This formalism could be easily extended to more than two masses.

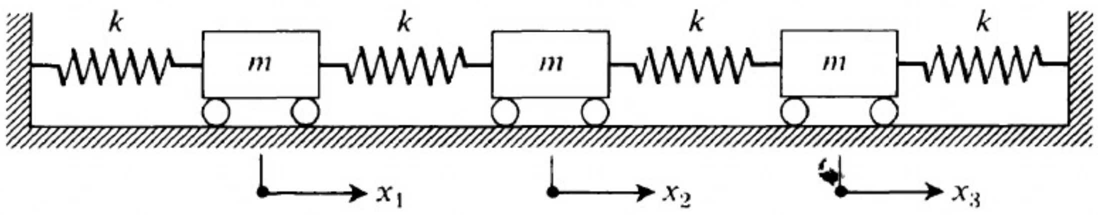


Figure 10.13

A. 運動方程式：

$$m\frac{d^2x_1}{dt^2} = k(x_2 - x_1) - kx_1 = -2kx_1 + kx_2$$

$$m\frac{d^2x_2}{dt^2} = k(x_3 - x_2) - k(x_2 - x_1) = -2kx_2 + kx_3 + kx_1$$

$$m\frac{d^2x_3}{dt^2} = -k(x_3 - x_2) - kx_3 = -2kx_3 + kx_2$$

B. 計算彈簧的彈力位能：

$$V = \frac{k}{2}[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2] = k(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

C. 若以矩陣語言來書寫：

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \boldsymbol{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

取$\boldsymbol{x} \equiv \boldsymbol{a}e^{i\omega t}$，代入得本徵值方程式：$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} = \lambda\omega_0^2\boldsymbol{a}$。特徵方程式為以下

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

取$\boldsymbol{x} \equiv \boldsymbol{a}e^{i\omega t}$，代入得本徵值方程式：$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} = \lambda\omega_0^2\boldsymbol{a}$。特徵方程式為以下行列式為零：

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = e^{i\omega t}\boldsymbol{a} = e^{i\omega t}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda\omega_0^2\boldsymbol{I})\cdot\boldsymbol{a} = 0$$

得代數方程式：

$$(2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)(2-4\lambda+\lambda^2) = 0$$

解得本徵值：

$$\lambda = 2, 2\pm\sqrt{2},\ \omega^2 = 2\omega_0^2, (2\pm\sqrt{2})\omega_0^2$$

將本徵值及對應的角頻率由小到大編號：

Order and label the angular frequencies from small to large.

$\omega_1 \sim \sqrt{2-\sqrt{2}}\,\omega_0$ $\omega_2 \sim \sqrt{2}\,\omega_0$ $\omega_3 \sim \sqrt{2+\sqrt{2}}\,\omega_0$

$\lambda = 2, \omega = \omega_2 \sim \sqrt{2}\omega_0$

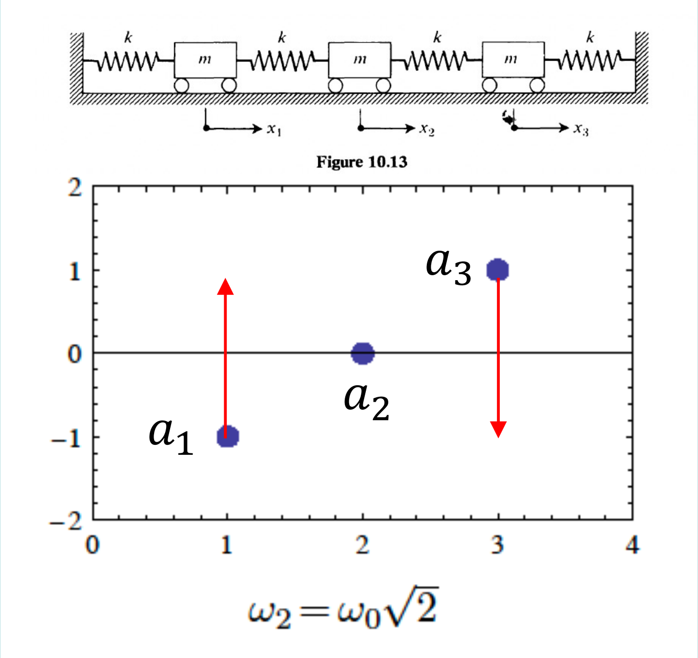
$(\boldsymbol{A} - 2\omega_0^2 \boldsymbol{I}) \cdot \boldsymbol{a} = 0$

$$\omega_0^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$-a_2 = 0$

$-a_1 - a_3 = 0$

$-a_2 = 0$



We can draw $a_1 : a_2 : a_3$ at their locations.

Red arrows indicate the SHM of individual particles.

$\boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ Eigenvector up to a constant

$\boldsymbol{x} = \boldsymbol{a}^{(2)} e^{i\sqrt{2}\omega_0 t}$ is one solution!

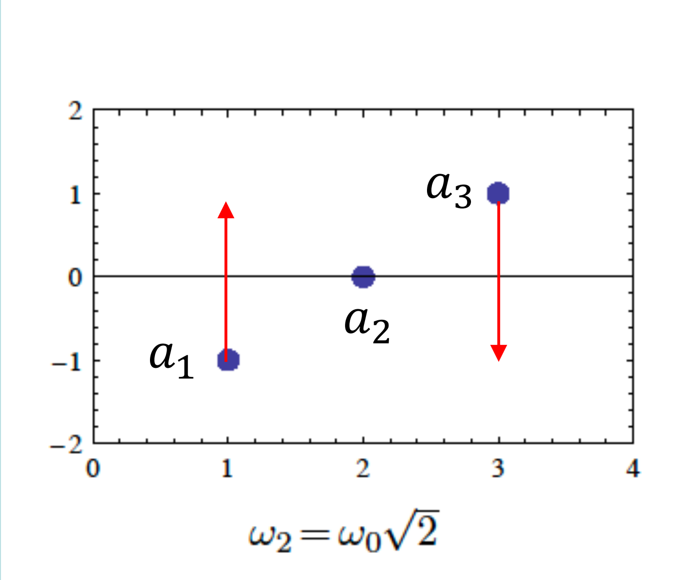
三個粒子位置$x_{1,2,3}$以$a_{1,2,3}$為振比例一起作同步(同ω)的簡諧運動。

Three particle's $x_{1,2,3}$ oscillate together by same ω, with amplitudes proportional to $a_{1,2,3}$.

$\boldsymbol{x}(t) \propto \boldsymbol{a}^{(2)}$ $x_1(t) : x_2(t) : x_3(t) = a_1 : a_2 : a_3 = -1 : 0 : 1$ At all t.

Looked individually every particle is doing a simple harmonic motion, with same ω.

At all t: $\boldsymbol{x}(t) \propto \boldsymbol{a}^{(2)}$



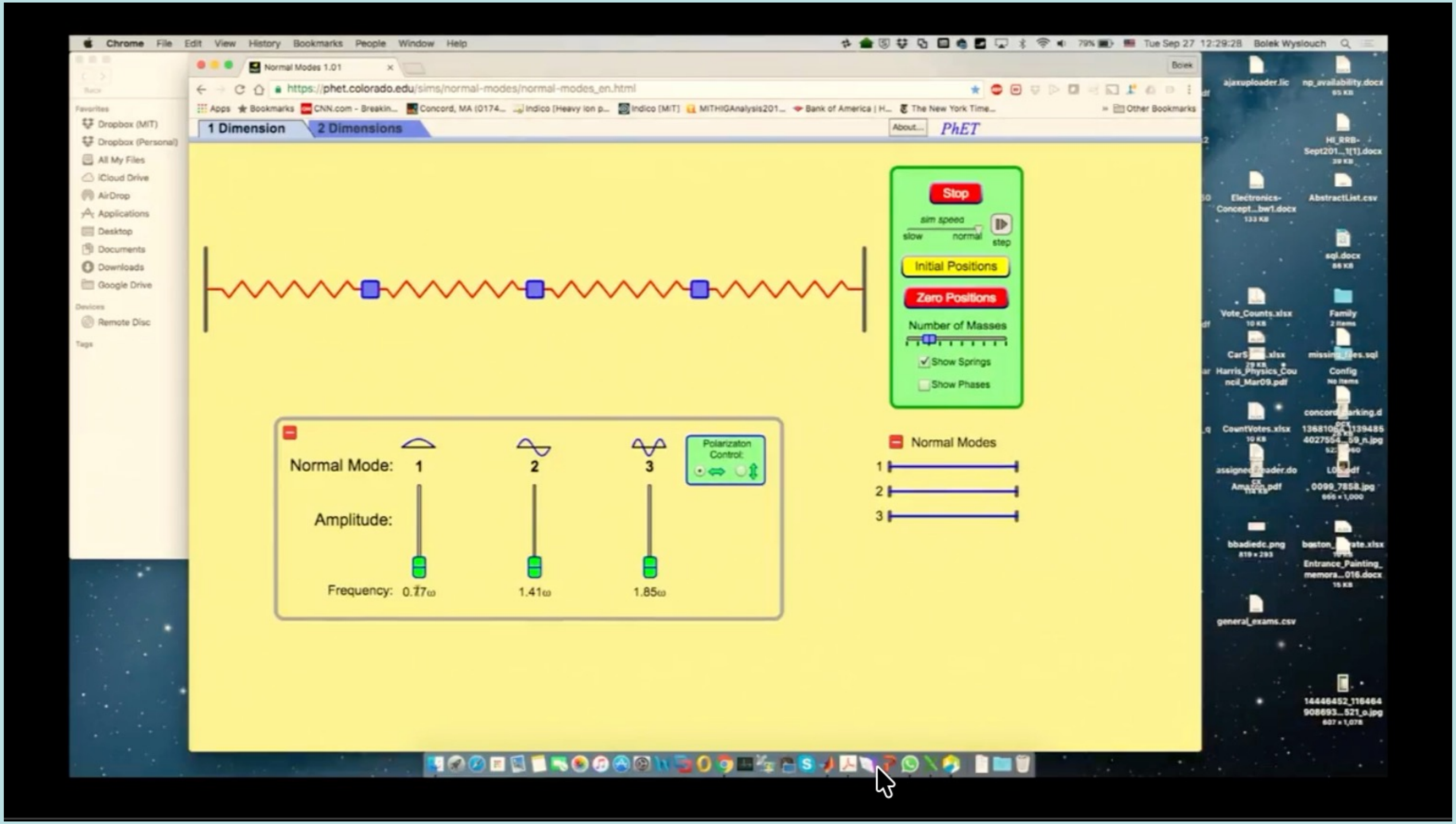
If the initial condition starts with this proportion, the mode will continue.

$$\boldsymbol{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_2 \boldsymbol{a}^{(2)} \sim f_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{x}'(0) = \begin{pmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{pmatrix} \propto \omega_{\mathbf{2}} g_2 \boldsymbol{a}^{(2)}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \boldsymbol{a}^{(2)}[f_2\cos(\omega_{\mathbf{2}} t) + g_2\sin(\omega_{\mathbf{2}} t)] = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\left[f_2\cos\left(\sqrt{2}\omega_{\mathbf{0}} t\right) + g_2\sin\left(\sqrt{2}\omega_{\mathbf{0}} t\right)\right]$$

這是一個可獨立振盪的模式。This is one independent oscillation mode.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Zero Positions
Number of Masses
Show Springs
Show Phases
Normal Mode:
Amplitude:
Frequency:
Polarization Control:
1
2
3
0.77ω
1.41ω
1.85ω
Normal Modes

$$\lambda = 2-\sqrt{2}, \omega = \omega_1 \sim \sqrt{2-\sqrt{2}}\omega_0:$$

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\sqrt{2}a_1 - a_2 = 0$$
$$-a_1 + \sqrt{2}a_2 - a_3 = 0$$
$$-a_2 + \sqrt{2}a_3 = 0$$

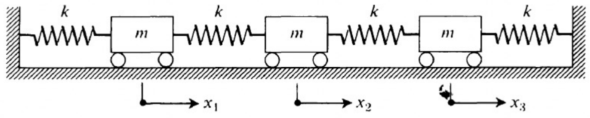
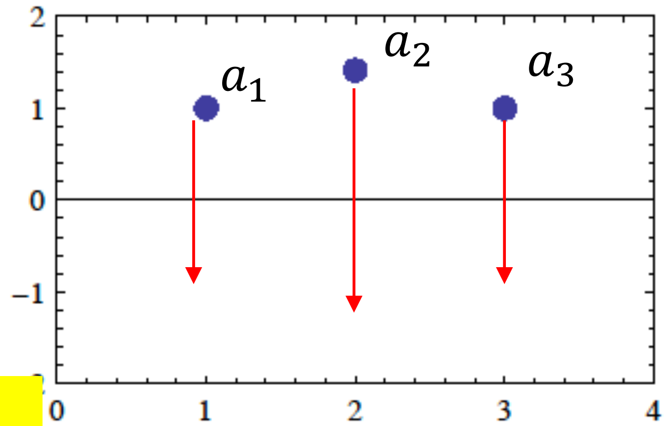


