Second Order Linear ODE Ordinary Differential Equation

$$y'' + P(x)y' + Q(x)y = f(x) \quad \longleftrightarrow \quad y' = f(y, x)$$

每一項只包含一次方的y函數，及一次方的一次或二次微分，以及一個已知x函數。

It consists of one power of y, first and second derivatives of y, and a known function of x.

Equation of Motion is most naturally a Second Order Linear ODE.

$$x'' + P(t)x' + Q(t)x = f(t)$$

Second Order Linear Ordinary Differential Equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

每一項只包含一次方的y函數，及一次方的一次或二次微分，以及一個已知x函數。

consists of one power of y, a first and second derivatives of y and a known function of x.

We’ll show with one solution of the homogeneous ODE, we can calculate general solutions.

Other than that, we don’t have general solutions for these ODE.

But if we restrict $P(x), Q(x)$ to constants such as:

$$y'' + a_1 y' + a_0 y = f(x)$$

we do have a procedure using complex number to write down the solutions.

These are called Second Order Linear ODE with constant coefficients.

Surprisingly, this systemic method can be extended to Linear ODE with constant coefficients of any finite order.

$$\sum_n a_n \cdot \frac{d^n y}{dx^n} = f(x)$$

Linear second order ODE 解法

$$y'' + P(x)y' + Q(x)y = f(x)$$

Variation of parameters
Wronskian

級數法 Series

Complex number 複變函數法

Linear ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

If we find one solution of the homogeneous ODE, we can calculate the general solution.

Linear ODE with constant coefficients

$$y'' + a_1 y' + a_0 y = f(x)$$

We can find general solutions using complex number exponential function $e^{\alpha t}$.

Second order Linear ODE

Inhomogeneous ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

⟷

Homogeneous ODE

$$y'' + P(x)y' + Q(x)y = 0$$

Any solutions y of Linear inhomogeneous ODE equals one solution y_2 of the inhomogeneous ODE plus the general solution y_1 of the homogeneous ODE.

$$y'' + P(x)y' + Q(x)y = f(x)$$

The difference $y - y_2$ between any two solutions of a Linear inhomogeneous ODE equals a solution of the homogeneous ODE.

$$y'' + P(x)y' + Q(x)y = f(x) \quad - \quad y_2'' + P(x)y_2' + Q(x)y_2 = f(x)$$

$$\Downarrow$$

$$(y - y_2)'' + P(x)(y - y_2)' + Q(x)(y - y_2) = 0$$ homogeneous ODE

$y - y_2$ equals a solution y_1 of the homogeneous ODE. Hence:

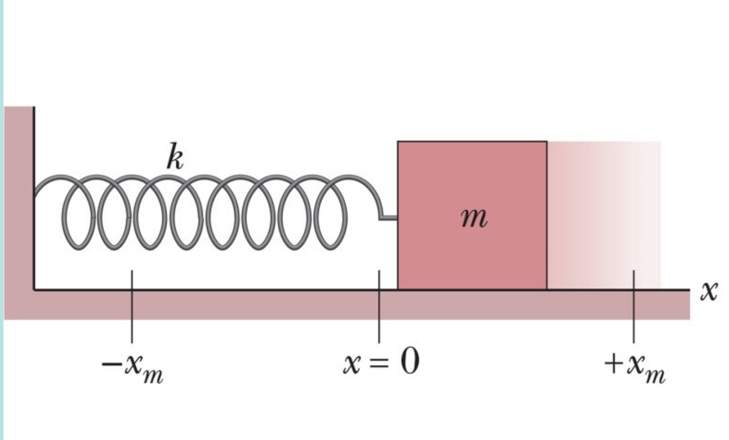
$$y = y_1 + y_2$$

Any solutions y of Linear inhomogeneous ODE equals one solution y_2 of the inhomogeneous ODE plus the general solution y_1 of the homogeneous ODE.

簡諧運動　　最簡單的週期運動

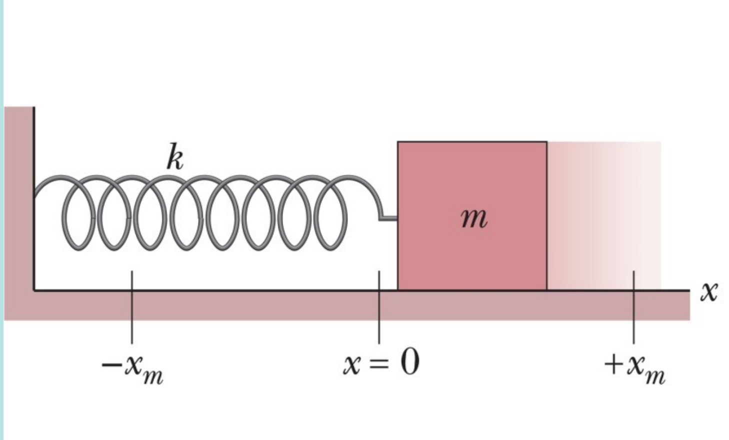
Simple Harmonic Motion SHM　　Periodic Motion

