System of 2nd order Linear ODE with constant coefficients

$$-\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X}$$

Matrix $\boldsymbol{A}$ is what determines the evolution of the system.

$$\boldsymbol{A} \equiv \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Assume that $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

$$\frac{d^2\boldsymbol{a}e^{i\omega t}}{dt^2} = \omega^2\boldsymbol{a}e^{i\omega t} = \boldsymbol{A}\cdot\boldsymbol{a}e^{i\omega t}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

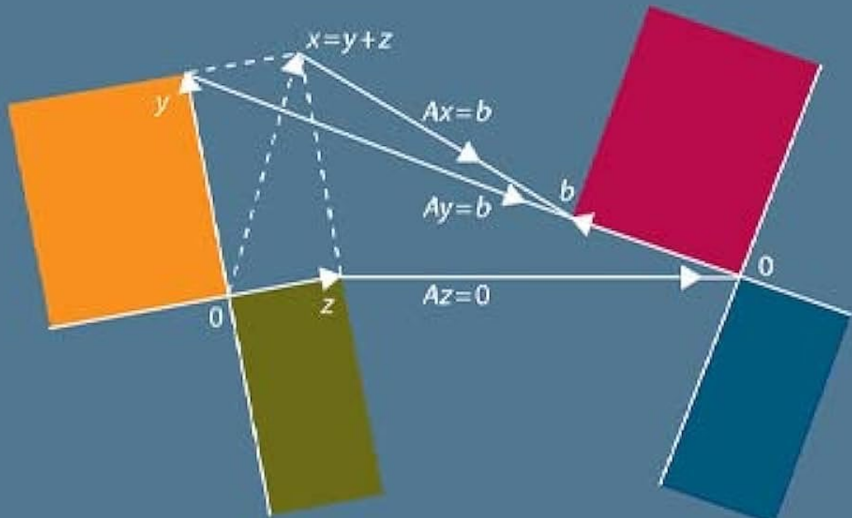
這是矩陣的本徵問題：線性代數Linear Algebra的中心問題。

This is the eigenvalue problem of a matrix, central to the subject Linear Algebra.

Introduction to

LINEAR ALGEBRA

SIXTH EDITION



GILBERT STRANG

行向量與矩陣的本徵值問題

以下將以數學的角度來討論，都是規則與定理。

以下或許抽象，但因此任何物理，只要能表示為行向量與矩陣，以下定理都適用。

Chapter 6

Eigenvalues and Eigenvectors

向量或行向量 vector or column vector

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

兩個數(→ N個數，實數或複數)：$a_i, i = 1,2$，組成的一組數學量。

Two numbers $a_i, i = 1,2$ (→ N) forms a mathematical quantity, called column vector。

空間向量就是例子，但行向量不一定是空間向量，可以是彈簧組的位移。

Space vectors are example, but columns need not be space vectors, $\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is not.

Like 3D vectors, Vectors can be multiplied by numbers, and two vectors can add up.

Linear combinations of vectors are still vectors. Vector space is also called linear space.

$\boldsymbol{v} = c_1\boldsymbol{a} + c_2\boldsymbol{b}$ 線性組合 Linear combinations

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + 2.0 \cdot \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \qquad \boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If the components of two vectors are proportional, we say they are in the same direction.

$$\boldsymbol{a} = c\boldsymbol{b}$$

$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ and $\begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$ are in the same direction.

向量vector可以定義長度：

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_1^2}$$

從此延伸，可以定義兩個向量的內積inner product：

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 \qquad |\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$$

行向量的transpose(轉置)稱為列row向量：

$$\boldsymbol{a}^{\mathrm{T}} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\mathrm{T}} = (a_1 \quad a_2)$$

The transpose of columns are called row vectors.

內積可以用行向量與列向量的矩陣乘積來寫：

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 = (a_1 \quad a_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}$$

A row times a column equals a number, the inner product.

兩個向量的內積為零，這兩個向量就彼此正交perpendicular：

$$\boldsymbol{a}^{\mathrm{T}} \boldsymbol{b} = 0$$

2×2 Matrix矩陣是兩組行向量(4個實數或複數)：$a_{ij}, i,j = 1,2$，組成的數學量。

$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ 也可以看成是兩組列向量。兩種看法都有用。

矩陣乘行向量得一行向量：A matrix times a column equals a column.

$$\boldsymbol{Sa} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$

$$(\boldsymbol{Sa})_i = \sum_{j=1}^{2} S_{ij}a_j \quad i = 1,2$$

此規則保證此乘積是線性的 $\boldsymbol{S}(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2) = c_1\boldsymbol{Sa}_1 + c_2\boldsymbol{Sa}_2$

列向量乘矩陣得一列向量： A row times a matrix equals a row.

$$\boldsymbol{a}^{\mathrm{T}}\boldsymbol{S} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} a_1S_{11} + a_2S_{21} & a_1S_{12} + a_2S_{22} \end{pmatrix}$$

$$\left(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{S}\right)_i = \sum_{j=1}^{2} a_jS_{ji} \; i = 1,2$$

矩陣乘矩陣還是矩陣： A matrix times a matrix equals a matrix.

$$\boldsymbol{SA} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S_{11}A_{11} + S_{12}A_{21} & S_{11}A_{12} + S_{12}A_{22} \\ S_{21}A_{11} + S_{22}A_{21} & S_{21}A_{12} + S_{22}A_{22} \end{pmatrix}$$

21 element of $\boldsymbol{SA}$ equals the inner product of the 2nd row of $\boldsymbol{S}$ and the 1st column of $\boldsymbol{A}$.

$$(\boldsymbol{SA})_{ik} = \sum_{j=1}^{2} S_{ij}A_{jk} \qquad i, k = 1,2$$

ik element of $\boldsymbol{SA}$ equals the inner product of the ith row of $\boldsymbol{S}$ and the kth column of $\boldsymbol{A}$.

