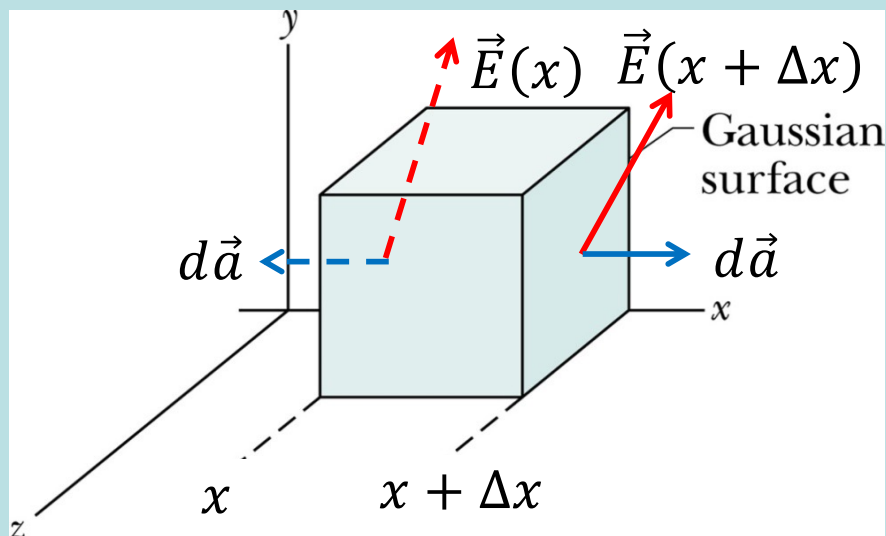


$$\Phi_E = \sum_i \Delta\Phi_{iE}$$

Let's calculate the flux of a vector field through a small cube:



Flux through a small cube
with sides along x, y, z axes



The flux on the two surfaces (right, left) perpendicular to x axis:

$$\vec{E} \cdot d\vec{a} \sim E_x \cdot da$$

$$E_x(x + \Delta x, y, z) \cdot \Delta y \Delta z - E_x(x, y, z) \cdot \Delta y \Delta z$$

The change in E_x , as only x change by Δx , equals partial derivative times Δx : $\frac{\partial E_x}{\partial x} \Delta x$.

$$= \left(\frac{\partial E_x}{\partial x} \right) \Delta x \cdot \Delta y \Delta z$$

Flux on two surfaces (top, bottom) perpendicular to the y axis:

$$\left(\frac{\partial E_y}{\partial y} \right) \Delta y \cdot \Delta x \Delta z$$

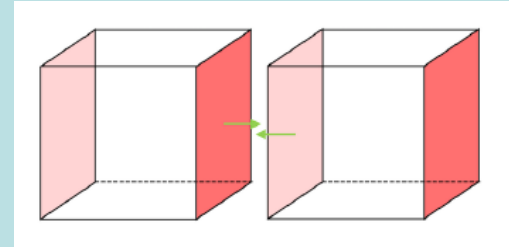
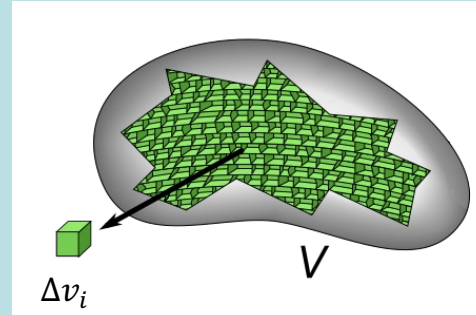
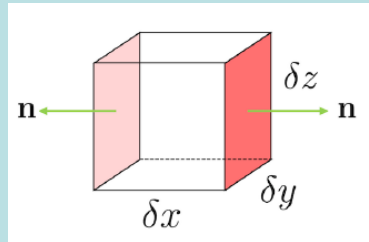
Flux on two surfaces (front, back) perpendicular to the z axis:

$$\left(\frac{\partial E_z}{\partial z} \right) \Delta z \cdot \Delta x \Delta y$$

Total flux:

$$\Phi_E = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \cdot \Delta x \Delta y \Delta z = (\vec{\nabla} \cdot \vec{E}) \cdot \Delta v$$

Shape does not matter!



Consider a small volume V and cut it into small cubes as $\Delta v_i \rightarrow 0$:

The flux through V equals the sum of the flux through all the cubes, since the contributions from interior faces cancel out.

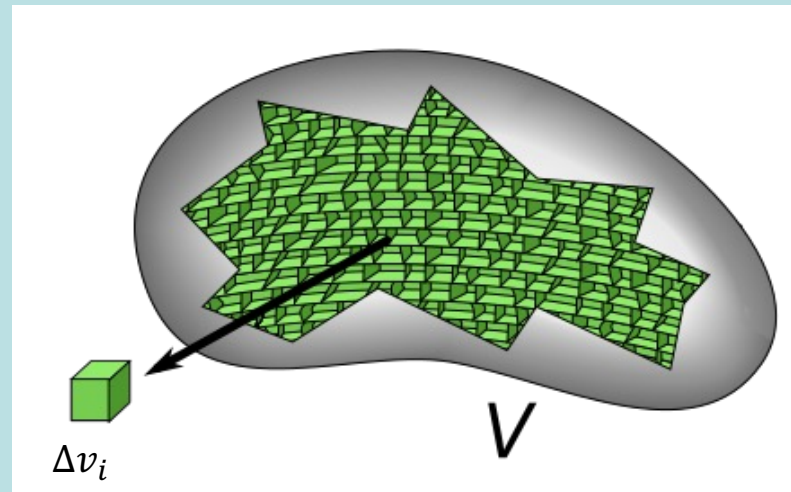
$$\Phi_E = \sum_i \Delta\Phi_{iE} = \sum_i (\vec{\nabla} \cdot \vec{E}_i) \cdot \Delta v_i$$

In case V is very small, $\vec{\nabla} \cdot \vec{E}$ does not change much,

$$\sim (\vec{\nabla} \cdot \vec{E}) \sum_i \Delta v_i = V (\vec{\nabla} \cdot \vec{E})$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\Phi_E}{V}$$

The meaning of divergence is flux of vector field per unit volume for small volume!



In case V is not small and $\vec{\nabla} \cdot \vec{E}$ does change,

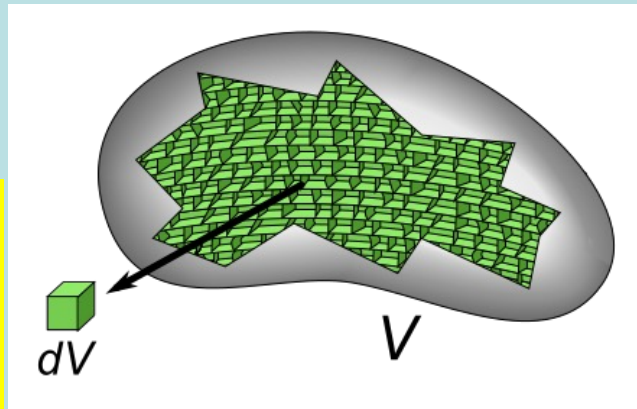
The sum of flux through cubes approaches volume integral as $\Delta v_i \rightarrow 0$.

$$\Phi_E = \sum_i \Delta\Phi_{iE} = \sum_i (\vec{\nabla} \cdot \vec{E}_i) \cdot \Delta v_i \rightarrow \int (\vec{\nabla} \cdot \vec{E}) dv$$

Flux through a surface equals volume integral of divergence inside the surface.

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E}) dv$$

$$\oint_S \vec{A} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{A}) dv$$



It is true for any vector field.

Divergence Theorem

Integration in a volume is related to quantities on the surface.

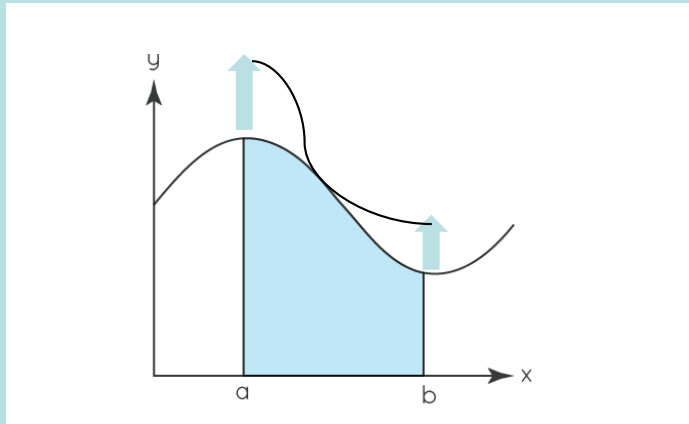
This is intrinsically generalization of fundamental theorem of calculus.

$$\int_a^b dx \frac{df}{dx} = f(b) - f(a)$$

The fundamental theorem of calculus is a remarkably surprising theorem.

$$\int_a^b dx \frac{df}{dx} = f(b) - f(a)$$

Right-hand side with function values on the boundary is fixed by left hand side computed from functions inside the boundaries.



Why can we simply raise $f(x)$ a bit just at the boundaries $x = a, b$?

The right-hand side will change but the left-hand side will not.

The key to fundamental theorem and Gauss law is continuity of functions.

If we raise continuous $f(x)$ at boundaries, it need be raised too nearby.

The left-hand side will be changed too.

Now Physics:

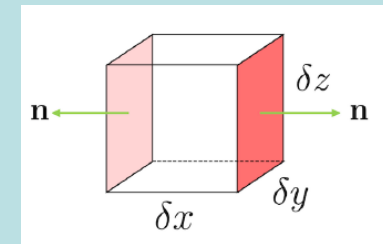


Convert Gauss's Law from integral formulation to differential formulation

Apply the integral formulation of Gauss's Law to this small cube:

Electric flux equals the enclosed charge.

$$\Phi_E = (\vec{\nabla} \cdot \vec{E}) \cdot \Delta v = \frac{q}{\epsilon_0} = \frac{\rho \cdot \Delta v}{\epsilon_0}$$



ρ is the charge density inside the cube, approximately a constant for small cube.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This is the **differential formulation of Gauss's Law**.

Note that it is an equation true for a point $\vec{r}$ (and for any point) in space.

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

Electric line accumulation of unit volume is proportional to charge density.

In reverse, take the differential formulation of Gauss's law.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Plug it into the divergence Theorem:

$$\oint \vec{E} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{E}) dv$$

We get the integral formulation of Gauss's law.

$$\oint \vec{E} \cdot d\vec{a} = \int \left(\frac{\rho}{\epsilon_0} \right) dv = \frac{q}{\epsilon_0}$$

Integral formulation of Gauss's law is equivalent to differential formulation.

Magnetic Gauss's law can also be converted into differential formulation.

$$\oint \vec{B} \cdot d\vec{a} = 0$$

Divergence Theorem

$$\oint \vec{B} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{B}) dv$$

Differential formulation: Divergence of the magnetic field is always zero.

$$\vec{\nabla} \cdot \vec{B} = 0$$



Maxwell Equations

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

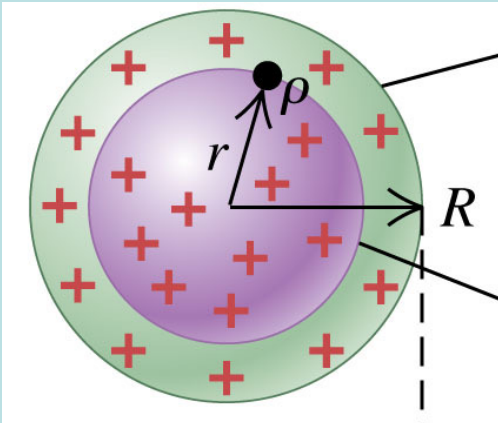


$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

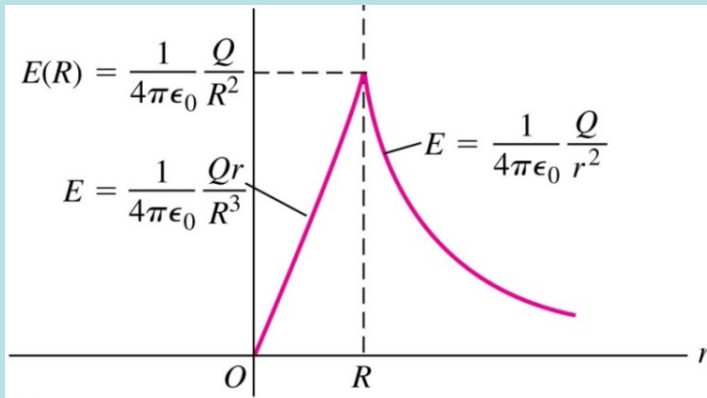


Within a uniform charge distribution : $r < R$

Charge density: ρ

$$E \cdot 4\pi r^2 = \frac{\rho \frac{4}{3} \pi r^3}{\epsilon_0}$$

$$E = \frac{\rho}{3\epsilon_0} r$$



Electric flux inside the ball:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{3\epsilon_0} \vec{\nabla} \cdot \vec{r} = \frac{\rho}{3\epsilon_0} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) = \frac{\rho}{\epsilon_0}$$