Simple Harmonic Oscillation SHO　　簡諧振盪



$F = -kx$

Equation of motion of SHM



$$F = -kx$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad -\omega^2 x$$

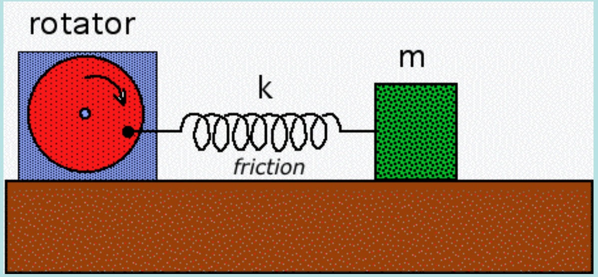
$$\omega = \sqrt{\frac{k}{m}}$$

This is a Homogeneous 2nd order Linear ODE.

$$y'' + a_1 y' + a_0 y = f(x)$$

外力下的震盪 Forced Oscillation

Assume that the additional force can be written as：$F_0 \cos \omega_D t$。



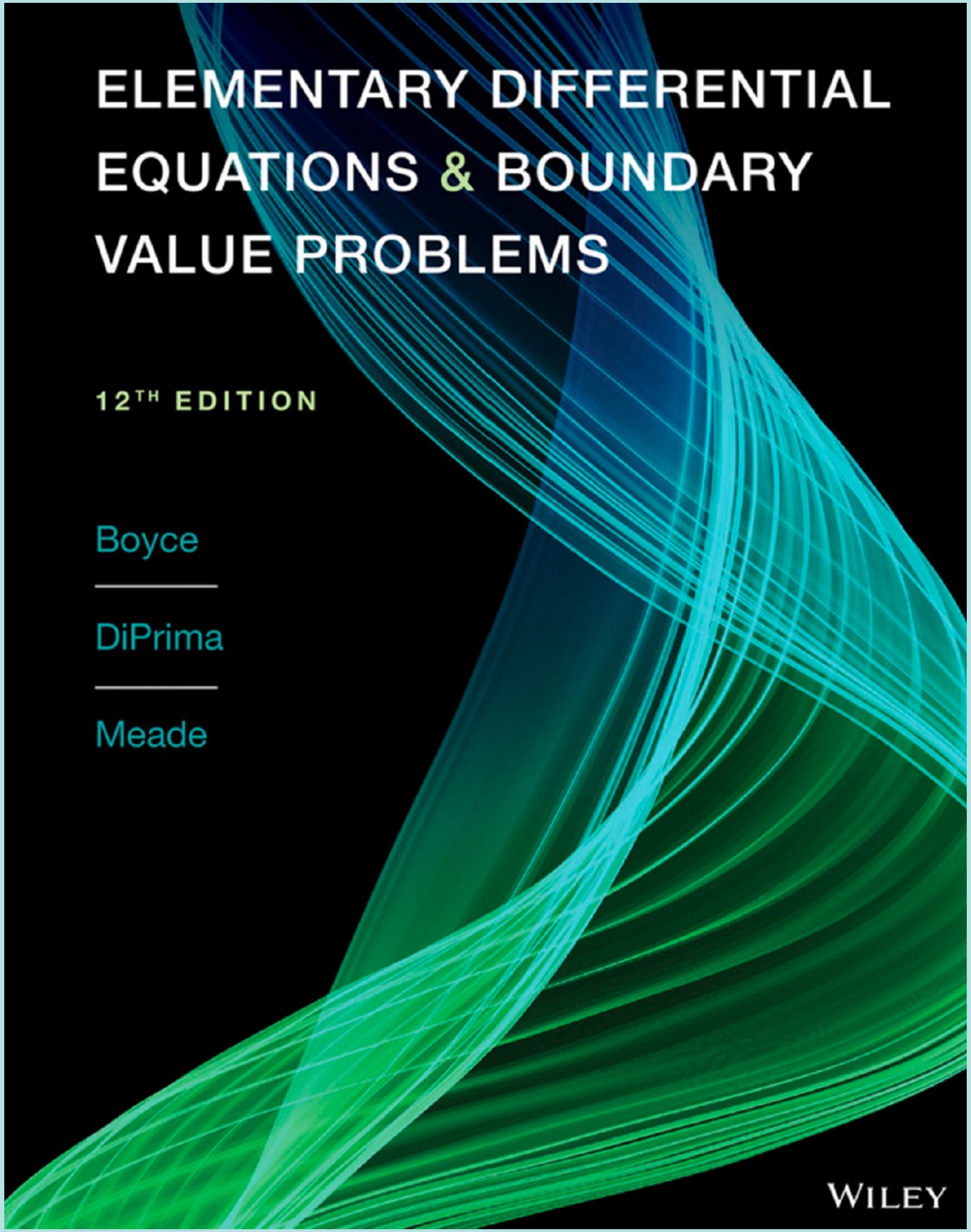
Equation of Motion contains one more term

$$m\frac{d^2x}{dt^2} = -kx + F_0 \cos \omega_D t$$

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{F_0}{m} \cos \omega_D t$$

$$y'' + a_1 y' + a_0 y = f(x)$$

This is an Inhomogeneous 2nd order Linear ODE with constant coefficients.