列向量乘矩陣乘行向量，等於列向量乘行向量，得到一個數：

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{Sa} = \sum_{i,j=1}^{2} b_i S_{ij} a_j$$

A row times a matrix times a column equals a number.

This is called quadratic form.

Arfken p96-99

1.4 Matrix Multiplication AB and CR

1 To multiply AB we need *row length for* A = *column length for* B.

2 The number in row i, column j of AB is **(row i of A) · (column j of B)**.

3 By columns: **A times column j of B produces column j of AB.**

4 Usually AB is different from BA. But always $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$.

5 If A has r independent columns in C, then $\boldsymbol{A} = \boldsymbol{CR} = (\boldsymbol{m} \times \boldsymbol{r})(\boldsymbol{r} \times \boldsymbol{n})$.

We know how to multiply a matrix A times a column vector $\boldsymbol{x}$ or $\boldsymbol{b}$. This section moves to matrix-matrix multiplication: **a matrix A times a matrix B**. The new rule builds on the old one, when the matrix B has columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_p$. We just multiply A times each of those p columns of B to find the p columns of AB.

Column j of AB equals A times column j of B

$$\textbf{If } B = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \end{bmatrix} \textbf{ then } AB = \begin{bmatrix} A\boldsymbol{b}_1 & \cdots & A\boldsymbol{b}_p \end{bmatrix} \tag{1}$$

To see that clearly, start with a 2 by 2 "exchange matrix" for B. So B has two columns $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. We multiply A times each column to produce a column of AB:

$$A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{4} \end{bmatrix} \quad A\boldsymbol{b}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} \quad \boldsymbol{AB} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$$

For this matrix B, the result of multiplying AB is to *exchange the columns of* A.

There is more to see when we multiply the same A by a full 2 by 2 matrix B:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} \end{bmatrix} \text{ has } A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{5} \\ \mathbf{7} \end{bmatrix} \text{ and } A\boldsymbol{b}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{6} \\ \mathbf{8} \end{bmatrix}$$

Here is the point. We can multiply $A\boldsymbol{b}_1$ (matrix times vector) the *row way* or the *column way*. The row way uses dot products of $\boldsymbol{b}_1$ with *every row of* A:

$$\begin{matrix}\textbf{Row way} \\ \textbf{Dot products}\end{matrix} \qquad A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot \boldsymbol{b}_1 \\ \text{row } 2 \cdot \boldsymbol{b}_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} \mathbf{19} \\ \mathbf{43} \end{bmatrix} \tag{2}$$

The column way uses a combination of the *columns* of A to find $A\boldsymbol{b}_1$. Same result:

$$\begin{matrix}\textbf{Column way} \\ \textbf{Combine columns}\end{matrix} \qquad A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \mathbf{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mathbf{7}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} \mathbf{19} \\ \mathbf{43} \end{bmatrix} \tag{3}$$

Both ways use the same 4 multiplications. With numbers like these, I think most people choose the row way. **To multiply AB, take the dot product of each row of A with each column of B.** When A has 2 rows and B has 2 columns, that means 4 dot products.

AB is usually different from BA

For $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, AB exchanged the columns of A. **But BA exchanges the rows of A!**

$$\boldsymbol{AB} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} \qquad \boldsymbol{BA} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \tag{6}$$

Matrix multiplication is not commutative. In general $\boldsymbol{BA \neq AB}$. Multiply A on the left for row operations on A, and multiply on the right by B for column operations on A.

AB times $C = A$ times BC

For matrix multiplication, **this associative law is true.** We are not willing to give up this extremely useful law. We can multiply AB first or we can multiply BC first.
The matrices stay in the order A, B, C and their sizes must be right for multiplication:

$\boldsymbol{A}$ is $\boldsymbol{m \times n}$ $\quad \boldsymbol{B}$ is $\boldsymbol{n \times p}$ $\quad \boldsymbol{C}$ is $\boldsymbol{p \times q}$. Then $\boldsymbol{AB}$ is $\boldsymbol{m \times p}$ and $\boldsymbol{(AB)C}$ is $\boldsymbol{m \times q}$.

We can test the law using the exchange matrix B on the rows and the columns of A:

$$\boldsymbol{(BA)B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{B(AB)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

So row operations on A can come *before or after* column operations on A.

Notice the meaning of $(AB)C = A(BC)$ when C is just a column vector $\boldsymbol{x}$. If that vector $\boldsymbol{x}$ has a single 1 in component j, then the associative law is $(AB)\boldsymbol{x} = A(B\boldsymbol{x})$. This tells us how to multiply matrices! The left side is **column j of AB**. The right side is **A times column j of B**. So their equality is exactly the rule for matrix multiplication that we saw in equation (1). It is simply the right rule.

Let me bring together the important facts about ABC and also A times $B + C$:

$$\boxed{\textbf{Associative} \quad \boldsymbol{(AB)C = A(BC)} \quad \text{and} \quad \textbf{Distributive} \quad \boldsymbol{A(B+C) = AB + AC}} \tag{7}$$

Inverse matrix

$$\boldsymbol{S}^{-1} = \frac{1}{\det \boldsymbol{S}} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

Transpose of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad \boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix} \quad \boldsymbol{A}_{ij}^{\mathrm{T}} = \boldsymbol{A}_{ji}$$

例如 $\boldsymbol{A}_{12}^{\mathrm{T}} = \boldsymbol{A}_{21}$

Symmetric Matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad \boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix}$$ Not symmetric

$$\boldsymbol{S} = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = \boldsymbol{S}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$$ Symmetric $\boldsymbol{S}_{ij} = \boldsymbol{S}_{ij}^{\mathrm{T}} = \boldsymbol{S}_{ji}$

例如 $\boldsymbol{S}_{12} = \boldsymbol{S}_{12}^{\mathrm{T}} = \boldsymbol{S}_{21}$

Arfken p99, 104-107

System of 2nd order Linear ODE with constant coefficients

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$$

Matrix $\boldsymbol{A}$ is what determines the evolution of the system.