Electric Field of a point charge

3. The electric field $\vec{E}$ of a point charge Q fixed at the origin can be written as:

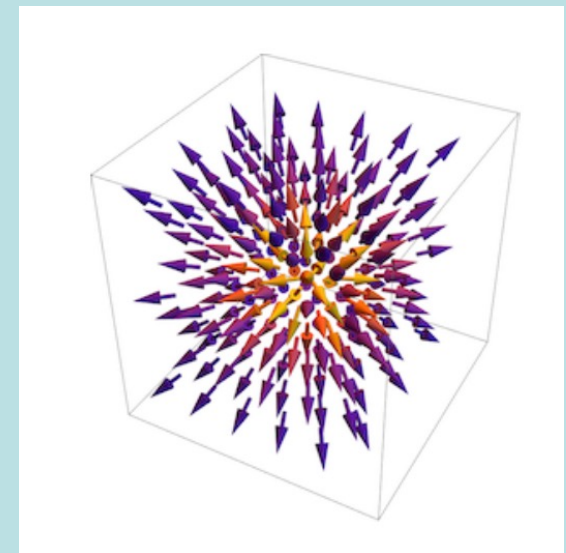
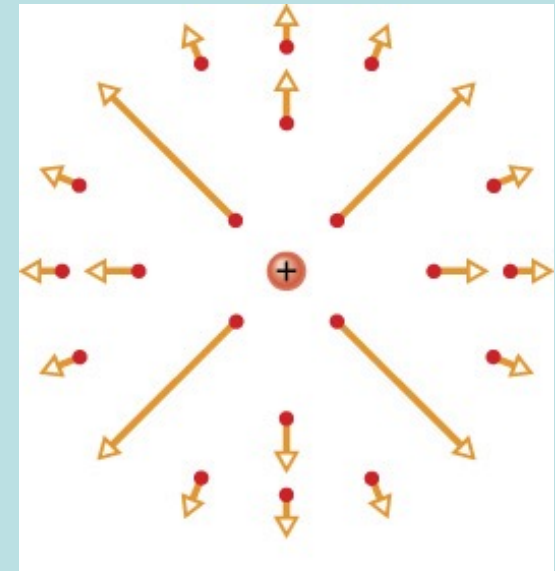
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

A. Calculate $\vec{\nabla} \times \vec{E}$.

B. Calculate $\vec{\nabla} \cdot \vec{E}$ and show it is equal to zero except at the origin $r = 0$.

$$\vec{\nabla} \cdot \vec{E} = \frac{Q}{4\pi\epsilon_0} \vec{\nabla} \cdot \left(\frac{1}{r^2} \hat{r} \right) = 0$$

except for $r = 0$.



更嚴格的證明：

Vector Calculus

University of Cambridge Part IA Mathematical Tripos

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3 Grad, Div and Curl

3.1 The Gradient

3.2 Div and Curl

3.2.1 Some Basic Properties

3.2.2 Conservative is Irrotational



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3 Grad, Div and Curl

In this section we're going to further develop the ways in which we can differentiate. We'll be particularly interested in how we can differentiate scalar and vector fields. Our definitions will be straightforward but, at least for the time being, we won't be able to offer the full intuition behind these ideas. Perhaps ironically, the full meaning of how to differentiate will become clear only in Section 4 where we also learn the corresponding different ways to integrate.

The 2d Divergence Theorem: Let $\mathbf{F}$ be a vector field in $\mathbb{R}^2$. Then

$$\int_D \nabla \cdot \mathbf{F} dA = \int_C \mathbf{F} \cdot \mathbf{n} ds \quad (4.6)$$

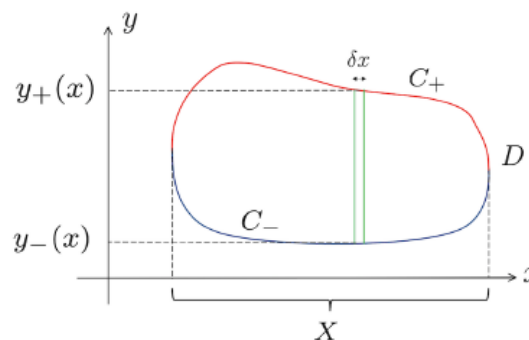
where D is a region in $\mathbb{R}^2$, bounded by the closed curve C and $\mathbf{n}$ is the outward normal to C .

Proof of the 2d Divergence Theorem: For simplicity, we'll assume that $\mathbf{F} = F(x, y) \hat{\mathbf{y}}$. The proof that we're about to give also works if $\mathbf{F}$ points solely in the $\hat{\mathbf{x}}$ direction, but a general $\mathbf{F}$ is just a linear sum of the two.

We then have

$$\int_D \nabla \cdot \mathbf{F} dA = \int_X dx \int_{y_-(x)}^{y_+(x)} dy \frac{\partial F}{\partial y}$$

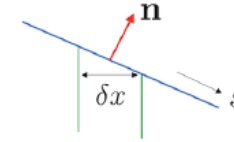
where, as the notation shows, we've chosen to do the area integral by first integrating over y , and then over x . We'll assume, for now, that the region D is convex, as shown in the figure, so that each $\int dy$ is over just a single interval with limits $y_{\pm}(x)$. These limits trace out an upper curve C_+ , shown



in red in the figure, and a lower curve C_- shown in blue. We then have

$$\int_D \nabla \cdot \mathbf{F} dA = \int_X dx \left(F(x, y_+(x)) - F(x, y_-(x)) \right)$$

We've succeeded in converting the area integral into an ordinary integral, but it's not quite of the line integral form that we need. The next part of the proof is to massage the integral over $\int dx$ into a line integral over $\int ds$. This is easily achieved if we look at the zoomed-in figure to the right. Along the upper curve C_+ , the normal $\mathbf{n}$ points upwards and makes an angle $\cos \theta = \hat{\mathbf{y}} \cdot \mathbf{n}$ with the vertical. Moving a small distance δs along the curve is equivalent to moving



$$\delta x = \cos \theta \delta s = \hat{\mathbf{y}} \cdot \mathbf{n} \delta s \quad \text{along } C_+$$

Along the lower curve, C_- , the normal $\mathbf{n}$ points downwards and so $\hat{\mathbf{y}} \cdot \mathbf{n}$ is negative. We then have

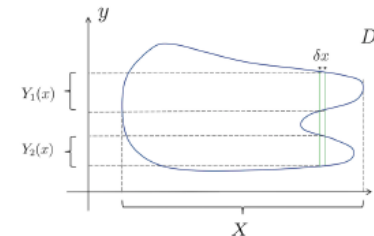
$$\delta x = -\hat{\mathbf{y}} \cdot \mathbf{n} \delta s \quad \text{along } C_-$$

The upshot is that we can write the area integral as

$$\begin{aligned} \int_D \nabla \cdot \mathbf{F} dA &= \int_X ds \left(\mathbf{n} \cdot \mathbf{F}(x, y_+(x)) + \mathbf{n} \cdot \mathbf{F}(x, y_-(x)) \right) \\ &= \int_{C_+} \mathbf{F} \cdot \mathbf{n} ds + \int_{C_-} \mathbf{F} \cdot \mathbf{n} ds \\ &= \int_C \mathbf{F} \cdot \mathbf{n} ds \end{aligned}$$

with $C = C_+ + C_- = \partial D$ the boundary of the region.

We're left with one small loophole to close: if the region D is not convex, then the range of the integral $\int dy$ may be over two or more disconnected intervals, as shown in the figure. In this case, the boundary curve decomposes into more pieces, but the basic strategy still holds. $\square$



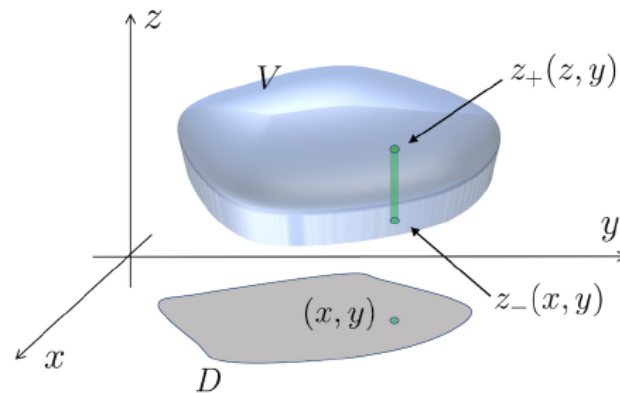


Figure 13. Performing the $\int dz$ integral for the proof of the 3d divergence theorem.

Proof of the 3d Divergence Theorem

The proof of the 3d (or, indeed, higher dimensional) divergence theorem follows using the same strategy. If we focus on $\mathbf{F} = F(x, y, z) \hat{\mathbf{z}}$ we have

$$\begin{aligned} \int_V \nabla \cdot \mathbf{F} dV &= \int_D dA \int_{z_-(x,y)}^{z_+(x,y)} dz \frac{\partial F}{\partial z} \\ &= \int_D dA \left(F(x, y, z_+(x, y)) - F(x, y, z_-(x, y)) \right) \end{aligned}$$

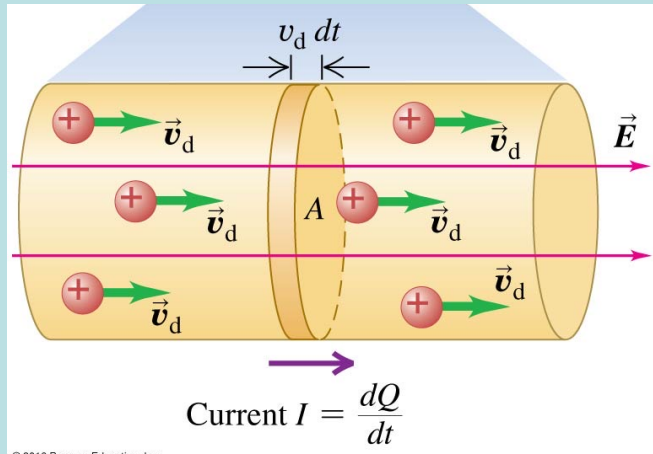
where the limits of the integral $z_{\pm}(x, y)$ are the upper and lower surfaces of the volume V . The area integral over D is an integral in the (x, y) plane, while to prove Gauss' theorem we need to convert this into a surface integral over $S = \partial V$. This step of the argument is the same as before: at any given point, the different between $dA = dx dy$ and dS is the angle $\cos \theta = \mathbf{n} \cdot \hat{\mathbf{z}}$ (up to a sign). This then gives the promised result (4.1). $\square$

電荷守恆 Conservation of Charge



Current is the total charge per unit time passing an area A :

$$i = \frac{q}{\Delta t}$$



Usually, i is proportional to A .

Define current density: current per unit area.

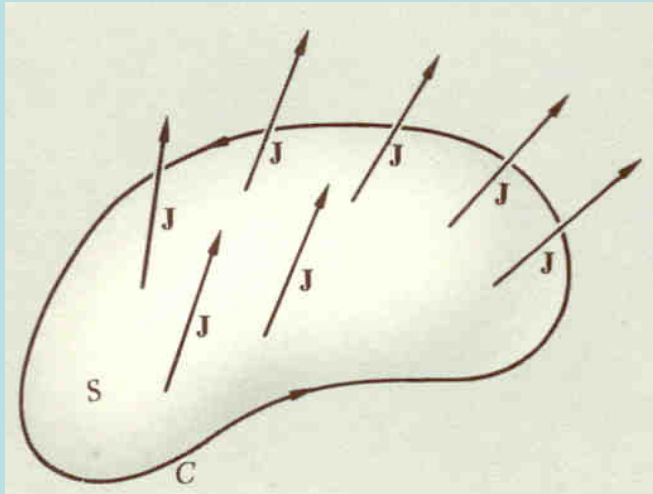
$$j = \frac{i}{A}$$

Identify the direction of $\vec{j}$ with charge motion.

$$j \rightarrow \vec{j}$$

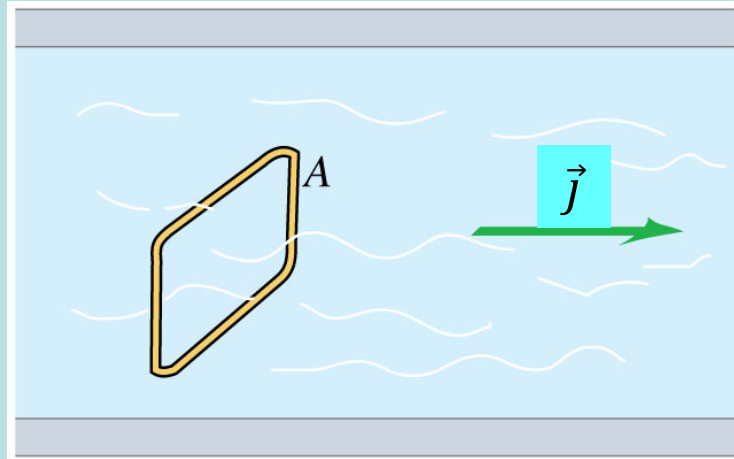
In a conductor $\vec{j}$ can be different at various location.

$$\vec{j} \rightarrow \vec{j}(\vec{r}) \quad \vec{j}(\vec{r}) \text{ is also a vector field as } E(\vec{r}) \circ$$





We can calculate current i through a surface by multiplying j and A .