ELEMENTARY DIFFERENTIAL EQUATIONS & BOUNDARY VALUE PROBLEMS

12TH EDITION

Boyce

DiPrima

Meade

WILEY

To discuss general properties of linear differential equations, it is helpful to introduce a **differential operator** notation. Let p and q be continuous functions on an open interval I—that is, for $\alpha < t < \beta$. The cases for $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

It is important to understand that the result of applying the operator L to a function ϕ is another function, which we refer to as $L[\phi]$. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin(3t)$, then

$$\begin{aligned} L[\phi](t) &= (\sin(3t))'' + t^2(\sin(3t))' + (1+t)\sin(3t) \\ &= -9\sin(3t) + 3t^2\cos(3t) + (1+t)\sin(3t). \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator, that is, $D[\phi] = \phi'$.

In this section we study the second-order linear homogeneous differential equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With equation (2) we associate a set of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second-order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first-order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 | Existence and Uniqueness Theorem

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists.*
2. The initial value problem has *only one* solution; that is, the solution is *unique.*
3. The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For instance, we found in Example 3.1.1 that the initial value problem

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by equation (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

For most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first-order and second-order linear differential equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

Let us now assume that y_1 and y_2 are two solutions of equation (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0,$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 | Principle of Superposition

If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any constant multiple of a solution of equation (2) is also a solution.

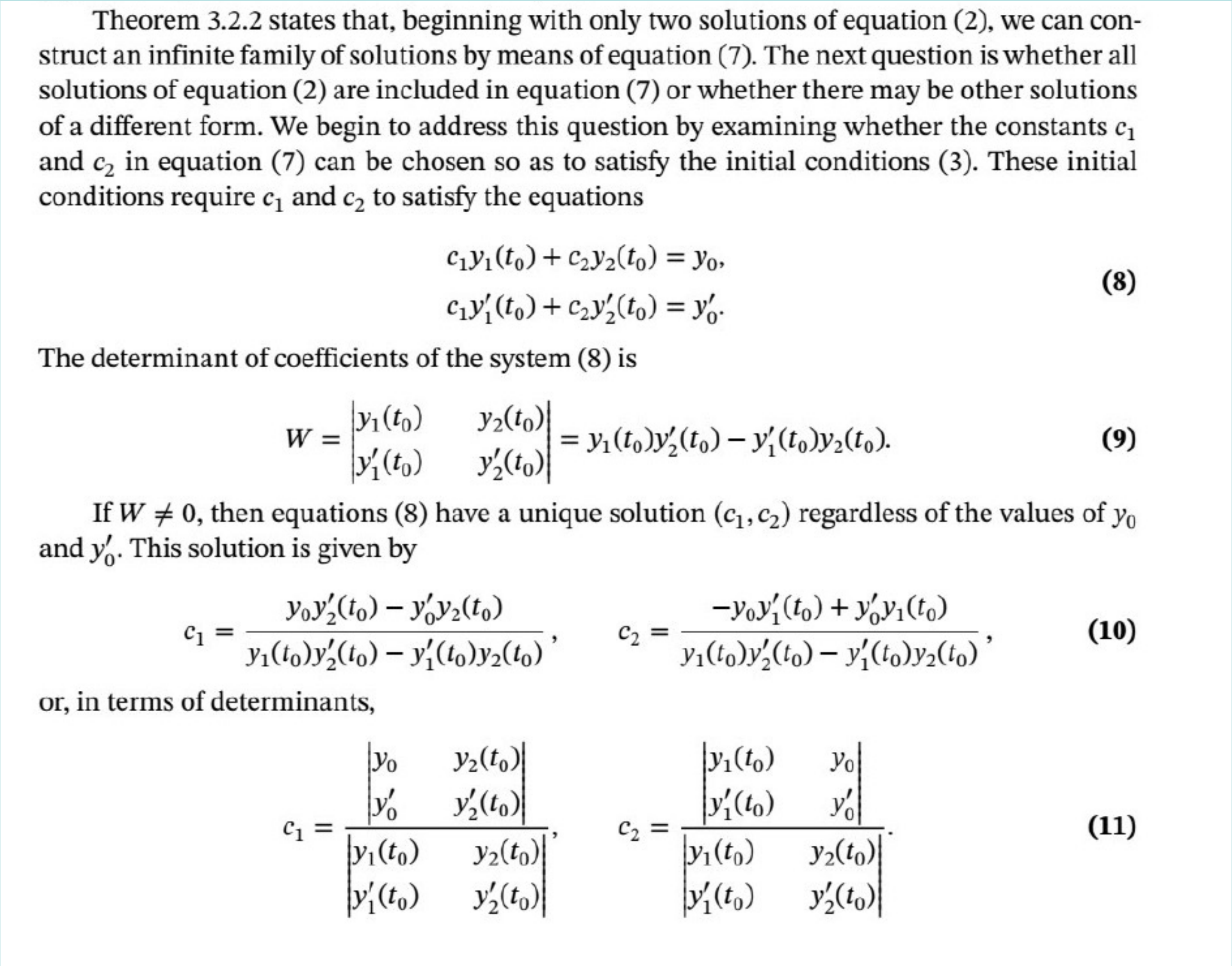
To prove Theorem 3.2.2, we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{7}$$

for y in equation (2). By calculating the indicated derivatives and rearranging terms, we obtain

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ &= c_1y_1'' + c_2y_2'' + c_1p(t)y_1' + c_2p(t)y_2' + c_1q(t)y_1 + c_2q(t)y_2 \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1L(y_1) + c_2L(y_2). \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, the function y as given by equation (7) satisfies the differential equation (2), and the proof of Theorem 3.2.2 is complete.