Figure 10.13



$$\boldsymbol{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\boldsymbol{x} = \boldsymbol{a}^{(1)} e^{i\sqrt{2-\sqrt{2}}\omega_0 t} \qquad \boldsymbol{x}(t) \propto \boldsymbol{a}^{(1)}$$

$$x_1(t): x_2(t): x_3(t) = a_1: a_2: a_3 = 1: \sqrt{2}: 1$$

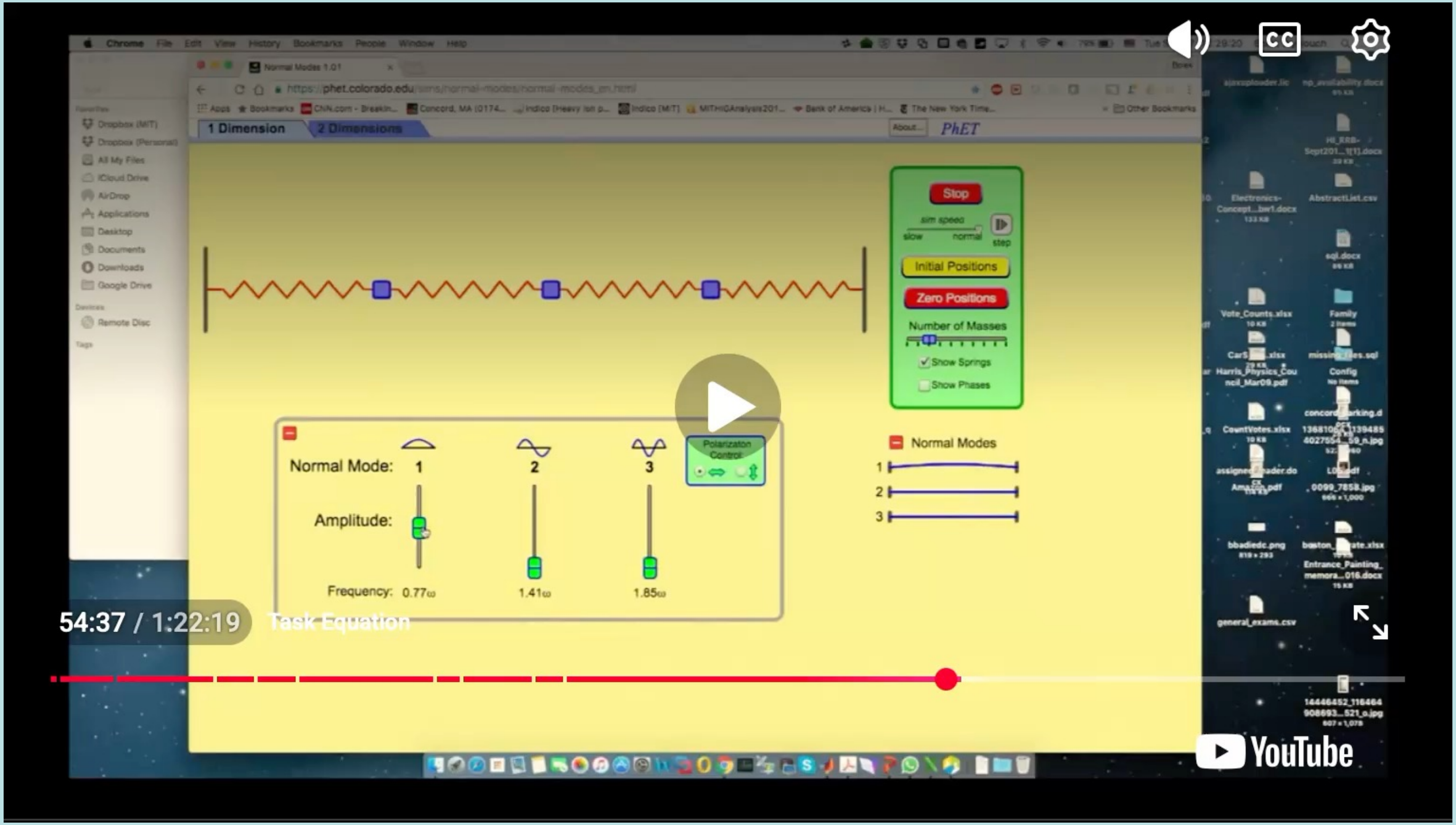
Three particle's $x_{1,2,3}$ oscillate as SHM by same ω, with amplitudes proportional to $a_{1,2,3}$.

$$\boldsymbol{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \propto \boldsymbol{a}^{(1)} \sim \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \boldsymbol{x}'(0) = \begin{pmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

以下式就是運動方程式與起始條件的解：

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \boldsymbol{a}^{(1)} \cos\sqrt{2-\sqrt{2}}\omega_0 t$$

這是另一個可獨立振盪的模式。This is another oscillation mode.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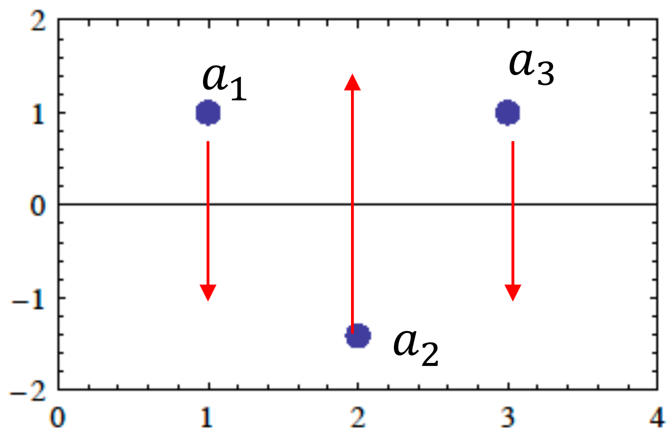
Amplitude:
Frequency:
0.77ω
1.41ω
1.85ω
Normal Modes
54:37 / 1:22:19
Task Equation
YouTube

$$\lambda = 2+\sqrt{2}, \omega = \omega_3 \sim \sqrt{2+\sqrt{2}}\omega_0$$

$$\sqrt{2}a_1 + a_2 = 0$$
$$a_1 + \sqrt{2}a_2 + a_3 = 0$$
$$a_2 + \sqrt{2}a_3 = 0$$



$\omega_3 = \omega_0\sqrt{2+\sqrt{2}}$

$$\boldsymbol{a}^{(3)} = c_3\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$$\boldsymbol{x} = \boldsymbol{a}^{(3)} e^{i\sqrt{2+\sqrt{2}}\omega_0 t} \qquad \boldsymbol{x}(t) \propto \boldsymbol{a}^{(3)}$$

$$x_1(t):x_2(t):x_3(t) = a_1:a_2:a_3 = 1:-\sqrt{2}:1$$

$$\boldsymbol{x}(t) = \begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix} \propto \boldsymbol{a}^{(3)}\cos\sqrt{2+\sqrt{2}}\,\omega_0 t$$

Three particle's $x_{1,2,3}$ oscillate together by same ω, with amplitudes proportional to $a_{1,2,3}$.

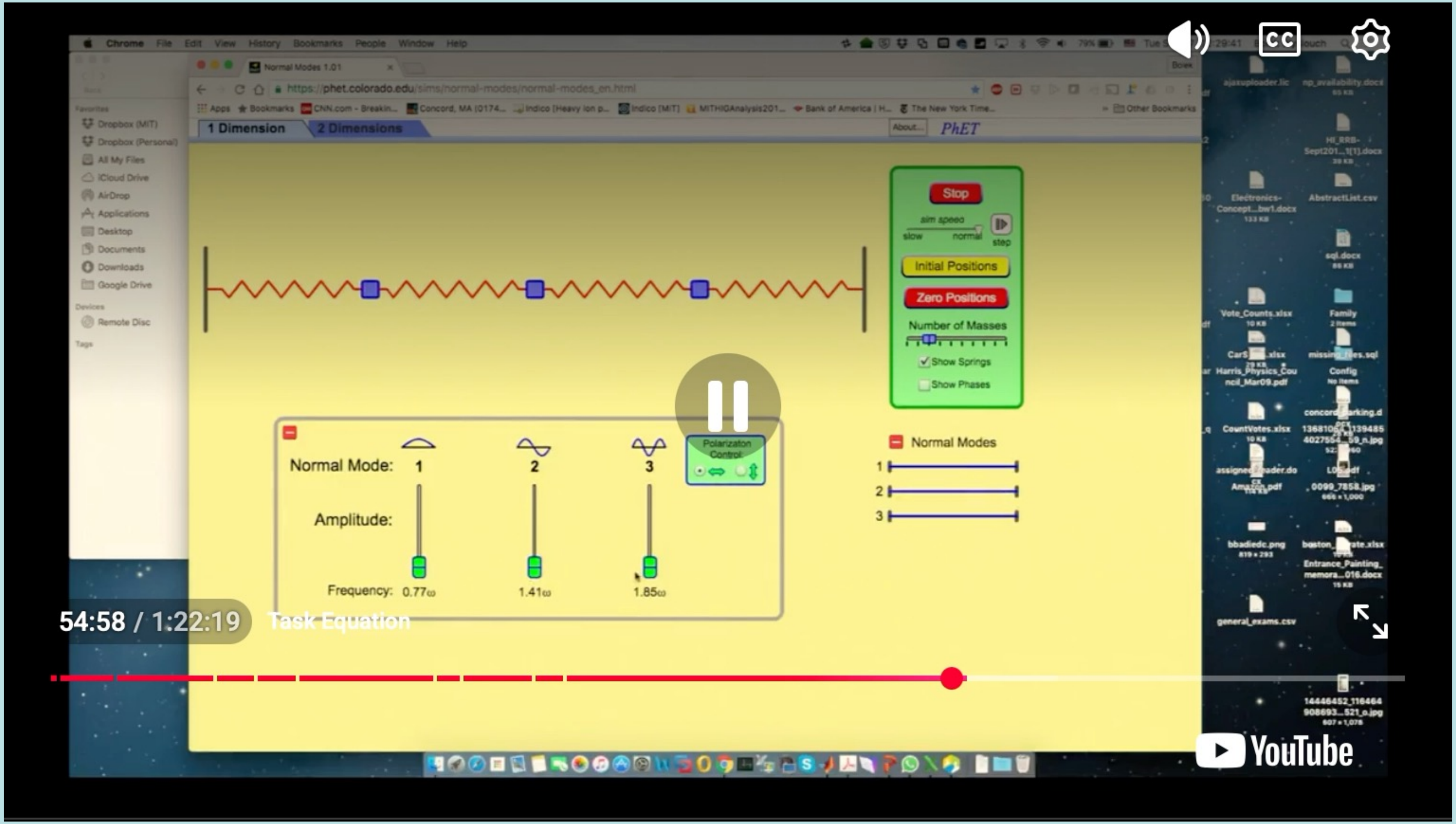
這是第三個可獨立振盪的模式。This is the third oscillation mode.

一個本徵向量對應一個可獨立振盪的模式。One eigenvalue one mode.

每一個運動模式(本徵向量$\boldsymbol{a}$)，有一個特定的簡諧振動頻率(本徵值)！

模式的數目，就是本徵向量數目，恰等於粒子數目。

The number of modes equals the number of particles.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Normal Mode:
Amplitude:
Frequency:
0.77ω
1.41ω
1.85ω
Polarization Control
Normal Modes
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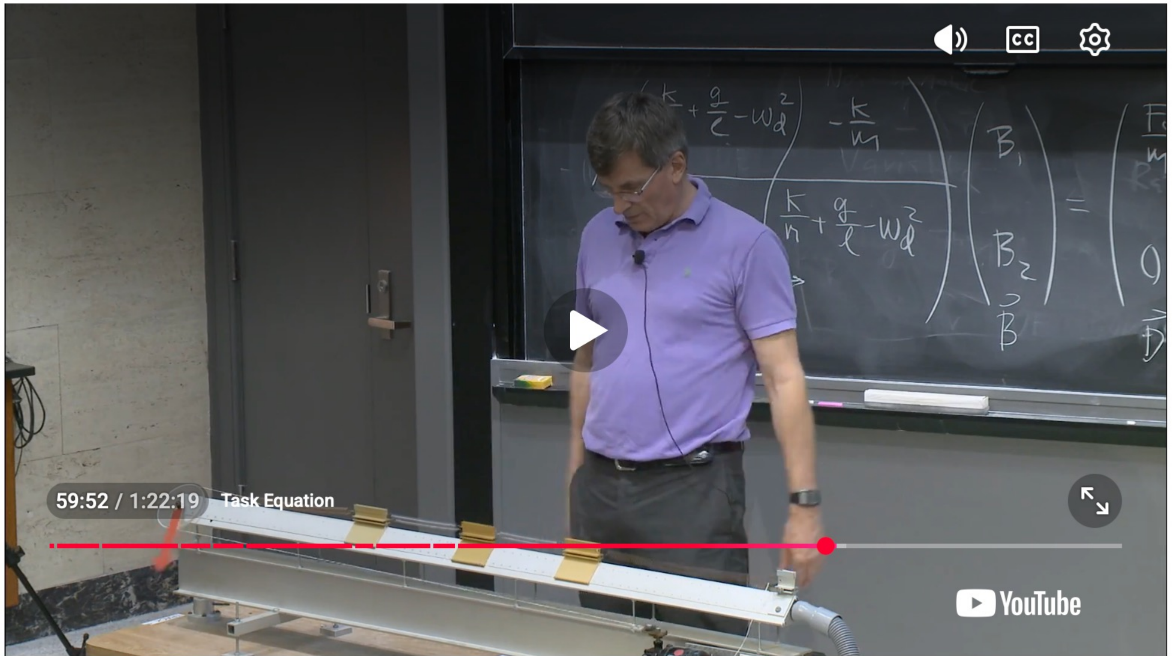
Task Equation
YouTube

Lecture 6: Driven Oscillators, Resonance

Coupled Air Carts



TRANSCRIPT

59:23 I'm not exactly at the right place, but I'm close.