If the surface is perpendicular to j :

$$j = \frac{i}{A} \quad i = jA$$

If the surface is not perpendicular to j :

We can always project it to $A_{\perp}$ that is $\perp \vec{j}$.

The current through A will pass $A_{\perp}$, too. ,

$$i = jA_{\perp} = j \cdot A \cos \theta = \vec{j} \cdot \vec{A}$$

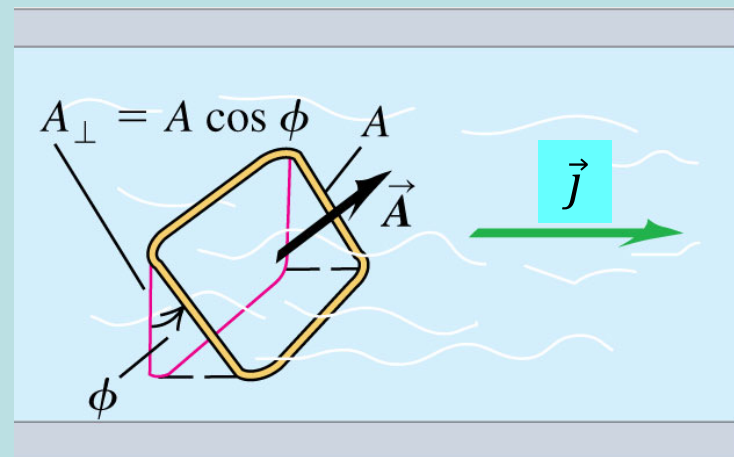
Current equals inner product of $\vec{j}$ and $\vec{A}$!

$$i = \vec{j} \cdot \vec{A}$$



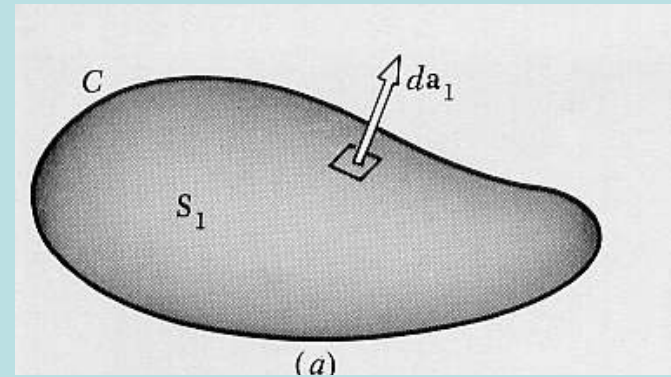
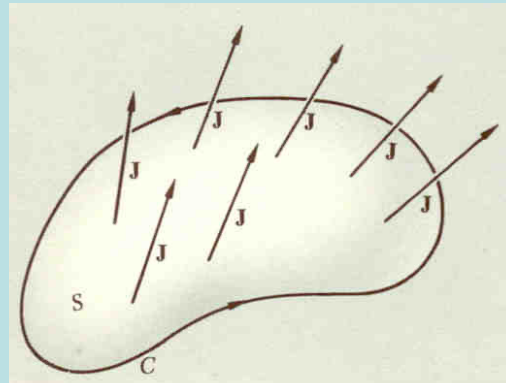
$$\Phi_E = \vec{E} \cdot \vec{A}$$

Current is the flux of $\vec{j}$!



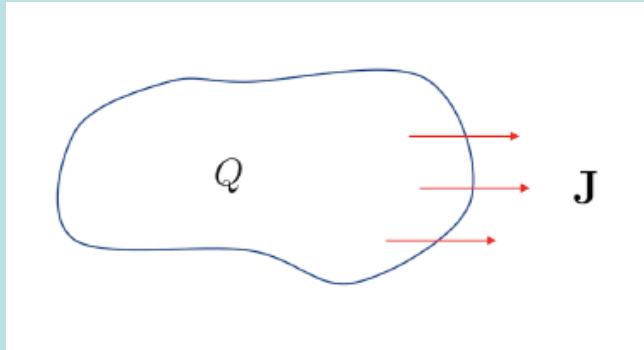


We can easily extend the discussion to a curved surface.



Current is the total flux of $\vec{j}$, ie. surface integral of $\vec{j}$!

$$i = \int \vec{j} \cdot d\vec{a}$$



Consider a very small volume ΔV

Current through its surface is the total flux of $\vec{J}$, which equals $\vec{\nabla} \cdot \vec{J}$ times volume:

$$(\vec{\nabla} \cdot \vec{J}) \cdot \Delta V$$

By charge conservation, current equals the time derivative of charge inside ΔV :

$$-\frac{dQ}{dt} = -\frac{\partial \rho}{\partial t} \cdot \Delta V$$

Cancel the factor of ΔV on both sides:

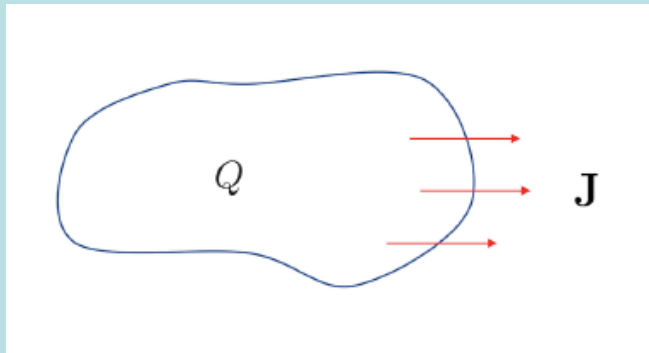
$$\vec{\nabla} \cdot \vec{J}(\vec{r}, t) = -\frac{\partial \rho(\vec{r}, t)}{\partial t}$$

Charge conservation law in differentiation form

If the volume is finite

$$i = \oint_S \vec{j} \cdot d\vec{a}$$

Current is the total flux of $\vec{j}$, surface integral of $\vec{j}$.



By charge conservation, current equals the time derivative of charge inside:

$$\oint_S \vec{j} \cdot d\vec{a} = -\frac{dQ}{dt}$$

Charge conservation law in integral form



Maxwell Equations

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$



$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

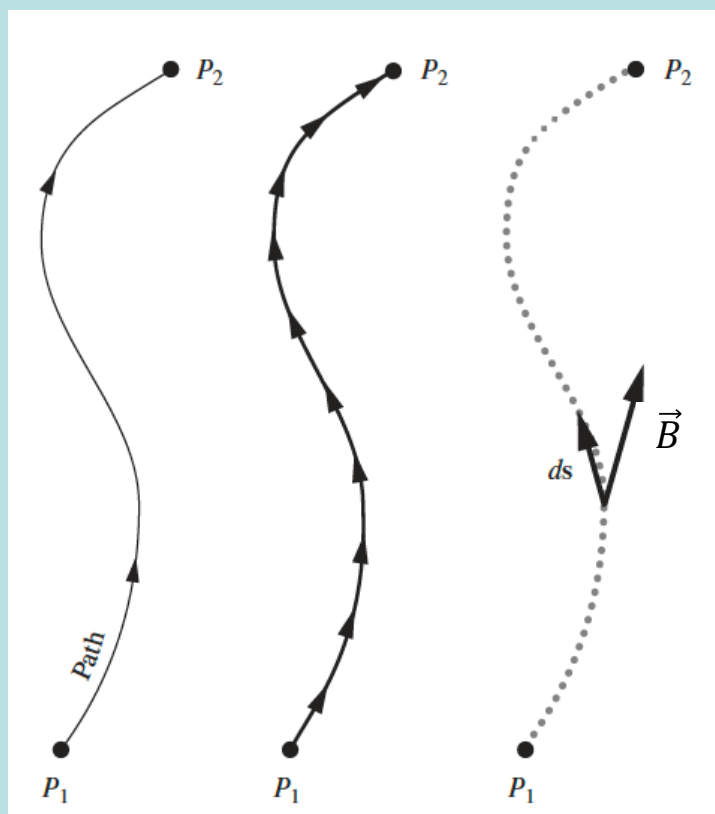
$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$

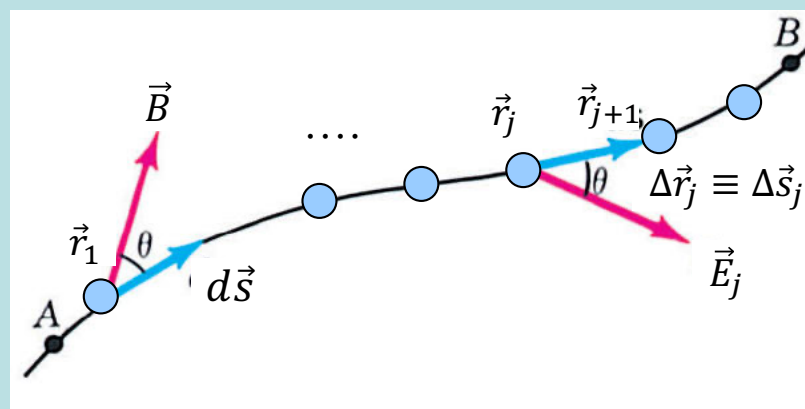
$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

Could we write the line integral in a differential formulation?

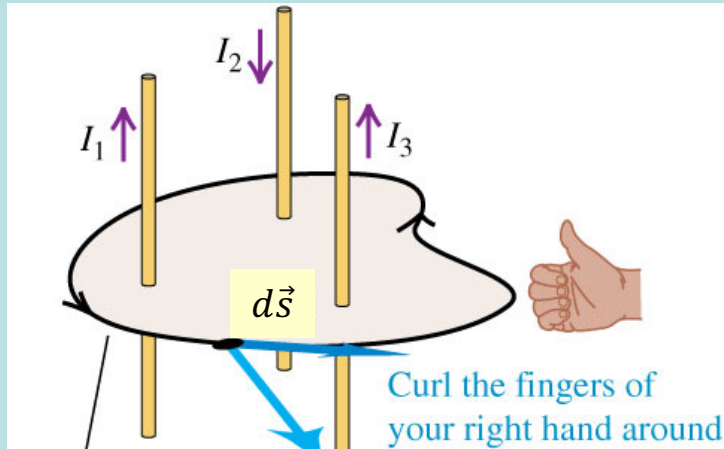


磁場 $\vec{B}$ 的線積分！



將一條路徑 $A \rightarrow B$ 分解為無限多段無限小的路徑 $\Delta \vec{s}$ 組合起來：
將每一無限小路徑 $\Delta \vec{s}$ 的 $\vec{B} \cdot d\vec{s}$ 加總，稱為 $\vec{B}$ 的線積分。

$$\int_{P_1}^{P_2} \vec{B} \cdot d\vec{s} = \lim_{\substack{N \rightarrow \infty \\ \Delta s \rightarrow 0}} \sum_{j=0}^N (\vec{B}_j \cdot \Delta \vec{s}_j)$$



對任一封閉曲線，磁場的線積分正比於該曲線所包圍的總電流（電流疊加）

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$

安培定律 **Ampere's Law**

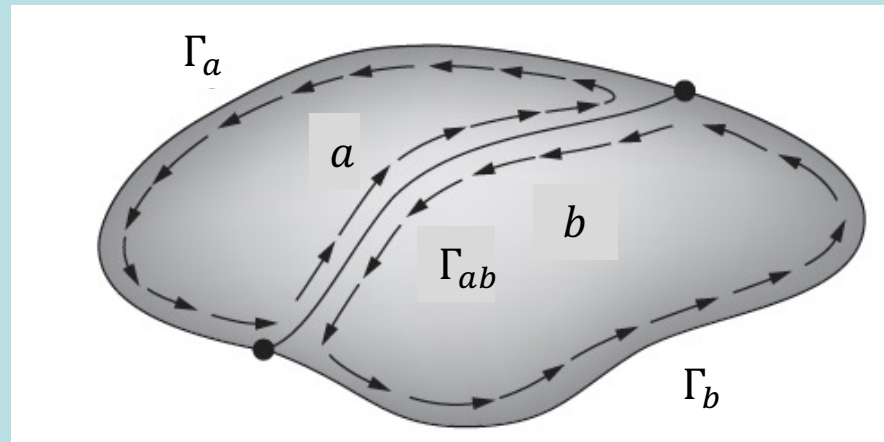
Curl of a vector field allows us calculate its line integral.

If a curved surface in space is divided into two regions $a + b$,
the boundary path Γ is decomposed into two sub-paths: $\Gamma = \Gamma_a + \Gamma_b$

Add the line integrals along Γ_a and Γ_b and contributions on interface Γ_{ab} cancel.
That is because on Γ_{ab} , path Γ_a and path Γ_b move in opposite directions.