Theorem 3.2.2 states that, beginning with only two solutions of equation (2), we can construct an infinite family of solutions by means of equation (7). The next question is whether all solutions of equation (2) are included in equation (7) or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in equation (7) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0'. \end{aligned} \tag{8}$$

The determinant of coefficients of the system (8) is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{9}$$

If $W \neq 0$, then equations (8) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. This solution is given by

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2, the linear combination $y = c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions (3) as well as the differential equation (2). Note that the denominator in the expressions for c_1 and c_2 is the nonzero determinant W.

On the other hand, if $W = 0$, then the denominators appearing in equations (10) and (11) are zero. In this case equations (8) have no solution unless y_0 and y_0' have values that also make the numerators in equations (10) and (11) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

The determinant W is called the **Wronskian[3] determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W[y_1, y_2](t_0)$ to stand for the expression on the right-hand side of equation (9), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument establishes the following result.

Theorem 3.2.3

Suppose that y_1 and y_2 are two solutions of equation (2)

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (3)

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

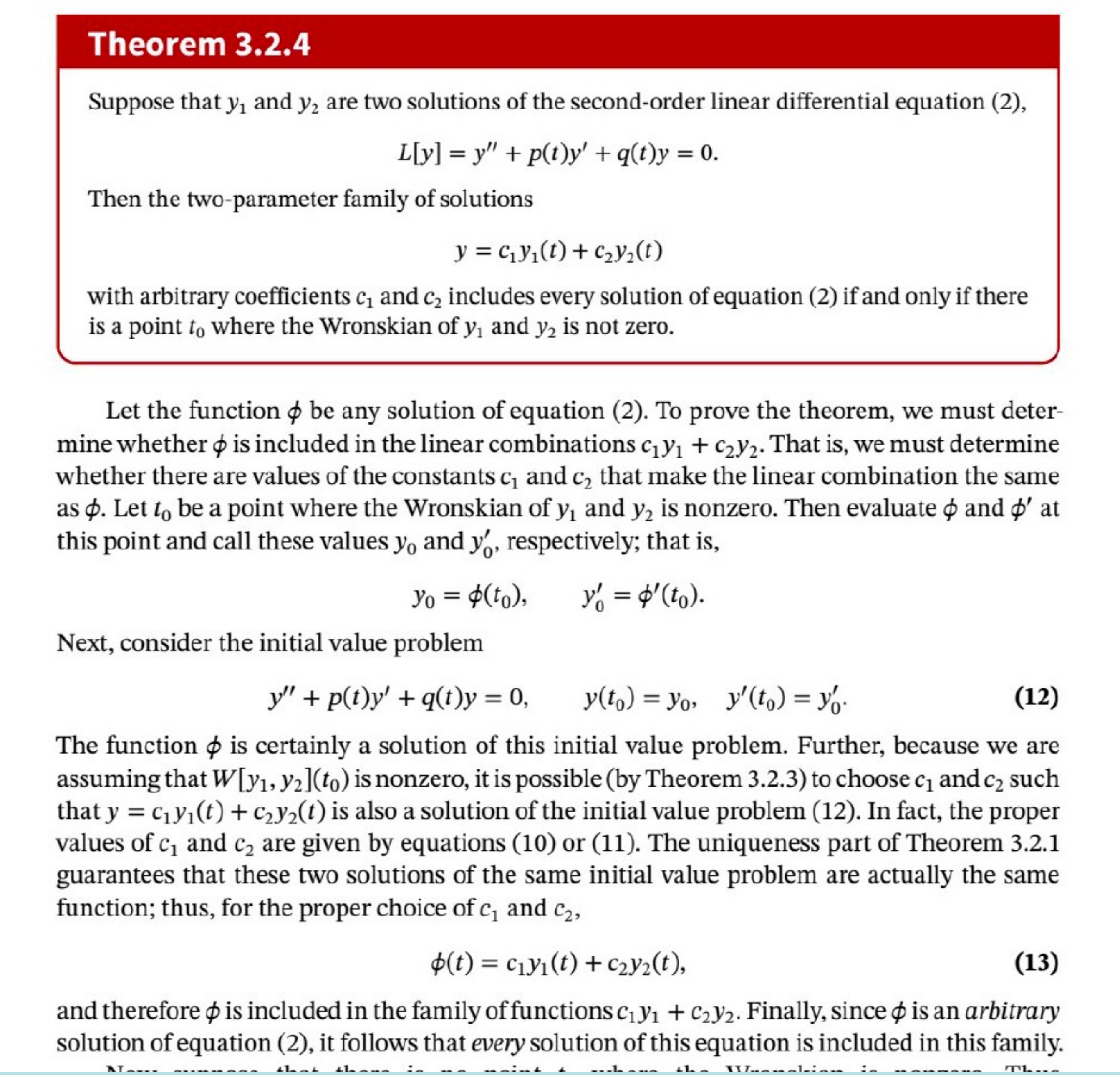
are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (2) and the initial conditions (3) if and only if the Wronskian

$$W[y_1, y_2] = y_1y_2' - y_1'y_2$$

is not zero at t_0.