Course Info

INSTRUCTOR

Prof. Yen-Jie Lee

DEPARTMENTS

Physics

AS TAUGHT IN

Fall 2016

LEVEL

Undergraduate

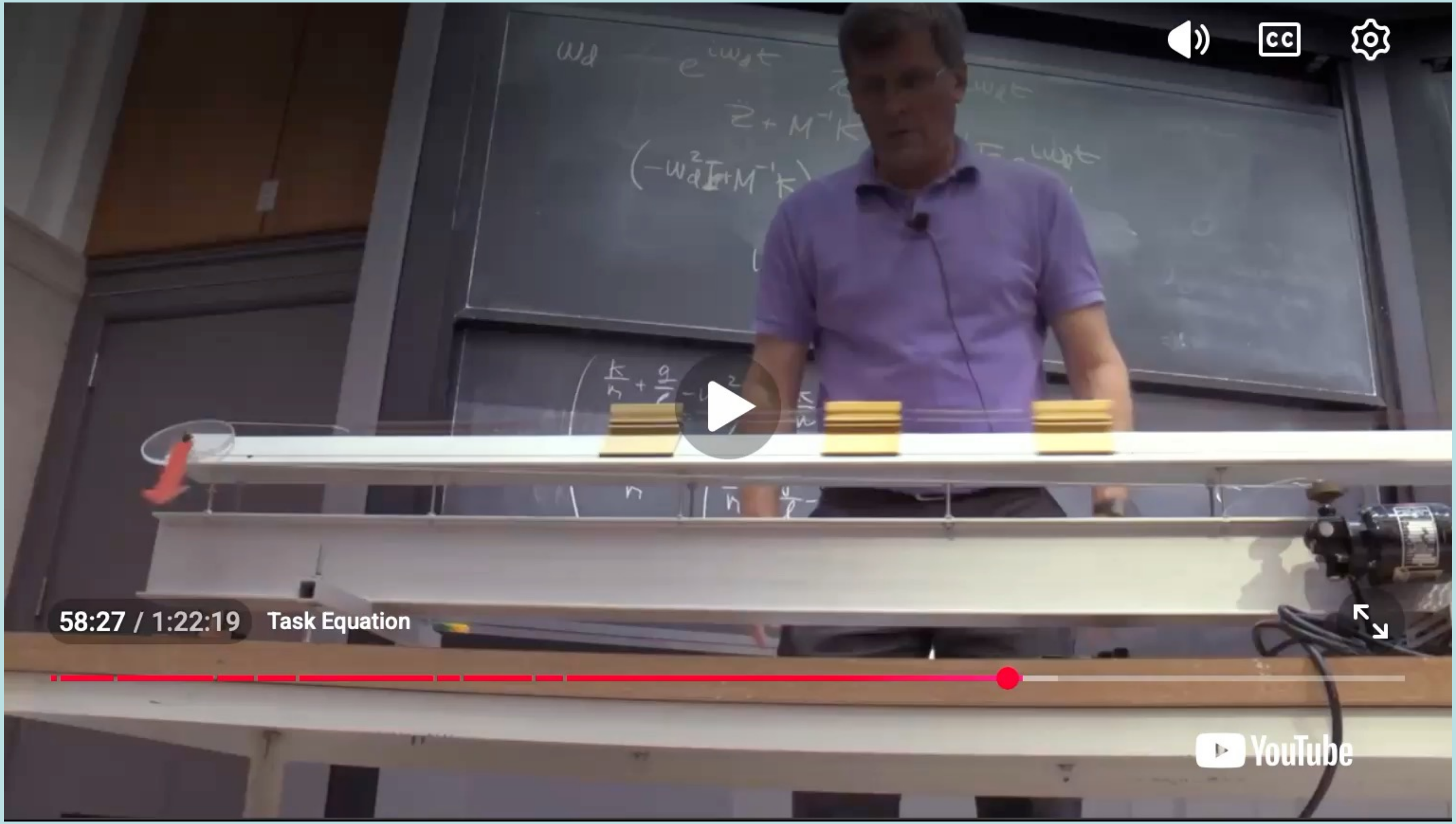
TOPICS

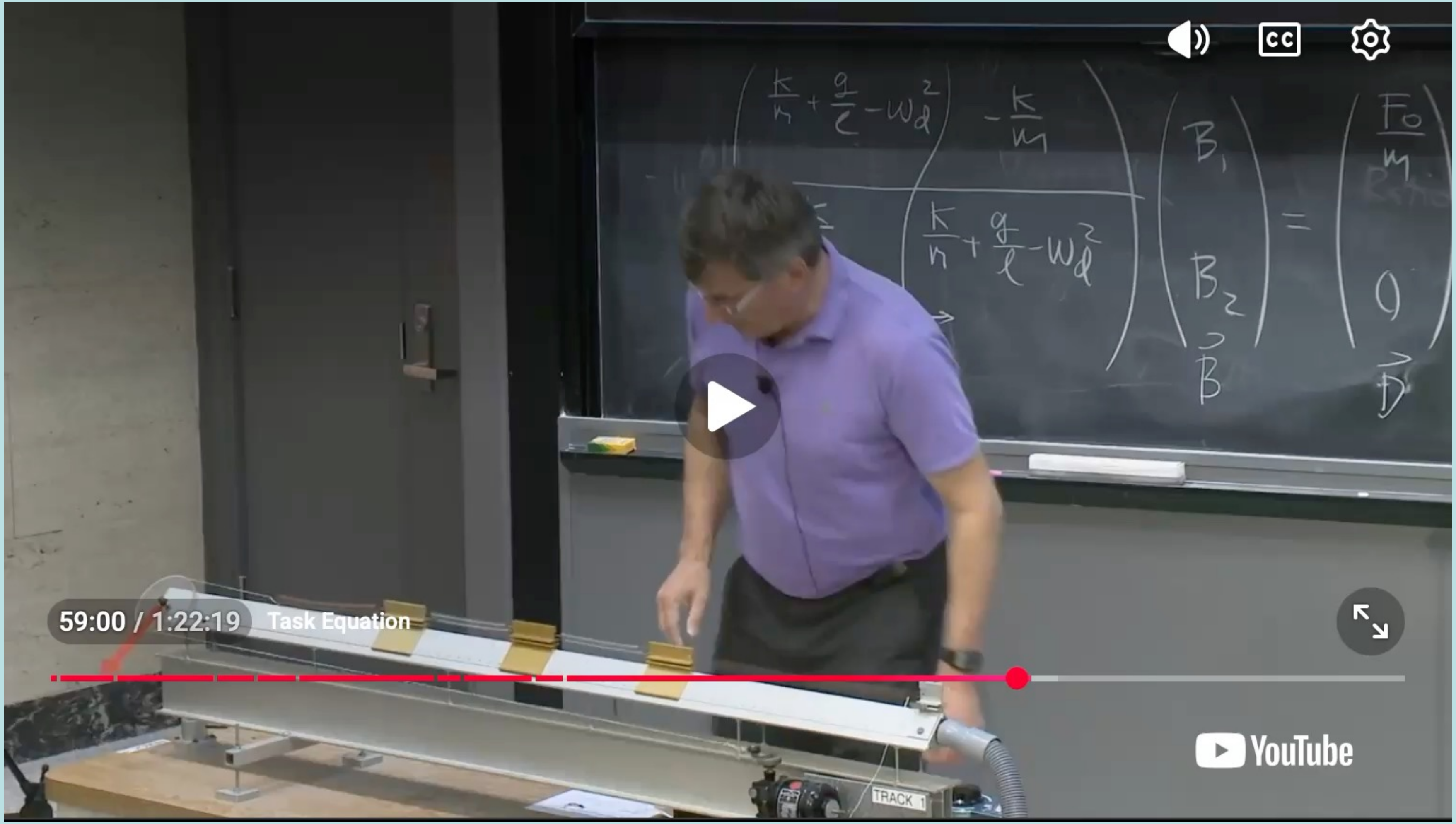
- Science
  - Physics
    - Atomic, Molecular, Optical Physics
    - Classical Mechanics
    - Electromagnetism

LEARNING RESOURCE TYPES

Exams

1st Mode

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Task Equation
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TRACK 1

有了以上三個模式的解，一般解將是三者的線性組合：

Linear Combinations of the 3 modes are solutions, too.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

三個未定常數$f_{1,2,3}$,將由三個起使條件$x_{1,2,3}(0)$ 決定：

3 unspecified constants $f_{1,2,3}$ will be determined by 3 initial conditions $x_{1,2,3}(0)$.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

Here I have chosen $\boldsymbol{a}^{(i)}$ to be of length 1. Hence the 3 are $\boldsymbol{a}^{(i)}$ orthonormal.

根據對稱矩陣本徵向量的展開定理，三個本徵向量彼此正交： $\boldsymbol{a}^{(m)T} \boldsymbol{a}^{(n)} = \delta_{mn}$

可以如上選$\boldsymbol{a}^{(n)}$使其向量長度為1。

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

$$\boldsymbol{a}^{(m)T} \boldsymbol{a}^{(n)} = \delta_{mn}$$

任一起始條件的行向量$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$，都可以展開成三個本徵向量的線性組合。

Any vector $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$ can be written as a linear combination of orthonormal vectors!

$f_{1,2,3}$可以解出：

$$\boldsymbol{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

The coefficients $f_{1,2,3}$ are simply the inner products of the vector and $\boldsymbol{a}^{(i)T}$.

另外三個未定常數 $g_{1,2,3}$則由另外三個起使條件$x'_{1,2,3}(0)$以類似方式決定。

The other coefficients $g_{1,2,3}$ are determined similarly by more initial conditions $x'_{1,2,3}(0)$.

起始各個配料依配方$f_{1,2,3}$收集 各個配料按分離烹煮$\cos(\omega_i t)$

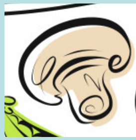
個自簡諧振蕩後，最後合體！

=

+

+

+



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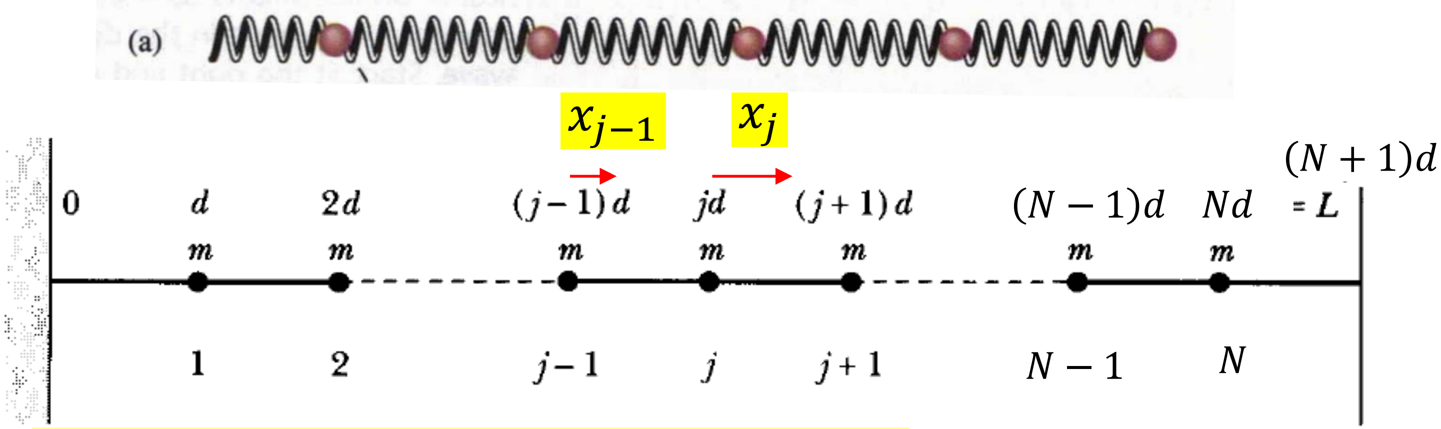
⋮

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i$$

$$\begin{pmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

Now extend the formalism to a large number N of particles.



$$m\frac{d^2x_j}{dt^2} = -k(x_j - x_{j-1}) + k(x_{j+1} - x_j)$$

$j = 1,2 \cdots N$

$$\frac{d^2x_j}{dt^2} = \frac{k}{m}x_{j-1} - \frac{2k}{m}x_j + \frac{k}{m}x_{j+1}$$

$$\omega_0^2 = \frac{k}{m}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

$$\boldsymbol{X} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A} \cdot \boldsymbol{X}$$

$$\boldsymbol{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

x_1, x_N二次微分的方程式中包含x_0, x_{N+1}。

如果假設邊界固定，第一式就還是正確的，這就稱為Boundary Condition.

$$x_0 = x_{N+1} = 0$$

先設x_i為複數，如下猜解並代入：

$\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$$

$-\boldsymbol{A}\cdot\boldsymbol{a} = -\omega^2\boldsymbol{a}$

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}$$

微分方程組被轉化為矩陣的本徵值問題。

$$-\omega_0^2\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}\cdot\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} = -\omega^2\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}$$

本徵值一般是以行列式為零的條件解出，但在此行列式的計算並不實際！

$N\times N$對稱正定矩陣應該洽有N個正本徵值，對應N個正本徵向量。

因此此系統有N個模式！

本徵方程式其實就是N個方程式，其中第j個方程式可寫成：

$\omega_0^2\left[-a_{j-1}+2a_j-a_{j+1}\right]=\omega^2 a_j$ 這次用個別式子想比用矩陣更容易。

如果本徵向量的元素如上述是等比數列，左手邊與右手邊的比例是定值，與j無關。

$a_j^{(p)} \sim e^{-ip\cdot j}$ $\frac{a_{j+1}}{a_j} \sim e^{-ip}$ $\frac{a_{j-1}}{a_j} \sim e^{ip}$ 這裏嘗試比例為e^{-ip}，p是一未定的常數。

整式除以a_j，得： $\omega_0^2\left[-e^{ip}+2-e^{-ip}\right]=\omega^2$

只要ω^2的值使此式正確， 第j個方程式就滿足。

但注意，此式中沒有j，因此只要ω^2的值使此式正確，所有N個方程式都滿足。

$$\boldsymbol{a}^{(p)}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} \sim \begin{pmatrix} e^{-ip} \\ e^{-i2p} \\ \vdots \\ e^{-ip\cdot j} \\ \vdots \\ e^{-ip\cdot N} \end{pmatrix}$$

的確就是$N\times N$矩陣$\boldsymbol{A}$的本徵向量！

滿足此式的ω^2值就是對應的本徵值。 $\omega_0^2\left[-e^{ip}+2-e^{-ip}\right]=\omega^2$

$$\omega_0^2\left[2-\left(e^{-ip}-e^{ip}\right)\right]=\omega^2$$

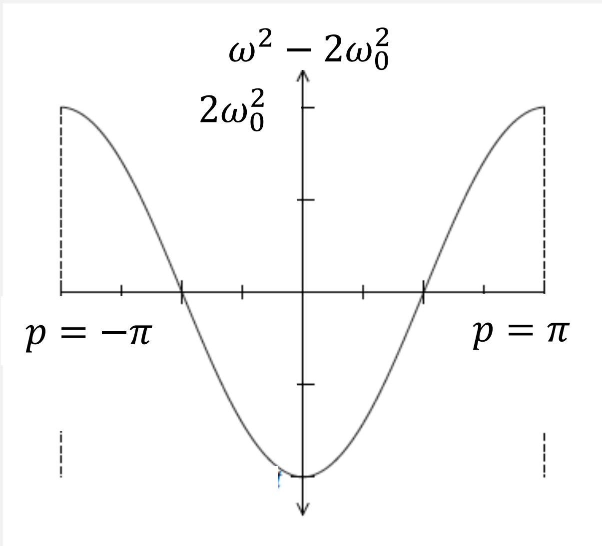
化簡此式就可以得到本徵值ω^2。

$$\omega^2=\omega_0^2(2-2\cos p)$$

注意本徵值ω^2都是正的。

一個p對應一特定的本徵值，p還待決定。

如果把等比數列的配方延伸到x_0, x_{N+1}，不是所有p都可滿足邊界條件。



$$x_0=x_{N+1}=0$$

$$\boldsymbol{X} = \boldsymbol{a}^{(p)} e^{i\omega t} = \begin{pmatrix} e^{-i0p} = 1 \\ e^{-ip} \\ \vdots \\ e^{-ijp} \\ \vdots \\ e^{-iNp} \\ e^{-i(N+1)p} \end{pmatrix} e^{i\omega t}$$