Line integral along Γ equals the sum of line integrals along Γ_a and Γ_b .

$$\oint_{\Gamma} \vec{B} \cdot d\vec{s} = \oint_{\Gamma_a} \vec{B} \cdot d\vec{s} + \oint_{\Gamma_b} \vec{B} \cdot d\vec{s}$$



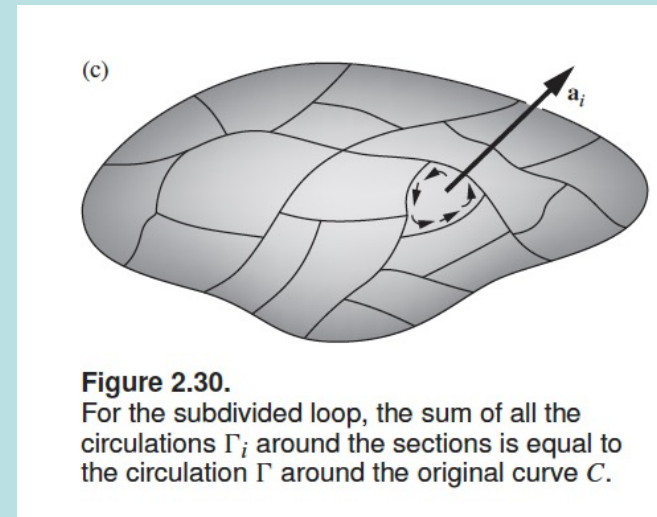
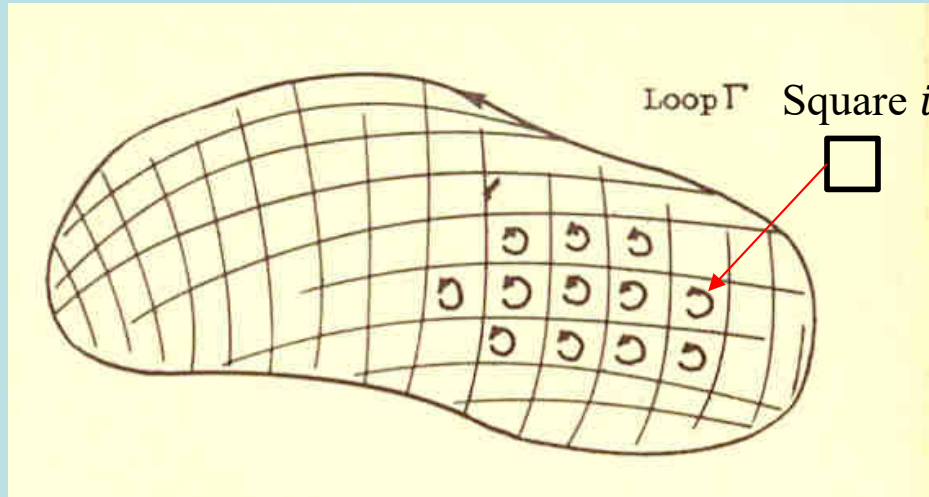
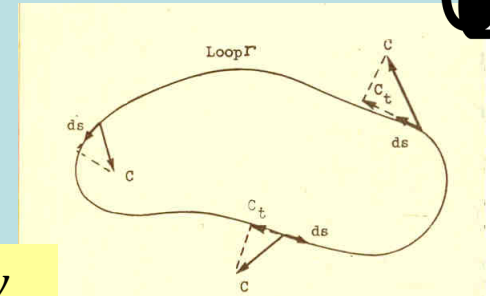
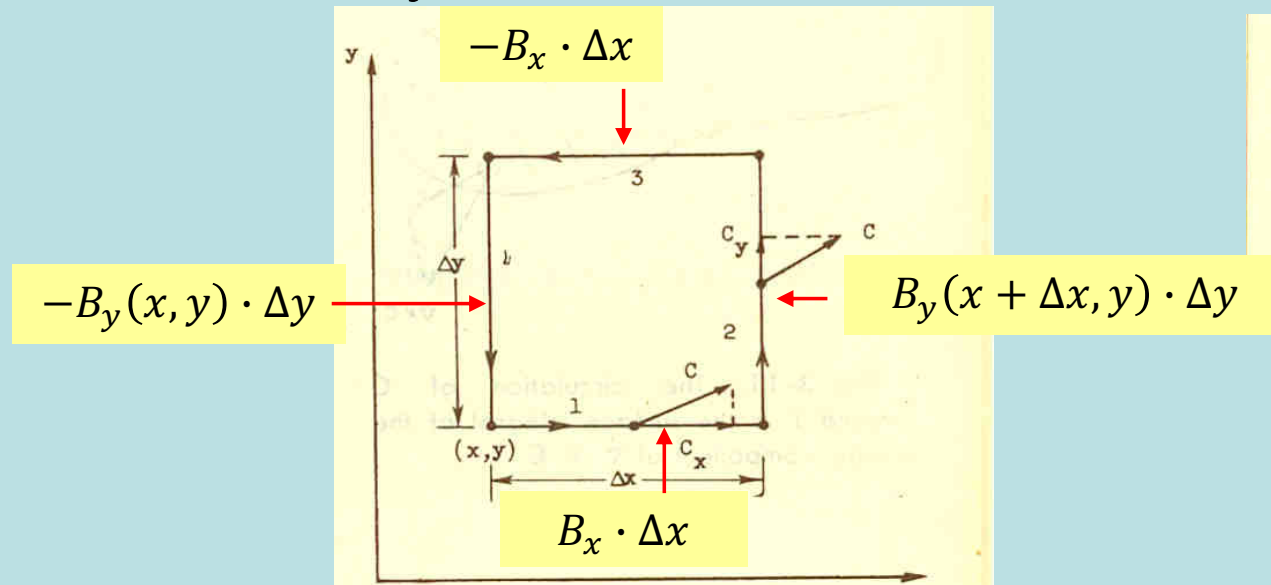


Figure 2.30.
For the subdivided loop, the sum of all the circulations Γ_i around the sections is equal to the circulation Γ around the original curve C .

Any path can be decomposed into the combination of a series of small squares.
Line integral along Γ equals the sum of line integrals along all the small squares.

$$\oint \vec{B} \cdot d\vec{s} = \sum_i \oint \vec{B}_i \cdot d\vec{s}_i$$

Calculate line integral $\oint \vec{B} \cdot d\vec{s}$ along a small square on the $x - y$ plane.



$$\oint \vec{B} \cdot d\vec{s} = [B_y(x + \Delta x, y)\Delta y - B_y(x, y)\Delta y] - [B_x(x, y + \Delta y)\Delta x + B_x(x, y)\Delta x]$$

注意：

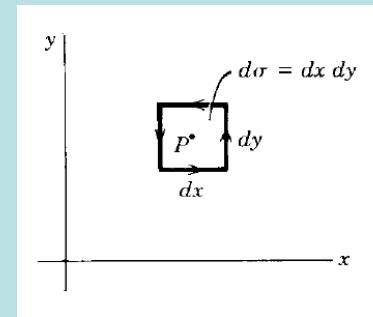
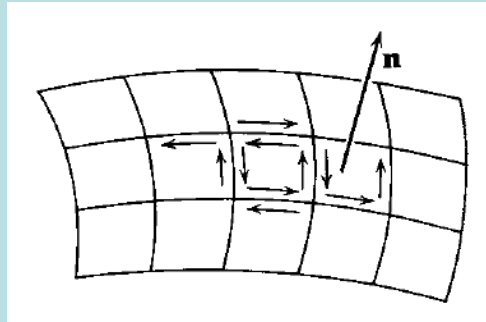
$$B_y(x + \Delta x, y) - B_y(x, y) \sim \frac{\partial B_y}{\partial x} \Delta x$$

$$= \frac{\partial B_y}{\partial x} \Delta x \Delta y - \frac{\partial B_x}{\partial y} \Delta y \Delta x = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \cdot \Delta x \Delta y = (\vec{\nabla} \times \vec{B})_z \cdot \Delta a = (\vec{\nabla} \times \vec{B}) \cdot \Delta \vec{a}$$

$$\oint \vec{B} \cdot d\vec{s} = (\vec{\nabla} \times \vec{B}) \cdot \Delta \vec{a}$$

$$(\vec{\nabla} \times \vec{B})_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

The line integral of $\vec{E}$ along a small square equals its curl times square area.



Consider an arbitrary loop Γ and decompose it into a series of small squares.

Line integral along Γ equals the sum of line integrals along squares composing Γ .

since contributions from intersections of neighboring squares will cancel.

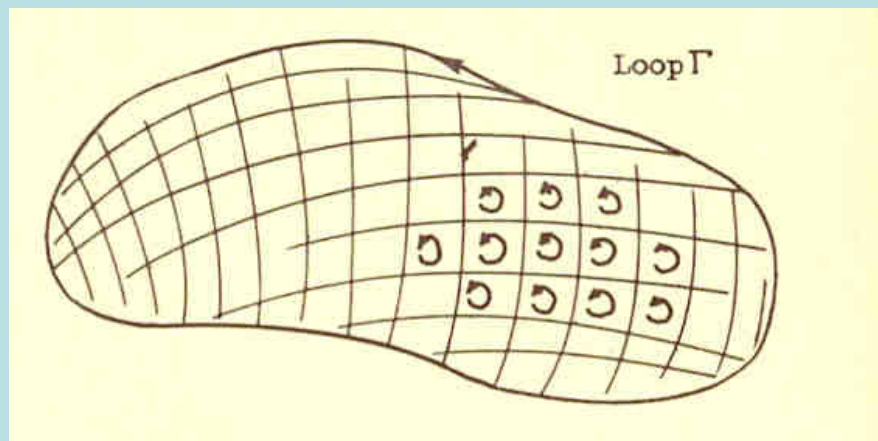
$$\oint \vec{B} \cdot d\vec{s} = \sum_i \oint \vec{B}_i \cdot d\vec{s}_i = \sum_i (\vec{\nabla} \times \vec{B}_i) \cdot \Delta \vec{a}_i$$

In case Γ is very small so that $\vec{\nabla} \times \vec{B}_i$ does not change much,

$$\sim (\vec{\nabla} \times \vec{B}) \sum_i \Delta \vec{a}_i = A (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times \vec{B} = \frac{\oint \vec{B} \cdot d\vec{s}}{A}$$

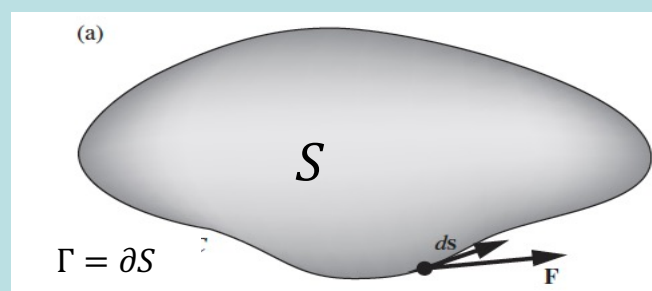
The meaning of curl is line integral of a vector field per unit area for small loop!



In case Γ is not small and $\vec{\nabla} \times \vec{B}_i$ does change,
sum of line integral along small squares approaches surface integral as $\Delta a_i \rightarrow 0$.

$$\oint_{\Gamma} \vec{B} \cdot d\vec{s} = \sum_i (\vec{\nabla} \times \vec{B}_i) \cdot \Delta \vec{a}_i \rightarrow \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$$

$$\oint_{\Gamma} \vec{B} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$$



Curve $\Gamma = \partial S$ is defined as the boundary of S .

Line integral of a vector field along $\Gamma = \partial S$, equals surface integral of curl on S .

Green's Theorem

4.3 Green's Theorem in the Plane

Let $P(x, y)$ and $Q(x, y)$ be smooth functions on $\mathbb{R}^2$. Then

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C Pdx + Qdy \quad (4.12)$$

where A is a bounded region in the plane and $C = \partial A$ is a piecewise smooth, non-intersecting closed curve which is traversed anti-clockwise.

Proof: Green's theorem is equivalent to the 2d divergence theorem (4.6). Let $\mathbf{F} = (Q, -P)$ be a vector field in $\mathbb{R}^2$. We then have

$$\int_A \nabla \cdot \mathbf{F} dA = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (4.13)$$

If $\mathbf{x}(s) = (x(s), y(s))$ is the parameterised curve C , then the tangent vector is $\mathbf{t}(s) = (x'(s), y'(s))$ and the normal vector $\mathbf{n} = (y'(s), -x'(s))$ obeys $\mathbf{n} \cdot \mathbf{t} = 0$.