Theorem 3.2.4

Suppose that y_1 and y_2 are two solutions of the second-order linear differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the two-parameter family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of equation (2) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Let the function ϕ be any solution of equation (2). To prove the theorem, we must determine whether ϕ is included in the linear combinations $c_1y_1 + c_2y_2$. That is, we must determine whether there are values of the constants c_1 and c_2 that make the linear combination the same as ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; that is,

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{12}$$

The function ϕ is certainly a solution of this initial value problem. Further, because we are assuming that $W[y_1, y_2](t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 such that $y = c_1y_1(t) + c_2y_2(t)$ is also a solution of the initial value problem (12). In fact, the proper values of c_1 and c_2 are given by equations (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1y_1(t) + c_2y_2(t), \tag{13}$$

and therefore ϕ is included in the family of functions $c_1y_1 + c_2y_2$. Finally, since ϕ is an *arbitrary* solution of equation (2), it follows that *every* solution of this equation is included in this family.

Now suppose that there is no point t_0 where the Wronskian is nonzero. Thus $W[y_1, y_2](t_0) = 0$ for every point t_0. Then (by Theorem 3.2.3) there are values of y_0 and y_0' such that no values of c_1 and c_2 satisfy the system (8). Select a pair of such values for y_0 and y_0' and choose the solution $\phi(t)$ of equation (2) that satisfies the initial condition (3). Observe that this initial value problem is guaranteed to have a solution by Theorem 3.2.1. However, this solution is not included in the family $y = c_1y_1 + c_2y_2$. Thus, in cases where $W[y_1, y_2](t_0) = 0$ for every t_0, the linear combinations of y_1 and y_2 do not include all solutions of equation (2). This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that the Wronskian of y_1 and y_2 is not everywhere zero if and only if the linear combination $c_1y_1 + c_2y_2$ contains all solutions of equation (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of equation (2). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of equation (2) if and only if their Wronskian is nonzero.

We can restate the result of Theorem 3.2.4 in slightly different language: to find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition.

Now that you have a little experience verifying the nonzero Wronskian condition for the examples from Section 3.1, the following example handles all second-order linear differential equations whose characteristic polynomial has two distinct real roots.

In several cases we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question arises as to whether a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 3.2.5

Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of equation (2) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of equation (2) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

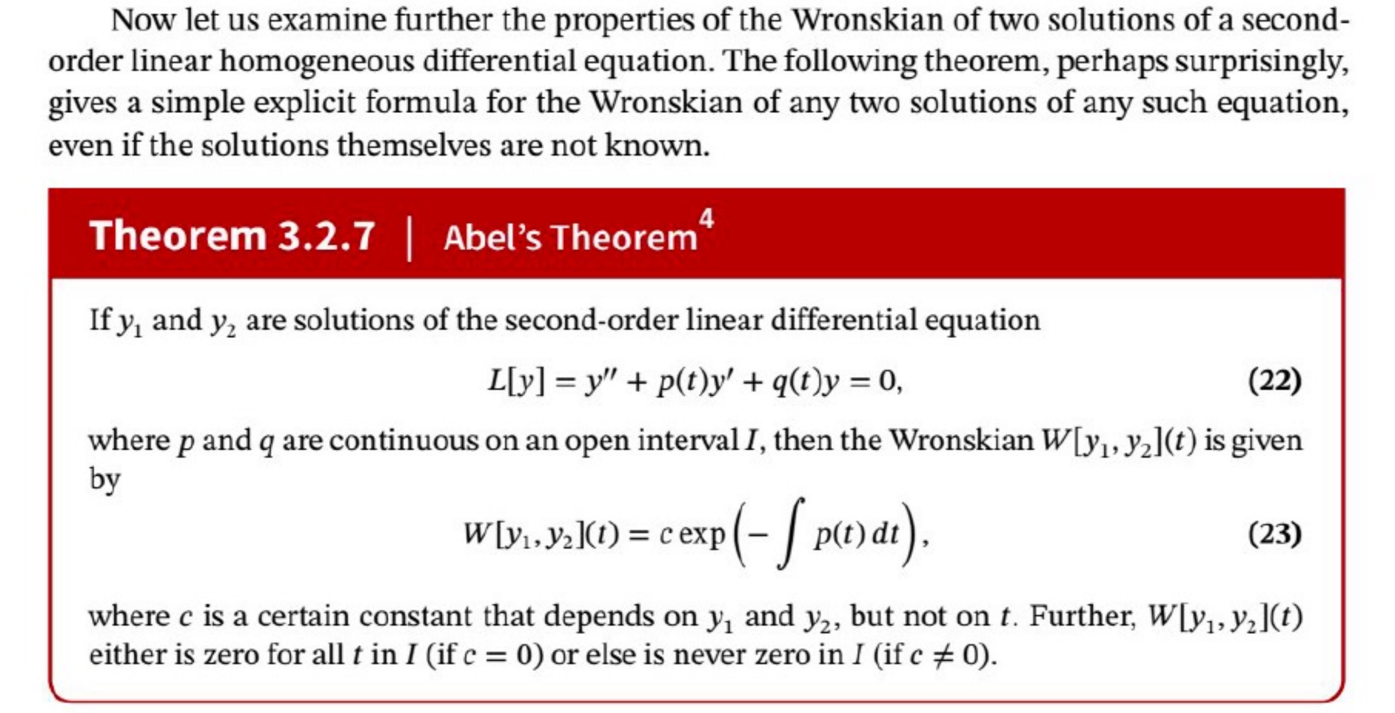
Then y_1 and y_2 form a fundamental set of solutions of equation (2).

