

Matthew D. Schwartz

Professor of Physics

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Teaching

I teach graduate and undergraduate students. When I teach, I like to write detailed lecture notes for my courses. I have done so for Quantum Field Theory (Physics 253a,b/254), Waves (Physics 15c), and Statistical Mechanics (Physics 181). The quantum field theory notes have been incorporated into a textbook [Quantum Field Theory and the Standard Model](#),

WAVES

Physics 15c, The Physics of Waves is a sophomore level course for physics majors, the third in the sequence after mechanics and electromagnetism. The course includes a tremendous number of real world applicatoins, such as to the physics of color, music and communication.

Here are the lecture notes for the Spring 2016 version of my course:

- [Lecture 1: Oscillators and linearity](#)
- [Lecture 2: Driven oscillators](#)
- [Lecture 3: Coupled oscillators](#)
- [Lecture 4: From oscillators to waves](#)
- [Lecture 5: Fourier series](#)
- [Lecture 6: Waves](#)
- [Lecture 7: Music](#)
- [Lecture 8: Fourier transforms](#)
- [Lecture 9: Reflection, Transmission and Impedance](#)
- [Lecture 10: Power](#)

Lecture 2: Driven oscillators

1 Introduction

We started last time to analyze the equation describing the motion of a damped-driven oscillator:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = F(t) \quad (1)$$

For small damping $\gamma \ll \omega_0$, we found solutions for $F(t) = 0$ of the form

$$x(t) = A e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi) \quad (2)$$

where the amplitude A and the phase ϕ are determined by initial conditions. Now we will see how to deal with $F(t)$.

We found the damped solution by guessing that an exponential $x(t) = A e^{\alpha t}$ should work, since its derivatives are all proportional to itself. Plugging this ansatz in with $F(t)$ we find

$$A e^{\alpha t} (\alpha^2 + \gamma \alpha + \omega_0^2) = F(t) \quad (3)$$

This will clearly not be solved for constant α unless $F(t)$ happens to be of the form $e^{\alpha t}$. The trick to solving this equation is to use linearity.

Let us suppose that we can write

$$F(t) = \sum_j c_j \cos(\omega_j t) \quad (4)$$

THE
Feynman
LECTURES ON PHYSICS

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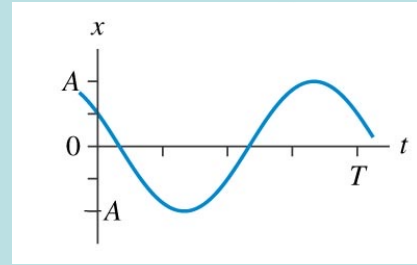
Resonance

23-1 Complex numbers and harmonic motion

In the present chapter we shall continue our discussion of the harmonic oscillator and, in particular, the forced harmonic oscillator, using a new technique in the analysis. In the preceding chapter we introduced the idea of complex numbers, which have real and imaginary parts and which can be represented on a diagram in which the ordinate represents the imaginary part and the abscissa represents the real part. If a is a complex number, we may write it as

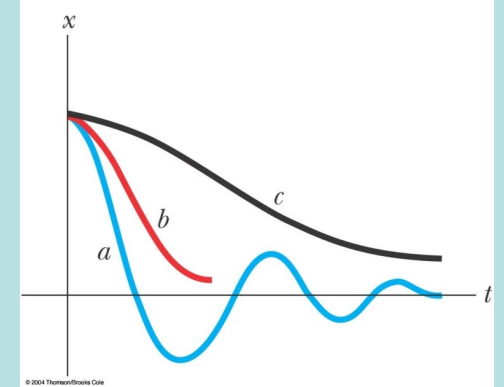
$$z = z_0 e^{\alpha t}$$

If α is imaginary, $z = z_0 e^{i\omega t}$



$$= A e^{i(\omega t + \phi)} = A \cos(\omega t + \phi) + iA \sin(\omega t + \phi)$$

$\text{Re}[z] = A \cos(\omega t + \phi)$ The solution is oscillating SHM.



If α is real, $z = z_0 e^{\pm bt}$

$\text{Re}[z] = A e^{\pm bt}$ The solution is exponentially decreasing or increasing.

If α is complex, $z = z_0 e^{\pm bt + i\omega t} = A e^{\pm bt} \cdot e^{i(\omega t + \phi)}$

$$\text{Re}[z] = A e^{\pm bt} \cdot \cos(\omega t + \phi)$$

The solution is an oscillation with exponentially decreasing amplitude.

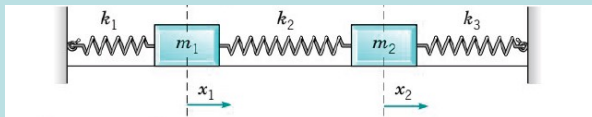
General solutions are linear combinations $z = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} + \dots + c_N e^{\alpha_N t}$

最後取實數部即可得實數解 Pick the real part $x = \text{Re } z$

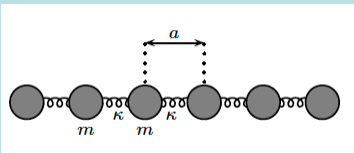
單體的運動方程式



多體且彼此耦合的運動方程式



$$\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$



Vibrations of a
One-Dimensional
Monatomic Chain

9



連續及大量極限

連續介質的波方程式

Ordinary Differential Equation

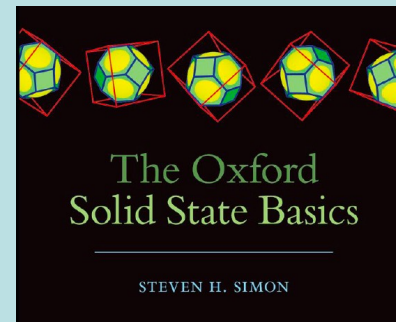
Coupled oscillation

System of ODE

Matrix and Linear Algebra

Eigenvalue problem of Matrix

Solid state Physics model



Partial Differential Equation

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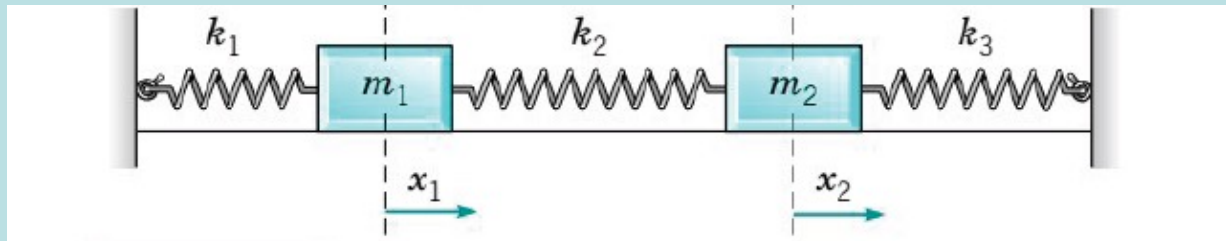
WAVES

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耦合振盪 coupled oscillation, Spring-Mass system



$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2 (x_2 - x_1) = -(k_1 + k_2) x_1 + k_2 x_2$$

The forces exerted on m_1 come from two springs.

Note that the change of length of the right hand-hand side spring: $x_2 - x_1$

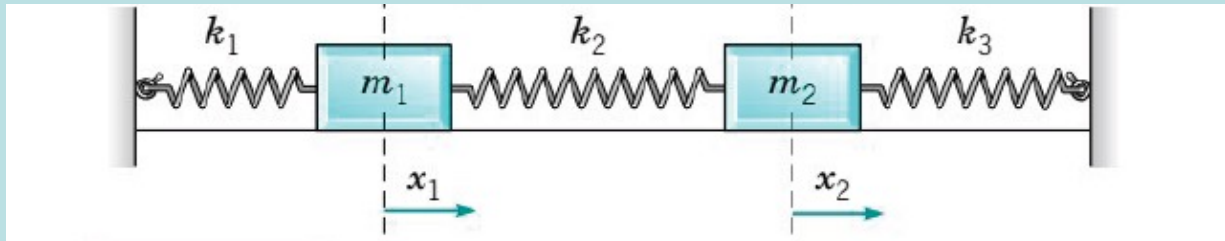
$$m_2 \frac{d^2 x_2}{dt^2} = -k_2 (x_2 - x_1) - k_3 x_2 = k_2 x_1 - (k_2 + k_3) x_2$$

The forces on the two particles are linear combinations of $x_{1,2}$. It is a **linear system**.

It is customary to move all the constants to the right hand side.

$$\frac{d^2 x_1}{dt^2} = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\frac{d^2 x_2}{dt^2} = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$



$$\frac{d^2 x_1}{dt^2} = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\frac{d^2 x_2}{dt^2} = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$

For simplicity of discussion, assume that : $k_1 = k_2 = k$ $m_1 = m_2 = m$

$$\frac{d^2 x_1}{dt^2} = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$$

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\frac{d^2 x_2}{dt^2} = \frac{k}{m} x_1 - \frac{2k}{m} x_2$$

$$\frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2$$

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

This is a System of 2nd order linear ODE.

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2$$

$$\frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2$$

Method 0

先設 $x_{1,2}$ 為複數，猜想其解如簡諧運動也可以寫成虛數的指數函數：

First elevate the real x 's into complex function.

And guess the solution is a complex number exponential function.

We will prove in the future that exponential functions of imaginary number work:

$$x_1 = a_1 \cdot e^{i\omega t}$$

$$x_2 = a_2 \cdot e^{i\omega t}$$

ω is a real unknown to be solved.

注意此猜想中， $x_{1,2}$ 是以同樣的方式一起隨時間振盪： $e^{i\omega t}$ 。

Note that in this guess, $x_{1,2}$ oscillate with time in identical manner: $e^{i\omega t}$.

將解代入方程式 Plug the above formula into the equation:

$$\omega^2 e^{i\omega t} a_1 = 2\omega_0^2 e^{i\omega t} a_1 - \omega_0^2 e^{i\omega t} a_2$$

$$2\omega_0^2 a_1 - \omega_0^2 a_2 = \omega^2 a_1$$

$$\omega^2 e^{i\omega t} a_2 = -\omega_0^2 e^{i\omega t} a_1 + 2\omega_0^2 e^{i\omega t} a_2$$

$$-\omega_0^2 a_1 + 2\omega_0^2 a_2 = \omega^2 a_2$$

微分方程組被轉化為 a_1, a_2 的一次代數方程組。

The original system of ODE is transformed into a system of algebraic Eq of a_1, a_2 .

$$2\omega_0^2 a_1 - \omega_0^2 a_2 = \omega^2 a_1$$

$$-\omega_0^2 a_1 + 2\omega_0^2 a_2 = \omega^2 a_2$$

Move all the terms to the left-hand side.

$$(2\omega_0^2 - \omega^2)a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 + (2\omega_0^2 - \omega^2)a_2 = 0$$

This is a homogeneous system of algebraic equation of a_1, a_2 .

$a_1 = a_2 = 0$ is a solution, but we want non-zero a_1, a_2 solution

According to Cramer's Rule, it has non-zero solution only if **the determinant is zero!**

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0 \quad (\omega^2 - 2\omega_0^2)^2 - (\omega_0^2)^2 = 0 \quad \text{Arfken p83-85, 88}$$

$$\omega^2 = \omega_0^2, 3\omega_0^2$$

未知數 ω^2 應該只有兩個可能！ The unknown ω^2 has only two options!

$$\omega = \omega_1 \equiv \omega_0$$

$$\omega = \omega_2 \equiv \sqrt{3}\omega_0$$

Only in these two cases is there non-zero a_1, a_2 solution.

When the determinant is zero, the two equations are linearly dependent. Arfken p88-91

They are essentially the same since one can be written as a multiple of the other.

對特定的 ω ，我們可以解出對應的 a_1, a_2 Let's find a_1, a_2 for specific ω .

$\omega = \omega_1 = \omega_0$ Plug in

$$\begin{aligned} (2\omega_0^2 - \omega^2)a_1 - \omega_0^2 a_2 &= 0 & (2\omega_0^2 - \omega_0^2)a_1 - \omega_0^2 a_2 &= 0 & \omega_0^2 a_1 - \omega_0^2 a_2 &= 0 & a_1 - a_2 &= 0 \\ -\omega_0^2 a_1 + (2\omega_0^2 - \omega^2)a_2 &= 0 & -\omega_0^2 a_1 + (2\omega_0^2 - \omega_0^2)a_2 &= 0 & -\omega_0^2 a_1 + \omega_0^2 a_2 &= 0 & -a_1 + a_2 &= 0 \end{aligned}$$

The second equation is minus the first one.

$a_1 = a_2 \equiv C_1 = A_1 e^{i\phi_1}$ This satisfies both equation.