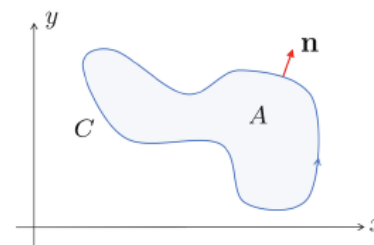
You'll need to do a little sketch to convince yourself that, as shown on the right, $\mathbf{n}$ is the outward pointing normal provided that the arc length s increases in the anti-clockwise direction. We then have

$$\mathbf{F} \cdot \mathbf{n} = Q \frac{dy}{ds} + P \frac{dx}{ds}$$

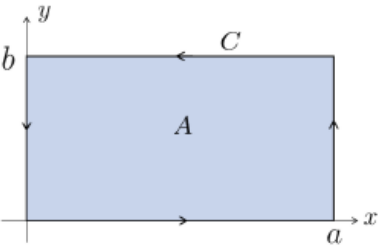
and so the integral around C is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C Pdx + Qdy \quad (4.14)$$

The 2d divergence theorem is the statement that the left-hand sides of (4.13) and (4.14) are equal; Green's theorem in the plane is the statement that the right-hand sides are equal. $\square$



Applied to a rectangular region, Green's theorem in the plane reduces to the fundamental theorem of calculus. We take the rectangular region to be $0 \leq x \leq a$ and $0 \leq y \leq b$. Then



$$\begin{aligned} \int_A -\frac{\partial P}{\partial y} dA &= -\int_0^a dx \int_0^b dy \frac{\partial P}{\partial y} \\ &= \int_0^a dx \left(-P(x, b) + P(x, 0) \right) = \int_C P dx \end{aligned}$$

where only the horizontal segments contribute, and the minus signs are such that C is traversed anti-clockwise. Meanwhile, we also have

$$\begin{aligned} \int_A \frac{\partial Q}{\partial x} dA &= \int_0^b dy \int_0^a dx \frac{\partial Q}{\partial x} \\ &= \int_0^b dy \left(Q(x, a) - Q(x, 0) \right) = \int_C Q dy \end{aligned}$$

where, this time, only the vertical segments contribute.

George Green (14 July 1793 – 31 May 1841) was a British [mathematical physicist](#) who wrote *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in 1828.^{[2][3]} The essay introduced several important concepts, among them a theorem similar to the modern [Green's theorem](#), the idea of [potential functions](#) as currently used in [physics](#), and the concept of what are now called [Green's functions](#). Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as [James Clerk Maxwell](#), [William Thomson](#), and others. His work on [potential theory](#) ran parallel to that of [Carl Friedrich Gauss](#).

Green's life story is remarkable in that he was almost entirely self-taught. He received only about one year of formal schooling as a child, between the ages of 8 and 9.

Early life [\[edit \]](#)

Green was born and lived for most of his life in the English town of [Sneinton](#), Nottinghamshire, now part of the city of [Nottingham](#). His father, also named George, was a baker who had built and owned a brick [windmill](#) used to grind grain.^[1]

In his youth, Green was described as having a frail constitution and a dislike for doing work in his father's bakery. He had no choice in the matter, however, and as was common for the time he likely began working daily to earn his living at the age of five.

Robert Goodacre's Academy [\[edit \]](#)

During this era it was common for only 25–50% of children in Nottingham to receive any schooling.^{[[citation needed](#)]} The majority of schools were Sunday schools, run by the

	George Green
Born	14 July 1793 Sneinton , Nottinghamshire, England
Died	31 May 1841 (aged 47) Nottingham , Nottinghamshire, England
Alma mater	Gonville and Caius College, Cambridge (BA, 1838)
Known for	Green measure Green's deformation tensor Green's function Green's identities Green's law Green's matrix Green's theorem Liouville–Green method
Scientific career	
Fields	Mathematics
Institutions	Gonville and Caius College, Cambridge ^[1]



Green's Mill in Sneinton, the mill

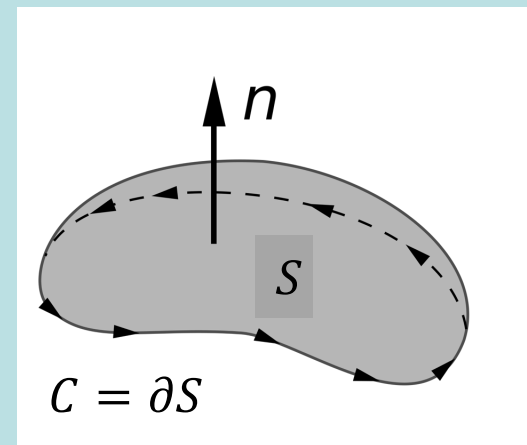
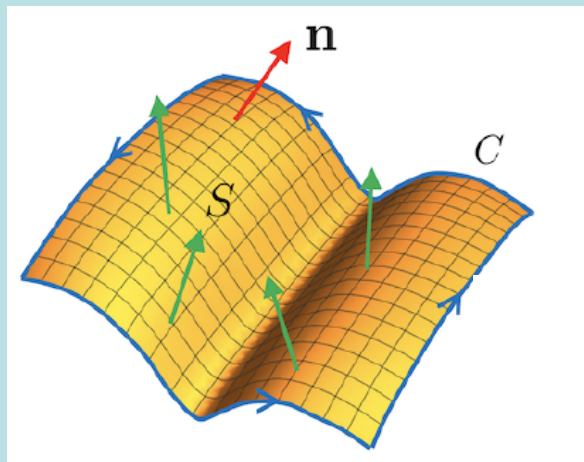
Green's theorem applies when loop $\Gamma = \partial S$ and surface S lie on the same plane. Similar Stokes' Theorem applies actually for arbitrary $C = \partial S$ and curved surface S .

4.4 Stokes' Theorem

Stokes' theorem is an extension of Green's theorem, but where the surface is no longer restricted to lie in a plane.

Let S be a smooth surface in $\mathbb{R}^3$ with boundary $C = \partial S$ a piecewise smooth curve. *Stokes' theorem* states that, for any smooth vector field $\mathbf{F}(\mathbf{x})$, we have

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \oint_C \vec{F} \cdot d\vec{s}$$



沿任一曲線 $C = \partial S$ 的向量場線積分，等於曲線內曲面 S ，該向量場旋度的面積分
Curl of a vector field allows us calculate its line integral.

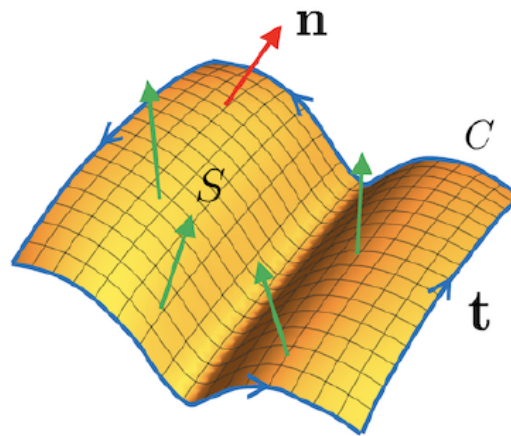


Figure 15. The surface S and bounding curve C for Stokes' theorem. The normal to the surface is shown (at one point) by the red arrow. The theorem invites us to compute the flux of a vector field $\mathbf{F}$, shown by the green arrows, through the surface, and compare it to the line integral around the boundary.

The orientations of S and C should be *compatible*. The former is determined by the choice of normal vector $\mathbf{n}$ to S ; the latter by the choice of tangent vector $\mathbf{t}$ to C . The two are said to be compatible if $\mathbf{t} \times \mathbf{n}$ points out of S . In practice, this means that if you orient the open surface so that $\mathbf{n}$ points towards you, then the orientation of C is anti-clockwise. The general set-up is shown in Figure 15.

Note that there will typically be many surfaces S that share the same boundary C . By Stokes' theorem, the integral of $\nabla \times \mathbf{F}$ over S must give the same answer for all such surfaces. The theorem also holds if the boundary ∂S consists of a number of disconnected components, again with their orientation determined by that of S .

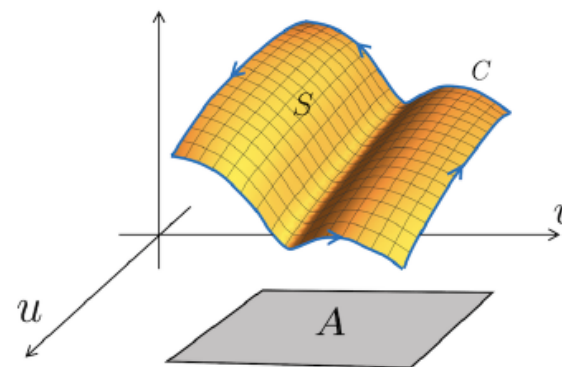
4.4.1 A Proof of Stokes' Theorem

It's clear that Stokes' theorem is a version of Green's theorem in the plane, but viewed through 3d glasses. Indeed, it's trivial to show that the latter follows from the former. Consider the vector field $\mathbf{F} = (P, Q, 0)$ in $\mathbb{R}^3$ and a surface S that lies flat in the $z = 0$ plane. The normal to this surface is $\mathbf{n} = \hat{\mathbf{z}}$, and we have

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$$

But Stokes' theorem then tells us that this can also be written as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C P dx + Q dy$$



However, with a little more work we can also show that the converse is true. In other words, we can lift Green's theorem out of the plane to find Stokes' theorem.

Consider a parameterised surface S defined by $\mathbf{x}(u, v)$ and denote the associated area in the (u, v) plane as A . We parameterise the boundary $C = \partial S$ as $\mathbf{x}(u(t), v(t))$ and the corresponding boundary ∂A as $(u(t), v(t))$. The key idea is to use Green's theorem in the (u, v) plane for the area A and then uplift this to prove Stokes theorem for the surface S .

We start by looking at the integral around the boundary. It is

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot \left(\frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv \right) = \int_{\partial A} F_u du + F_v dv$$

where $F_u = \mathbf{F} \cdot \partial \mathbf{x} / \partial u$ and $F_v = \mathbf{F} \cdot \partial \mathbf{x} / \partial v$. Now we're in a position to invoke Green's theorem, in the form

$$\int_{\partial A} F_u du + F_v dv = \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) dA$$

Now our task is clear. We should look at the partial derivatives on the right hand side. We just need to be careful about what thing depends on what thing:

$$\frac{\partial F_v}{\partial u} = \frac{\partial}{\partial u} \left(\mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial v} \right) = \frac{\partial}{\partial u} \left(F_i \frac{\partial x^i}{\partial v} \right) = \left(\frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial u} \right) \frac{\partial x^i}{\partial v} + F_i \frac{\partial^2 x^i}{\partial u \partial v}$$

Meanwhile, we have

$$\frac{\partial F_u}{\partial v} = \frac{\partial}{\partial v} \left(\mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial u} \right) = \frac{\partial}{\partial v} \left(F_i \frac{\partial x^i}{\partial u} \right) = \left(\frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial v} \right) \frac{\partial x^i}{\partial u} + F_i \frac{\partial^2 x^i}{\partial v \partial u}$$

Subtracting the second expression from the first, the second derivative terms cancel, leaving us with

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} \left(\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i} \right) = (\delta_{jk} \delta_{il} - \delta_{jl} \delta_{ik}) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j}$$

At this point we wield everyone's favourite index notation identity

$$\epsilon_{jip} \epsilon_{pkl} = \delta_{jk} \delta_{il} - \delta_{jl} \delta_{ik}$$

We then have

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \epsilon_{jip} \epsilon_{pkl} \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j} = (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right)$$

Now we're done. Following through the chain of identities above, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{x} &= \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) dudv \\ &= \int_A (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right) dudv \\ &= \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\end{aligned}$$

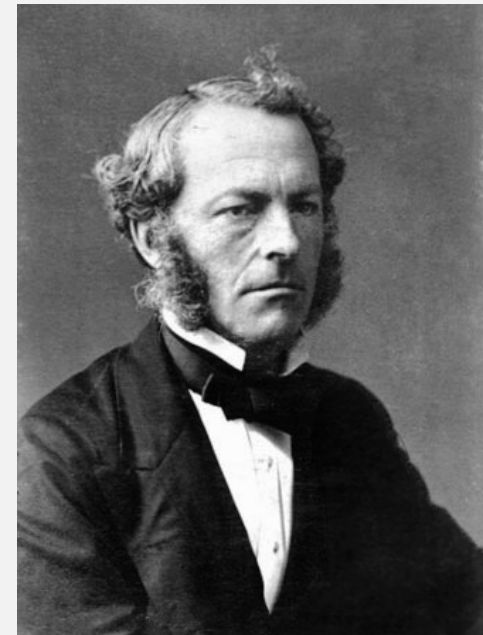
This is Stokes' theorem. □

8. If X, Y, Z be functions of the rectangular co-ordinates x, y, z , dS an element of any limited surface, l, m, n the cosines of the inclinations of the normal at dS to the axes, ds an element of the bounding line, shew that

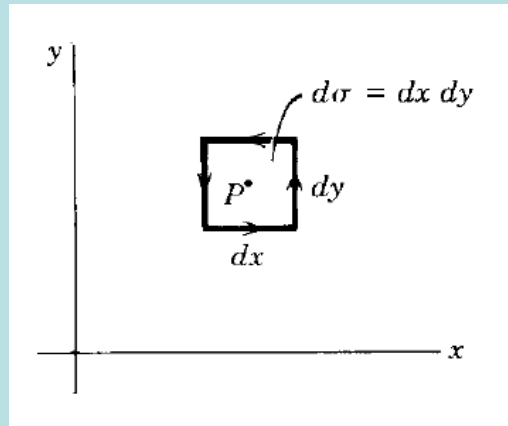
$$\begin{aligned}\iint \left\{ l \left(\frac{dZ}{dy} - \frac{dY}{dx} \right) + m \left(\frac{dX}{dz} - \frac{dZ}{dx} \right) + n \left(\frac{dY}{dx} - \frac{dX}{dy} \right) \right\} dS \\ = \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds,\end{aligned}$$

the differential coefficients of X, Y, Z being partial, and the single integral being taken all round the perimeter of the surface.