$$a_j^{(p)} \sim e^{-ip\cdot j} = \cos pj + i \sin pj$$

因矩陣$\boldsymbol{A}$是實數矩陣，本徵向量$\boldsymbol{a}$可取其實數部或虛數部都滿足本徵向量方程式：

取其虛數部則可以直接滿足左側的邊界條件： $x_0 = 0$ ➡ $a_0 = 0$

$a_j^{(p)} \sim e^{-ip\cdot j}$ ➡ $a_j^{(p)} = \sin pj$ ➡ $a_0 = 0$

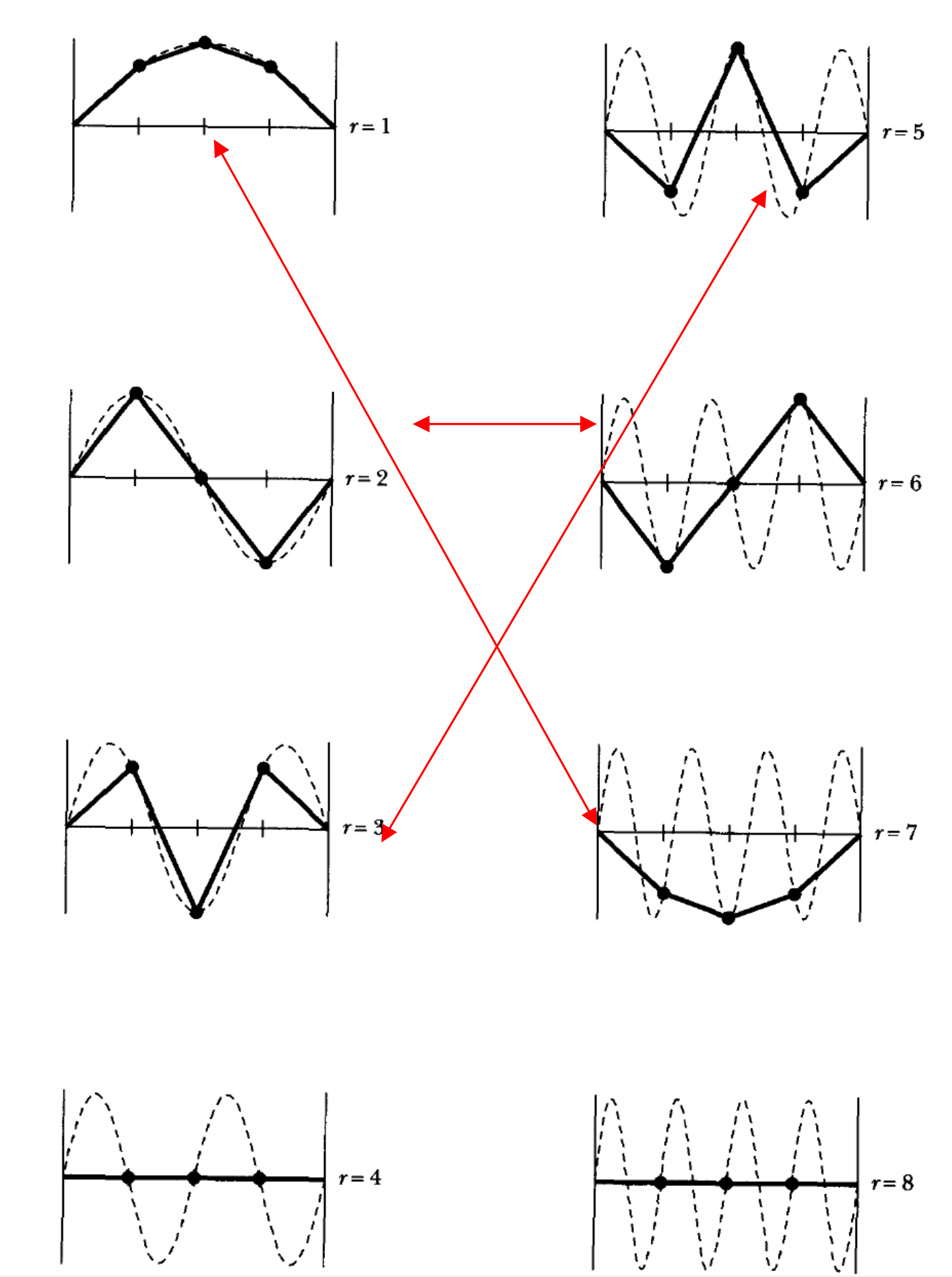
$$\boldsymbol{a}^{(p)} = \sim \begin{pmatrix} 0 \\ \sin p \\ \vdots \\ \sin jp \\ \vdots \\ \sin Np \\ \sin(N+1)p \end{pmatrix}$$

模式本徵向量的元素都是正弦函數值，角度以等差p增加。

若要滿足右側的邊界條件： $a_{N+1} = \sin(N+1)p = 0$

$p \cdot (N+1) = m\pi$ m是任一自然數。

$p = \dfrac{m\pi}{N+1}$ 此角度的等差p是離散的，以m標定： $a_j^{(m)} = \sin \dfrac{m\pi}{N+1} \cdot j$



但還是有無限多個本徵值，這是不可能的。

$p = \frac{m\pi}{N+1}$　　$a_j^{(m)} = \sin pj$

$m = N+1$對應$a_j = 0$。非合理本徵向量。

$m = N+2$與$m = N$對應的p之和為2π，

對應相似的$a_j^{(N+2)} = -a_j^{(N)}$。

其餘$m > N+2$都對應$2N+2-m < N$，

其p和為2π: $a_j^{(m)} = -a_j^{(2N+2-m)}$

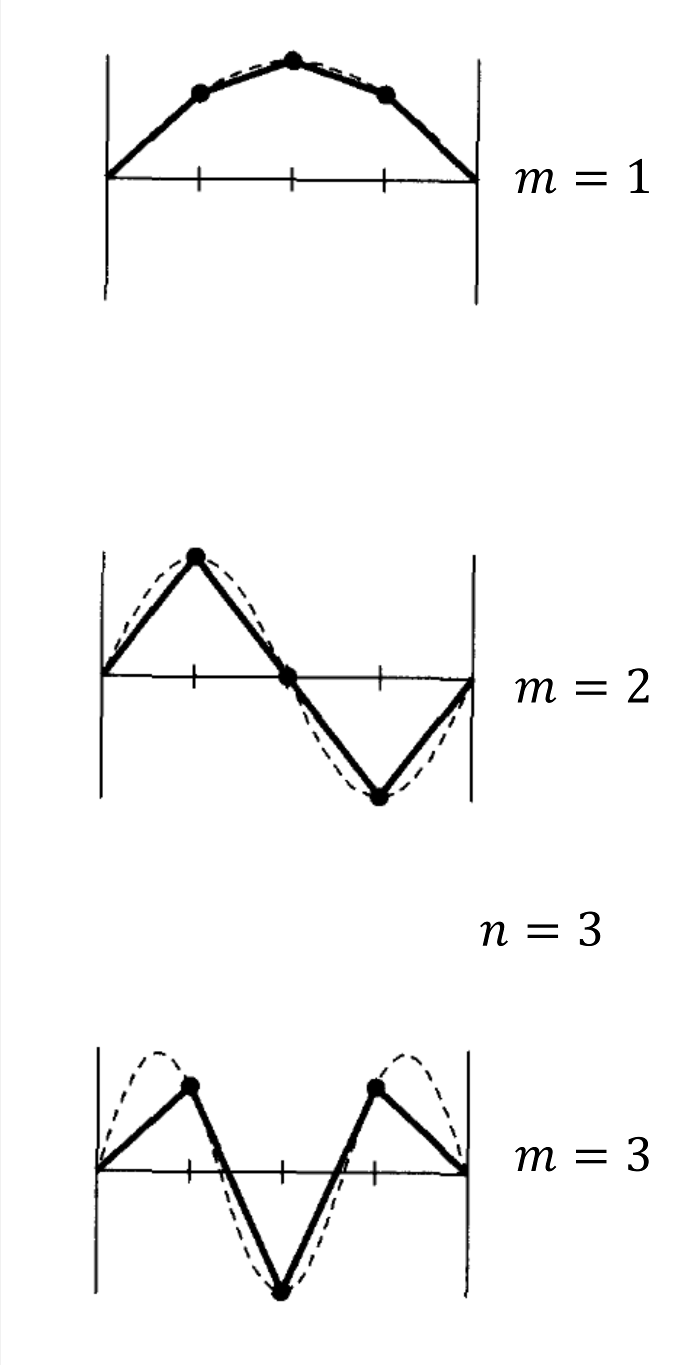
因此m只能取以下這些值，共N個。

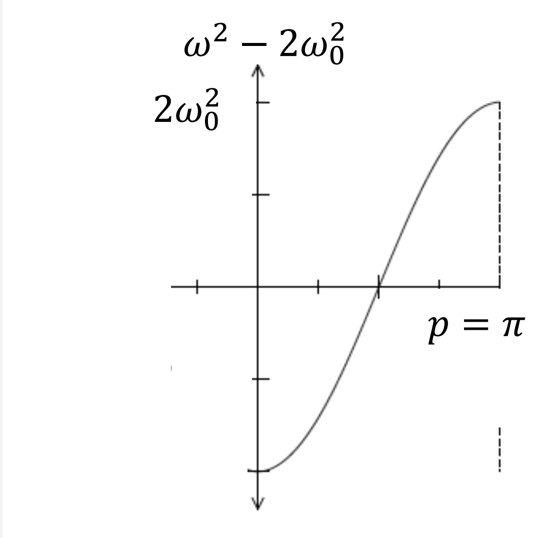
$m = 1,2,3 \cdots N$　本徵向量才是獨立的！

如預期洽有N個本徵值。

一個p對應一特定的本徵值及本徵向量→一個m對應一本徵值及本徵向量。

Summary





一個自然數m對應一本徵值及一本徵向量。

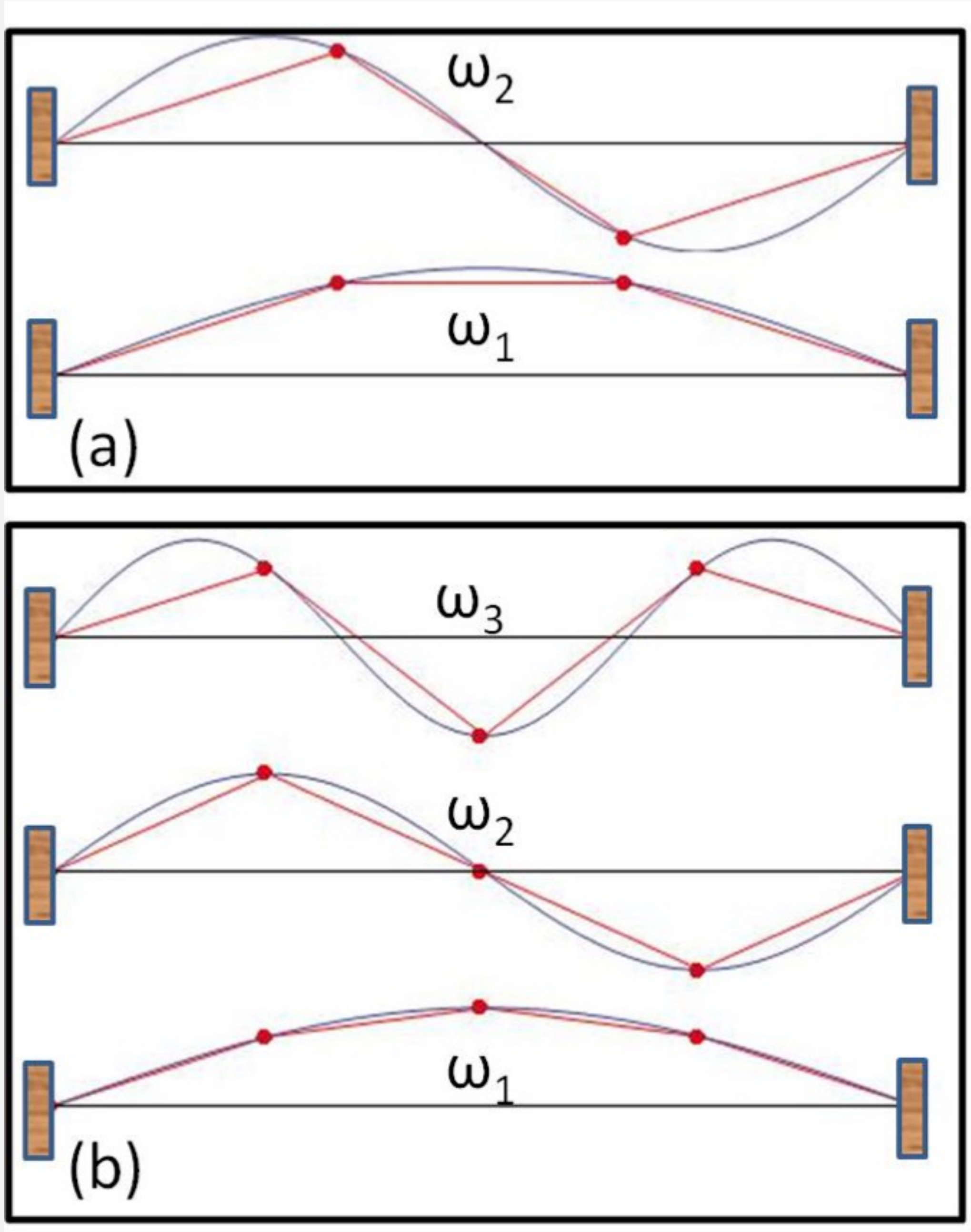
$$p = \frac{m\pi}{N+1} \qquad m = 1,2,3 \cdots N$$

$$\boldsymbol{a}^{(p)} \text{ or } \boldsymbol{a}^{(m)} = \begin{pmatrix} \sin p \\ \vdots \\ \sin(j-1)p \\ \sin jp \\ \sin(j+1)p \\ \vdots \\ \sin Np \end{pmatrix}$$