First observe that the *existence* of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

Note that the potentially difficult part of this proof, demonstrating the existence of a pair of solutions, is taken care of by reference to Theorem 3.2.1. Note also that Theorem 3.2.5 does not address the question of how to find the solutions y_1 and y_2 by solving the specified initial value problems. Nevertheless, it may be reassuring to know that a fundamental set of solutions always exists.



Now let us examine further the properties of the Wronskian of two solutions of a second-order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 3.2.7 | Abel's Theorem[4]

If y_1 and y_2 are solutions of the second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right), \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem, we start by noting that y_1 and y_2 satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0,$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Niels Henrik Abel 1802-1829

If we multiply the first equation by $-y_2$, multiply the second by y_1, and add the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{25}$$

Next, we let $W(t) = W[y_1, y_2](t)$ and observe that

$$W' = y_1y_2'' - y_1''y_2. \tag{26}$$

Then we can write equation (25) in the form

$$W' + p(t)W = 0. \tag{27}$$

Equation (27) can be solved immediately since it is both a first-order linear differential equation (Section 2.1) and a separable differential equation (Section 2.2). Thus

$$W(t) = c\exp\left(-\int p(t)\,dt\right), \tag{28}$$

where c is a constant.

The value of c depends on which pair of solutions of equation (22) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t. This completes the proof of Theorem 3.2.7.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

Let's first start with Homogeneous Linear ODE： $y'' + P(x)y' + Q(x)y = 0$

It is called linear because of this theorem：

If we could find two function $y_1(x), y_2(x)$ satisfying $y'' + P(x)y' + Q(x)y = 0$

any linear combination $C_1y_1(x) + C_2y_2(x)$ would also satisfy the equation.

$$C_1 \quad y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad + \quad C_2 \quad y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$(C_1y_1 + C_2y_2)'' + P(x)(C_1y_1 + C_2y_2)' + Q(x)(C_1y_1 + C_2y_2) = 0$$

QED

如果已找到兩個函數$y_1(x), y_2(x)$都滿足方程式，我們就得到無限多個解！

If we could find two function $y_1(x), y_2(x)$, we get infinite number of solutions.

微分方程式的解需要讓自己挪出足夠的空間，這樣才能滿足起始條件。

That is natural. Then we can fit the initial condition.

可以證明 It could be proven that：

We could always find two independent solutions $y_1(x), y_2(x)$.

一定可以找到兩個函數$y_1(x), y_2(x)$都滿足方程式： $y'' + P(x)y' + Q(x)y = 0$

而且$y_1(x), y_2(x)$不是彼此的倍數，稱為彼此線性獨立。

All solutions can then be written as： $C_1y_1(x) + C_2y_2(x)$

C_1, C_2 would be determined by $y'(0), y(0)$。

There is a powerful theorem:

Arfken p362

If we have one solution $y_1(x)$ $\quad y'' + P(x)y' + Q(x)y = 0$

we can always compute another independent solution $y_2(x)$.

The key is Wronskian function $W(x)$, defined as:

$$W(x) \equiv y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

$W(x)$ satisfies a simple 1st order ODE, which could be easily solved.

$$\frac{dW}{dx} = y_1(x)y_2''(x) + y_1'(x)y_2'(x) - y_2(x)y_1''(x) - y_2'(x)y_1'(x)$$

$$= y_1(x)y_2''(x) - y_2(x)y_1''(x)$$

$$y_1 \times \quad y_2'' + P(x)y_2' + Q(x)y_2 = 0 \quad - \quad y_2 \times \quad y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$\frac{dW}{dx} + P(x)y_1y_2' - P(x)y_2y_1' = \frac{dW}{dx} + P(x)W = 0$$

$$\frac{dW}{dx} + P(x)W = 0$$

$$W = W(0)\exp\left(-\int_0^x P(x') \cdot dx'\right)$$

Józef Maria Hoene-Wroński

This is a separable equation.

$$\frac{dy}{dx} + p(x)y = 0$$

$$\int \frac{1}{y} dy = -\int p(x) \cdot dx + C'$$

$$\ln y = -\int p(x) \cdot dx + C'$$

$$y = C \exp\left(-\int_0^x P(x') \cdot dx' \right)$$

There is one more formula:

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}$$

If Wronskian $W(x)$ and y_1 is known, $\frac{y_2}{y_1}$ could be obtained by integration!

$$\frac{y_2}{y_1} = C + \int dx \frac{W}{y_1^2}$$

$$y_2 = Cy_1 + W(0) y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$

$$y_2 \sim y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$

由 y_1我們計算出另一個線性獨立的函數$y_2(x)$，也滿足此方程式。

From $y_1(x)$ we compute a second independent solution $y_2(x)$.