Figure 16. You may now turn the page... the original version of Stokes' theorem, set as an exam question.



George Stokes 1819-1903



Apply Ampere's law to a small square and let the size approaches zero:

Current is the flux of $\vec{j}$! $i = \vec{j} \cdot d\vec{a}$

$$\oint \vec{B} \cdot d\vec{s} = (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 i = \mu_0 \vec{j} \cdot d\vec{a}$$



$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \quad \text{differential formulation of Ampere's Law}$$

Plug Differential Formulation into the Mathematical Stoke's Law:



$$\oint \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$$

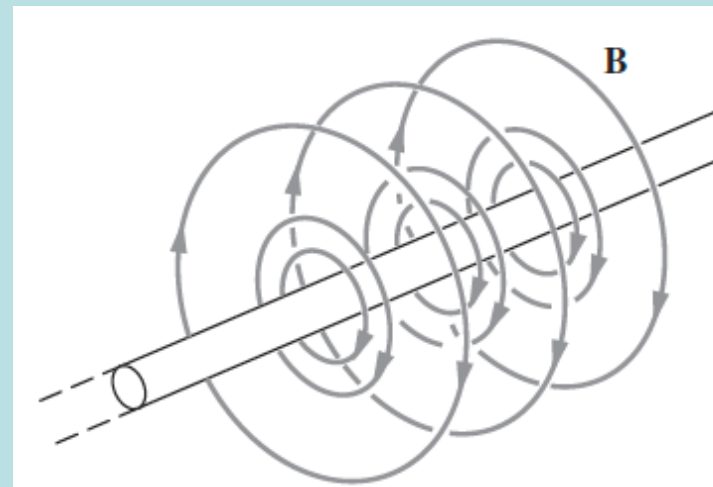
$$\oint \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \int (\mu_0 \vec{J}) \cdot d\vec{a} = \mu_0 i$$

Integral Formulation

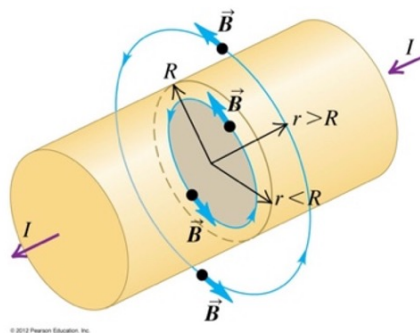
長直導線外的磁場

$$\vec{\nabla} \times \vec{B} = 0$$

except for $r = 0$.



5. The magnetic field inside a cylindrical current along the z axis with radius R and constant current density j can be written as ($r < R$):



$$\vec{B} = \frac{\mu_0 j}{2} (y, -x, 0)$$

Calculate $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$ and check it is consistent with Maxwell Equations.

$$\oint \vec{E} \cdot d\vec{s} = 0$$

Integral Formulation

Apply it to a small square and let the size approaches zero:



$$\vec{\nabla} \times \vec{E} = 0$$

Differential Formulation

$$\oint \vec{E} \cdot d\vec{s} = (\vec{\nabla} \times \vec{E}) \cdot \Delta \vec{a}$$

Maxwell Equations For Static Fields



$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$

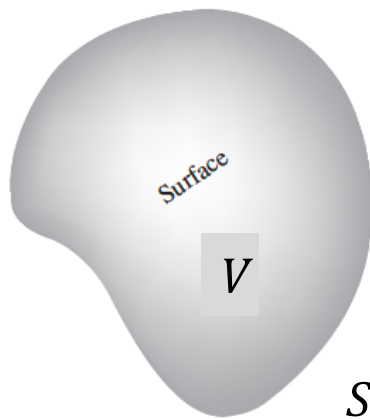
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$



散度透過高斯定律對應到面積分！旋度透過史塔克定律對應到線積分！

Summary

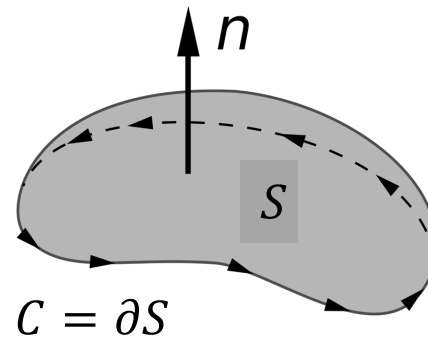
GAUSS



$$\oint_S \vec{F} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{F}) dv$$

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \nabla \cdot \mathbf{F} \end{aligned}$$

STOKES

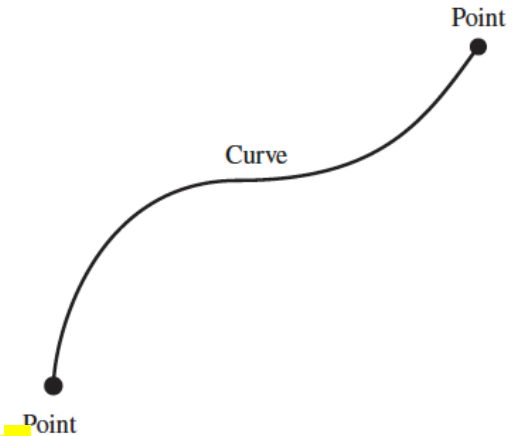


$$\oint_C \vec{A} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

In Cartesian coordinates, with $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$:

$$\begin{aligned} \text{curl } \mathbf{A} &= \hat{x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \\ &\quad + \hat{y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ &\quad + \hat{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= \nabla \times \mathbf{A} \end{aligned}$$

GRAD



Points enclose curve

$$\phi_2 - \phi_1 = \int_{\text{curve}} \text{grad } \phi \cdot ds$$

$$\begin{aligned} \text{grad } \phi &= \hat{x} \frac{\partial \phi}{\partial x} + \hat{y} \frac{\partial \phi}{\partial y} + \hat{z} \frac{\partial \phi}{\partial z} \\ &= \nabla \phi \end{aligned}$$

散度就是一個小體積的單位體積電場總通量！

旋度等於沿無限小的正方形的場線積分乘上正方形面積！

Time changing magnetic field will generate electric field by induction.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

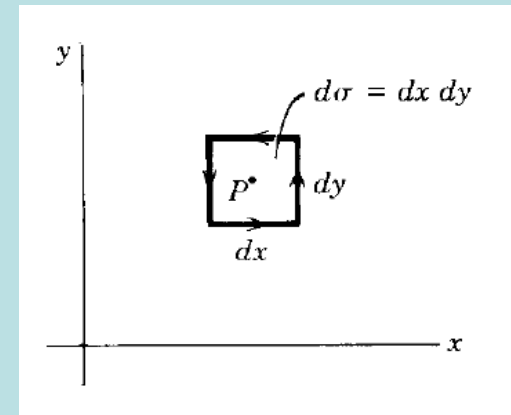
Faraday's Law



Apply Faraday's Law to a small square:

For a small square, fields do not change much.

$$\oint \vec{E} \cdot d\vec{s} = (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$



$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law in differential formulation

Complete Maxwell Equations

積分形式

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

微分形式

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

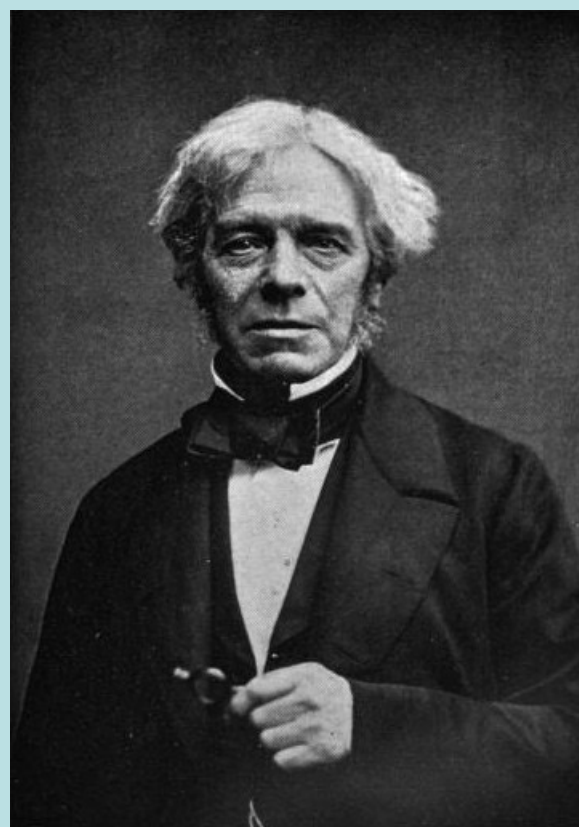
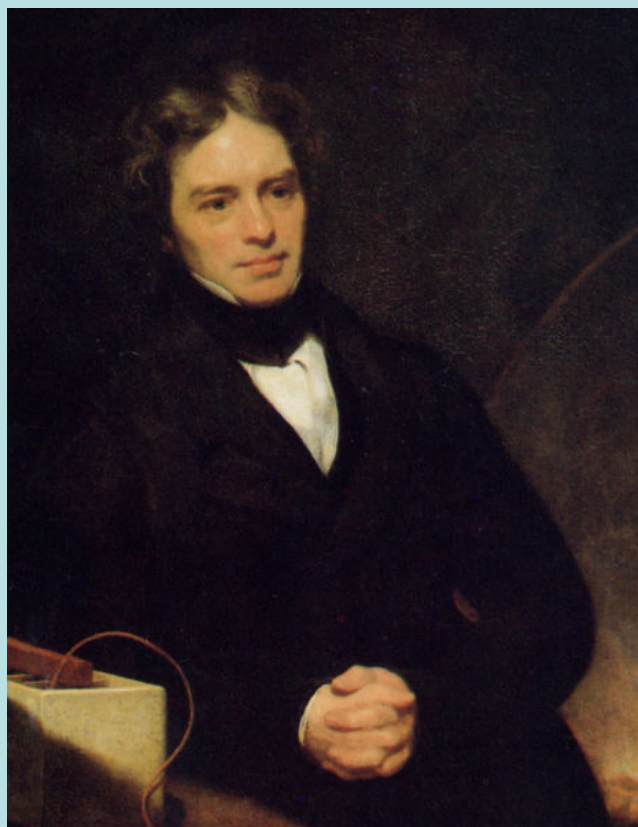
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$