複變函數法 $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

Eigenvalue problem of matrix

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

這是線性代數Linear Algebra的中心問題。

Eigenvalue problem of matrix

$$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$$

這代表矩陣$\boldsymbol{A}$乘在行向量$\boldsymbol{a}$，與一常數乘法沒有差異，並未改變$\boldsymbol{a}$的方向。

This means multiplying a column by a matrix does not change its direction.

任一矩陣$\boldsymbol{A}$有特定的行向量$\boldsymbol{a}$，及對應的常數λ滿足上式。

For any specific matrix, there are specific columns $\boldsymbol{a}$ with specific constant λ that satisfy the condition.

行向量$\boldsymbol{a}$稱為矩陣$\boldsymbol{A}$的本徵向量eigenvector，常數λ稱為本徵值eigenvalue。

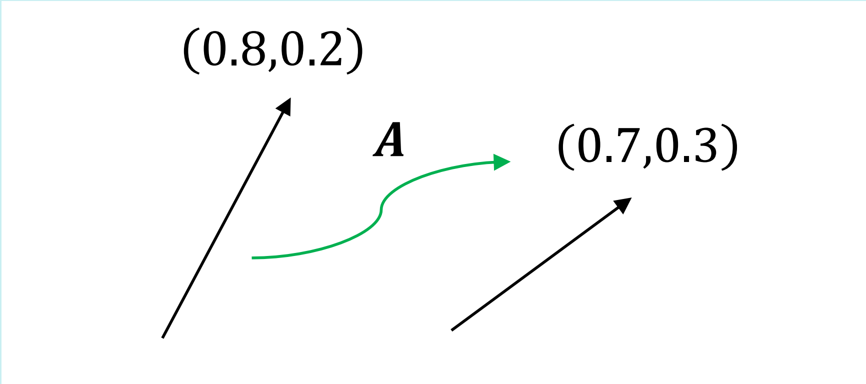
$\boldsymbol{a}$’s are called eigenvectors while $\lambda' s$ called eigenvalues of the matrix $\boldsymbol{A}$.

To find $\boldsymbol{a}$’s and $\lambda' s$ is called eigenvalue problem of the matrix $\boldsymbol{A}$.

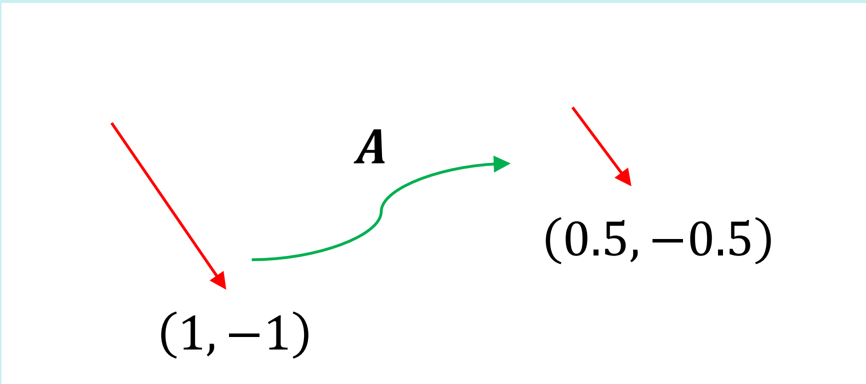
For example, take $\boldsymbol{A}$ as a 2×2 matrix. $\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$

In general, $\boldsymbol{A}$ times a vector will produce another vector not in the same direction.

$$\boldsymbol{A}\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \neq c \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$$



But there are two special vectors that matrix multiplication is just a number product.



$$\boldsymbol{A} \cdot \boldsymbol{a}^{(2)} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For them, $\boldsymbol{A}$ times the vector will produce a vector in the same direction.

$\boldsymbol{A}$ times its eigenvector will produce a vector in the same direction.

Solving eigenvalue problem of matrix

$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$ The procedure is general.

$$\boldsymbol{S}^{-1} = \frac{1}{\det \boldsymbol{S}} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a} = 0$$

$\boldsymbol{A} - \lambda \boldsymbol{I}$ is also a matrix. If it has an inverse, the vector $\boldsymbol{a}$ can only be zero:

$$\boldsymbol{a} = (\boldsymbol{A} - \lambda \boldsymbol{I})^{-1} \cdot 0 = 0$$

For λ to be an eigenvalue, the condition is $\boldsymbol{A} - \lambda \boldsymbol{I}$ has no inverse and hence the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ is zero.

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} 0.8 - \lambda & 0.3 \\ 0.2 & 0.7 - \lambda \end{vmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right) = 0$$

$\lambda = \lambda_1 = 1$ or $\lambda_2 = \frac{1}{2}$ are eigenvalues of $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

對特定的λ，我們可以解出對應的$\boldsymbol{a}$:

For a specific λ, we can solve the corresponding columns $\boldsymbol{a}$.

$$\lambda = \lambda_1 = 1$$

$$(\boldsymbol{A} - \lambda\boldsymbol{I}) \cdot \boldsymbol{a}^{(1)} = \begin{pmatrix} 0.8 - 1 & 0.3 \\ 0.2 & 0.7 - 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{a}_1^{(1)} \\ \boldsymbol{a}_2^{(1)} \end{pmatrix} = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1^{(1)} \\ \boldsymbol{a}_2^{(1)} \end{pmatrix} = 0$$

$$0.2\boldsymbol{a}_1^{(1)} = 0.3\boldsymbol{a}_2^{(1)}$$

Eigenvector has one undetermined constant.