如果可以一個函數$y_1(x)$ 滿足方程式： $y'' + P(x)y' + Q(x)y = 0$

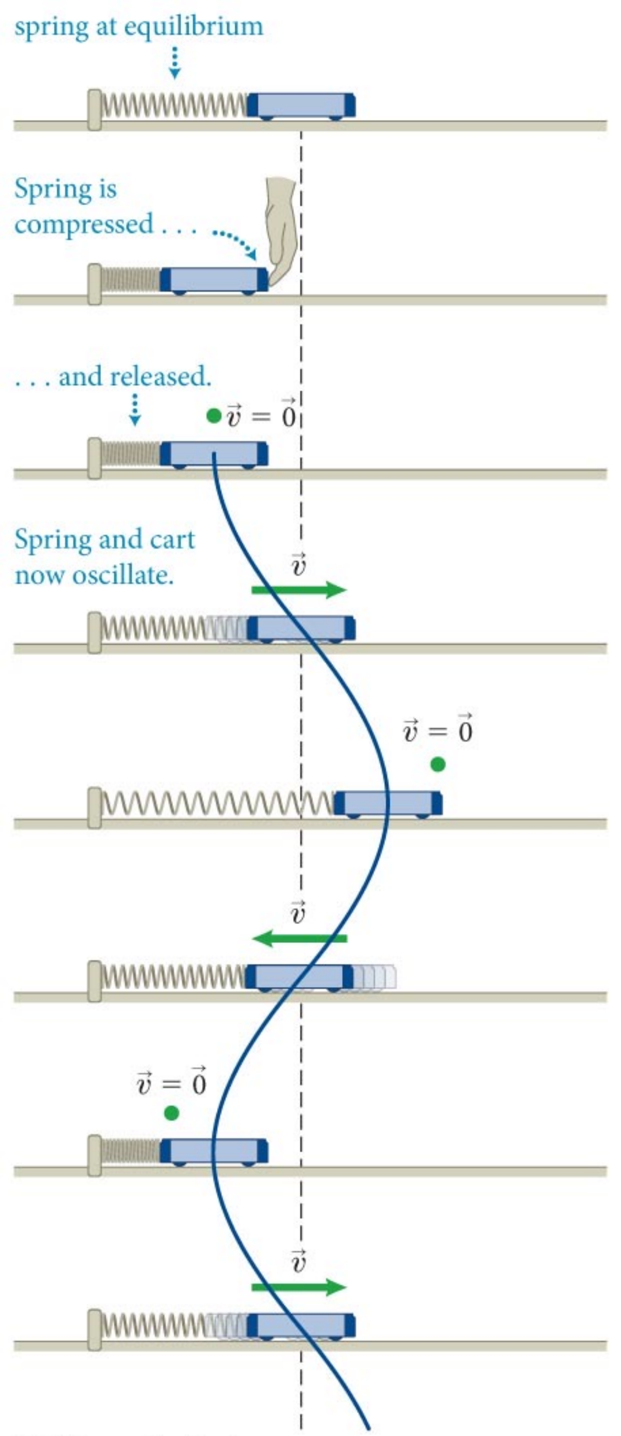
由 y_1我們計算出另一個線性獨立的函數$y_2(x)$，也滿足此方程式。

所有的解都可以寫成： $C_1y_1(x) + C_2y_2(x)$

C_1, C_2可以由起始條件$y'(0), y(0)$唯一決定。

因此，加上起始條件後， $y'' + P(x)y' + Q(x)y = 0$ 只有唯一解。

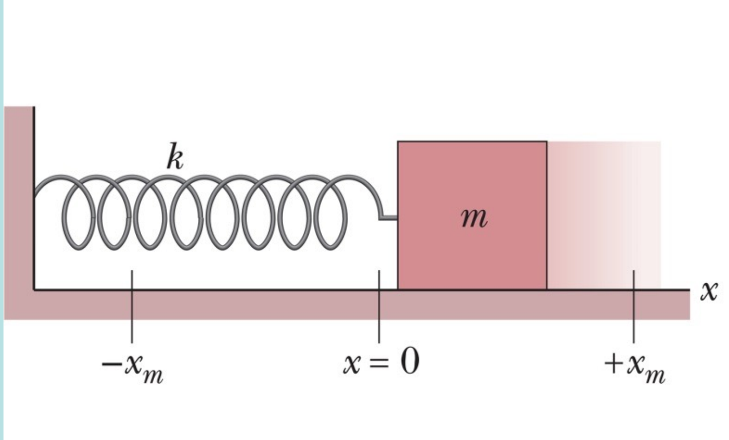
Simple Harmonic Motion SHM



簡諧運動　　最簡單的週期運動

Simple Harmonic Motion　　Periodic Motion

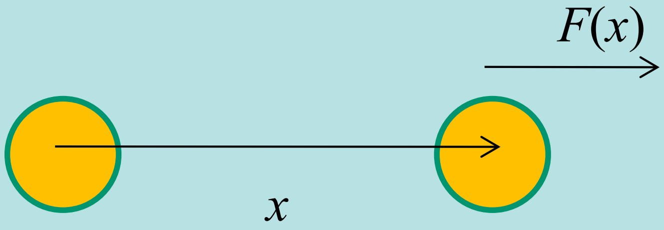
Simple Harmonic Oscillation　簡諧振盪



$F = -kx$

Any small range motion around an equilibrium point is a SHM！

原子力 Take atomic force as an example.