電場的發明者 Faraday 1791-1861



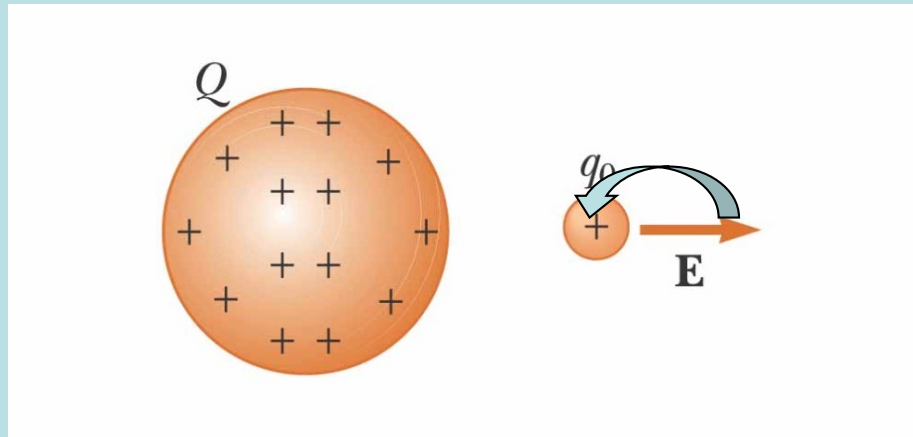
在對岸英國的法拉第憑直覺，反對靜電力是超距力，因而引進電場的概念。



Your obly. & faithful Servant
M^r Faraday

超距力並不是正確的概念！

電場的引進使得電力有機會可以不再是超距力



Locality在地主義：一個地方發生的事物，在當時只會影響它附近的事物。



積分形式

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

微分形式

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

左式的路徑內磁場的變化似乎與整條路徑上的電場有關。

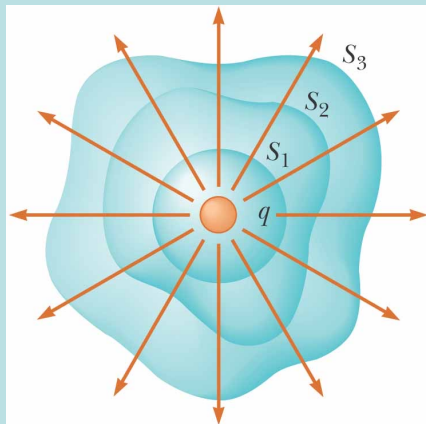
Locality在地主義並不確定。但這只是表象。

微分形式的法拉第定律只關乎一個點上的場，

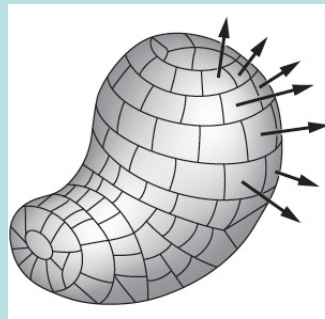
一個點上磁場的變化等於此點上電場旋度。完全在地！

馬克思威爾方程式原來完全遵守法拉第期待的Locality在地主義。

Shapes of the field lines

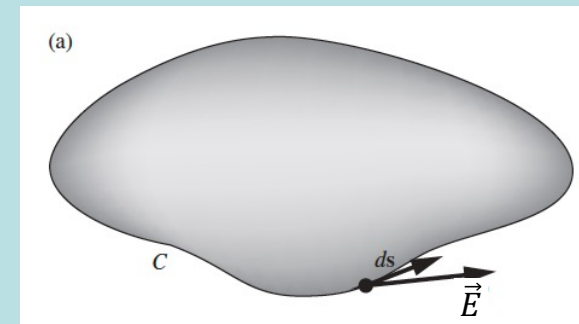


放射狀的場線



面積分

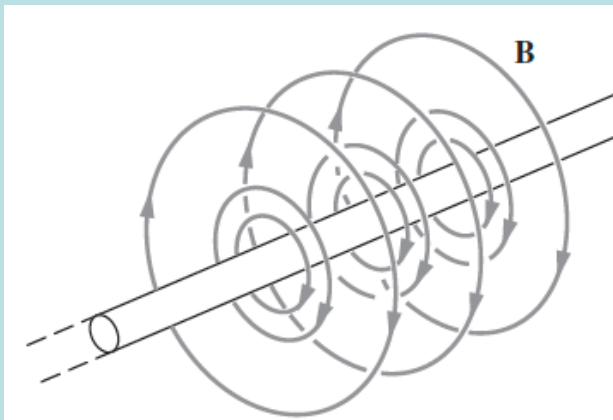
$$\oint \vec{E}_{\text{放射}} \cdot d\vec{A}$$



線積分

$$\oint \vec{E}_{\text{放射}} \cdot d\vec{s} = 0$$

面積分會補抓放射狀的場線！



漩渦狀的場線

$$\oint \vec{E}_{\text{漩渦}} \cdot d\vec{A} = 0$$

$$\oint E_{\text{漩渦}} \cdot ds$$

線積分會補抓漩渦狀的場線！

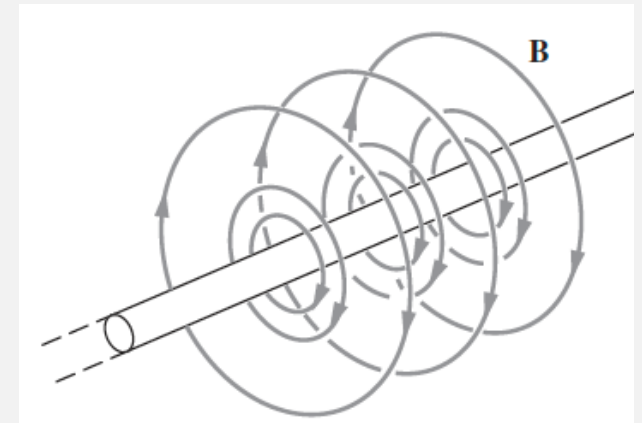
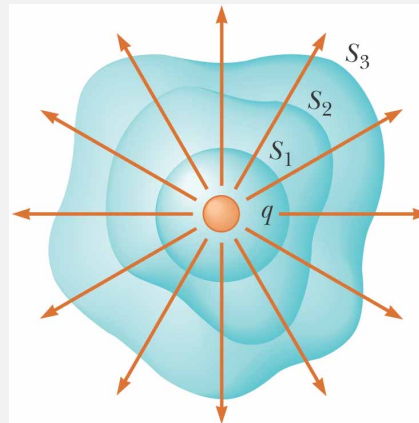
Electrostatic field and Magnetostatics field are quite different.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$



Curl of electrostatic field is zero and divergence of magnetostatics field is zero.

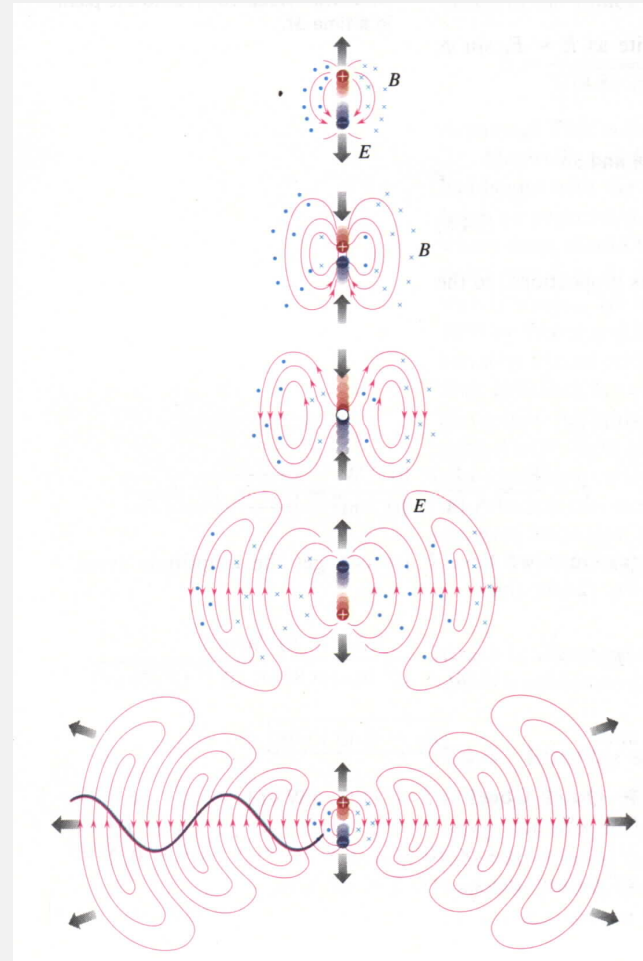
In vacuum (no charge and current),
electric field and magnetic can only come from induction.

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

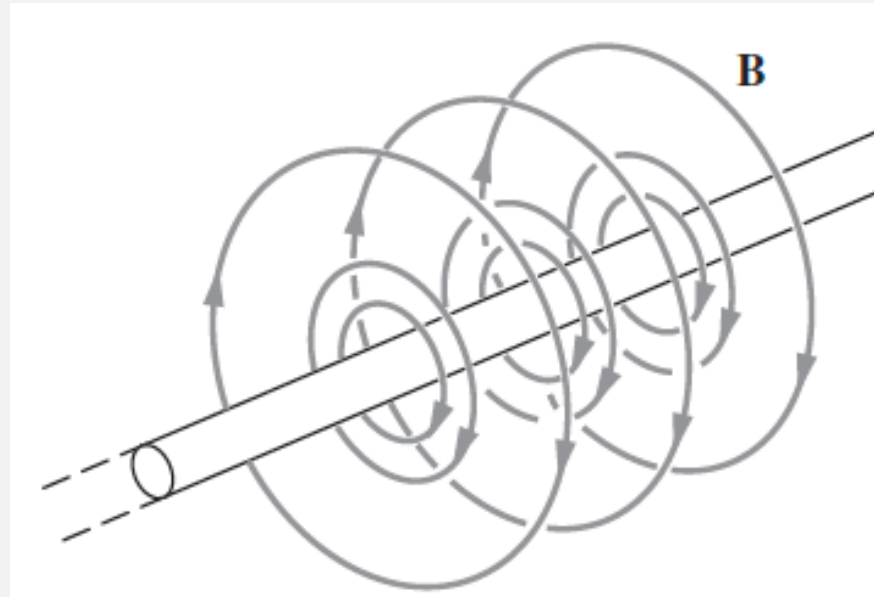
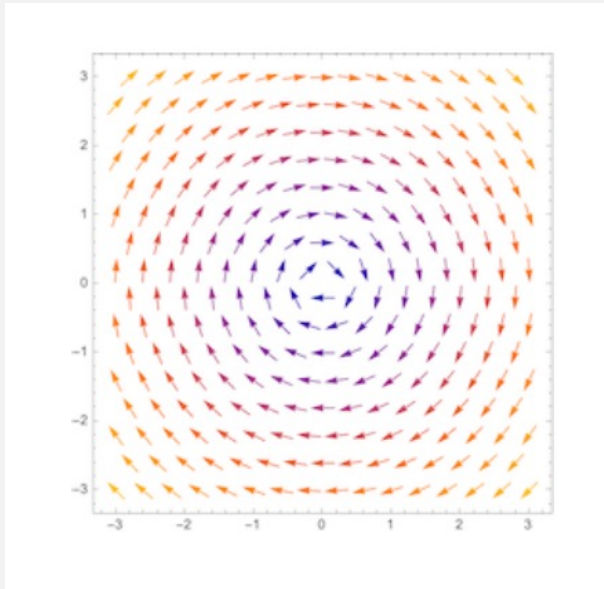
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$



Divergence of electric field and magnetic fields are both zero.

以靜磁場與靜電場的場線來理解是最方便的！



靜磁場是零散度，靜磁場線是一條一條的封閉曲線（不一定是圓）。

沒有起點與終點，也就沒有磁荷了！

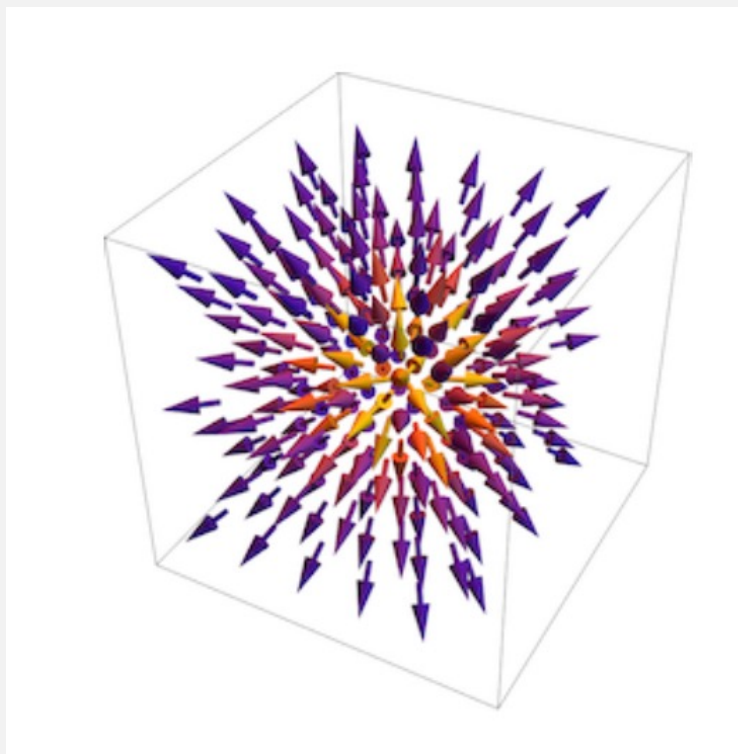
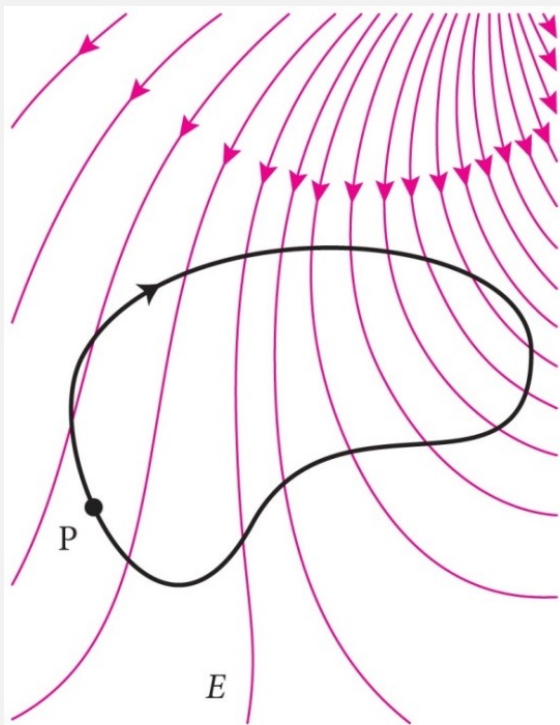
我們可以把它們稱為**漩渦狀**的場線。

漩渦狀場線沒有起點或末點，因此對任一高斯面，進入的場線一定離開。

散度永遠為零！也稱為無散度場**Divergence Free Field**.