Therefore, we find that the solution is:

$$x_1 = a_1 \cdot e^{i\omega_1 t} = C_1 \cdot e^{i\omega_0 t} = A_1 e^{i(\omega_0 t + \phi_1)}$$

$$x_2 = C_1 \cdot e^{i\omega_0 t} = A_1 e^{i(\omega_0 t + \phi_1)}$$

取實數部，就得到原方程組的解：Take the real part:

$$x_1 = x_2 = \text{Re } A_1 e^{i(\omega_0 t + \phi_1)} = A_1 \cos(\omega_0 t + \phi_1)$$

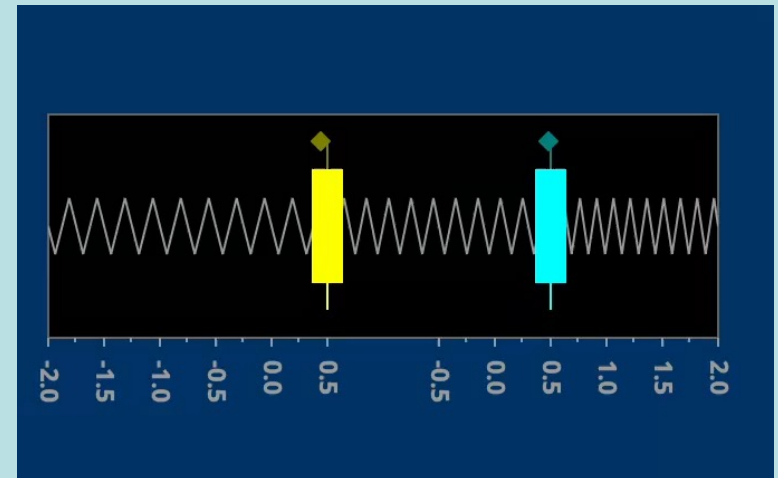
$$x'_1 = x'_2 = -\omega_0 A_1 \sin(\omega_0 t + \phi_1)$$

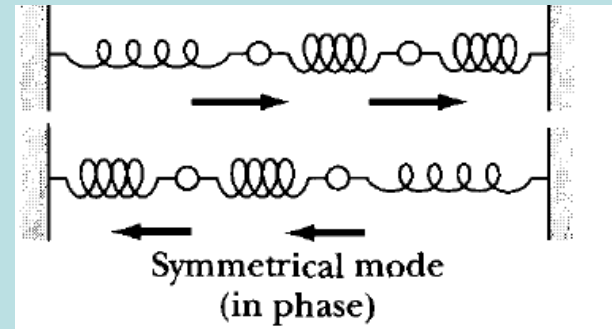
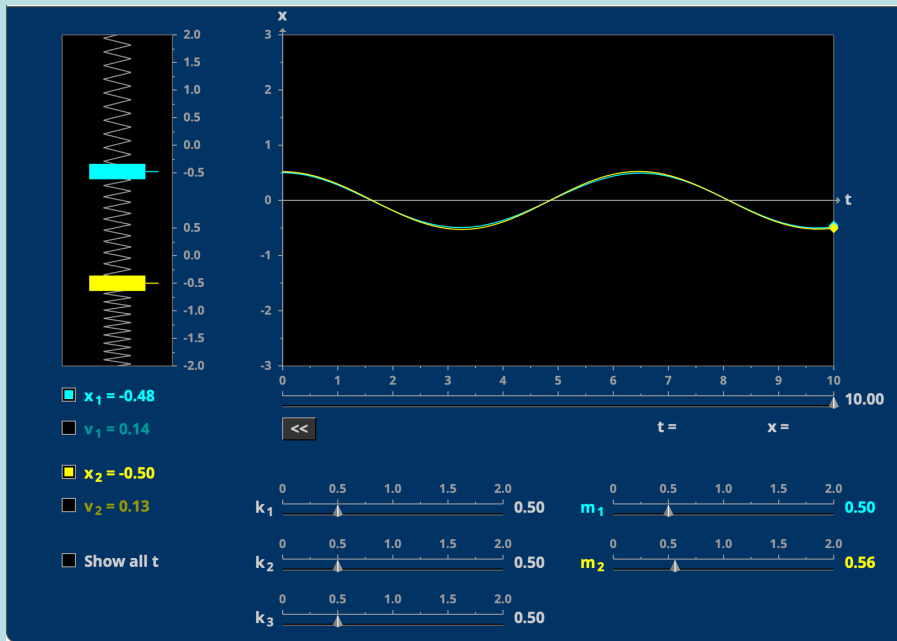
The two move and oscillate together! $x_1 = x_2$ 。

這樣的運動就稱為一個模式，對稱模式。

This is called a mode: in this case Symmetric mode.

If initial condition is symmetric : $x_1(0) = x_2(0), x'_1(0) = x'_2(0)$, symmetric mode will ensue and continue.



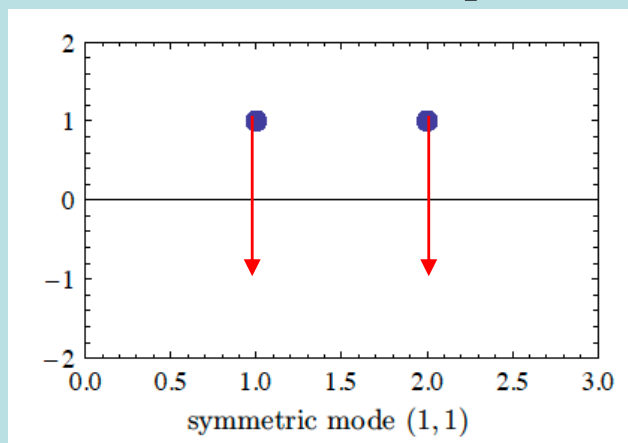


此模式是由 a_1, a_2 標定，畫出 a_1, a_2 可以代表此模式。

This mode is designated by a_1, a_2 . You can draw a_1, a_2 to represent the mode.

兩個粒子 x_1, x_2 是以 a_1, a_2 為振幅一起作同步(同 ω)的簡諧運動。

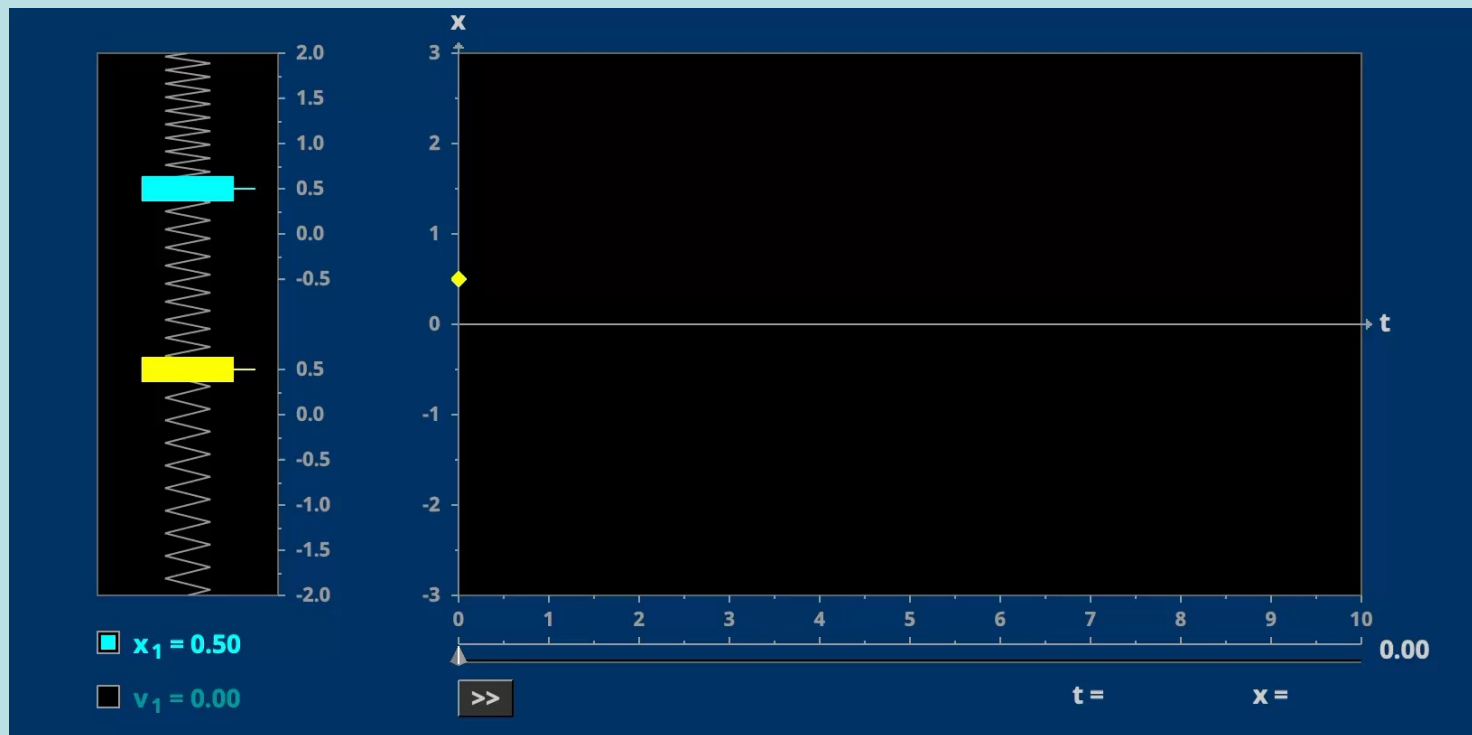
In the mode, the two particles take a_1, a_2 as amplitudes, oscillate together (same ω).



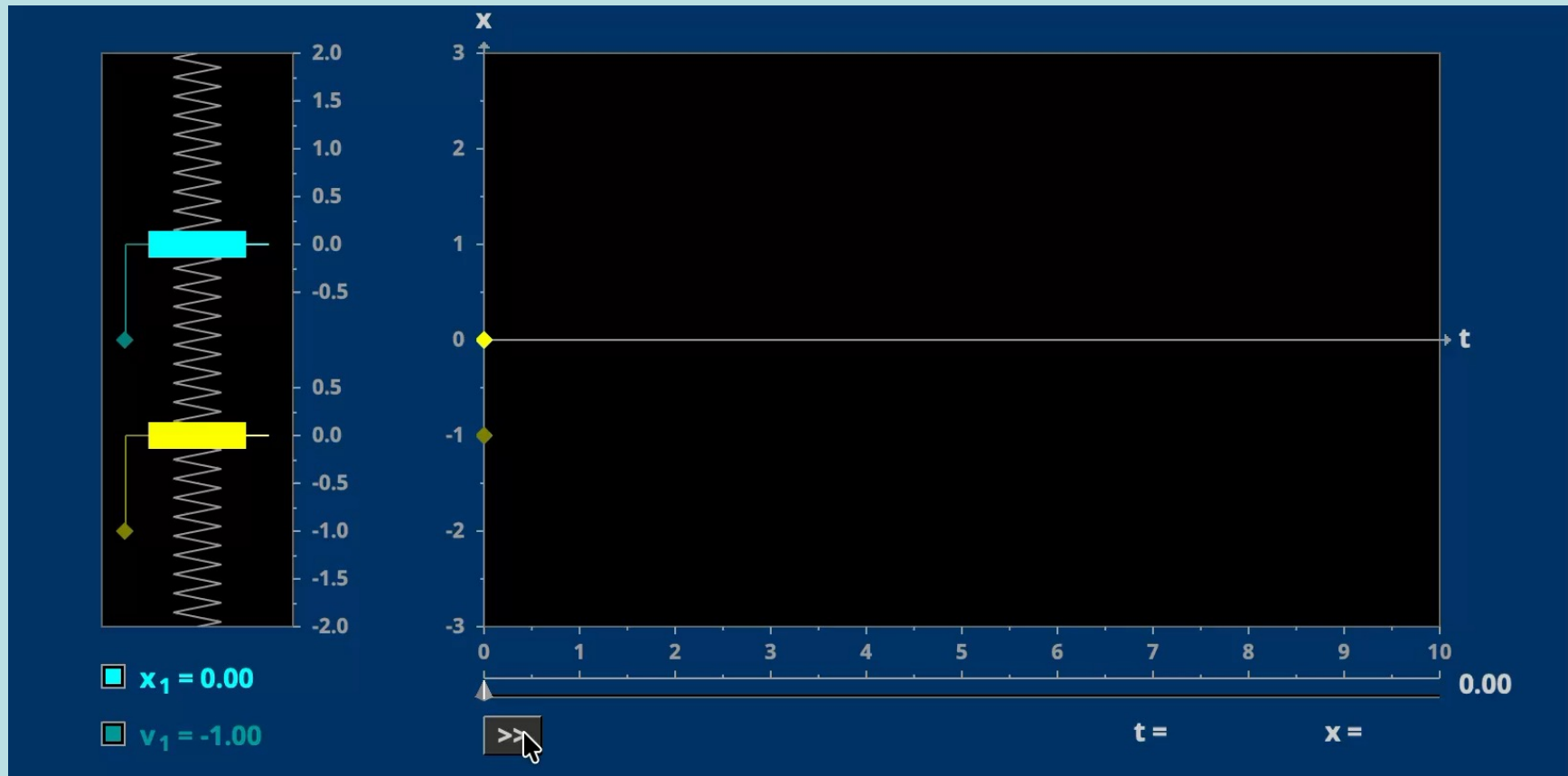
$$a_1 = a_2$$

$$x_1 = a_1 \cdot A_1 \cos(\omega_0 t + \phi_1)$$

$$x_2 = a_2 \cdot A_1 \cos(\omega_0 t + \phi_1)$$



$$x_1(0) = x_2(0) = A, x_1'(0) = x_2'(0) = 0$$



$$x_1(0) = x_2(0) = 0, x_1'(0) = x_2'(0) = B$$

$$\omega = \omega_2 = \sqrt{3}\omega_0$$

$$(2\omega_0^2 - 3\omega_0^2)a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 - \omega_0^2 a_2 = 0$$

$$a_1 + a_2 = 0$$

$$-\omega_0^2 a_1 + (2\omega_0^2 - 3\omega_0^2)a_2 = 0$$

$$-\omega_0^2 a_1 - \omega_0^2 a_2 = 0$$

$$a_1 + a_2 = 0$$

The second equation is the same as the first one.