$$\boldsymbol{a}^{(1)} = c_1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

同理：

$$\lambda = \lambda_2 = \frac{1}{2}$$

$$(\boldsymbol{A} - \lambda\boldsymbol{I}) \cdot \boldsymbol{a}^{(2)} = \begin{pmatrix} 0.8 - 0.5 & 0.3 \\ 0.2 & 0.7 - 0.5 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{a}_1^{(2)} \\ \boldsymbol{a}_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1^{(2)} \\ \boldsymbol{a}_2^{(2)} \end{pmatrix} = 0$$

$$0.3\boldsymbol{a}_1^{(2)} = -0.3\boldsymbol{a}_2^{(2)}$$

$$\boldsymbol{a}^{(2)} = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boldsymbol{A}$ times the eigenvector will produce a vector in the same direction.

$$\boldsymbol{A} \cdot \boldsymbol{a}^{(1)} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = 1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\boldsymbol{A} \cdot \boldsymbol{a}^{(2)} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For the two special eigenvectors, matrix multiplication is just a number product.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.
3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = 0$ ***to find an eigenvector*** $\boldsymbol{x}$.

我們再試一個矩陣，這次是一對稱矩陣！

Solving eigenvalue problem of matrix

$$\boldsymbol{S} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\boldsymbol{S} \cdot \boldsymbol{u} = \lambda \boldsymbol{u}$$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u} = 0$$

If it has an inverse, the vector $\boldsymbol{u}$ can only be zero:

$$\boldsymbol{u} = (\boldsymbol{S} - \lambda \boldsymbol{I})^{-1} \cdot 0 = 0$$

For λ to be an eigenvalue, the condition is $\boldsymbol{S} - \lambda \boldsymbol{I}$ has no inverse and hence the determinant of $\boldsymbol{S} - \lambda \boldsymbol{I}$ is zero.

$$\textbf{det}\,(\boldsymbol{S} - \lambda \boldsymbol{I}) = 0$$

$$\textbf{det}\,(\boldsymbol{S} - \lambda \boldsymbol{I}) = \textbf{det} \begin{bmatrix} 9-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = \lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$

$\lambda = \lambda_1 = 0 \text{ or } \lambda_2 = 10$ are eigenvalues.

對特定的λ，我們可以解出對應的$\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}$

$\lambda = \lambda_1 = 0$

$$\boldsymbol{u}^{(1)} \equiv \begin{pmatrix} \boldsymbol{u}_1^{(1)} \\ \boldsymbol{u}_2^{(1)} \end{pmatrix}$$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u}^{(1)} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{u}_1^{(1)} \\ \boldsymbol{u}_2^{(1)} \end{pmatrix} = 0$$

$$3\boldsymbol{u}_1^{(1)} = -\boldsymbol{u}_2^{(1)}$$

Eigenvector has one undetermined constant.

$$\boldsymbol{u}^{(1)} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

同理：

$\lambda = \lambda_2 = 10$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u}^{(2)} = \begin{pmatrix} 9-10 & 3 \\ 3 & 1-10 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{u}_1^{(2)} \\ \boldsymbol{u}_2^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1^{(2)} \\ \boldsymbol{u}_2^{(2)} \end{pmatrix} = 0$$

$$\boldsymbol{u}_1^{(2)} = 3\boldsymbol{u}_2^{(2)}$$

$$\boldsymbol{u}^{(2)} = c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

There are two beautiful theorems that illustrate the utility of eigenvectors

The eigenvectors and eigenvalues are representations of a matrix!

Theorem: All vectors can be written as the linear combination of eigenvectors $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$.

$$\boldsymbol{X} = c_1\boldsymbol{a}^{(1)} + c_2\boldsymbol{a}^{(2)}$$

These are two equations for two unknowns $c_{1,2}$.

As long as $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ are not in the same directions, $c_{1,2}$ can be solved.

$$\boldsymbol{X} = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = c_1\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$c_1 = 1, c_2 = 0.2$$

$$\boldsymbol{X} = \boldsymbol{a}^{(1)} + 2\boldsymbol{a}^{(2)}$$

Then the action of $\boldsymbol{A}$ on any vector $\boldsymbol{X}$ can be easily written down:

$$\boldsymbol{AX} = \boldsymbol{A}\left(c_1\boldsymbol{a}^{(1)} + c_2\boldsymbol{a}^{(2)}\right) = c_1\boldsymbol{Aa}^{(1)} + c_2\boldsymbol{Aa}^{(2)} = c_1\lambda_1\boldsymbol{a}^{(1)} + c_2\lambda_2\boldsymbol{a}^{(2)}$$

For example:

$$\boldsymbol{AX} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \boldsymbol{A} \cdot \boldsymbol{a}^{(1)} + 2\boldsymbol{A} \cdot \boldsymbol{a}^{(2)}$$

$$= 1\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + 0.2 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

The information in the eigenvectors and eigenvalues alone can represent the matrix!