$$\omega_m^2 = \omega_0^2(2 - 2\cos p)$$

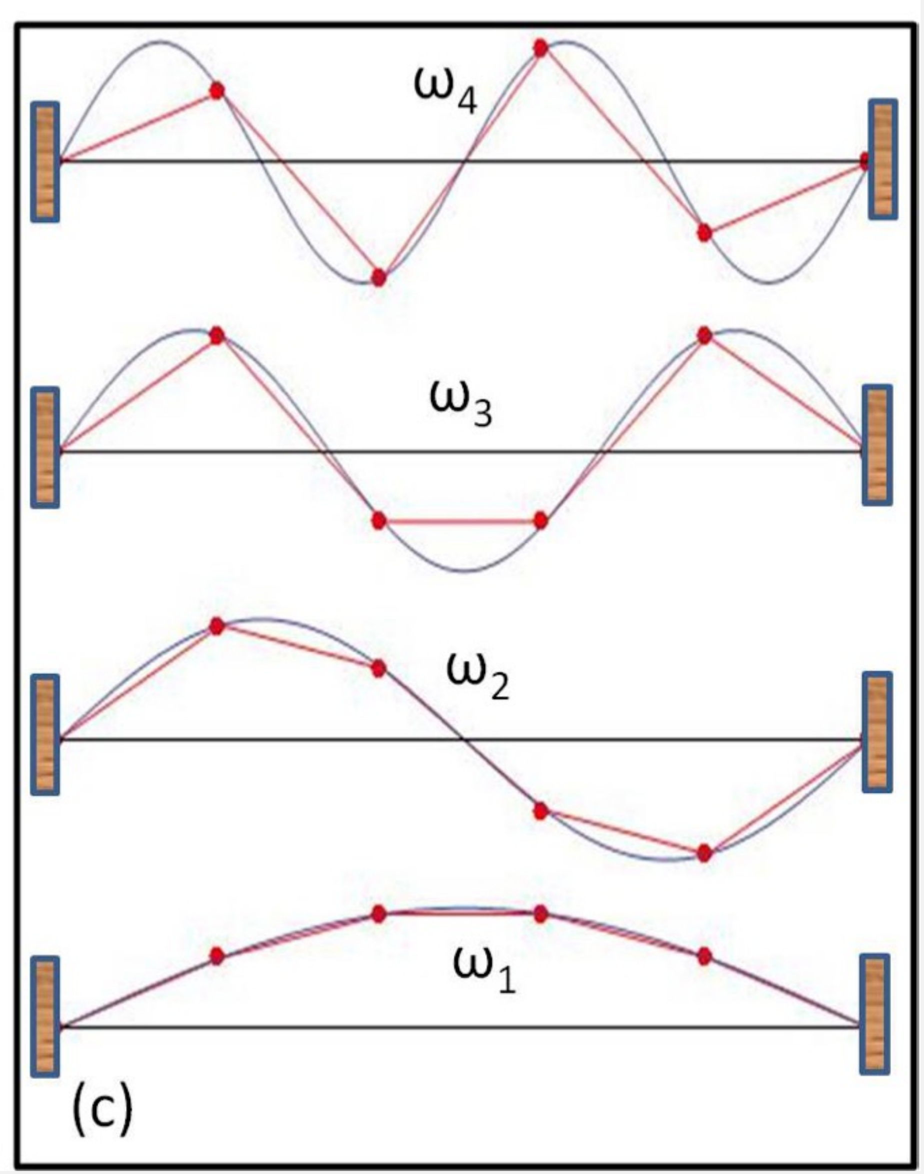
$$\boldsymbol{X} = \boldsymbol{a}^{(m)} C e^{i\omega_m t} \to \boldsymbol{a}^{(m)} \cdot [A_m \cos \omega_m t]$$

一個自然數m對應一個獨立振盪模式(駐波)。



$N = 2$

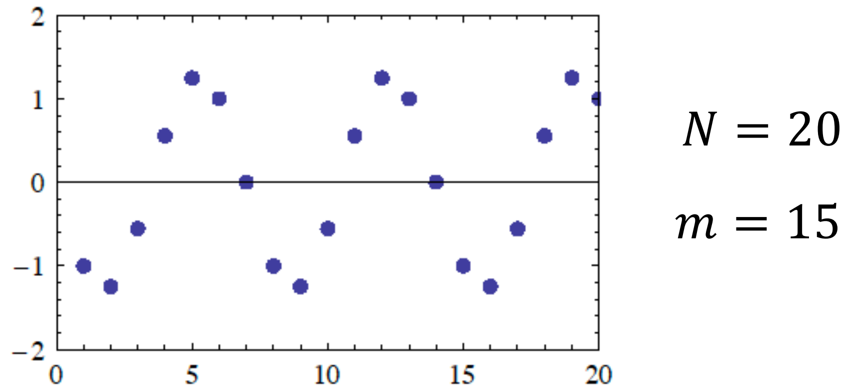
$N = 3$



$N = 4$

$$\boldsymbol{a}^{(p)} \text{ or } \boldsymbol{a}^{(m)} = \begin{pmatrix} \sin p \\ \vdots \\ \sin(j-1)p \\ \sin jp \\ \sin(j+1)p \\ \vdots \\ \sin Np \end{pmatrix}$$

本徵向量的元素恰為正弦函數值，而角度以等距隨粒子編號j增加。



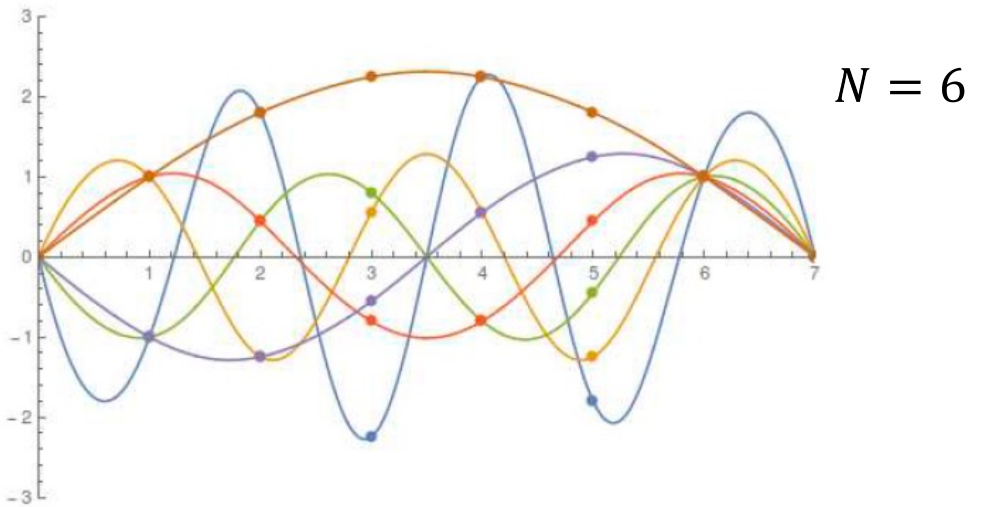


Figure 5. Same as Figure 4, but with interpolations extended to $n = 0$ and $n = 7$ to show the boundary conditions.

本徵向量的元素恰為正弦函數值，而角度以等距隨粒子編號j增加。

有了所有模式的解，一般解將是所有模式的線性組合：

$$\boldsymbol{X} = \sum_{m=1}^{N} \boldsymbol{a}^{(m)} \cdot [f_m \cos(\omega_m t) + g_m \sin(\omega_m t)] = \sum_{m=1}^{N} \boldsymbol{a}^{(m)} \cdot [A_m \cos(\omega_m t + \phi_m)]$$

這個推導的關鍵是下列對本徵向量的猜想：

這不是偶然。來自彈簧組的重覆性。

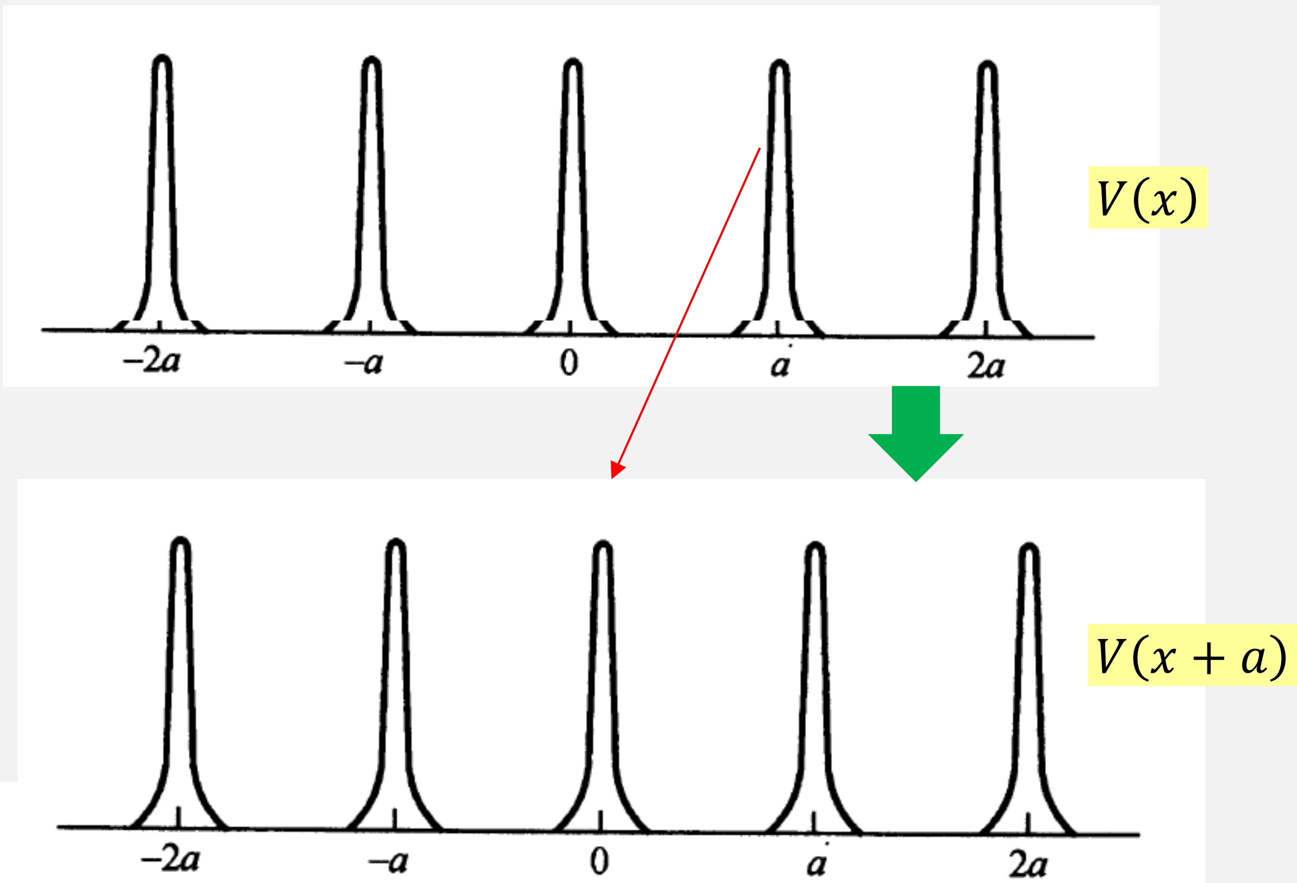
將彈簧組平移一組，彈簧組除了邊界完全不變。

這是為何同樣的運動方程式適用於所有的x_j。

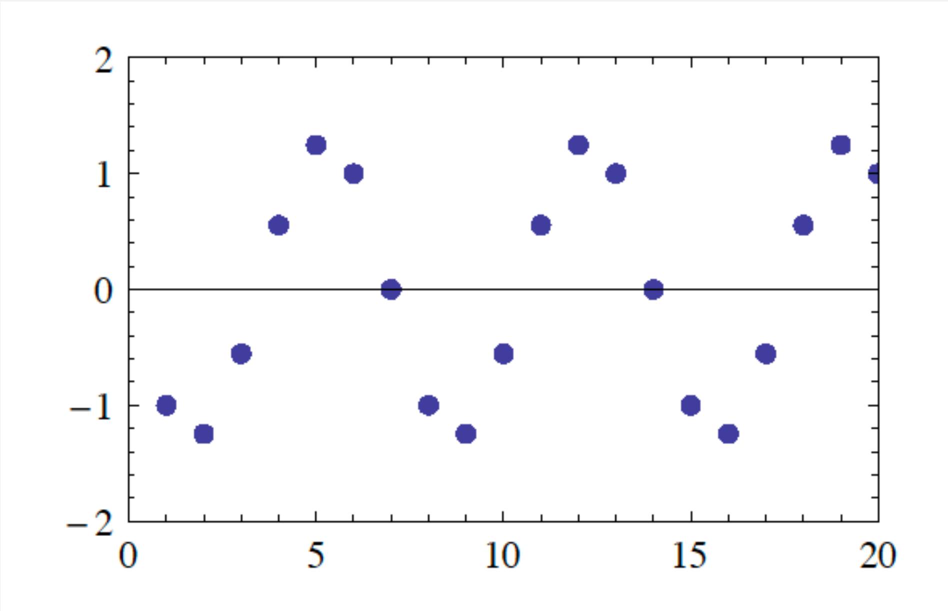
$$\boldsymbol{a}^{(p)} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} \sim \begin{pmatrix} e^{-ip} \\ e^{-i2p} \\ \vdots \\ e^{-ip\cdot j} \\ \vdots \\ e^{-ip\cdot N} \end{pmatrix}$$

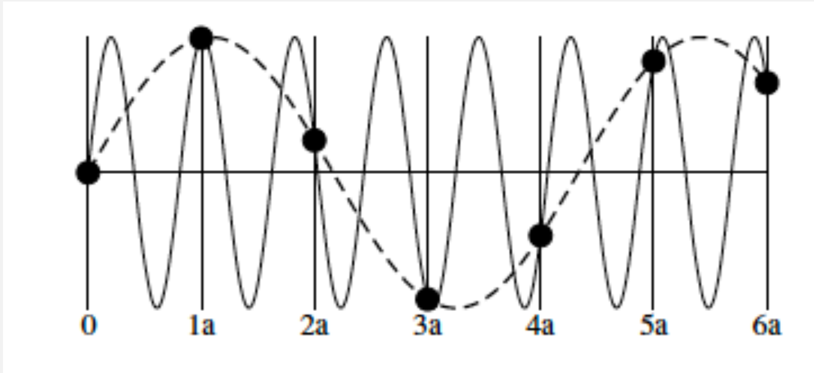
$$\frac{d^2 x_j}{dt^2} = \frac{k}{m} x_{j-1} - \frac{2k}{m} x_j + \frac{k}{m} x_{j+1}$$

這樣的平移對稱決定了本徵向量必須是虛數指數函數的樣式： $a_j^{(p)} \sim e^{-ip\cdot j}$

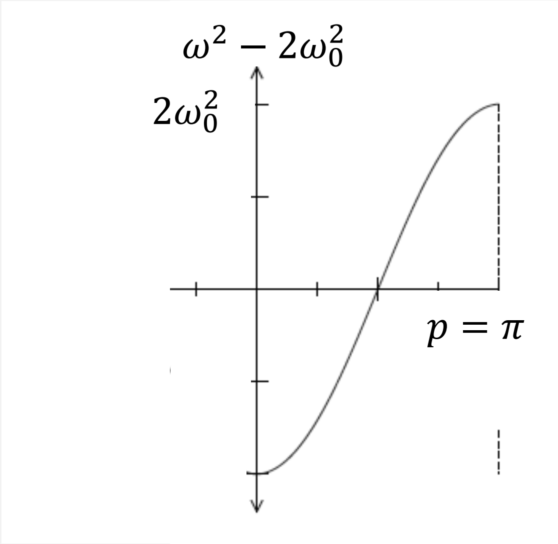


$$a_j^{(p)} \sim \mathrm{Im}\, e^{-ip\cdot j} = \sin jp \rightarrow \sin \frac{p}{d} \cdot jd = \sin kx$$