$$a_1 = -a_2 \equiv C_2 = A_2 e^{i\phi_2}$$

The solution is:

$$x_1 = a_1 e^{i\omega_2 t} = C_2 e^{i\sqrt{3}\omega_0 t} = A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)}$$

$$x_2 = -x_1 = A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)}$$

Now taking the real part would give us the solution.

$$x_1 = \text{Re } A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)} = A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

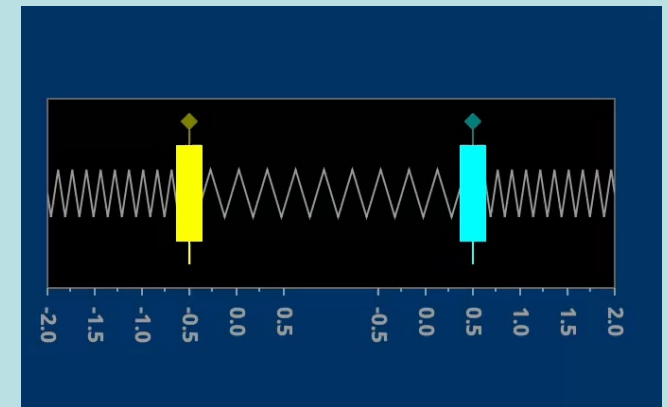
$$x_2 = -x_1 = -A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

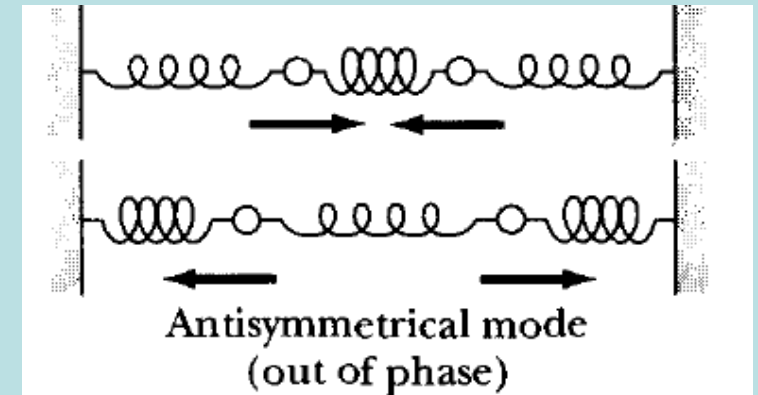
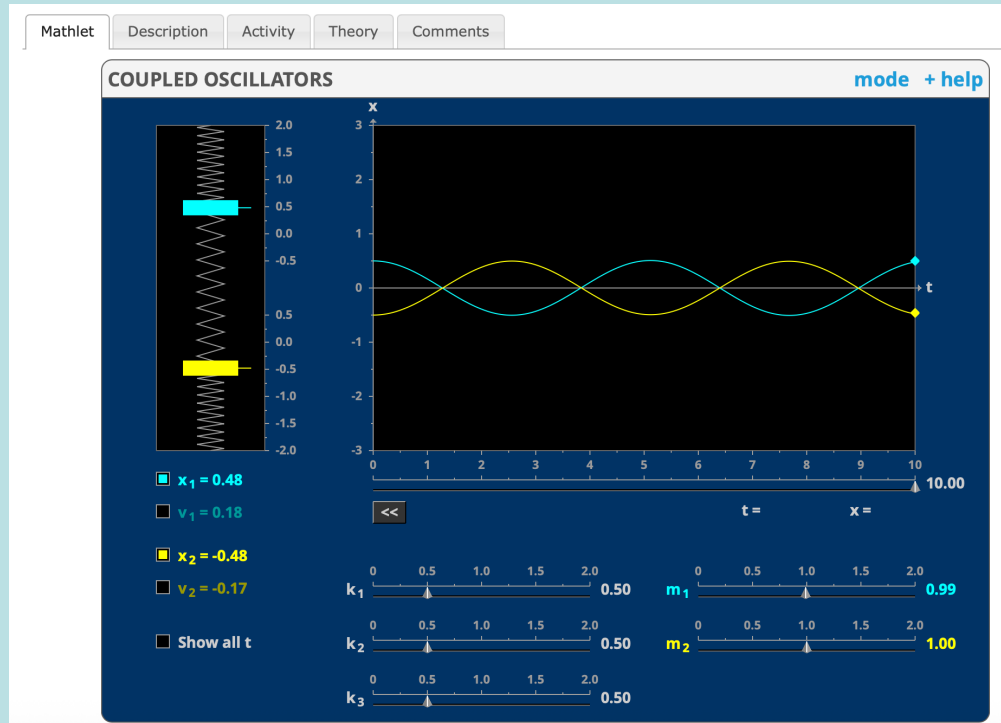
$$x_1' = -x_2' = \sqrt{3}\omega_0 A_2 \sin(\sqrt{3}\omega_0 t + \phi_1)$$

The two move opposite! $x_2 = -x_1$.

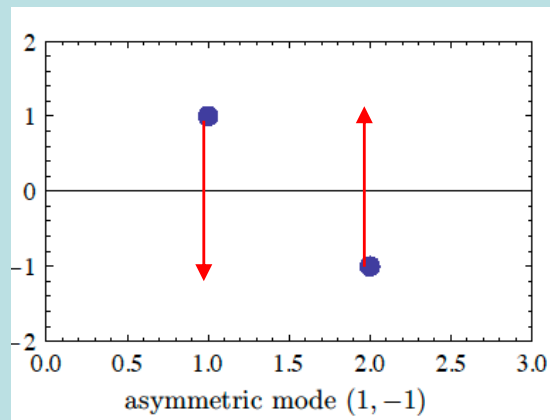
這樣的運動就稱為反對稱模式。 This is called Antisymmetric Mode.

If initial condition is symmetric $x_1(0) = -x_2(0)$, $x_1'(0) = -x_2'(0)$ antisymmetric mode will ensue and continue.





兩個粒子 x_1, x_2 是以 a_1, a_2 為振幅一起作同步(同 ω) 的簡諧運動。
 a_1, a_2 一正一負，因此 x_1, x_2 的振盪有相角差 π 。

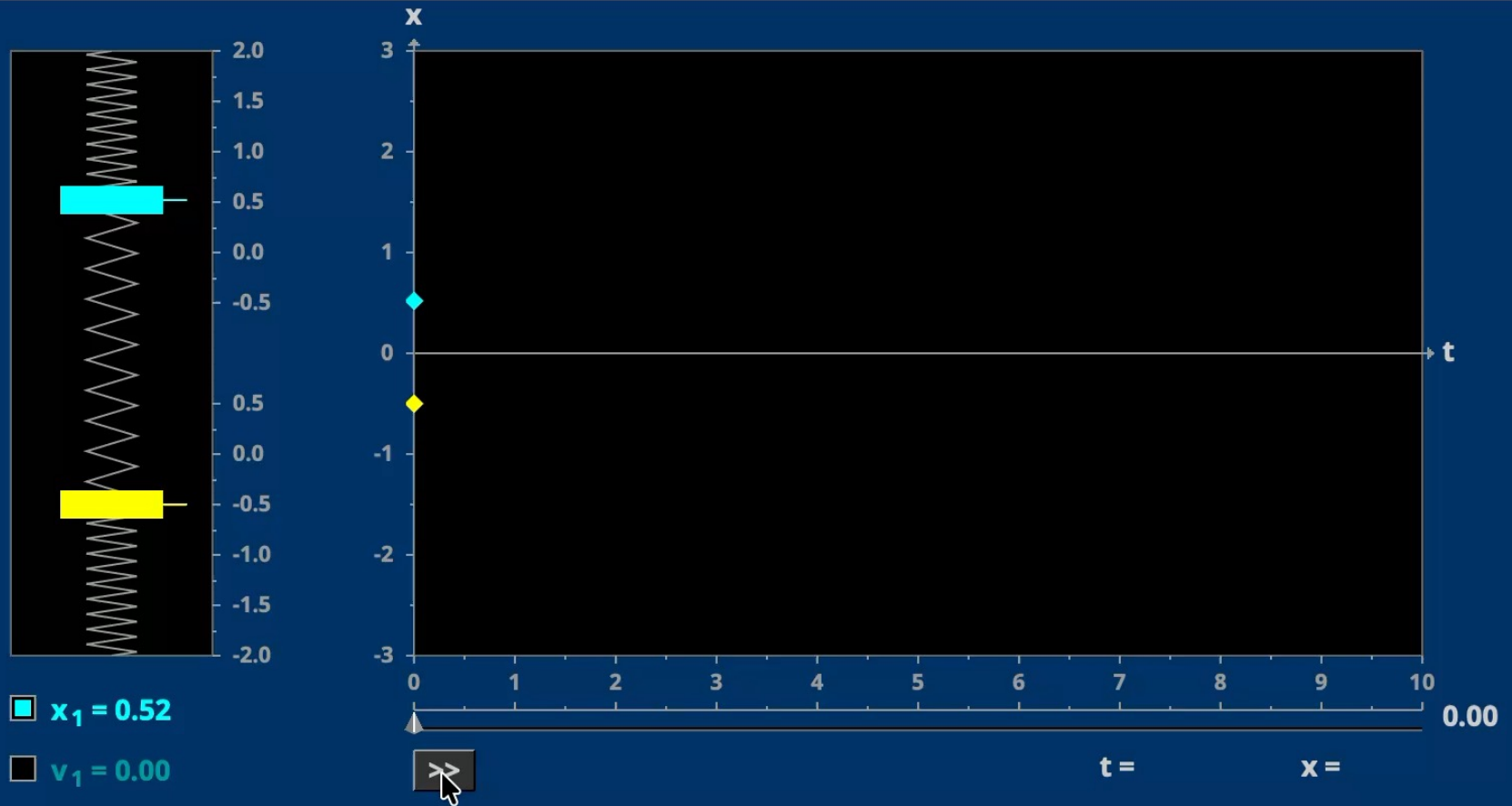


$$a_1 = -a_2$$

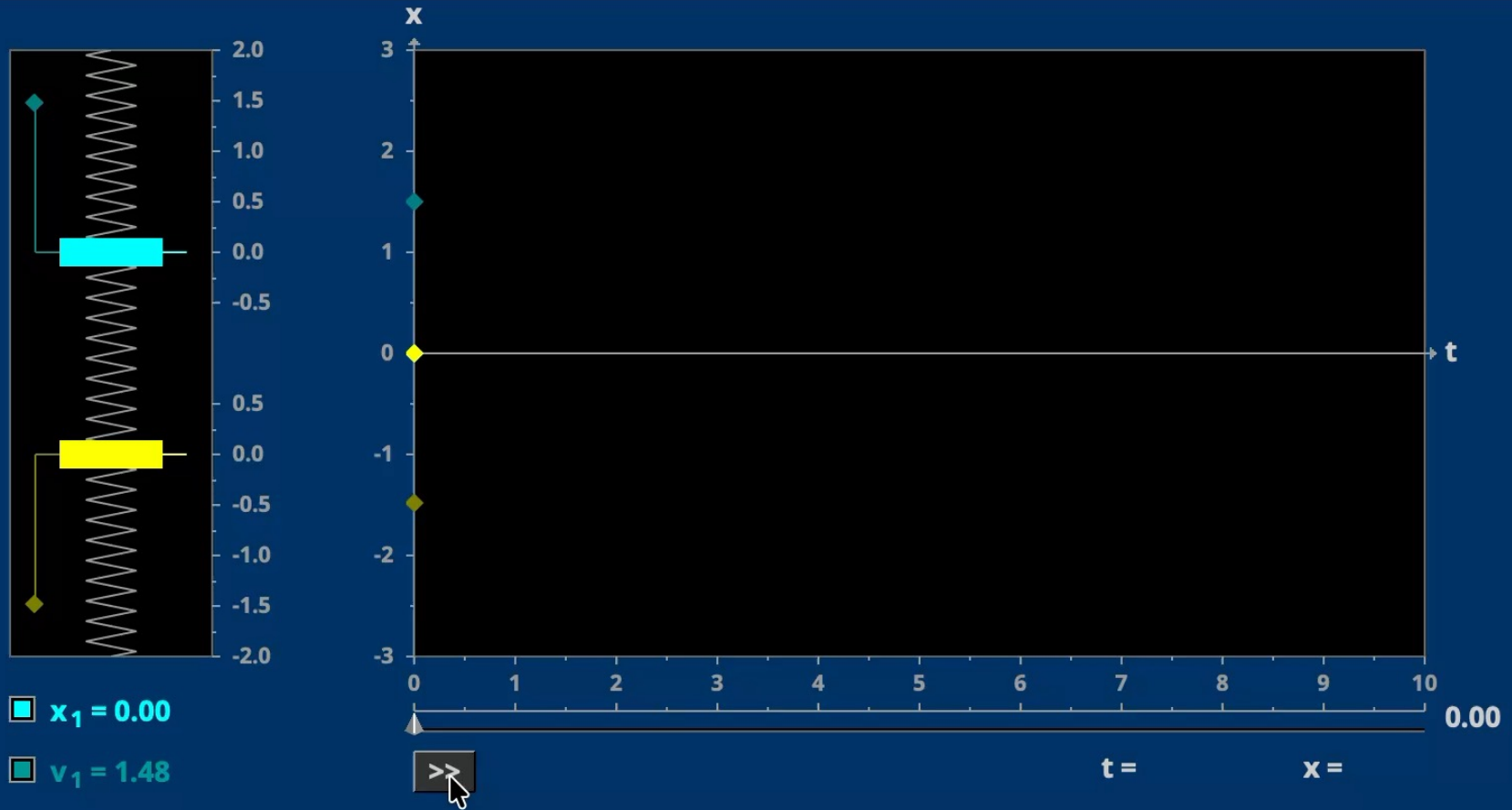
$$x_1 = a_1 \cdot A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = a_2 \cdot A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$= -a_1 \cdot A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$



$$x_1(0) = -x_2(0) = A, x_1'(0) = -x_2'(0) = 0$$



$$x_1(0) = -x_2(0) = 0, x_1'(0) = -x_2'(0) = B$$

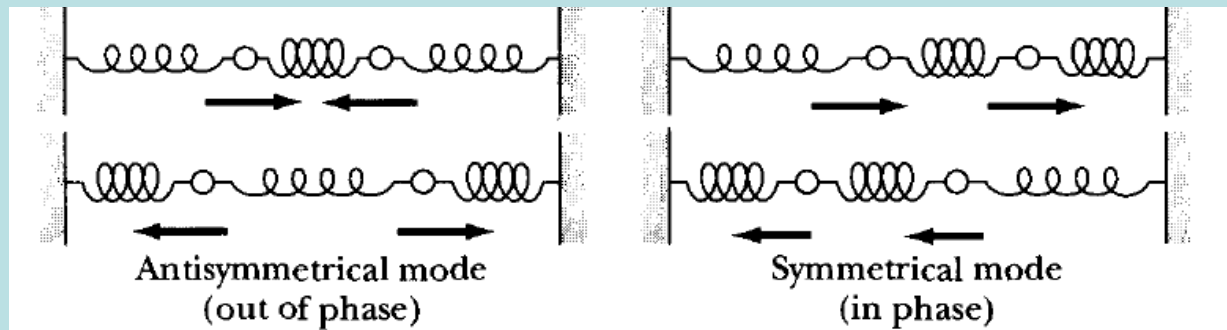
We have found two oscillating solutions for Coupled system.