一個矩陣的本徵值與本徵向量就足夠能代表該矩陣。

Theorem: The eigenvectors $\boldsymbol{a}$ of $\boldsymbol{A}$ are also eigenvectors of $\boldsymbol{A}^n$ with the eigenvalue λ^n.

$$\boldsymbol{A}\cdot\boldsymbol{a}=\lambda\boldsymbol{a} \Rightarrow \boldsymbol{A}^n\cdot\boldsymbol{a}=\lambda^n\boldsymbol{a}$$

$$\boldsymbol{A}^n\boldsymbol{a}=\boldsymbol{A}^{n-1}\boldsymbol{A}\boldsymbol{a}=\boldsymbol{A}^{n-1}\cdot\lambda\boldsymbol{a}=\lambda\boldsymbol{A}^{n-2}\cdot\boldsymbol{A}\boldsymbol{a}=\lambda\boldsymbol{A}^{n-2}\cdot\lambda\boldsymbol{a}\cdots=\lambda^n\boldsymbol{a}$$

The action of $\boldsymbol{A}^n$ on any vector $\boldsymbol{X}$ can also be written down similarly:

$$\boldsymbol{A}^n\boldsymbol{X}=c_1\boldsymbol{A}^n\boldsymbol{a}^{(1)}+c_2\boldsymbol{A}^n\boldsymbol{a}^{(2)}=c_1\lambda_1^n\boldsymbol{a}^{(1)}+c_2\lambda_2^n\boldsymbol{a}^{(2)}$$

$$\boldsymbol{A}^{100}\begin{pmatrix}0.8\\0.2\end{pmatrix}=1^{100}\begin{pmatrix}0.6\\0.4\end{pmatrix}+\left(\frac{1}{2}\right)^{100}0.2\cdot\begin{pmatrix}1\\-1\end{pmatrix}\sim\begin{pmatrix}0.6\\0.4\end{pmatrix}$$

This matrix is called a Markov matrix, with an eigenvalue = 1, the other < 1.

The repeated action of a Markov matrix will push any vector into the eigenvector with λ = 1.

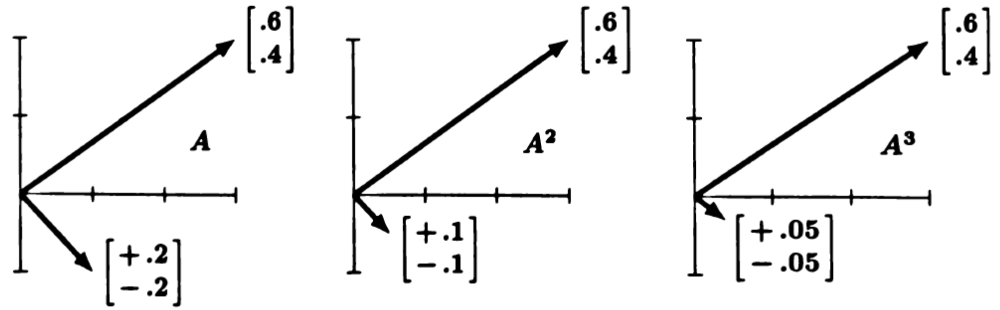


Figure 6.1: The first columns of A, A^2, A^3 are $\begin{bmatrix}.8\\.2\end{bmatrix}, \begin{bmatrix}.7\\.3\end{bmatrix}, \begin{bmatrix}.65\\.35\end{bmatrix}$ approaching $\begin{bmatrix}.6\\.4\end{bmatrix}$.

Diagonalizing a matrix 對角化

$\boldsymbol{A}\cdot\boldsymbol{a}^{(1)}=\lambda_1\boldsymbol{a}^{(1)}$ $\quad$ $\boldsymbol{A}\cdot\boldsymbol{a}^{(2)}=\lambda_2\boldsymbol{a}^{(2)}$

$\boldsymbol{a}_1,\boldsymbol{a}_2$ are column vectors and we can use them to form a matrix.

$$\boldsymbol{U}\equiv\begin{pmatrix}\boldsymbol{a}^{(1)} & \boldsymbol{a}^{(2)}\end{pmatrix}=\left(\begin{pmatrix}\boldsymbol{a}_1^{(1)}\\ \boldsymbol{a}_2^{(1)}\end{pmatrix}\quad\begin{pmatrix}\boldsymbol{a}_1^{(2)}\\ \boldsymbol{a}_2^{(2)}\end{pmatrix}\right)=\begin{pmatrix}\boldsymbol{a}_1^{(1)} & \boldsymbol{a}_2^{(2)}\\ \boldsymbol{a}_2^{(1)} & \boldsymbol{a}_2^{(2)}\end{pmatrix}\equiv\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

$$\boldsymbol{AU}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\left(\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}\quad\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}\right)=\begin{pmatrix}\boldsymbol{Aa}^{(1)} & \boldsymbol{Aa}^{(2)}\end{pmatrix}=\begin{pmatrix}\lambda_1\boldsymbol{a}^{(1)} & \lambda_2\boldsymbol{a}^{(2)}\end{pmatrix}$$

$$=\begin{pmatrix}\lambda_1 a_{11} & \lambda_2 a_{12}\\ \lambda_1 a_{21} & \lambda_2 a_{22}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\boldsymbol{U}\cdot\boldsymbol{\Lambda}$$

$\boldsymbol{AU}=\boldsymbol{U\Lambda}$ 兩邊左乘$\boldsymbol{U}$的反矩陣$\boldsymbol{U}^{-1}$ Multiply both sides on the left by $\boldsymbol{U}^{-1}$：

$\boldsymbol{U}^{-1}\cdot\boldsymbol{A}\cdot\boldsymbol{U}=\boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ is diagonal with eigenvalues as diagonal elements.

$\boldsymbol{U}^{-1}\boldsymbol{AU}$是一個對角矩陣，矩陣元素就是本徵值！

$\boldsymbol{AU = U\Lambda}$ 兩邊右乘$\boldsymbol{U^{-1}}$ Multiply both sides on the right by $\boldsymbol{U^{-1}}$：

$$\boldsymbol{A = U\Lambda U^{-1} = U}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\boldsymbol{U^{-1}}$$

Any Matrix can be decomposed into 3 factors, involving just eigenvalues and eigenvectors.