看起來就像一個正弦波的波形！

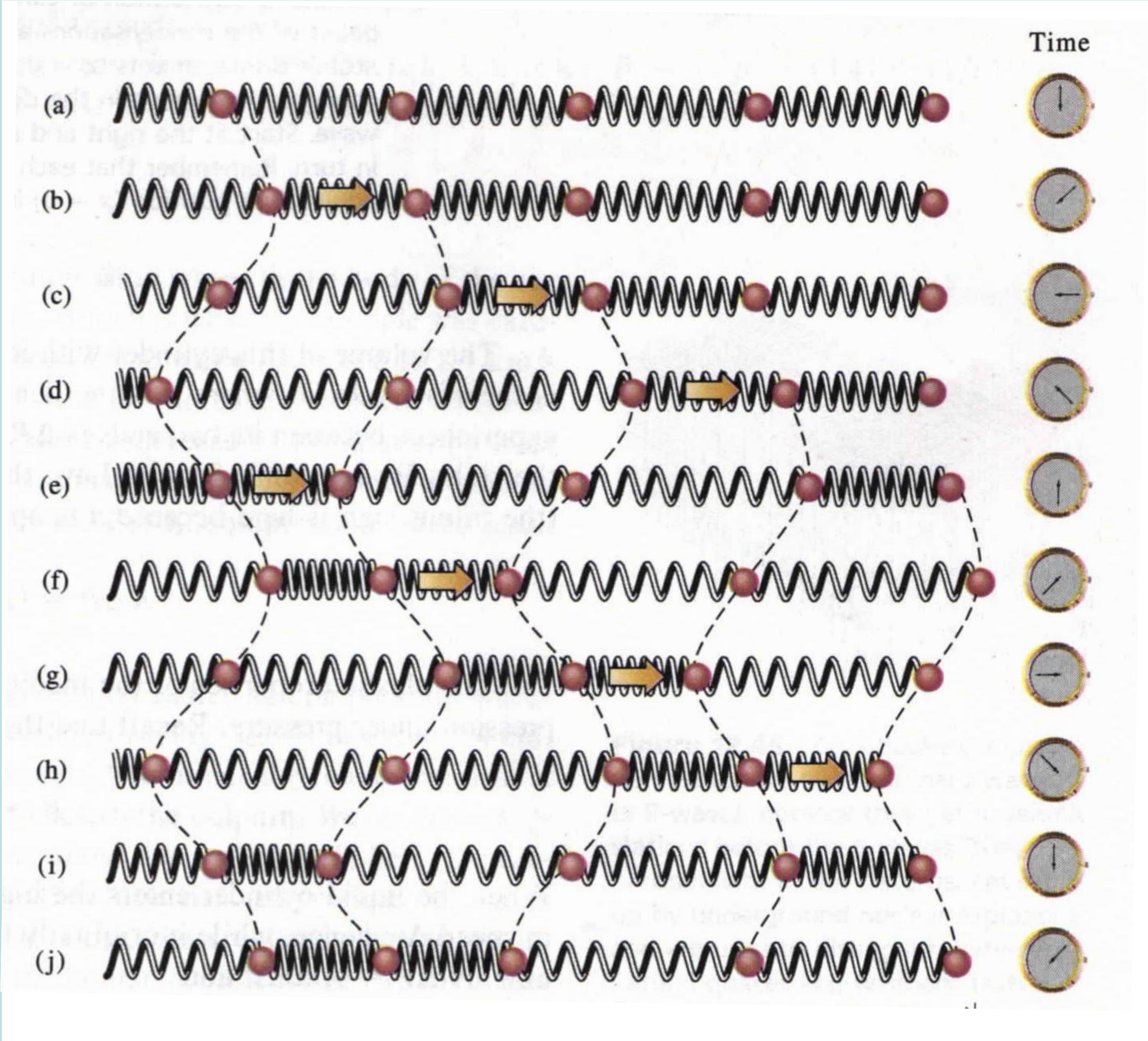


一個自然數m對應一本徵值及一本徵向量。

$p = \dfrac{m\pi}{N+1}$ $m = 1,2,3 \cdots N$

$\omega_m^2 = \omega_0^2(2 - 2\cos p)$

$$\omega_m^2 - 2\omega_0^2 = 2\omega_0^2 \cos p \approx 2\omega_0^2\left(1 + \frac{1}{2}p^2\right) \quad p \ll 1$$


Time
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)

Continuum limit

$$m\frac{d^2x_j}{dt^2} = -k(x_j - x_{j-1}) + k(x_{j+1} - x_j)$$

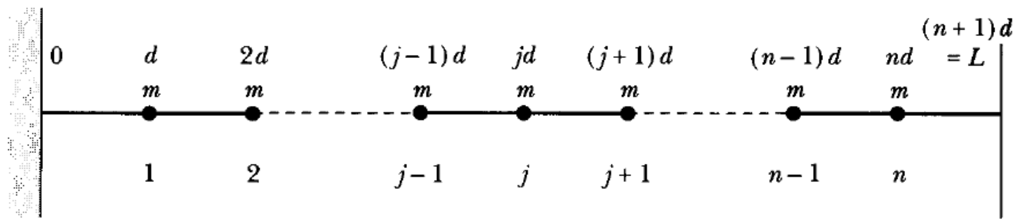
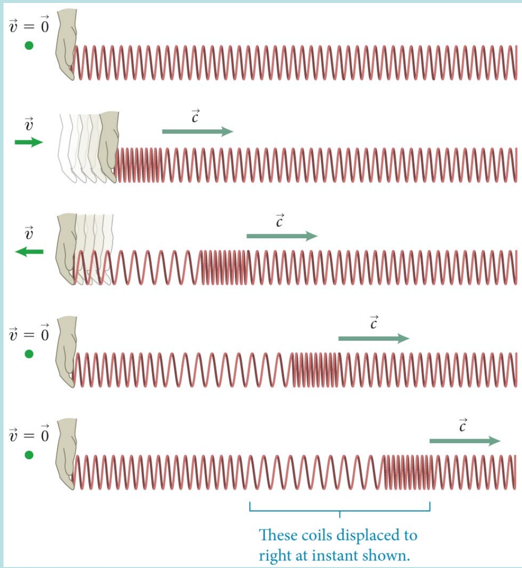
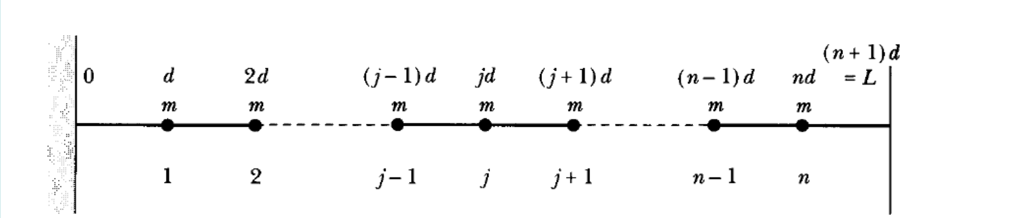


FIGURE 12-9 A schematic of the loaded string. In equilibrium, identical masses are spaced equidistantly. The ends of the string are fixed.

Wave





$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$$

$$m\frac{d^2x_j}{dt^2} = +k(x_{j+1}-x_j) - k(x_j - x_{j-1})$$

$$\boldsymbol{X} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

當粒子數量太龐大，用行向量就不是那麼方便，反而用函數$\phi(x)$來描述更好。

When the number N of particles is huge, column vector is no longer convenient.

We'd rather convert it into a function of the original position $x = dj$

with $\phi(dj) \equiv x_j$ the displacement of the jth particles.

$\phi(dj) \equiv x_j$ 　　$\phi(dj)$就是第j個粒子的位移。

此時函數$\phi(x)$還只有在有粒子處 $x = dj$ 有定義，

This function $\phi(x)$ is defined only at discreet locations $x = dj$.

但當$N \to \infty, d \to 0$，有粒子處的位置就近乎是連續的變數：

But when $N \to \infty, d \to 0$, discreet locations becomes continuous variable x: $dj \to x$

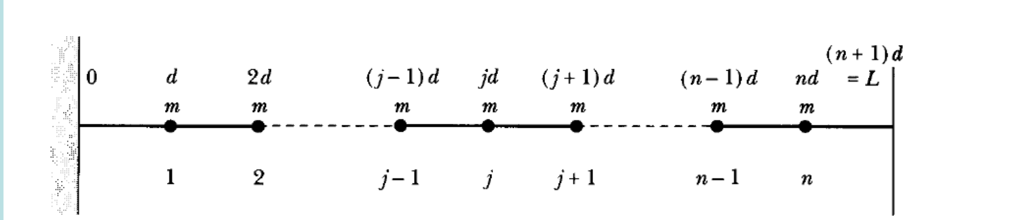
$\phi(dj) \to \phi(x)$ 　$\phi(x)$真的成為x的函數，而且$\phi(x,t)$是雙變數函數。稱為波函數。

$\phi(x)$ becomes a real function of x.

It is time dependent: $\phi(x,t)$. We get a two-variable function, called wavefunction.

$$m\frac{d^2x_j}{dt^2} = +k\left(x_{j+1} - x_j\right) - k\left(x_j - x_{j-1}\right)$$

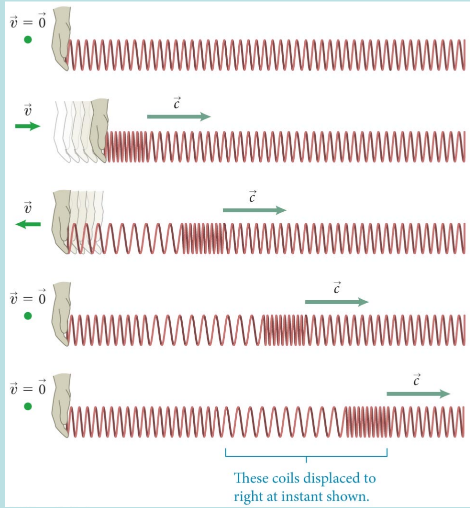
$$\phi(dj) \equiv x_j$$



Let's find the equation for wavefunction from the equation of motion of coupled oscillation.

All you need to do is plugging the corresponding quantities.

$$m\frac{d^2\phi(dj)}{dt^2} = +k\left(\phi(dj+d) - \phi(dj)\right) - k\left(\phi(dj) - \phi(dj-d)\right)$$



Divide the difference of displacements by the separation d.

$$= +kd\frac{\Delta\phi(dj)}{d} - kd\frac{\Delta\phi(dj-d)}{d}$$

$$\rightarrow kd\left[\frac{d\phi}{dx}(dj) - \frac{d\phi}{dx}(dj-d)\right] \qquad = kd^2\frac{\Delta\frac{d\phi}{dx}}{d}$$

$$= kd^2\frac{d^2\phi}{dx^2}(dj) \quad \Rightarrow \quad \frac{d^2\phi}{dt^2}(x) = \frac{kd^2}{m}\frac{d^2\phi}{dx^2}(x)$$

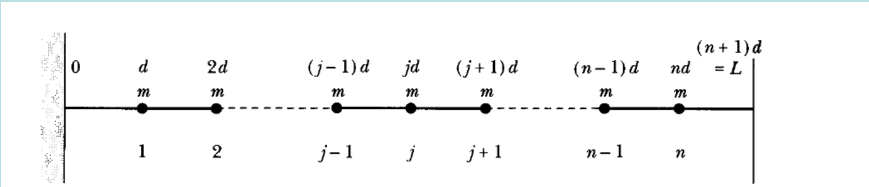
$$v = \sqrt{\frac{kd}{m/d}} \equiv \sqrt{\frac{\tau}{\mu}}$$

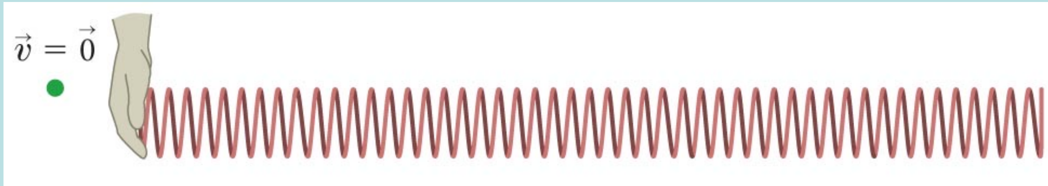
$\frac{d^2\phi}{dt^2}$ is differentiated as $x = dj$ is fixed. That is the partial differentiation $\frac{\partial^2\phi}{\partial t^2}$.

Same for $\frac{d^2\phi}{dx^2}$, differentiated with fixed t.

$$\Rightarrow \quad \frac{\partial^2\phi}{\partial t^2}(x,t) = v^2\frac{\partial^2\phi}{\partial x^2}(x,t)$$

對照表





dj $\quad N \to \infty, d \to 0 \quad$ x

$x_j(t)$ Continuous limit $\phi(dj,t) = \phi(x,t)$ Wave function

$$\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A} \cdot \boldsymbol{X}$$

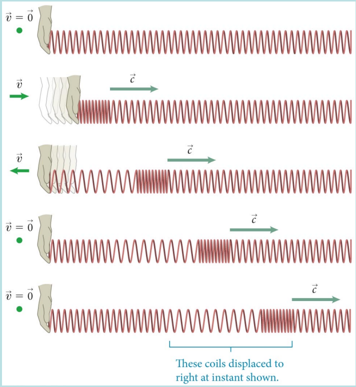
$$\frac{\partial^2 \phi}{\partial t^2}(x,t) = v^2 \frac{\partial^2 \phi}{\partial x^2}(x,t)$$

Wave Equation

This is a partial differential equation PDE.