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{B}_{\text{漩渦}} \cdot d\vec{A} = 0$$



庫倫電場是由點狀源頭發出的場線，可以稱為放射狀，
它是保守力場，**放射狀**場線對線積分貢獻永遠為零！

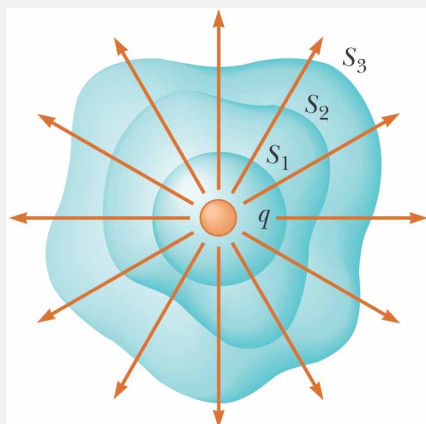
對任一封閉曲線

$$\oint \vec{E}_{\text{放射}} \cdot d\vec{s} = 0$$

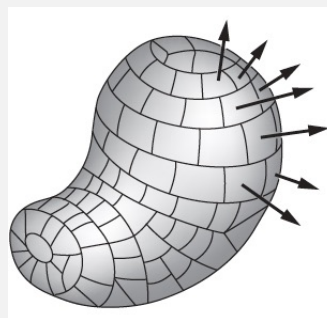
因此

$$\vec{\nabla} \times \vec{E} = 0$$

又稱為無旋度場 **Irrotational Field**.



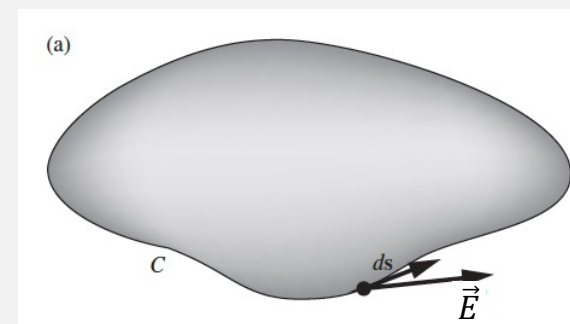
放射狀的場線



面積分

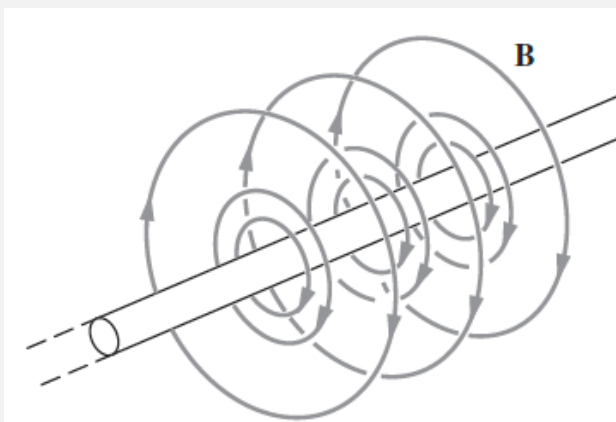
$$\oint \vec{E}_{\text{放射}} \cdot d\vec{A}$$

面積分會補抓放射狀的場線！



線積分

$$\oint \vec{E}_{\text{放射}} \cdot d\vec{s} = 0$$

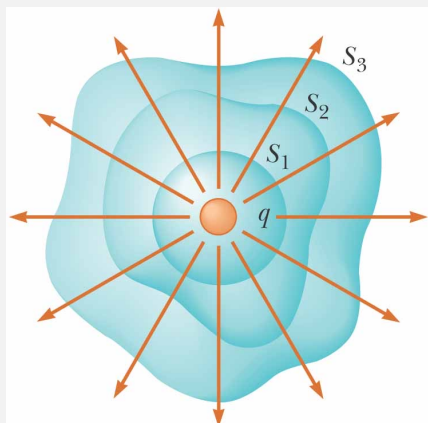


漩渦狀的場線

$$\oint \vec{E}_{\text{漩渦}} \cdot d\vec{A} = 0$$

$$\oint E_{\text{漩渦}} \cdot ds$$

線積分會補抓漩渦狀的場線！



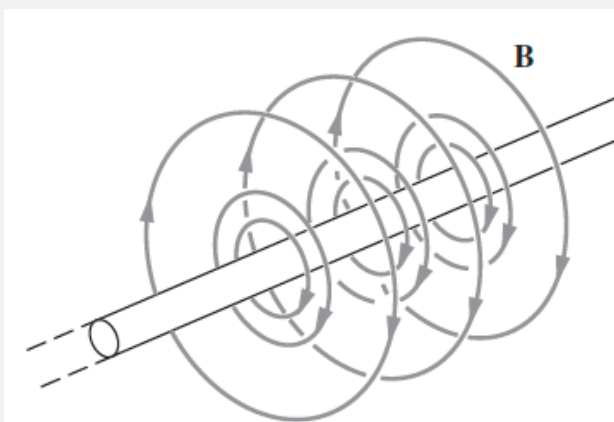
精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

圖像：

放射狀的場線，

稱為無旋度場 Irrotational Field



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線，

稱為無散度場 Divergence Free Field.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

電荷是靜電場的來源，產生的靜電場是放射狀，**無旋度場 Irrotational Field**。

電流是靜磁場的來源，產生的靜磁場是漩渦狀，**稱為無散度場 Divergence Free Field**。

這兩種場分別各有一個定理！這兩種場都可以寫成另一個場的特定微分！

Potentials

$\vec{\nabla} \times \vec{E} = 0$ 電場線積分為零，所以有電位存在，是電位的梯度： $\vec{E} = \vec{\nabla} V$

定理：任一無散度向量場 $\vec{F}$ ， $\vec{\nabla} \times \vec{F} = 0$

等價於：此一向量場 $\vec{F}$ 可以寫成一個純量場 ϕ 的梯度： $\vec{F} = \vec{\nabla} \phi$

倒過來的陳述的證明：

純量場 ϕ 的梯度，必無散度：

$$\vec{\nabla} \times (\vec{\nabla} \phi) = 0$$

向量外積的規則顯示梯度的旋度必為零！ $\vec{\nabla} \times \vec{\nabla} \sim 0$ ，電場滿足：

$$\vec{\nabla} \times \vec{F} = 0$$

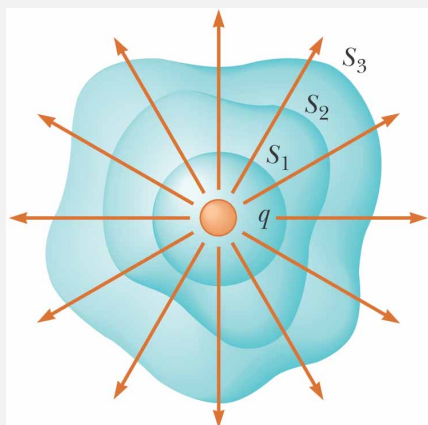
正向的證明：根據 Stokes Theorem

$$\vec{\nabla} \times \vec{F} = 0 \quad \longrightarrow \quad \oint \vec{F} \cdot d\vec{s} = 0$$

是一個保守力場，因此可以寫下位能 ϕ ：

$$\Delta \phi = \int \vec{F} \cdot d\vec{s}$$

如同電場為電位的梯度， $\vec{F}$ 也是 ϕ 的梯度： $\vec{F} = \vec{\nabla} \phi$ 得證！



精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

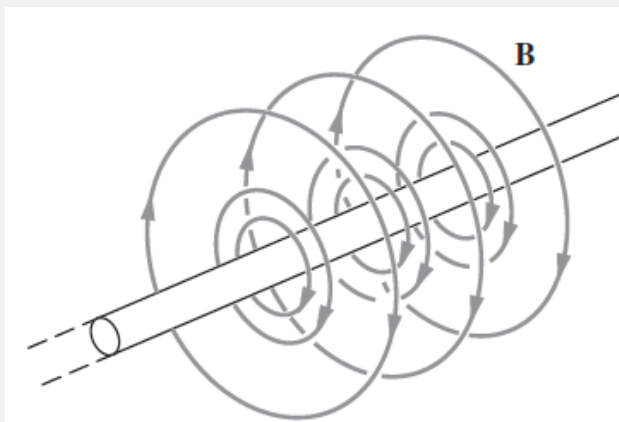
圖像：

放射狀的場線

另一個場的微分：

$$\vec{E} = \vec{\nabla} \phi$$

稱為無旋度場 Irrotational Field



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線

?

稱為無散度場 Divergence Free Field.

定理：任一無散度向量場 $\vec{B}$ ，

$$\vec{\nabla} \cdot \vec{B} = 0$$

等價於此一向量場 $\vec{B}$ 可以寫成一個向量場 $\vec{A}$ 的旋度：

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

倒過來的陳述的證明：

向量外積的規則顯示旋度的散度必為零！向量場 $\vec{A}$ 的旋度的散度為零。

$$\vec{\nabla} \cdot (\vec{\nabla} \times)$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

正向的證明：我們可以直接定義向量場 $\vec{A}$ ，見下一頁。

This time, we will prove the converse statement by explicitly exhibiting a vector potential $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$. We pick some arbitrary point $\mathbf{x}_0 = (x_0, y_0, z_0)$ and then construct the following vector field

$$\mathbf{A}(\mathbf{x}) = \left(\int_{z_0}^z F_y(x, y, z') dz' , \int_{x_0}^x F_z(x', y, z_0) dx' - \int_{z_0}^z F_x(x, y, z') dz' , 0 \right) \quad (3.6)$$

Since $A_z = 0$, the definition of the curl (3.4) becomes

$$\nabla \times \mathbf{A} = \left(-\frac{\partial A_y}{\partial z} , \frac{\partial A_x}{\partial z} , \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Using the ansatz (3.6), we find that the first two components of $\nabla \times \mathbf{A}$ immediately give what we want

$$(\nabla \times \mathbf{A})_x = F_x(x, y, z) \quad \text{and} \quad (\nabla \times \mathbf{A})_y = F_y(x, y, z)$$

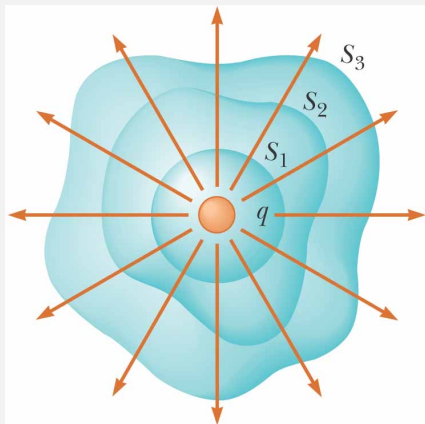
both of which follow from the fundamental theorem of calculus. Meanwhile, we still have a little work ahead of us for the final component

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) - \int_{z_0}^z \frac{\partial F_x}{\partial x}(x, y, z') dz' - \int_{z_0}^z \frac{\partial F_y}{\partial y}(x, y, z') dz'$$

At this point we use the fact that $\mathbf{F}$ is solenoidal, so $\nabla \cdot \mathbf{F} = 0$ and so $\partial F_z / \partial z' = -(\partial F_x / \partial x + \partial F_y / \partial y)$. We then have

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) + \int_{z_0}^z \frac{\partial F_z}{\partial z'}(x, y, z') dz' = F_z(x, y, z)$$

This is the result we want. □



精確的定義：

$$\vec{\nabla} \times \vec{E} = 0$$

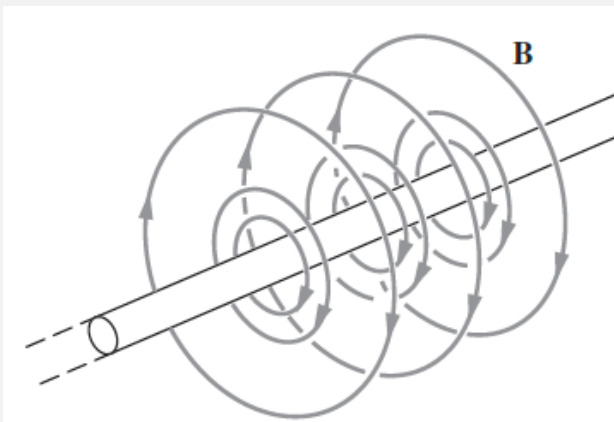
圖像：

放射狀的場線

另一個場的微分：

$$\vec{E} = \vec{\nabla} \phi$$

稱為無旋度場 Irrotational Field



$$\vec{\nabla} \cdot \vec{B} = 0$$

漩渦狀的場線

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

稱為無散度場 Divergence Free Field.

所以 $\phi, \vec{A}$ 可以代替 $\vec{E}, \vec{B}$ ，作為討論電磁現象的物理場。

在量子力學中 $\phi, \vec{A}$ 甚至比 $\vec{E}, \vec{B}$ 更真實。

以上分類不只適用於電磁場。