Info. of eigenvalues   Info. of eigenvectors

$$\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix} = \begin{bmatrix}0.6 & 1\\ 0.4 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}1 & 1\\ 0.4 & -0.6\end{bmatrix}$$

一個矩陣的本徵值與本徵向量就足夠能代表該矩陣。

對角矩陣的冪次很容易計算 It is easy to compute the powers of a diagonal matrix:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$$

$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$ This formula is useful for the following calculation!

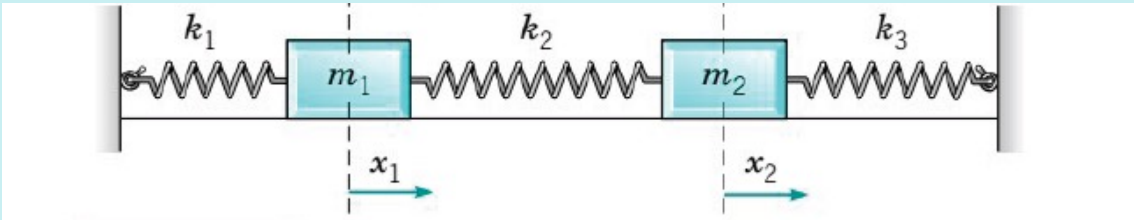
$$\boldsymbol{A}^k = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} \cdot \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} \cdot \cdots \cdot \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}^k\boldsymbol{U}^{-1} = \boldsymbol{U}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}\boldsymbol{U}^{-1}$$

矩陣的指數函數很容易定義。We can define the exponential function of a matrix.

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \cdots$$

$$= \boldsymbol{U}\begin{pmatrix} 1 + \lambda_1^1 + \frac{\lambda_1^2}{2!} + \frac{\lambda_1^3}{3!} + \cdots & 0 \\ 0 & 1 + \lambda_2^1 + \frac{\lambda_2^2}{2!} + \frac{\lambda_2^3}{3!} + \cdots \end{pmatrix}\boldsymbol{U}^{-1} = \boldsymbol{U}\begin{pmatrix} \boldsymbol{e}^{\lambda_1} & 0 \\ 0 & \boldsymbol{e}^{\lambda_2} \end{pmatrix}\boldsymbol{U}^{-1}$$

$$e^{\boldsymbol{A}} = \boldsymbol{U}\begin{pmatrix} \boldsymbol{e}^{\lambda_1} & 0 \\ 0 & \boldsymbol{e}^{\lambda_2} \end{pmatrix}\boldsymbol{U}^{-1}$$



考慮一般的耦合振盪為例：

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} = \lambda\boldsymbol{a}$$

$$\boldsymbol{A} \equiv \begin{pmatrix} \frac{k_1+k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2+k_3}{m} \end{pmatrix}$$

Every eigenvalue corresponds to a single oscillation mode.

But are you sure that the eigenvalues ω_1^2, ω_2^2 are both positive? $\omega_1^2, \omega_2^2 >?\, 0$

If $\omega^2 < 0$, $e^{i\omega t}$ is an exponential function instead of oscillating function.

Will prove if matrix $\boldsymbol{S}$ is symmetric and positive definite, eigenvalues ω^2 are both positive.

And for coupled system the matrix $\boldsymbol{S}$ is symmetric and positive definite.

對稱矩陣

定理：一個對稱矩陣的本徵值都是實數，且不同本徵值的本徵向量彼此正交。

All n eigenvalues λ of a symmetric matrix S are real.

The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

反例：$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

$\boldsymbol{a}^{(1)} = c_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ $\boldsymbol{a}^{(2)} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boldsymbol{a}^{(1)} \cdot \boldsymbol{a}^{(2)} \neq \boldsymbol{0}$

$$\boldsymbol{S} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

這就是對稱矩陣。

$$\lambda = \lambda_1 = 0$$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u}^{(1)} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$3u_{11} = u_{21}$$

$$\boldsymbol{u}^{(1)} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

可以選常數c_1使向量長度為1:

Choose c_1 so that the length equals 1:

$$\boldsymbol{u}^{(1)} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda = \lambda_2 = 10$$

$$\boldsymbol{u}^{(2)} = c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \to \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

很容易確認這兩個本徵向量是彼此正交。 $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ are orthogonal.

$$\boldsymbol{u}^{(2)T} \boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} \cdot \boldsymbol{u}^{(1)} = 0$$

$$\boldsymbol{u}^{(m)T} \boldsymbol{u}^{(n)} = \boldsymbol{u}^{(m)} \cdot \boldsymbol{u}^{(n)} = \delta_{mn}$$

這兩個向量稱為orthonormal.

All n eigenvalues λ's of a symmetric matrix $\boldsymbol{S}$ are real.

Proof:

$$\boldsymbol{Su} = \lambda\boldsymbol{u}$$

Take inner product of the both sides with the complex conjugate of eigenvector $\boldsymbol{u}^*$.

$$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su} = \lambda\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{u}$$

$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{u} = u_1^* u_1 + u_2^* u_2$ is real by definition. 這是複數行向量的內積，定義必定為實數，

要確認$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$是否是實數，取他的Complex conjugate

$$\left(\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}\right)^* = \left(\sum_{i,j=1}^{2} u_i^* S_{ij} u_j\right)^* = \sum_{i,j=1}^{2} u_i S_{ij}^* u_j^* = \sum_{i,j=1}^{2} u_j^* S_{ij} u_i = \sum_{i,j=1}^{2} u_j^* S_{ji} u_i = \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$$

$\boldsymbol{S}$ is real實數 $\boldsymbol{S}$ is 對稱 symmetric

$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$ is real. Hence λ's are real. And $\boldsymbol{u}$'s are real too.