彈性介質縱波的波方程式

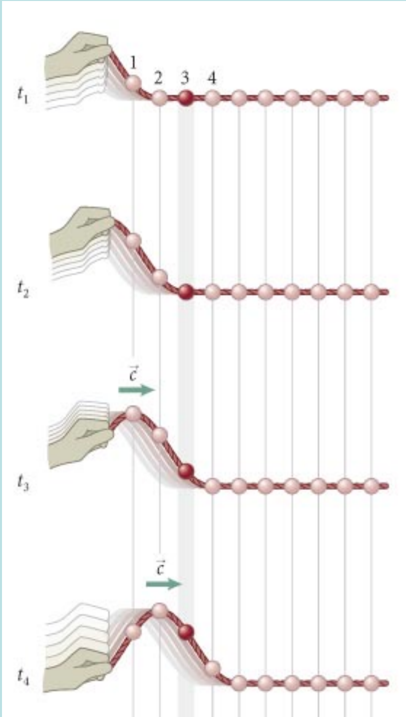
$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2}$$

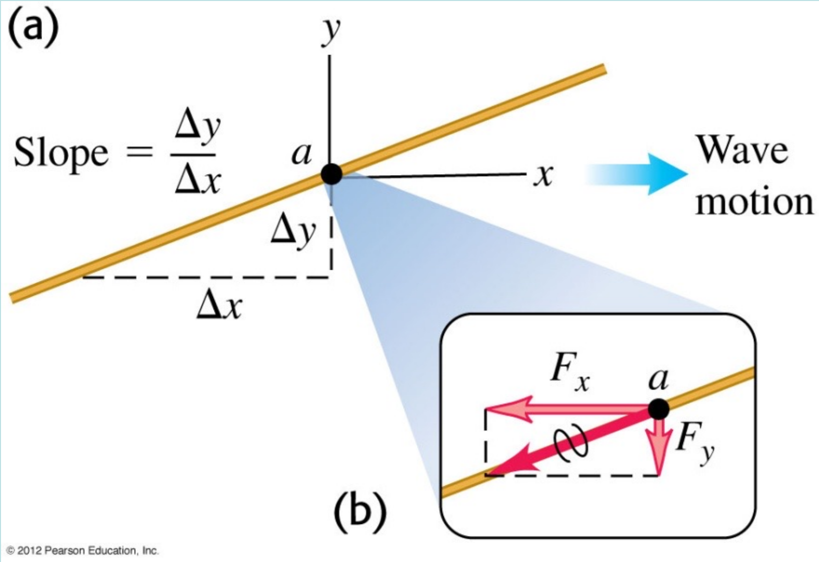


彈性介質橫波的波方程式

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{\tau}{\mu}}$$



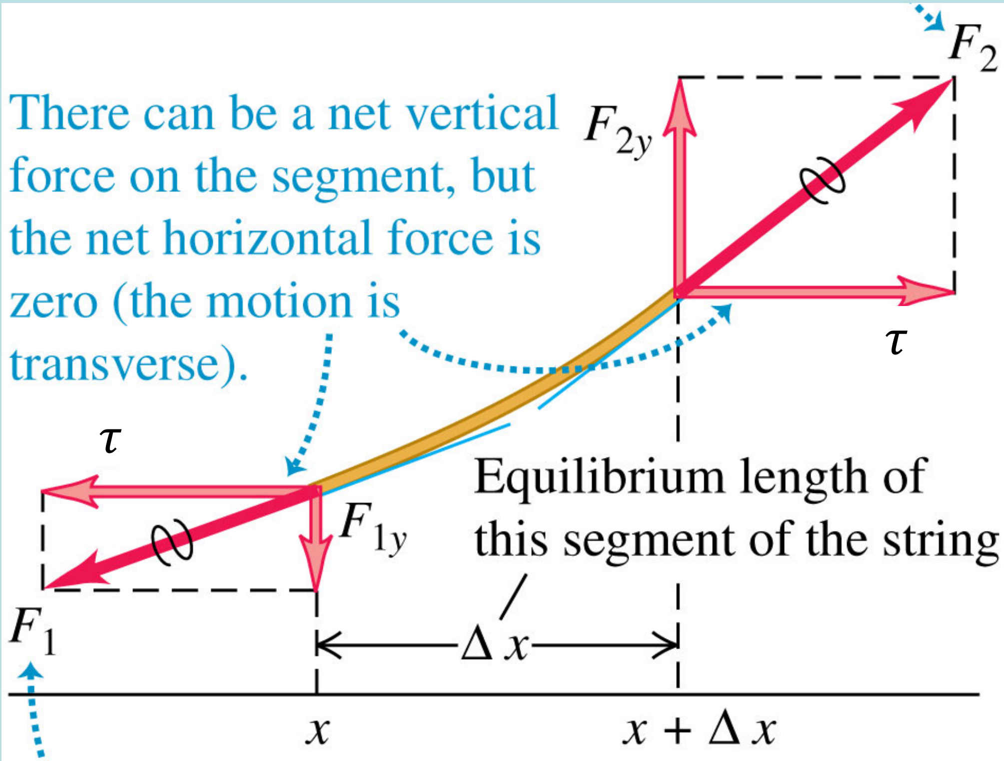


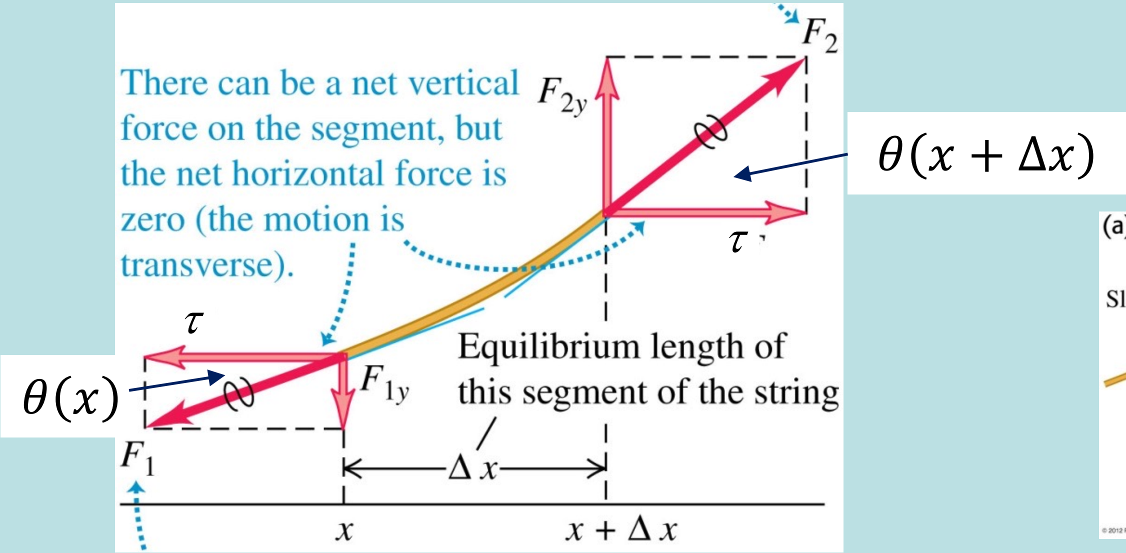
因為弦段在水平方向不動

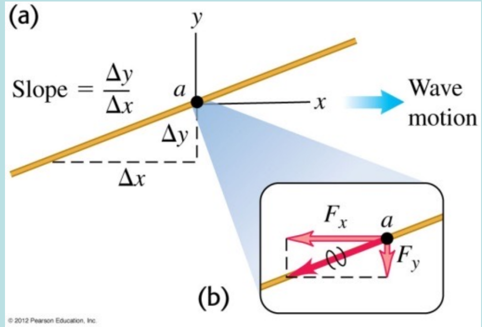
弦段來自兩邊受力的水平分量抵消，

若波幅度不大，此力的水平分量，

就等於弦無波時的張力 τ！







來自左端垂直受力： $F_{1y} = \tau \tan\theta(x)$

右端垂直受力： $F_{2y} = \tau \tan\theta(x+\Delta x)$

垂直總受力

$$\tan\theta = \frac{\partial y}{\partial x}(x,t)$$

斜率隨 x 座標之變化

$$F_{2y} - F_{1y} = \tau\tan\theta(x+\Delta x) - \tau\tan\theta(x) \sim \tau\cdot\left[\frac{\partial y}{\partial x}(x+\Delta x,t) - \frac{\partial y}{\partial x}(x,t)\right] = \tau\cdot\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\cdot\Delta x$$

一小段弦的垂直受力必須等於垂直加速度！

$$\tau\frac{\partial^2 y}{\partial x^2}\cdot\Delta x = (\mu\Delta x)\cdot\frac{\partial^2 y}{\partial t^2}$$

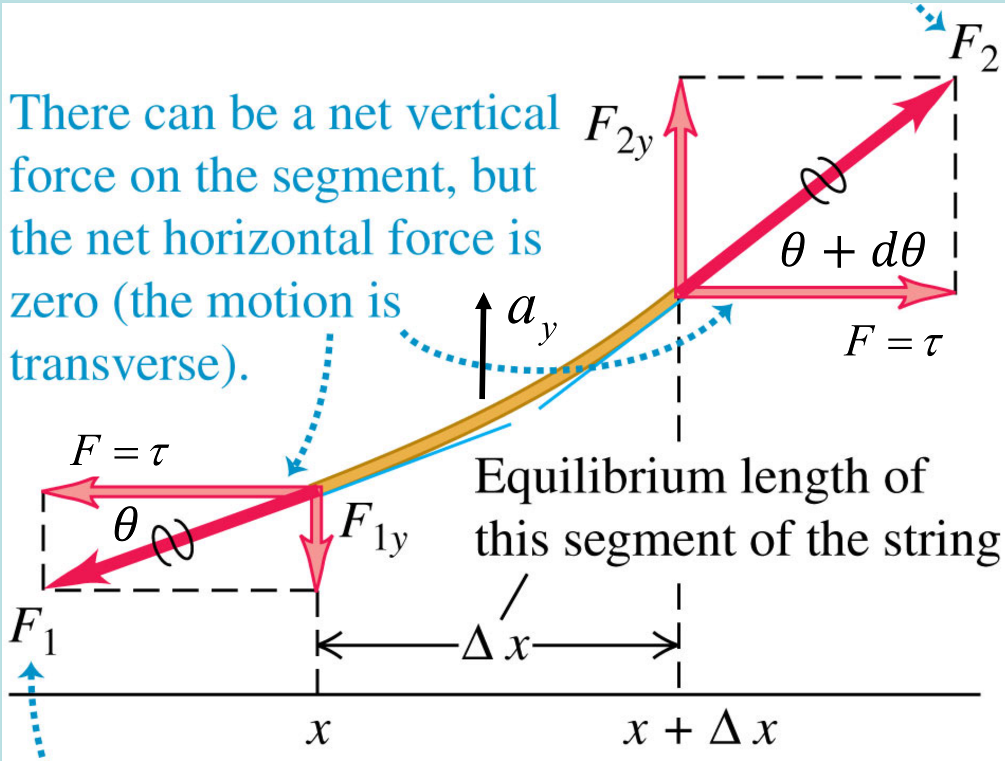
質量

垂直加速度

x 座標變化

斜率隨 x 座標之變化率

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2}$$



一小段弦的垂直受力必須等於垂直加速度！

$$F_{2y} - F_{1y} = \tau \tan(\theta + d\theta) - \tau \tan\theta \sim \tau \cdot \left[\frac{\partial y}{\partial x}(x + \Delta x, t) - \frac{\partial y}{\partial x}(x, t)\right] = \tau \cdot \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \cdot \Delta x$$

$$\tau \frac{\partial^2 y}{\partial x^2} \cdot \Delta x = (\mu \Delta x) \cdot \frac{\partial^2 y}{\partial t^2}$$

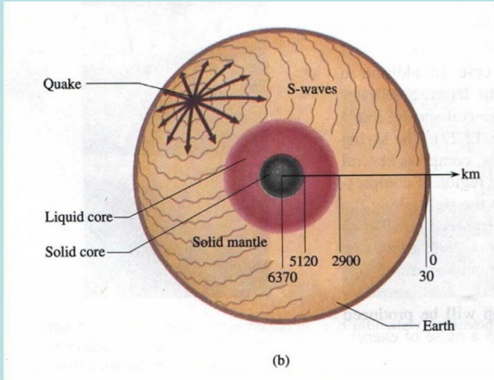
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{\tau} \frac{\partial^2 y}{\partial t^2}$$

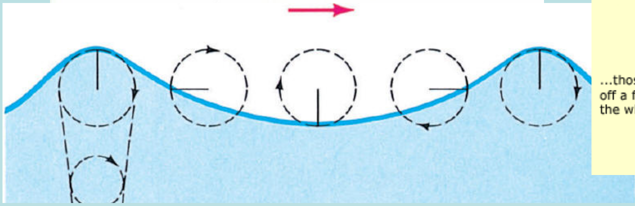
$$v = \sqrt{\frac{\tau}{\mu}}$$

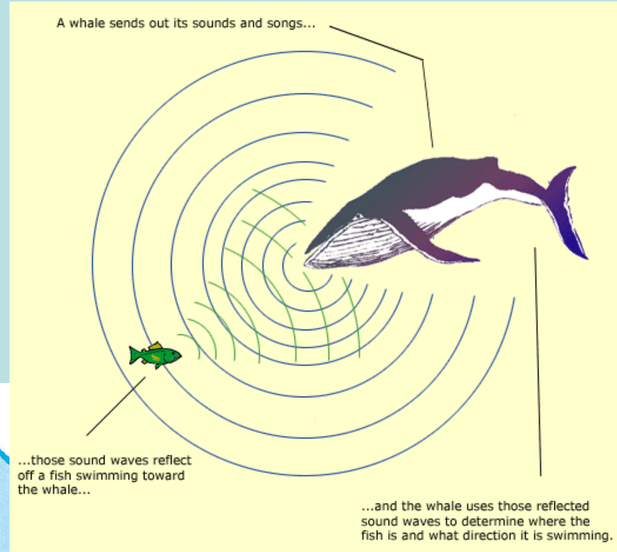
波方程式 Wave Equation

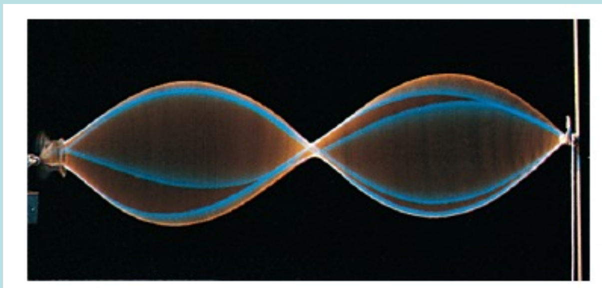
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{\tau}{\mu}}$$