$$x_1 = -x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$x_1 = x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$\omega_2 = \sqrt{3}\omega_0$$

$$\omega_1 = \omega_0$$



If initial conditions follow the condition of the mode,

$$x_1(0) = -x_2(0), x'_1(0) = -x'_2(0)$$

$$x_1(0) = x_2(0), x'_1(0) = x'_2(0)$$

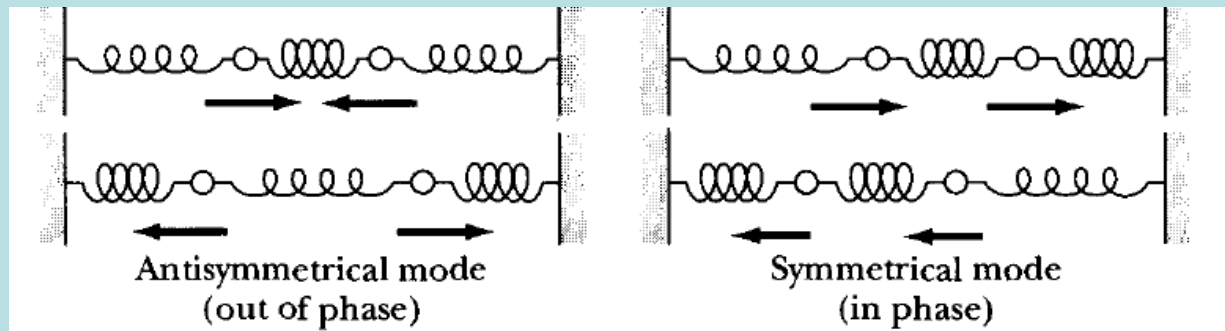
the system will continue oscillating as SHM according to the mode!

$$x_1 = -x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$x_1 = x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$\omega_2 = \sqrt{3}\omega_0$$

$$\omega_1 = \omega_0$$



$$x_1(0) = -x_2(0), x'_1(0) = -x'_2(0)$$

$$x_1(0) = x_2(0), x'_1(0) = x'_2(0)$$

中間的彈簧的中點沒有移動，等同固定。
 粒子都只受一條加上半條彈簧的力，

中間的彈簧並沒有長度的變化。
 所以兩個粒子都只受一條彈簧的力，

$$\omega_2 = \sqrt{\frac{k + 2k}{m}} = \sqrt{3}\omega_0$$

$$\omega_1 = \sqrt{\frac{k}{m}} = \omega_0$$

A general solution is a sum of the two modes.

一般起始條件下，彈簧組運動就是兩種模式的疊加！

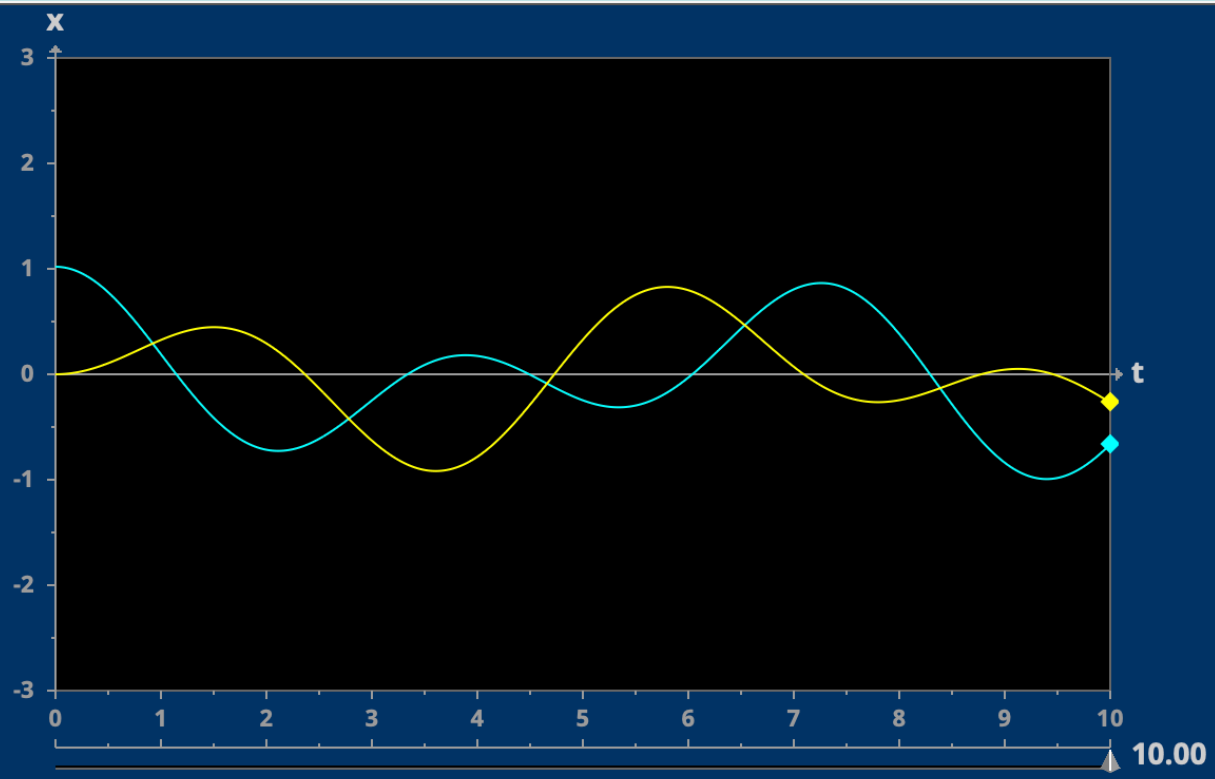
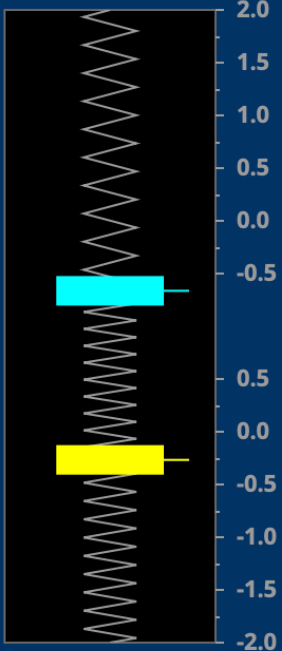
There are four undetermined constants $A_{1,2}, \phi_{1,2}$.

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

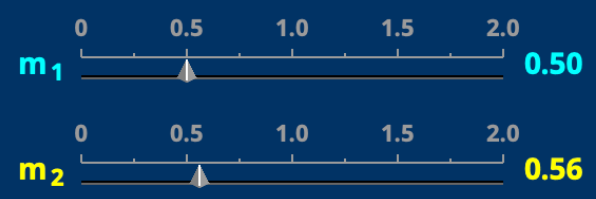
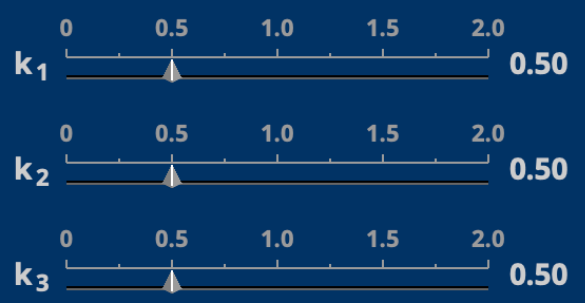
There are also four initial conditions $x_{1,2}(0), x'_{1,2}(0)$.

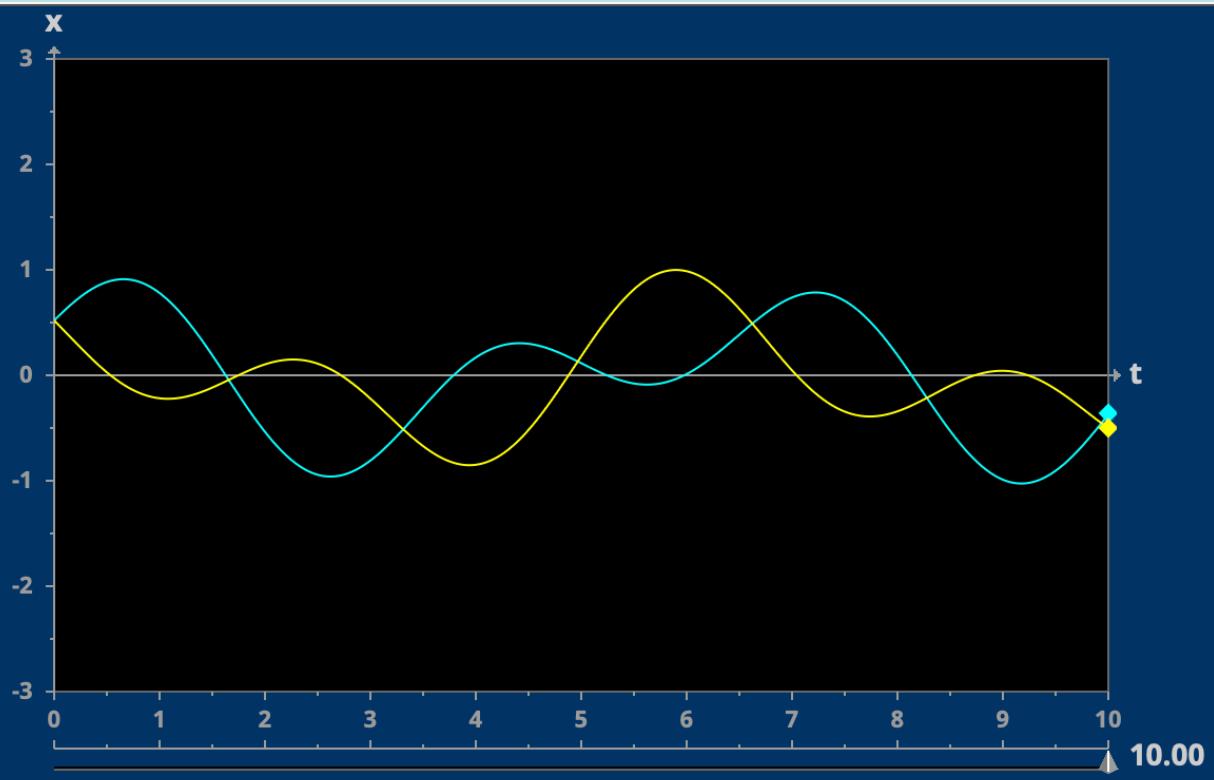
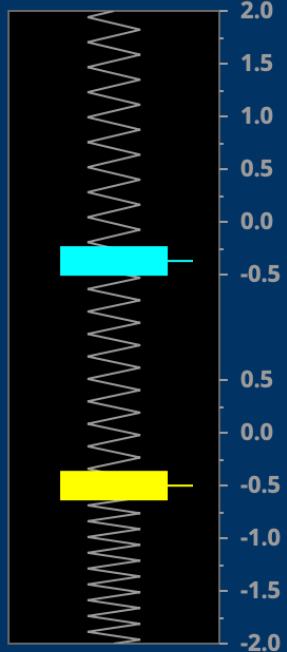
It would be the one unique solution.