λ是實數，$\boldsymbol{S}$是實數，解$\boldsymbol{u}$的方程式都是實數，$\boldsymbol{u}$自然就是實數。

The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

Proof:

Consider two eigenvectors with different eigenvalues:

$$\boldsymbol{S}\boldsymbol{u}^{(1)} = \lambda_1 \boldsymbol{u}^{(1)}$$

將第一式與$\boldsymbol{u}^{(2)}$作內積，可得：Take inner product of the both sides with $\boldsymbol{u}^{(2)}$.

$$\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(1)} = \lambda_1 \boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)}$$

左邊可以寫成: $S_{ij} = S_{ji}$

$$\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(1)} = \sum_{i,j=1}^{2} u_{2i}S_{ij}u_{1j} = \sum_{i,j=1}^{2} u_{1j}S_{ji}u_{2i} = \boldsymbol{u}^{(1)T}\cdot\boldsymbol{S}\boldsymbol{u}^{(2)} = \boldsymbol{u}^{(1)T}\lambda_2\boldsymbol{u}^{(2)} = \lambda_2\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)}$$

$$\boldsymbol{S}\boldsymbol{u}^{(2)} = \lambda_2 \boldsymbol{u}^{(2)}$$

$$\lambda_1\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)} = \lambda_2\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)}$$

$$\mathbf{0} = (\lambda_1 - \lambda_2)\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)}$$

右式兩本徵向量的內積必須為零，彼此正交！

$\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)}\cdot\boldsymbol{u}^{(1)} = 0$ 本徵向量的正交性

可以選本徵向量的長度都為1，

$\boldsymbol{u}^{(m)T}\boldsymbol{u}^{(n)} = \boldsymbol{u}^{(m)}\cdot\boldsymbol{u}^{(n)} = \delta_{mn}$ 這兩個向量稱為orthonormal.

展開定理

定理：若有兩個orthonormal的向量，任何向量都可展開成他們的線性組合，

With two orthonormal vectors , any vector can be written as their linear combination.

且係數很容易寫下！with the coefficients easily computed.

給定任一向量：$\boldsymbol{v}$，若展開可以如下：

$$\boldsymbol{v} = c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}$$

取向量$\boldsymbol{v}$與本徵向量$\boldsymbol{u_1}$的內積：

$$\boldsymbol{u}^{(1)T}\boldsymbol{v} = \boldsymbol{u}^{(1)T}\left(c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}\right) = c_1\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(2)}$$

代入正交關係：

$$\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(2)} = 0 \qquad \boldsymbol{u}^{(1)T}\boldsymbol{u}^{(1)} = 1$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{v} = \boldsymbol{u}^{(1)} \cdot \boldsymbol{v} = c_1$$

同理：

$$\boldsymbol{u}^{(2)T}\boldsymbol{v} = \boldsymbol{u}^{(2)} \cdot \boldsymbol{v} = c_2$$

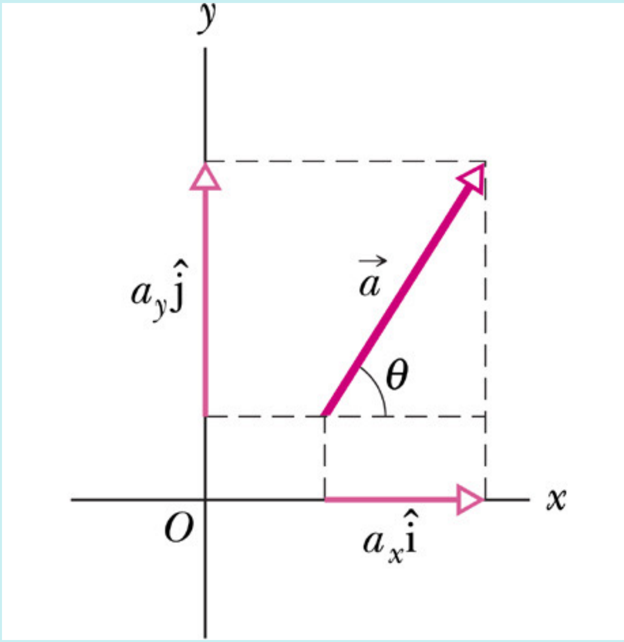
展開定理

給定任一向量：$\boldsymbol{v}$，可以展開如下：

$$\boldsymbol{v} = c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{v} = \boldsymbol{u}^{(1)} \cdot \boldsymbol{v} = c_1$$

$$\boldsymbol{u}^{(2)T}\boldsymbol{v} = \boldsymbol{u}^{(2)} \cdot \boldsymbol{v} = c_2$$



向量可以用它的分量表示！

把一向量投影在選取的座標軸上即是它的分量！

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (a_x, a_y)$ 分量法

$a_x = \hat{\imath} \cdot \vec{a}, \qquad a_y = \hat{\jmath} \cdot \vec{a}$

這兩個orthonormal的本徵向量$\boldsymbol{u}_1, \boldsymbol{u}_2$，類似組成一組座標軸的單位向量$\hat{\imath}, \hat{\jmath}$ ！

使對稱矩陣$\boldsymbol{S}$對角化的矩陣$\boldsymbol{U}$，有很好的性質：

$$\boldsymbol{U} \equiv (\boldsymbol{u}^{(1)} \quad \boldsymbol{u}^{(2)}) = \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\right)$$

$$\boldsymbol{U}^{\mathrm{T}} \equiv \begin{pmatrix} (a_{11} & a_{21}) \\ (a_{12} & a_{22}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \end{pmatrix}$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(2)} = 0$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(1)} = 1$$

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} \equiv \begin{pmatrix} (a_{11} & a_{21}) \\ (a_{12} & a_{22}) \end{pmatrix} \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\right) = \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \end{pmatrix} (\boldsymbol{u}^{(1)} \quad \boldsymbol{u}^{(2)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{U}^{-1} = \boldsymbol{U}^{\mathrm{T}}$$