所有波動現象滿足的運動方程式

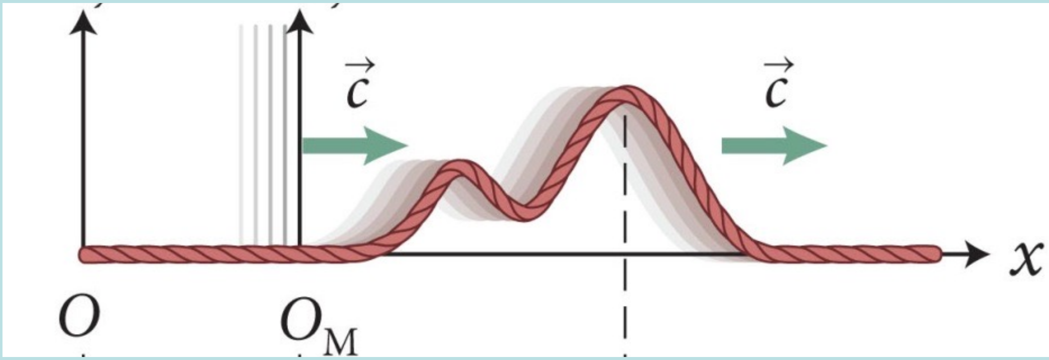
弦整體波函數所滿足的牛頓運動方程式就是波方程式

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{\tau}{\mu}}$$

如果不考慮邊界，

以下證明：這個方程式有以下的解：$f(x - vt)$



$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ 波方程式

代入我們猜出來的波動解，此解的意涵就是它只是x'的函數：

$$y(x,t) = f(x - vt) = f(x')$$ $$x' \equiv x - vt$$

$$\frac{\partial y}{\partial x} = \frac{df}{dx'}\cdot\frac{dx'}{dx} = \frac{df}{dx'}$$ 固定 t

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{dx}\frac{df}{dx'} = \frac{dx'}{dx}\cdot\frac{d}{dx'}\frac{df}{dx'} = \frac{d^2 f}{dx'^2}$$

$$\frac{\partial y}{\partial t} = \frac{df}{dx'}\cdot\frac{dx'}{dt} = \frac{df}{dx'}\cdot(-v)$$ 固定 x

$$\frac{\partial^2 y}{\partial t^2} = \frac{d}{dt}\frac{df}{dx'}\cdot(-v) = \frac{dx'}{dt}\cdot\frac{d}{dx'}\frac{df}{dx'}\cdot(-v) = \frac{d^2 f}{dx'^2}\cdot(-v)^2$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{\tau}{\mu}}$$

我們猜出來的波動解的確是波方程式的解！ 波速真的是常數

x'將時間與空間連鎖在一起，於是時間微分與空間微分也是連鎖的。

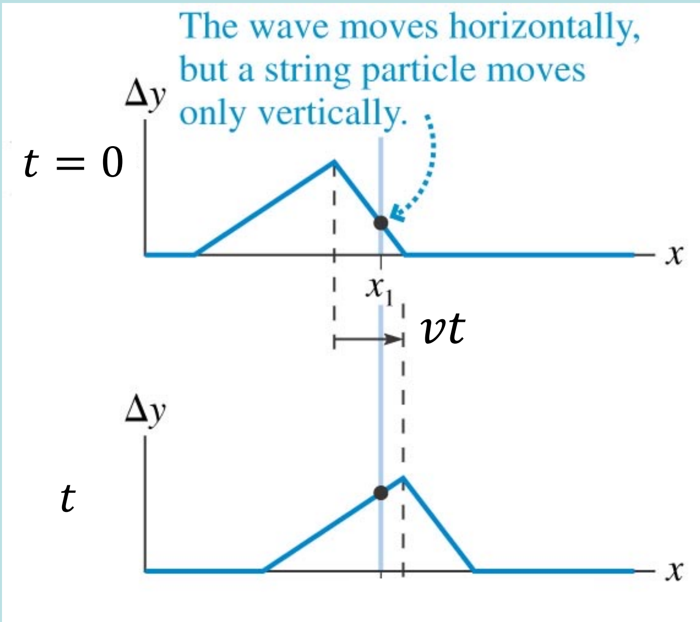
函數$f(x)$就是時間為零時的波型。 $y(x,t) = f(x - vt)$

$$y(x,t) = y(x - vt, 0) = f(x - vt)$$

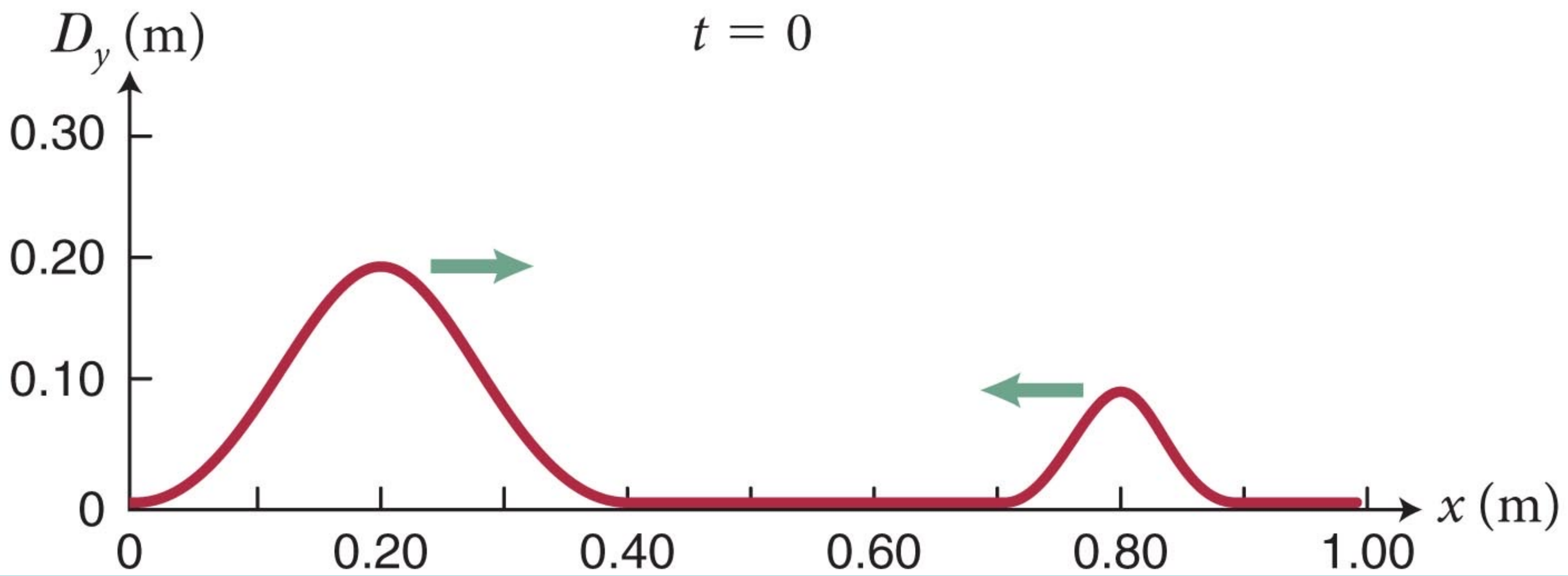
在x處一段弦，時間為t時的位移，

等於時間為零時，位置值等於$x - vt$處的弦的位移。

可見波形是以等速v移動。



向 $-x$ 移動的波，將 v 以 $-v$ 取代即可

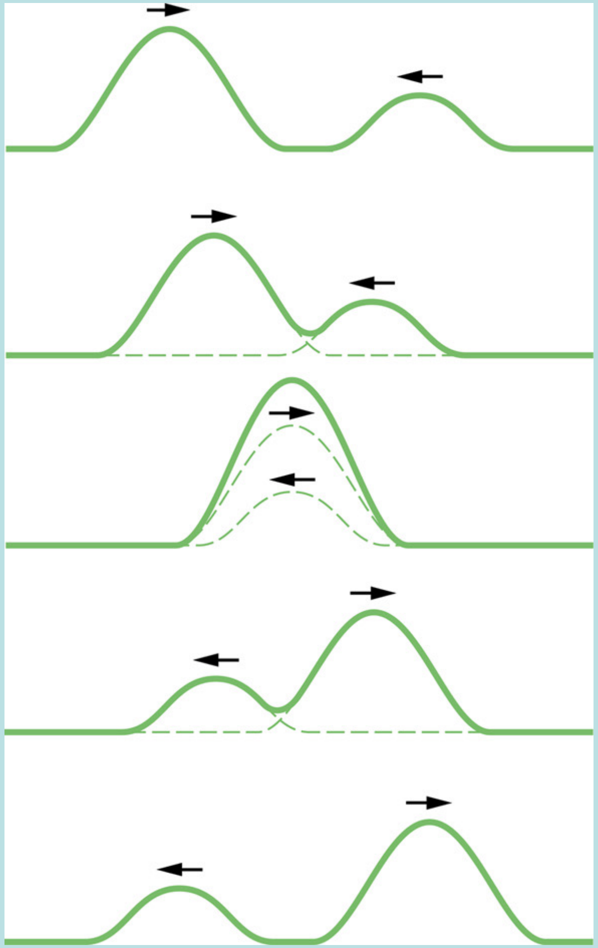


$$y(x,t) = g(x + vt)$$

疊加定理

波方程式是線性方程式：兩個波方程式的解的和依舊是波方程式的解：

$$\frac{\partial^2 y_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y_1}{\partial t^2} \qquad \frac{\partial^2 y_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y_2}{\partial t^2} \quad \rightarrow \quad \frac{\partial^2 (y_1 + y_2)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 (y_1 + y_2)}{\partial t^2}$$



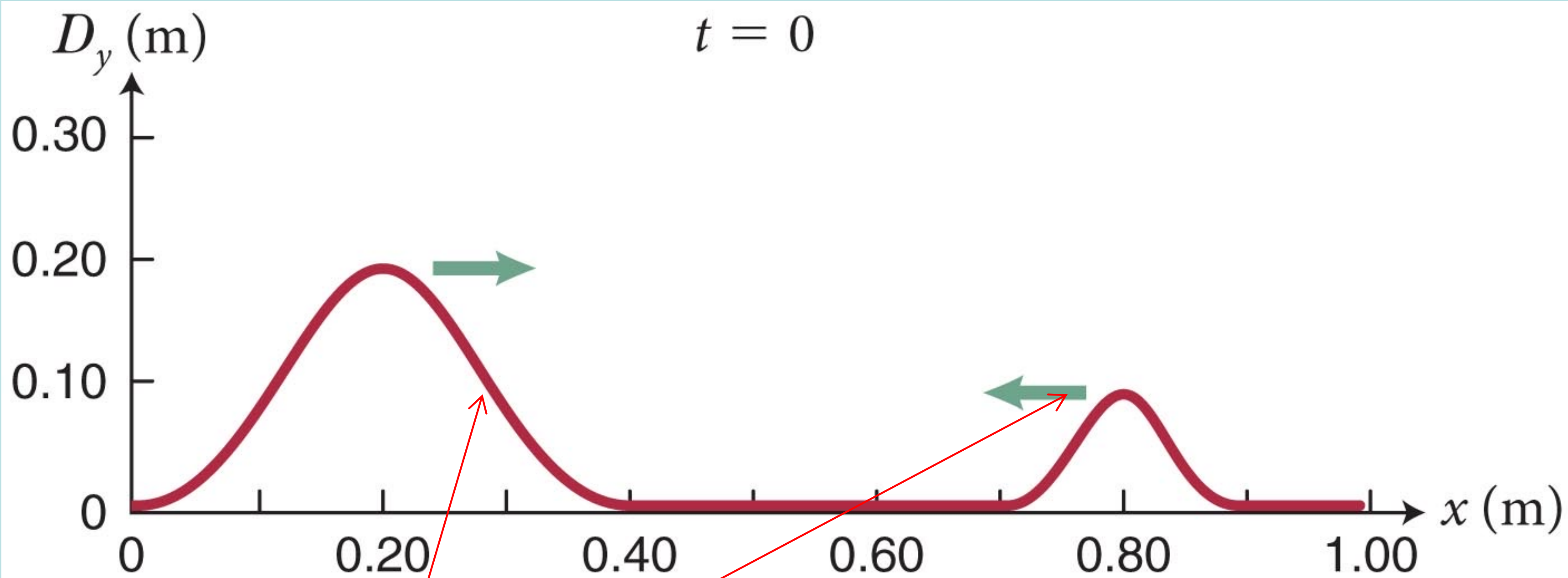
y_1+y_2依舊是波方程式的解：

兩個分立的波重疊時，只要將兩個波函數相加即可。

之後若又分立，原來重疊前的波型不變。

模擬（WaveForm-Impulse)

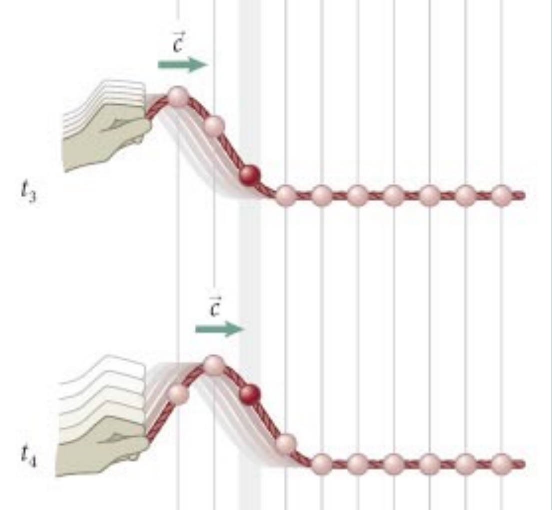
弦波波函數的最普遍解d'Alembert solution： $y(x,t) = f(x-vt) + g(x+vt)$



而 $f(x)$（$g(x)$）即為向右（左）傳播的波的瞬間波型

Jean le Rond d'Alembert 1717-83

不考慮邊界的話，



波方程式其實是一群粒子的運動方程式。

它的唯一解是由所有粒子的初位置及初速度兩個起始條件完全決定。

在此這就對應：起始的弦位移$y(x,0)$，起始的弦垂直方向速度$\frac{\partial y}{\partial t}(x,0)$。

由$y(x,0)$及$\frac{\partial y}{\partial t}(x,0)$正好可以解出這兩個函數：

$$y(x,0)=f(x)+g(x)$$

$$y(x,0),\frac{\partial y}{\partial t}(x,0)\leftrightarrow f(x),g(x)$$

$$\frac{\partial y}{\partial t}(x,0)=v[f'(x)-g'(x)]$$

因此由$f(x),g(x)$得到滿足方程式及起始條件的唯一解：

$$y(x,t)=f(x-vt)+g(x+vt)$$

不考慮邊界時的波方程式

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

解：$f(x - vt) + g(x + vt)$

在這些波函數的解之中，時間與空間並不獨立，而是連鎖在一起 x'，

對空間的偏微分與時間的偏微分基本上都是函數 f 對 x' 的常微分

因此對空間的兩次偏微分與對時間的兩次偏微分成正比。

波動的特徵皆來自此方程式：

波型以定速傳播

波型在傳播過程中不變形

疊加定律

以上結果適用於任何滿足波方程式的波動現象！

單體的運動方程式

Ordinary Differential Equation

多體且彼此耦合的運動方程式

System of ODE

Matrix and Linear Algebra

Eigenvalue problem of Matrix

連續及大量極限

連續介質的波方程式

Partial Differential Equation