- $x_1 = -0.66$
- $v_1 = 1.00$
- $x_2 = -0.27$
- $v_2 = -0.61$
- Show all t

<< t = x =





$x_1 = -0.37$

$v_1 = 1.28$

$x_2 = -0.50$

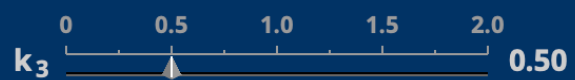
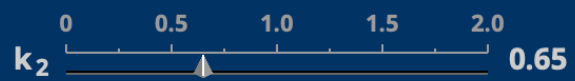
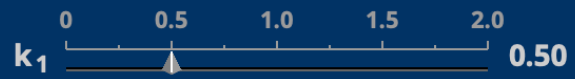
$v_2 = -0.71$

Show all t

<<

t =

x =



Setup: 2 masses

Reset Positions

Reset Masses

Reset Springs

Stopped

Elastic Collisions

Lissajous Figures

Mouse: Pull string

Simulation Speed

Damping

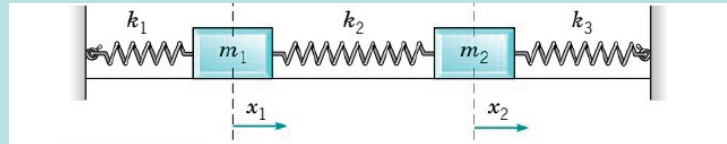
Number of Loads

<http://www.falstad.com>

Normal Modes: Magnitudes

Normal Modes: Phases

Method I



線性方程式組的右手邊可以以矩陣及行向量的乘積來簡化符號！

The right-hand side can be simplified as a multiplication of matrix and column.

$$\frac{d^2 x_1}{dt^2} = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$$

$$\frac{d^2 x_2}{dt^2} = \frac{k}{m} x_1 - \frac{2k}{m} x_2$$



$$\begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = - \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

矩陣乘行向量得一行向量： multiplication of a matrix and a column is a column.

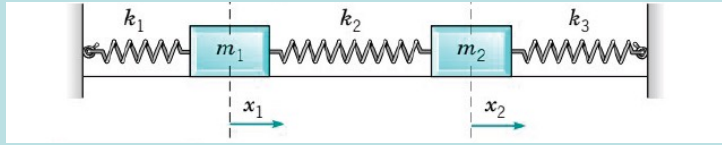
$$(\mathbf{S}\mathbf{a})_i = \sum_{j=1}^2 S_{ij}a_j = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$

此規則保證此乘積是線性的 $\mathbf{S}(c_1\mathbf{a}_1 + c_2\mathbf{a}_2) = c_1\mathbf{S}\mathbf{a}_1 + c_2\mathbf{S}\mathbf{a}_2$

矩陣乘矩陣還是矩陣： multiplication of two matrices and is a matrix.

$$(\mathbf{S}\mathbf{A})_{mn} = \sum_{j=1}^2 S_{mj}A_{jn} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S_{11}A_{11} + S_{12}A_{21} & S_{11}A_{12} + S_{12}A_{22} \\ S_{21}A_{11} + S_{22}A_{21} & S_{21}A_{12} + S_{22}A_{22} \end{pmatrix}$$

Arfken p96-99



$$\begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = - \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We can use simplified notations to write the equation.

行向量 Column vector

矩陣 Matrix

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} \equiv \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \equiv \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}$$

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

兩粒子位置自然組成一行向量。

$$\frac{d^2}{dt^2} \mathbf{x} = \begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix}$$

$$- \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{A} \cdot \mathbf{x}$$

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

$$-A \cdot x = \frac{d^2}{dt^2} x$$

Elevate the real x 's into complex function. Guess the solutions are the same exponential:

The solution can be written as a constant column vector \mathbf{a} times the usual $e^{i\omega t}$.

ω and \mathbf{a} are to be determined.

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{a} e^{i\omega t} \quad \mathbf{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$x_1 = a_1 e^{i\omega t} \quad x_2 = a_2 e^{i\omega t} \quad \text{Key idea is the two particle oscillate with one frequency.}$$

Plug the guess into Eq.

$$-A \cdot \mathbf{a} e^{i\omega t} = -\omega^2 \mathbf{a} e^{i\omega t}$$

$$A \cdot \mathbf{a} = \omega^2 \mathbf{a}$$

$$\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The differential Equation of $\mathbf{x}(t)$ is transformed into algebraic equation of \mathbf{a} .

This is the eigenvalue problem of Matrix A . ω^2 is the eigenvalue while \mathbf{a} the eigenvector.

微分方程組被轉化為矩陣 A 的本徵值 ω 問題， \mathbf{a} 稱為本徵向量。

To solve eigenvalue problem:

$$\mathbf{A} \cdot \mathbf{a} = \omega^2 \mathbf{a}$$

$$\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$(\mathbf{A} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$$

$$\begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

If the matrix $(\mathbf{A} - \omega^2)$ has an inverse matrix $(\mathbf{A} - \omega^2)^{-1}$, \mathbf{a} equals 0.

$$\mathbf{a} = (\mathbf{A} - \omega^2)^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is not what we want.

Unless, **determinant** $|\mathbf{A} - \omega^2| = 0$. Then inverse matrix $(\mathbf{A} - \omega^2)^{-1}$ does not exist.

Only in this case would \mathbf{a} have nontrivial solutions.

$$|\mathbf{A} - \omega^2 \mathbf{I}| = 0$$

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$(\omega^2 - 2\omega_0^2)^2 - (\omega_0^2)^2 = 0$$

$$\omega^2 - 2\omega_0^2 = \pm\omega_0^2$$

$$\omega^2 = \omega_0^2, 3\omega_0^2$$

Inverse matrix

$$\mathbf{S}^{-1} = \frac{1}{\det \mathbf{S}} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$\omega^2 = \omega_0^2, 3\omega_0^2$ 本徵值 ω^2 有兩個，各自對應一本徵向量 \mathbf{a} 。

Two eigenvalues ω^2 , each corresponding to one eigenvector \mathbf{a} , determined by this formula:

For $\omega^2 = \omega_0^2$

$$\omega = \omega_1 \equiv \omega_0$$

$$\begin{pmatrix} 2\omega_0^2 - \omega_1^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega_1^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega_0^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1 - a_2 = 0$$

$\mathbf{a}^{(1)} \equiv a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Index to denote the first mode

$$\mathbf{x} = \mathbf{a}^{(1)} e^{i\omega_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_1 e^{i\omega_0 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 e^{i(\omega_0 t + \phi_1)}$$

$$a_1 \equiv A_1 e^{i\phi_1}$$

Take the real part:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_0 t + \phi_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [f_1 \cos(\omega_0 t) + g_1 \sin(\omega_0 t)]$$

$$x_1 = A_1 \cos(\omega_0 t + \phi_1)$$

$$x_2 = x_1 = A_1 \cos(\omega_0 t + \phi_1)$$

$$\frac{x_1}{x_2} = \frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}} = 1$$

This mode is designated by $\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}$.

Note that in this mode, the ratio of $\frac{x_1}{x_2}$ at any time equals the ratio of $\frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}}$.

$$\omega^2 = 3\omega_0^2$$

$$\omega = \omega_2 \equiv \sqrt{3}\omega_0$$

$$\begin{pmatrix} 2\omega_0^2 - \omega_2^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega_2^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega_0^2 \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1 + a_2 = 0$$

$$\mathbf{a}^{(2)} \leftarrow a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{Index to denote the second mode}$$

$$\mathbf{x} = \mathbf{a}^{(2)} e^{i\omega_2 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} a_1 e^{i\sqrt{3}\omega_0 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)}$$

$$a_1 \equiv A_2 e^{i\phi_2}$$

Take the real part:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\sqrt{3}\omega_0 t + \phi_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} [f_2 \cos(\sqrt{3}\omega_0 t) + g_2 \sin(\sqrt{3}\omega_0 t)]$$

$$x_1 = A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = -x_1 = -A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$\frac{x_1}{x_2} = \frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}} = -1$$

In this mode, the ratio of $\frac{x_1}{x_2}$ at any time equals the ratio of $\frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}} = -1$.

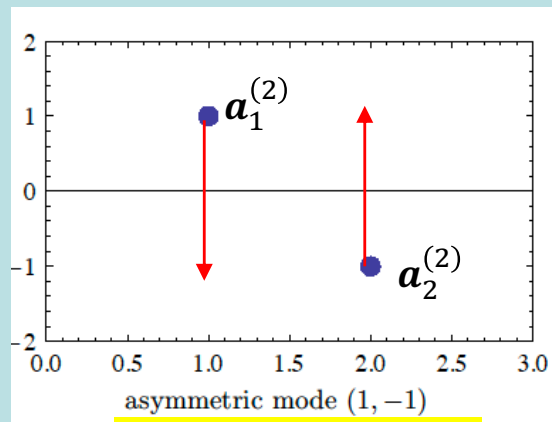
一個本徵向量就對應一個可獨立振盪的模式。

One eigenvalue ω^2 with one eigenvector \mathbf{a} corresponds to one oscillating mode.

We have found two oscillating solutions for Coupled system.

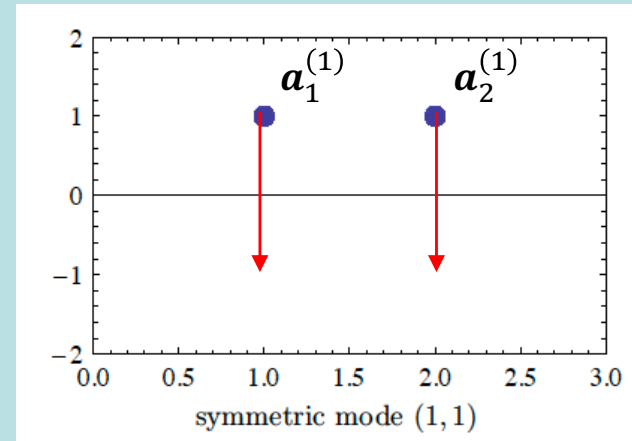
In these modes, the ratio of $\frac{x_1}{x_2}$ at any time equals the ratio of $\frac{\mathbf{a}_1^{(1,2)}}{\mathbf{a}_2^{(1,2)}}$.

We can plot \mathbf{a} as follows to represent a mode:



$$\omega_2 = \sqrt{3}\omega_0$$

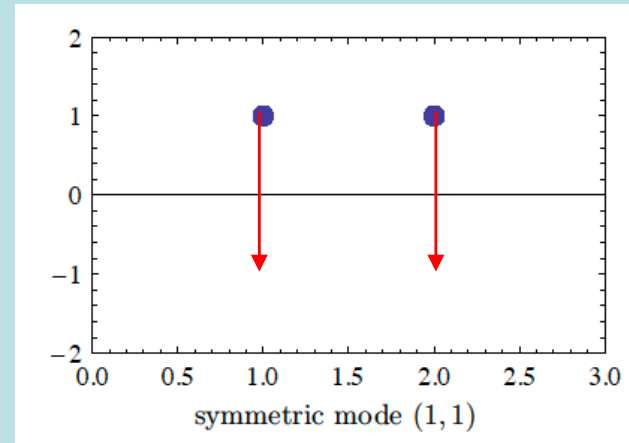
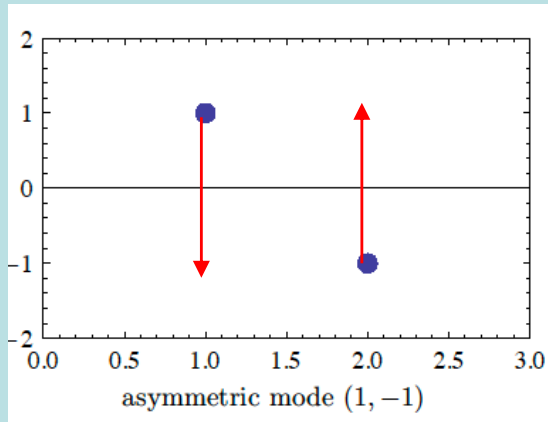
$$\frac{x_1}{x_2} = \frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}} = -1$$



$$\omega_1 = \omega_0$$

$$\frac{x_1}{x_2} = \frac{\mathbf{a}_1^{(1)}}{\mathbf{a}_2^{(1)}} = 1$$

We have found two oscillating solutions for Coupled system.