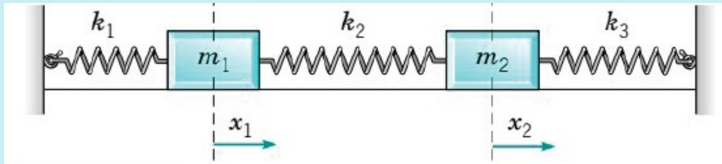
定理：因此任一對稱矩陣$\boldsymbol{S}$可以表示為對角化的矩陣，夾在$\boldsymbol{U}, \boldsymbol{U}^{\boldsymbol{T}}$之間。

$$\boldsymbol{S} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{T} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^{T}$$

$\boldsymbol{U}^{\boldsymbol{T}}$比$\boldsymbol{U}^{-1}$更容易寫！

Positive Definite 正定矩陣與二次型

$$\begin{pmatrix} \dfrac{d^2x_1}{dt^2} \\ \dfrac{d^2x_2}{dt^2} \end{pmatrix} = -\begin{pmatrix} \dfrac{k_1+k_2}{m} & -\dfrac{k_2}{m} \\ -\dfrac{k_2}{m} & \dfrac{k_2+k_3}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \boldsymbol{S}\cdot\boldsymbol{x}$$



彈簧組物理還可以以能量討論：

$$V = \frac{1}{2}k_2(x_2-x_1)^2 + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_3x_2^2$$

$$= \frac{1}{2}(k_1+k_2)x_1^2 + k_2x_1x_2 + \frac{1}{2}(k_2+k_3)x_2^2$$

$$= m(x_1,x_2)\begin{pmatrix} \dfrac{k_1+k_2}{m} & -\dfrac{k_2}{m} \\ -\dfrac{k_2}{m} & \dfrac{k_2+k_3}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv m\,\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$$

Quadratic form 二次型 of the matrix $\boldsymbol{A}$.

要確定ω有兩個實數解，本徵值λ必須是正實數才行！

To be sure that ω has two real solutions, eigenvalue λ must be positive.

若Quadratic form 二次型 $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$ 恆為正值，此矩陣$\boldsymbol{S}$為正定矩陣。

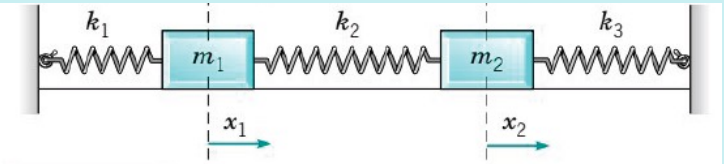
If Quadratic form $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$ is always positive, the matrix $\boldsymbol{S}$ is Positive Definite 正定.

Theorem: if $\boldsymbol{S}$ is positive definite, all the eigenvalues are positive.

Pick $\boldsymbol{x} = \boldsymbol{u}_1$. $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{S}\boldsymbol{u}_1$ is known to be positive。

$$\boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(1)} = \lambda_1\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(1)} = \lambda_1$$

λ_1 is positive and so is λ_2.



彈簧組位能恆為正，因此$\boldsymbol{S}$為正定矩陣：本徵值λ都是正數。

For coupled oscillation, potential is alwayspositive and hence $\boldsymbol{S}$ is positive definite.

$$V = \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_3x_2^2 = m\,\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$$

Both eigenvalues are positive. $\omega = \sqrt{\lambda}$ are both real.

模式頻率：$\omega = \sqrt{\lambda}$ 都是實數。

Theorem: If all the eigenvalues of $\boldsymbol{S}$ are positive, $\boldsymbol{S}$ is positive definite. $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x} > 0$.

$\boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(1)} = \lambda_1 \boldsymbol{u}^{(1)T}\boldsymbol{u}^{(1)} = \lambda_1$ $\quad$ $\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(2)} = \lambda_2$

Any vector $\boldsymbol{x}$ can be expanded as

$$\boldsymbol{x} = c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}$$

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x} = \left(c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}\right)^{\mathrm{T}}\boldsymbol{S}\left(c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}\right)$$

$$= c_1^2\boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(1)} + c_2^2\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(2)} + c_1c_2\left(\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(1)} + \boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(2)}\right)$$

$$= c_1^2\boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(1)} + c_2^2\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(2)} + c_1c_2\left(\lambda_1\boldsymbol{u}^{(2)T}\boldsymbol{u}^{(1)} + \lambda_2\boldsymbol{u}^{(1)T}\boldsymbol{u}^{(2)}\right)$$

已知這兩個本徵向量是彼此正交。

$$= c_1^2\boldsymbol{u}^{(1)T}\boldsymbol{S}\boldsymbol{u}^{(1)} + c_2^2\boldsymbol{u}^{(2)T}\boldsymbol{S}\boldsymbol{u}^{(2)} = c_1^2\lambda_1 + c_2^2\lambda_2 > \boldsymbol{0}$$

$\boldsymbol{S}$ is positive definite. $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x} > 0$.

That $\boldsymbol{S}$ is positive definite is equivalent to that all the eigenvalues of $\boldsymbol{S}$ are positive.

若$\boldsymbol{S}$為正定對稱矩陣：If $\boldsymbol{S}$ is a positive definite symmetric matrix:

$$\boldsymbol{S} = \boldsymbol{U\Lambda U^T} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^T$$

$$= \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^T = \boldsymbol{U}\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}\boldsymbol{U}^T$$

$$= \left[\boldsymbol{U}\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}\right]\left[\boldsymbol{U}\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}\right]^{\boldsymbol{T}} = \boldsymbol{AA^T}$$

定理：正定對稱矩陣永遠可以寫成： $\boldsymbol{S} = \boldsymbol{AA^T}$