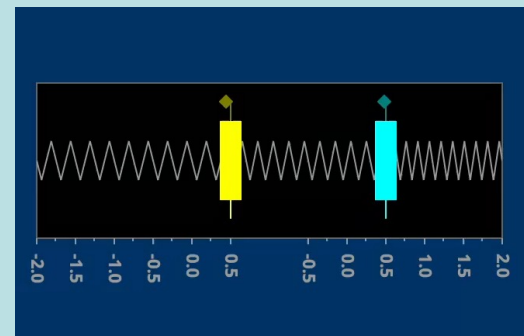
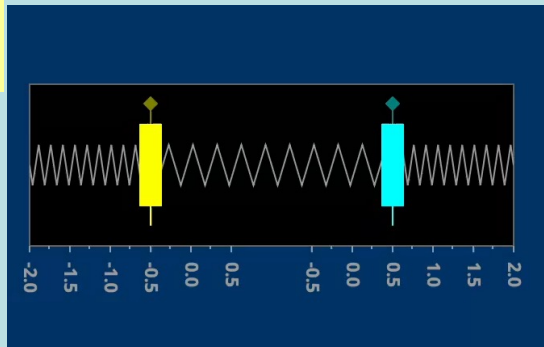


$$\frac{x_1}{x_2} = \frac{a_1^{(1)}}{a_2^{(1)}} = -1$$

$$\omega_2 = \sqrt{3}\omega_0$$

$$\omega_1 = \omega_0$$

$$\frac{x_1}{x_2} = \frac{a_1^{(1)}}{a_2^{(1)}} = 1$$



If initial conditions follow the condition of the mode,

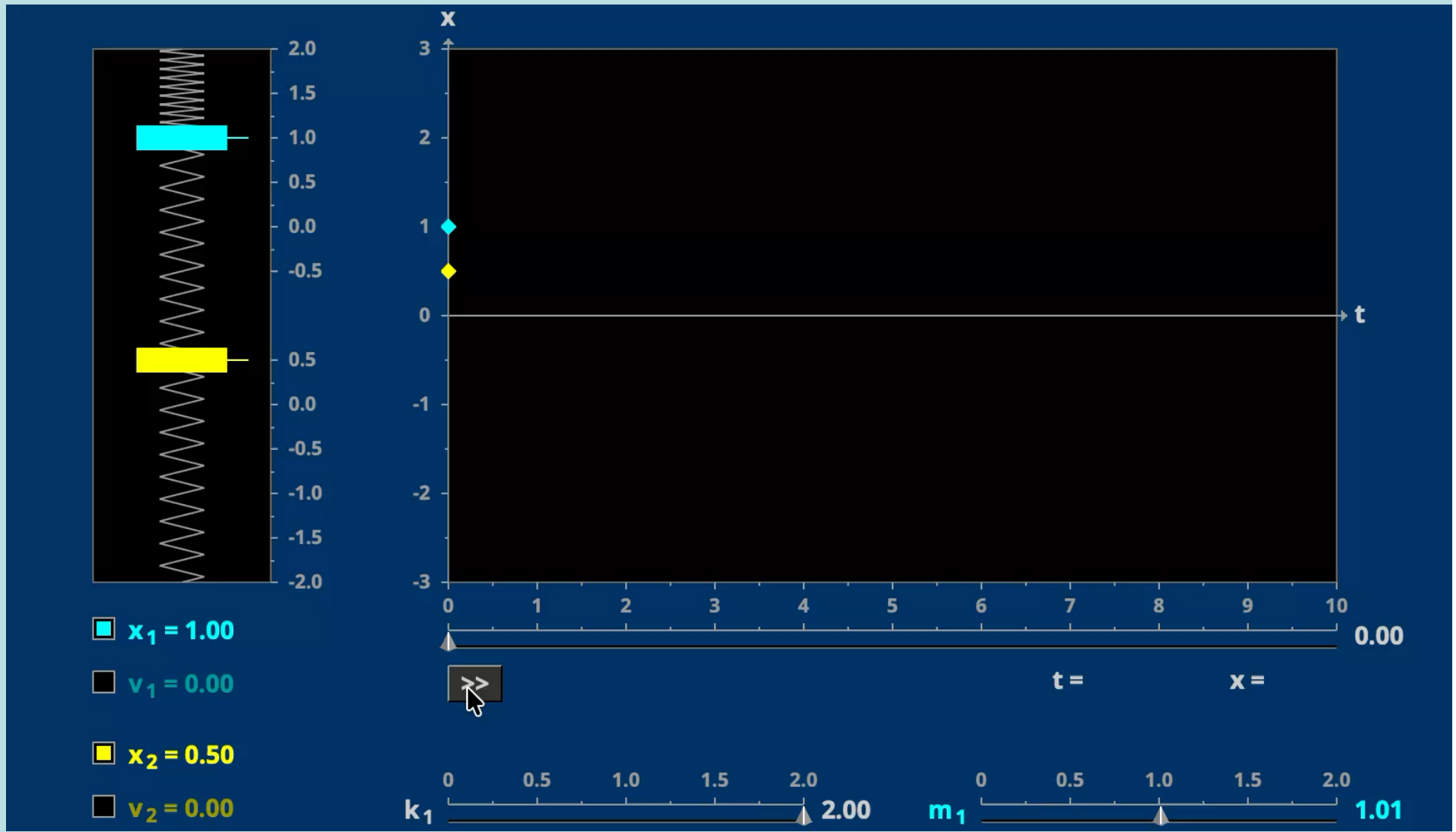
$$\frac{x_1(0)}{x_2(0)} = \frac{a_1^{(2)}}{a_2^{(2)}} = -1$$

$$\frac{x_1(0)}{x_2(0)} = \frac{a_1^{(1)}}{a_2^{(1)}} = 1$$

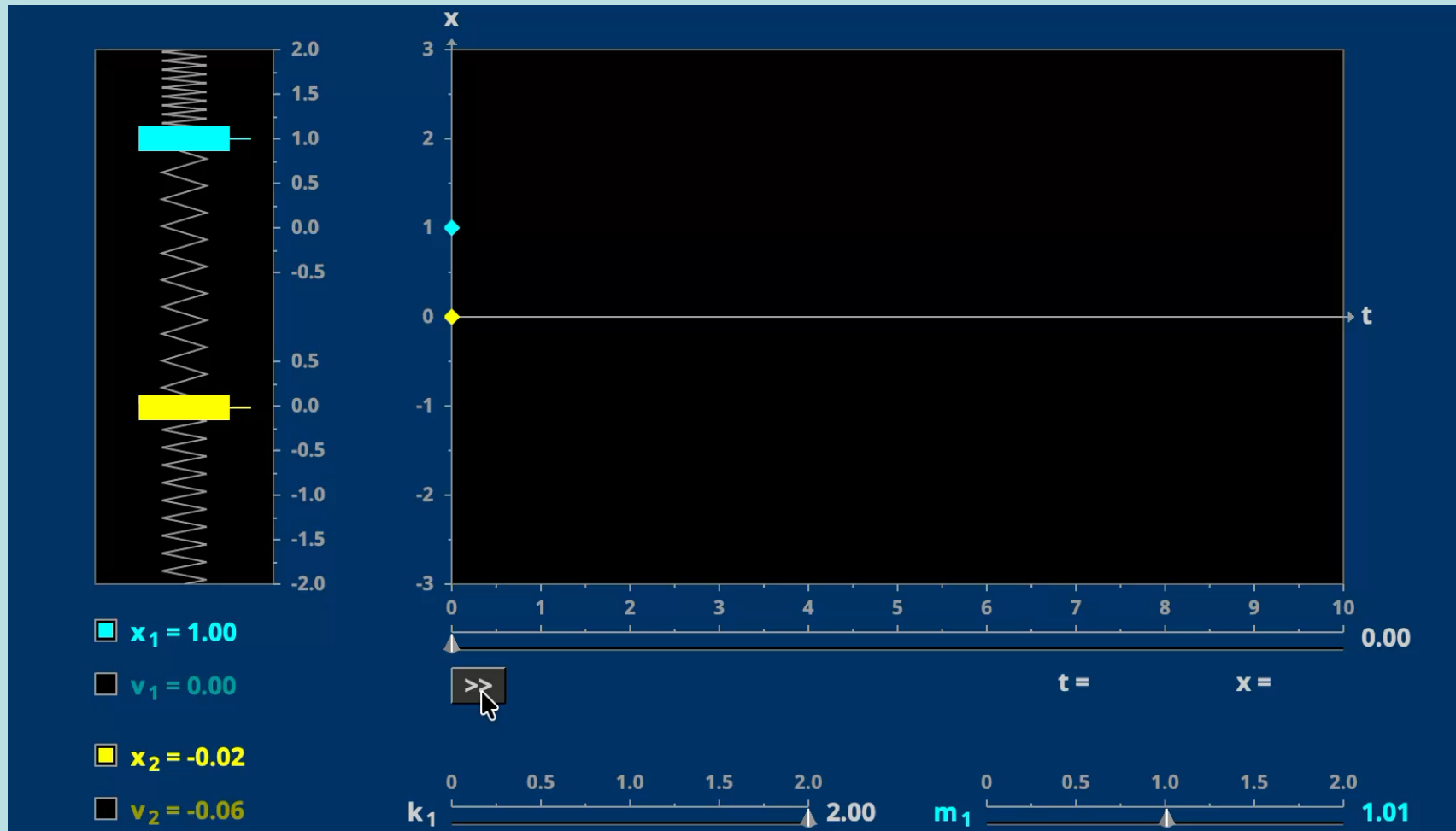
the system will continue oscillating as SHM according to the mode!

We usually say: one mode is excited while the other is not.

Is it possible both modes are excited in the beginning? Of course.



$$x_1(0) = 2x_2(0), x_1'(0) = x_2'(0) = 0$$



$$x_1(0) = A, x_2(0) = 0, x_1'(0) = x_2'(0) = 0$$

有了以上兩個模式的解，一般解將是兩者的線性組合：

With two mode solutions, any linear combinations of the two modes are also solutions of the system of linear ODE. There are four unspecified constants $f_{1,2}, g_{1,2}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [f_1 \cos(\omega_0 t) + g_1 \sin(\omega_0 t)] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} [f_2 \cos(\sqrt{3}\omega_0 t) + g_2 \sin(\sqrt{3}\omega_0 t)]$$

四個未定常數將由四個起使條件決定。例如：

The four unspecified constants will be determined by four initial conditions.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} f_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} f_2$$



$$x_1(0) = f_1 + f_2$$

$$x_2(0) = f_1 - f_2$$

因此 $f_{1,2}$ 可以唯一決定。 $f_{1,2}$ will now be uniquely specified.

$$\begin{pmatrix} x'_1(0) \\ x'_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} g_1 \omega_0 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} g_2 \sqrt{3} \omega_0$$



$$x'_1(0) = g_1 \omega_0 + g_2 \sqrt{3} \omega_0$$

$$x'_2(0) = g_1 \omega_0 - g_2 \sqrt{3} \omega_0$$

$g_{1,2}$ 也可以唯一決定。 $g_{1,2}$ will also be uniquely specified.

Note that $g_{1,2}$ will vanish if initial velocities $x'_{1,2}(0)$ are both zero.

於是微分方程組加起始條件，求解完成。

Our result satisfy both system of ODE and initial condition. It is the unique solution.

For example: if

$$x_1(0) = 2A, x_2(0) = 0, x_1'(0) = x_2'(0) = 0$$

$$g_1 = g_2 = 0$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2A \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} f_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} f_2 \rightarrow f_1 + f_2 = 2A$$

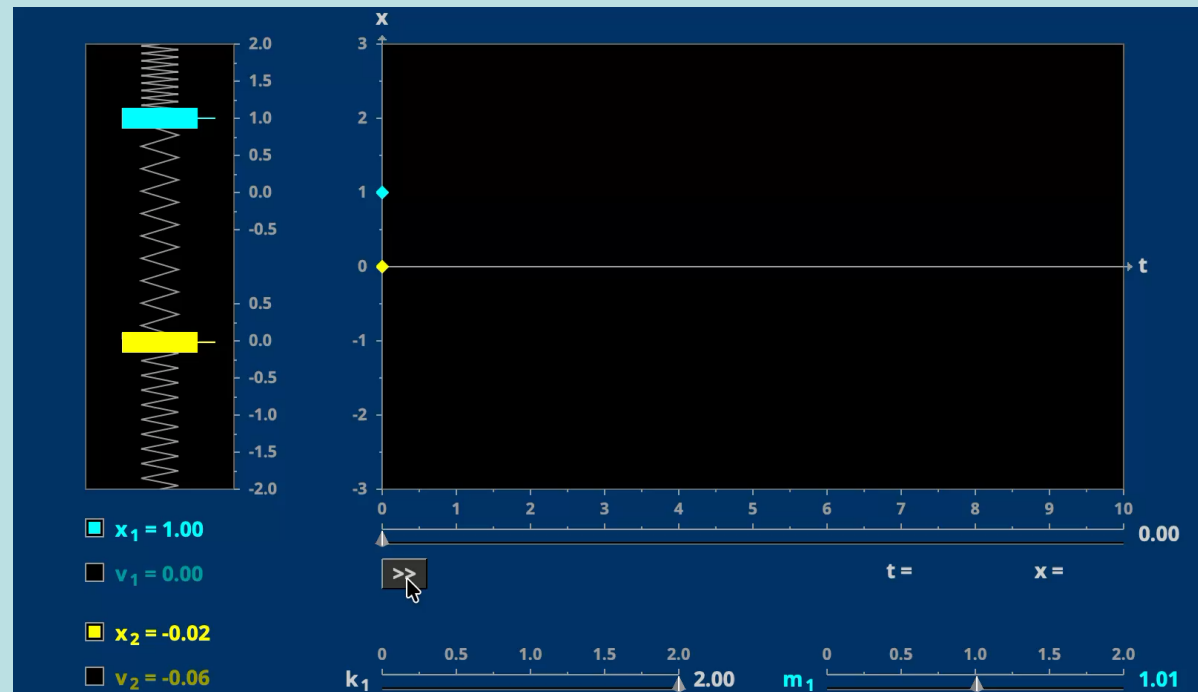
$$f_1 = f_2 = A$$

$$f_1 - f_2 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [A \cos(\omega_0 t)] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} [A \cos(\sqrt{3} \omega_0 t)]$$

$$x_1 = A [\cos(\omega_0 t) + \cos(\sqrt{3} \omega_0 t)]$$

$$x_2 = A [\cos(\omega_0 t) - \cos(\sqrt{3} \omega_0 t)]$$



There is another way to look at this solution.

Method II

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_+ \equiv x_1 + x_2 = 2A_1 \cos(\omega_0 t + \phi_1)$$

Generalized Coordinates 廣義座標

$$x_- \equiv x_1 - x_2 = 2A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

廣義座標可以完全決定真實座標！

x_+, x_- are pure simple harmonic oscillators with respective angular frequencies.

從廣義座標 x_+, x_- 滿足的運動方程式的角度來看：

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2 \quad + \quad \frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2 \quad \Rightarrow \quad \frac{d^2(x_1 + x_2)}{dt^2} = -\omega_0^2 x_1 - \omega_0^2 x_2$$

$$\frac{d^2 x_+}{dt^2} = -\omega_0^2 x_+$$

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2 \quad - \quad \frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2 \quad \Rightarrow \quad \frac{d^2 x_-}{dt^2} = -3\omega_0^2 x_-$$

x_+, x_- are called **Normal Coordinates**. 模式座標

$$\mathbf{x} \equiv \begin{pmatrix} x_+ \\ x_- \end{pmatrix}$$

$$-\mathbf{A} \cdot \mathbf{x} = \frac{d^2}{dt^2} \mathbf{x}$$

$$\mathbf{A} \equiv \begin{pmatrix} \omega_0^2 & 0 \\ 0 & 3\omega_0^2 \end{pmatrix}$$

For normal coordinates, **A** is now diagonal. Off-diagonal elements means coupling.

x_+, x_- are uncoupled, independent SHM.

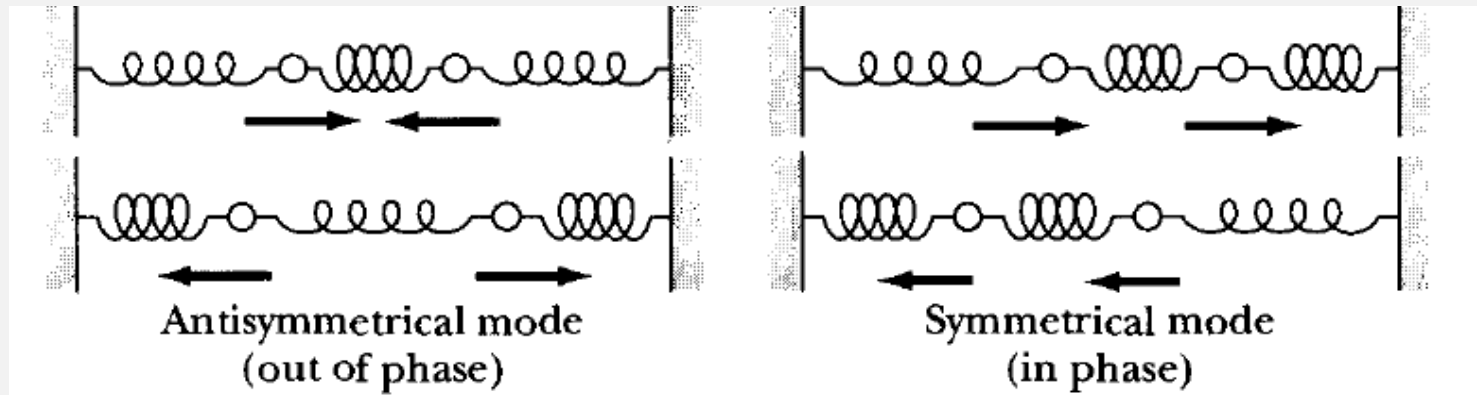
在這兩個模式，廣義座標非常簡單。

$$x_1 = -x_2 = A_2 \cos(\sqrt{3}\omega_0 t + \phi_1)$$

$$x_1 = x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$x_+ = 0, x_- = 2A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

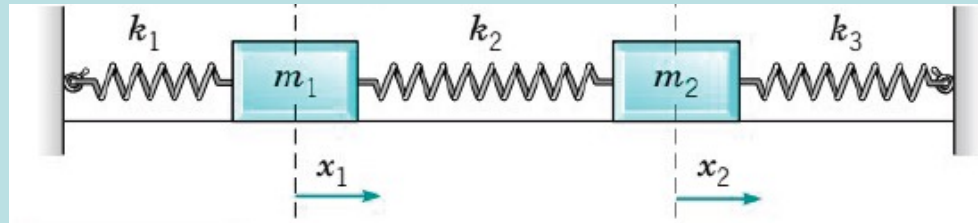
$$x_+ = A_1 \cos(\omega_0 t + \phi_1), x_- = 0$$



可以說這兩個模式座標就是來標定這兩個模式的。

This formalism could be easily extended to the more general cases.

The force constants k_1, k_2, k_3 could be different.



$$\frac{d^2 x_1}{dt^2} = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

行向量 Column vector
矩陣 Matrix

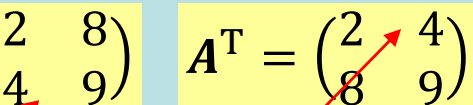
$$\frac{d^2 x_2}{dt^2} = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$

$$\mathbf{S} \equiv \begin{pmatrix} \frac{k_1 + k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} \end{pmatrix} \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

$$\frac{d^2}{dt^2} \mathbf{x} = \begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = - \begin{pmatrix} \frac{k_1 + k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \mathbf{S} \cdot \mathbf{x}$$

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{S} \cdot \mathbf{x} \quad \mathbf{S} \text{ is a symmetric Matrix.}$$

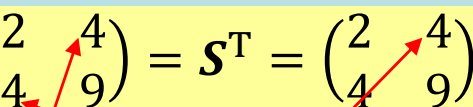
To define symmetric matrix, first define the Transpose of matrix A

$$A = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix} \quad A_{ij}^T = A_{ji}$$


例如 $A_{12}^T = A_{21}$

Symmetric Matrix S $S^T = S$

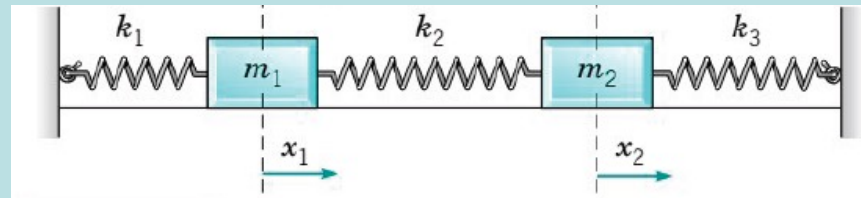
$$A = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix} \quad \text{Not symmetric}$$

$$S = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = S^T = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} \quad \text{Symmetric} \quad S_{ij} = S_{ij}^T = S_{ji}$$


例如 $S_{12} = S_{12}^T = S_{21}$

Arfken p99, 104-107

$$-\mathbf{S} \cdot \mathbf{x} = \frac{d^2}{dt^2} \mathbf{x}$$



Method I

Elevate the real x 's into complex function. Guess the solutions are the same exponential:

The solution can be written as a constant column vector \mathbf{a} times the usual $e^{i\omega t}$.

ω and \mathbf{a} are to be determined.

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{a} e^{i\omega t}$$

$$\mathbf{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Plug the guess into Eq.

$$-\mathbf{S} \cdot \mathbf{a} e^{i\omega t} = -\omega^2 \mathbf{a} e^{i\omega t}$$

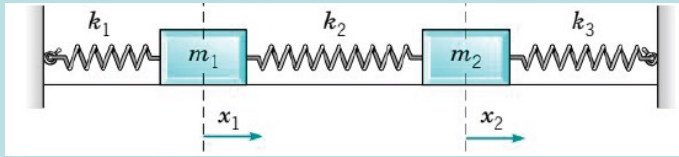
$$\mathbf{S} \cdot \mathbf{a} = \omega^2 \mathbf{a}$$

$$(\mathbf{S} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$$

$$\begin{pmatrix} \frac{k_1 + k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{k_1 + k_2}{m} - \omega^2 & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

This is the eigenvalue problem of Matrix \mathbf{S} . ω^2 is the eigenvalue while \mathbf{a} the eigenvector.



$$(\mathbf{S} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$$

$$\begin{pmatrix} \frac{k_1 + k_2}{m} - \omega^2 & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Unless, **determinant** $|\mathbf{S} - \omega^2 \mathbf{I}| = 0$. Then inverse matrix $(\mathbf{S} - \omega^2 \mathbf{I})^{-1}$ does not exist.

Only in this case would \mathbf{a} have nontrivial solutions.

$$|\mathbf{S} - \omega^2 \mathbf{I}| = 0$$

$$\begin{vmatrix} \frac{k_1 + k_2}{m} - \omega^2 & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2 + k_3}{m} - \omega^2 \end{vmatrix} = 0$$

This is a second order equation for ω^2 usually with two solutions ω_1^2, ω_2^2 .

Plug $\omega^2 = \omega_1^2, \omega_2^2$ into

$(\mathbf{S} - \omega^2 \mathbf{I}) \cdot \mathbf{a} = 0$ to solve the two columns vector \mathbf{a} :

$$\mathbf{a}^{(1)} = \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \end{pmatrix}$$

$$\mathbf{a}^{(2)} = \begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix}$$

We have found two oscillating mode solutions for the coupled system.

$$\mathbf{a}^{(1)} e^{i\omega_1 t} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} e^{i\omega_1 t}$$

$$\mathbf{a}^{(2)} e^{i\omega_2 t} = \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} e^{i\omega_2 t}$$

Since the ODE's are linear, any linear combinations of the two modes are solutions.

$$\mathbf{x} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} \cdot e^{i\omega_1 t} \cdot (c_1 e^{i\phi_1}) + \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} \cdot e^{i\omega_2 t} \cdot (c_2 e^{i\phi_2})$$

Take the real part.

Two complex coefficients

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} c_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} c_2 \cos(\omega_2 t + \phi_2)$$

It is clear that, for example:

$$[f_2 \cos(\omega_2 t) + g_2 \sin(\omega_2 t)]$$

$$x_1 = \mathbf{a}_1^{(1)} c_1 \cos(\omega_1 t + \phi_1) + \mathbf{a}_1^{(2)} c_2 \cos(\omega_2 t + \phi_2)$$

$$x_2 = \mathbf{a}_2^{(1)} c_1 \cos(\omega_1 t + \phi_1) + \mathbf{a}_2^{(2)} c_2 \cos(\omega_2 t + \phi_2)$$

Again, four unspecified constants $c_{1,2}, \phi_{1,2}$ will be determined by four initial conditions:

$$x_1(0), x_2(0), x_1'(0), x_2'(0)$$

We have solved the coupled oscillation problem for two particles.

It is easier to fit initial conditions by another form of cosine and sine solution.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} [f_1 \cos(\omega_0 t) + g_1 \sin(\omega_0 t)] + \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} [f_2 \cos(\sqrt{3}\omega_0 t) + g_2 \sin(\sqrt{3}\omega_0 t)]$$

四個未定常數將由四個起使條件決定。例如：

The four unspecified constants will be determined by four initial conditions.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} f_1 + \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} f_2$$



$$x_1(0) = \mathbf{a}_1^{(1)} f_1 + \mathbf{a}_1^{(2)} f_2$$

$$x_2(0) = \mathbf{a}_2^{(1)} f_1 - \mathbf{a}_2^{(2)} f_2$$

因此 $f_{1,2}$ 可以唯一決定。 $f_{1,2}$ will now be uniquely specified.

$$\begin{pmatrix} x'_1(0) \\ x'_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} g_1 \omega_0 + \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} g_2 \sqrt{3} \omega_0$$



$$x'_1(0) = \mathbf{a}_1^{(1)} g_1 \omega_0 + \mathbf{a}_1^{(2)} g_2 \sqrt{3} \omega_0$$

$$x'_2(0) = \mathbf{a}_2^{(1)} g_1 \omega_0 - \mathbf{a}_2^{(2)} g_2 \sqrt{3} \omega_0$$

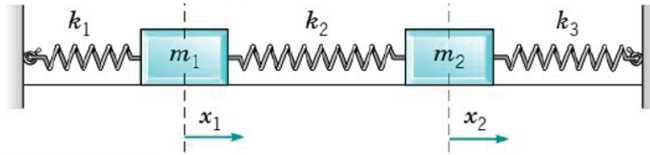
$g_{1,2}$ 也可以唯一決定。 $g_{1,2}$ will also be uniquely specified.

Note that $g_{1,2}$ will vanish if initial velocities $x'_{1,2}(0)$ are both zero.

於是微分方程組加起始條件，求解完成。

Our result satisfy both system of ODE and initial condition. It is the unique solution.

2. Consider a coupled oscillation of two particles as shown below:



Homework III

with equations of motion:

$$\frac{d^2 x_1}{dt^2} = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2, \quad \frac{d^2 x_2}{dt^2} = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$

which can be written in the notations of matrices:

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

Assume that the matrix \mathbf{A} equals:

$$\mathbf{A} \equiv \omega_0^2 \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

The general solutions can be written as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix} \cos(\omega_1 t + \phi_1) + c_2 \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

- A. Find the numbers $\omega_1, \begin{pmatrix} \mathbf{a}_1^{(1)} \\ \mathbf{a}_2^{(1)} \end{pmatrix}$ and $\omega_2, \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{a}_2^{(2)} \end{pmatrix}$.
- B. If the initial condition is $x_1(0) = a_m, x_2(0) = 0, x_1'(0) = x_2'(0) = 0$, find the solution. Hint: $\phi_1 = \phi_2 = 0$.

But are you sure that the eigenvalues ω_1^2, ω_2^2 are both positive? $\omega_1^2, \omega_2^2 >? 0$

If $\omega^2 < 0$, $e^{i\omega t}$ is an exponential function instead of oscillating function.

Will prove if matrix \mathbf{S} is symmetric and positive definite, eigenvalues ω^2 are both positive.

And for coupled system the matrix \mathbf{S} is symmetric and positive definite.

對稱矩陣

定理：一個對稱矩陣的本徵值都是實數，且不同本徵值的本徵向量彼此正交。

All n eigenvalues λ of a **symmetric matrix** S are real.

The n eigenvectors \mathbf{u} can be chosen to be orthogonal.

要確定 ω 有兩個實數解，本徵值 λ 必須是正實數才行！

Positive Definite 正定 Matrix

若一對稱矩陣 \mathbf{S} 的本徵值都是正數，此矩陣稱為正定矩陣。

證明：若Quadratic form 二次型 $\mathbf{x}^T \mathbf{S} \mathbf{x}$ 恆為正值，此矩陣 \mathbf{S} 為正定矩陣。

取 $\mathbf{x} = \mathbf{u}_1$ 。已知 $\mathbf{u}_1^T \mathbf{S} \mathbf{u}_1$ 為正。

$$\mathbf{u}^{(1)T} \mathbf{S} \mathbf{u}^{(1)} = \lambda_1 \mathbf{u}^{(1)T} \mathbf{u}^{(1)} = \lambda_1$$

因此 λ_1 為正。同理 λ_2 亦為正。矩陣 \mathbf{S} 本徵值皆為正，為正定矩陣。

彈簧組位能恆為正，因此 \mathbf{S} 為正定矩陣：本徵值 λ 都是正數。

$$V = \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 = m \mathbf{x}^T \mathbf{S} \mathbf{x}$$

模式頻率： $\omega = \sqrt{\lambda}$ 都是實數。

This formalism could be also easily extended to more than two masses.

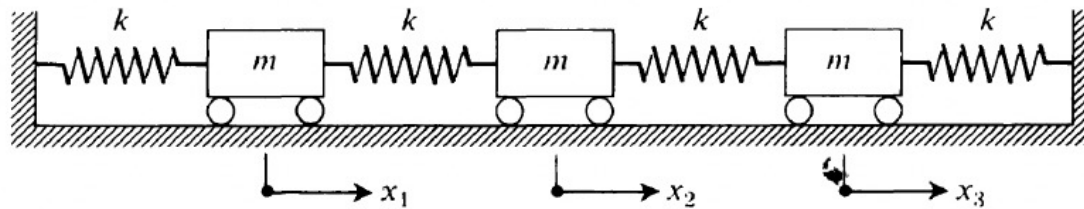


Figure 10.13

A. 運動方程式：

$$m \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - kx_1 = -2kx_1 + kx_2$$

$$m \frac{d^2 x_2}{dt^2} = k(x_3 - x_2) - k(x_2 - x_1) = -2kx_2 + kx_3 + kx_1$$

$$m \frac{d^2 x_3}{dt^2} = -k(x_3 - x_2) - kx_3 = -2kx_3 + kx_2$$

B. 計算彈簧的彈力位能：

$$V = \frac{k}{2} [x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2] = k(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

C. 若以矩陣語言來書寫：

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

取 $\mathbf{x} \equiv \mathbf{a}e^{i\omega t}$ ，代入得本徵值方程式： $\mathbf{A} \cdot \mathbf{a} = \omega^2 \mathbf{a} = \lambda \omega_0^2 \mathbf{a}$ 。特徵方程式為以下

$$\frac{d^2}{dt^2} \mathbf{x} = -\mathbf{A} \cdot \mathbf{x}$$

取 $\mathbf{x} \equiv \mathbf{a}e^{i\omega t}$ ，代入得本徵值方程式： $\mathbf{A} \cdot \mathbf{a} = \omega^2 \mathbf{a} = \lambda \omega_0^2 \mathbf{a}$ 。特徵方程式為以下行列式為零：

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

得代數方程式：

$$(2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)(2-4\lambda+\lambda^2) = 0$$

解得本徵值：

$$\lambda = 2, 2 \pm \sqrt{2}, \omega^2 = 2\omega_0^2, (2 \pm \sqrt{2})\omega_0^2$$

將本徵值及對應的角頻率由小到大編號：

$$\omega_1 \sim \sqrt{2 - \sqrt{2}}\omega_0$$

$$\omega_2 \sim \sqrt{2}\omega_0$$

$$\omega_3 \sim \sqrt{2 + \sqrt{2}}\omega_0$$

D. 計算本徵向量：

For $\lambda = 2$: 解聯立方程組： $(\mathbf{A} - \lambda\omega_0^2) \cdot \mathbf{a} = 0$

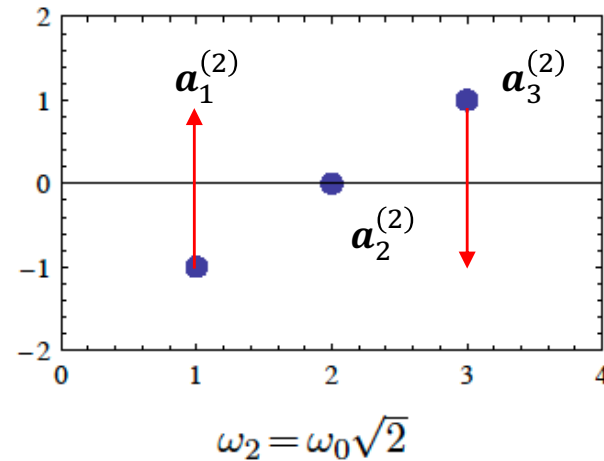
可以寫成：

$$\omega_2 \sim \sqrt{2}\omega_0$$

$$\begin{aligned} -a_2 &= 0 \\ -a_1 - a_3 &= 0 \\ -a_2 &= 0 \end{aligned}$$

可以得： $\mathbf{a}^{(2)} \sim \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\mathbf{x} = \mathbf{a}^{(2)} e^{i\omega_2 t}$$



兩個粒子 x_1, x_2 以 a_1, a_2 為振幅一起作同步(同 ω)的簡諧運動。

a_1, a_3 一正一負，因此 x_1, x_3 的振盪有相角差 π 。 a_2 為零，在此模式 x_2 是靜止的。

若起始條件滿足，位移行向量正比於本徵向量：

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \propto \mathbf{a}^{(2)} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{x}'(0) = \begin{pmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

以下式就是運動方程式與起始條件的解，接下來位移就會維持此比例關係。

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \mathbf{a}^{(2)} \cos \omega_2 t \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos \omega_2 t$$

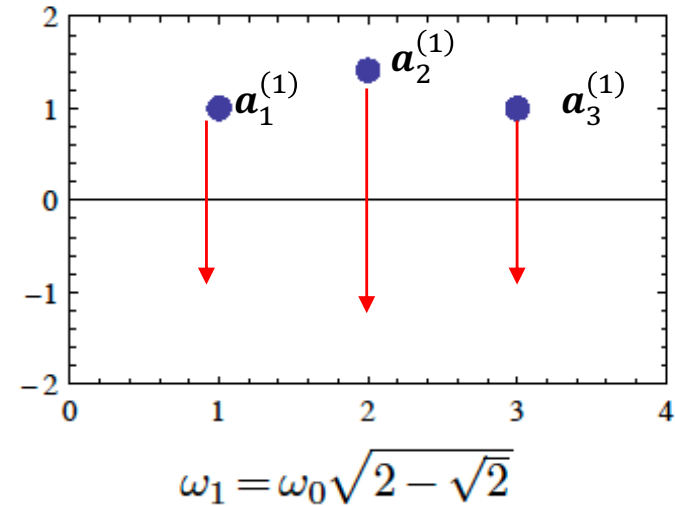
這是一個可獨立振盪的模式。

For $\lambda = 2 - \sqrt{2}$:

$$\omega_1 \sim \sqrt{2 - \sqrt{2}} \omega_0$$

$$\begin{aligned}\sqrt{2}a_1 - a_2 &= 0 \\ -a_1 + \sqrt{2}a_2 - a_3 &= 0 \\ -a_2 + \sqrt{2}a_3 &= 0\end{aligned}$$

$$\mathbf{a}^{(1)} \sim \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \sin \frac{\pi}{4} \\ \sin \frac{\pi}{2} \\ \sin \frac{3\pi}{4} \end{pmatrix}$$



$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \propto \mathbf{a}^{(1)} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{x}'(0) = \begin{pmatrix} x'_1(0) \\ x'_2(0) \\ x'_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

以下式就是運動方程式與起始條件的解：

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \mathbf{a}^{(1)} \cos \omega_1 t$$

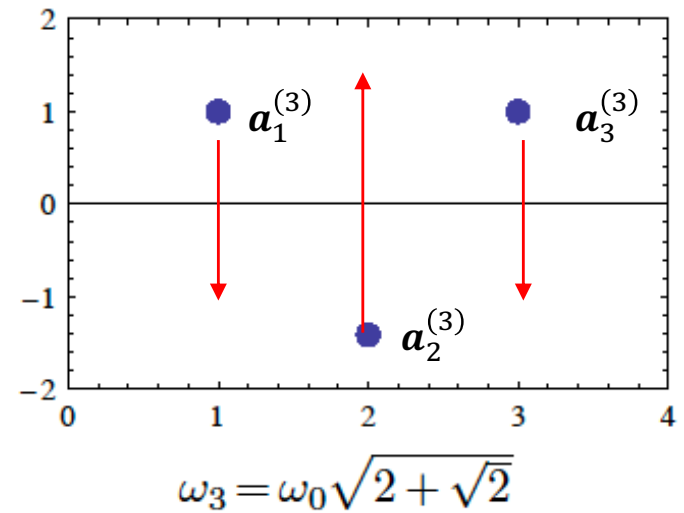
這是另一個可獨立振盪的模式。

For $\lambda = 2 + \sqrt{2}$:

$$\omega_3 \sim \sqrt{2 + \sqrt{2}} \omega_0$$

$$\mathbf{a}^{(3)} \sim \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\begin{aligned} \sqrt{2}a_1 + a_2 &= 0 \\ a_1 + \sqrt{2}a_2 + a_3 &= 0 \\ a_2 + \sqrt{2}a_3 &= 0 \end{aligned}$$



$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \propto \mathbf{a}^{(3)} \cos \omega_3 t$$

位移行向量若一開始就與一本徵向量成正比，接下來位移就會維持此比例關係。

一個本徵向量對應一個可獨立振盪的模式。

每一個運動模式(本徵向量 \mathbf{a})，有一個特定的簡諧振動頻率(本徵值)！

模式的數目，就是本徵向量數目，洽等於粒子數目。

有了以上三個模式的解，一般解將是三者的線性組合：

$$\omega_1 \sim \sqrt{2 - \sqrt{2}}\omega_0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

$$\omega_2 \sim \sqrt{2}\omega_0$$

三個未定常數 $f_{1,2,3}$ 將由三個起使條件 $x_{1,2,3}(0)$ 決定：

$$\omega_3 \sim \sqrt{2 + \sqrt{2}}\omega_0$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} f_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

根據對稱矩陣本徵向量的展開定理，三個本徵向量彼此正交： $\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \delta_{mn}$

可以如上選 $\mathbf{a}^{(n)}$ 使其向量長度為1。

任一起始條件的行向量 $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$ ，都可以展開成三個本徵向量的線性組合。

$f_{1,2,3}$ 可以解出：

$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

另外三個未定常數 $g_{1,2,3}$ 則由另外三個起使條件 $x'_{1,2,3}(0)$ 以類似方式決定。

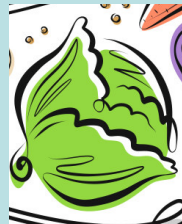
起始各個配料依配方 $f_{1,2,3}$ 收集



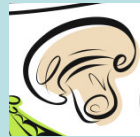
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⋮

各個配料按分離烹煮 $\cos(\omega_i t)$

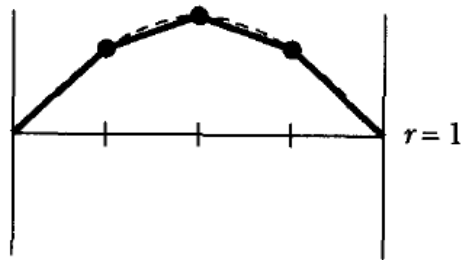
個自簡諧振蕩後，最後合體！



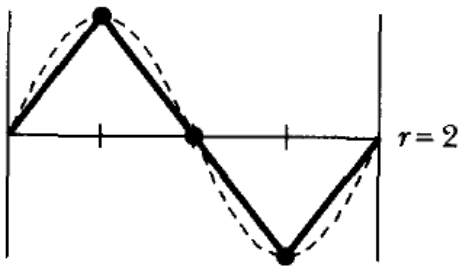
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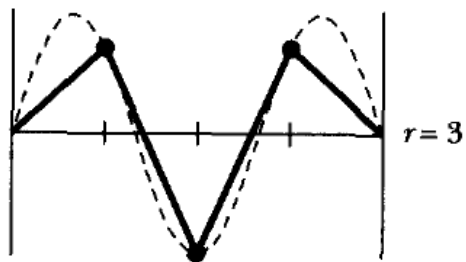
三個本徵向量的元素似乎滿足簡單的規則：都是正弦函數，角度以等差 $\frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}$ 增加。



$$\mathbf{a}^{(1)} \sim \begin{pmatrix} \sin \frac{\pi}{4} \\ \sin \frac{\pi}{2} \\ \sin \frac{3\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



$$\mathbf{a}^{(2)} \sim \begin{pmatrix} \sin \frac{\pi}{2} \\ \sin \pi \\ \sin \frac{3\pi}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$



$$\mathbf{a}^{(3)} \sim \begin{pmatrix} \sin \frac{3\pi}{4} \\ \sin \frac{3\pi}{2} \\ \sin \frac{9\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$