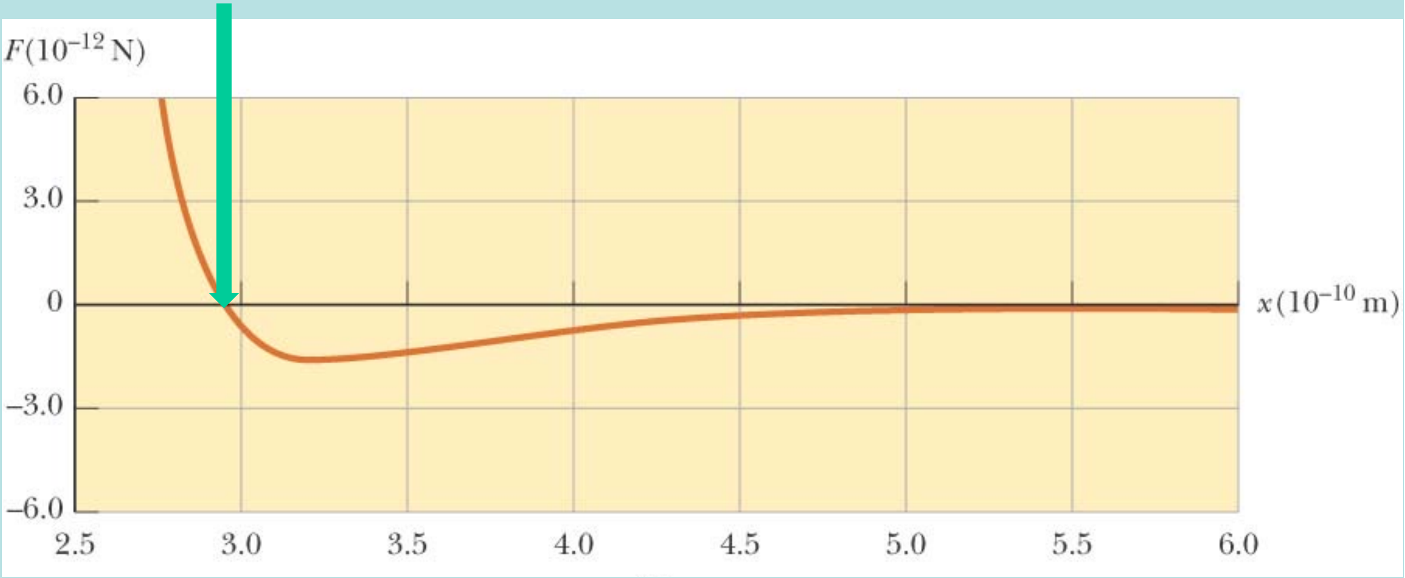


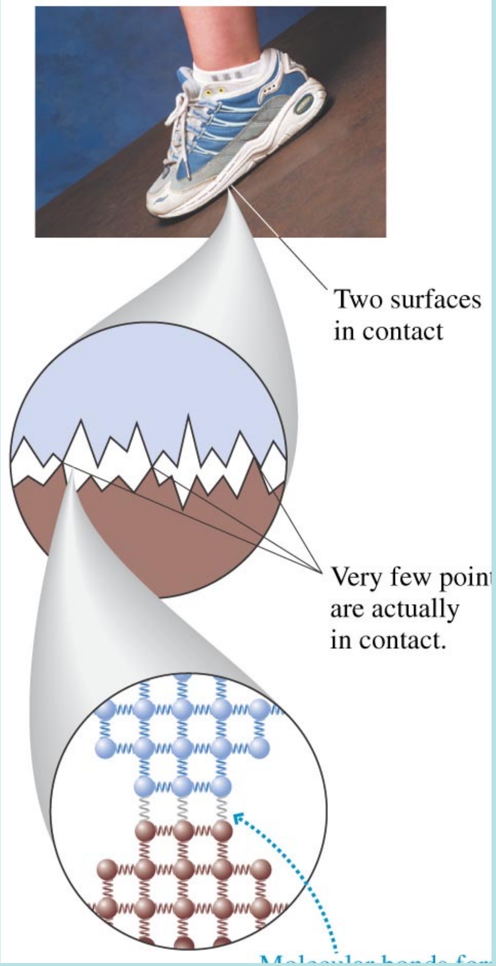
正力為排斥力，力為負則為吸引力

Positive force indicates repulsion, negative attraction.

原子力有一平衡點，力剛好為零：

There is an Equilibrium Point where the force vanishes.





摩擦力的根源是原子力

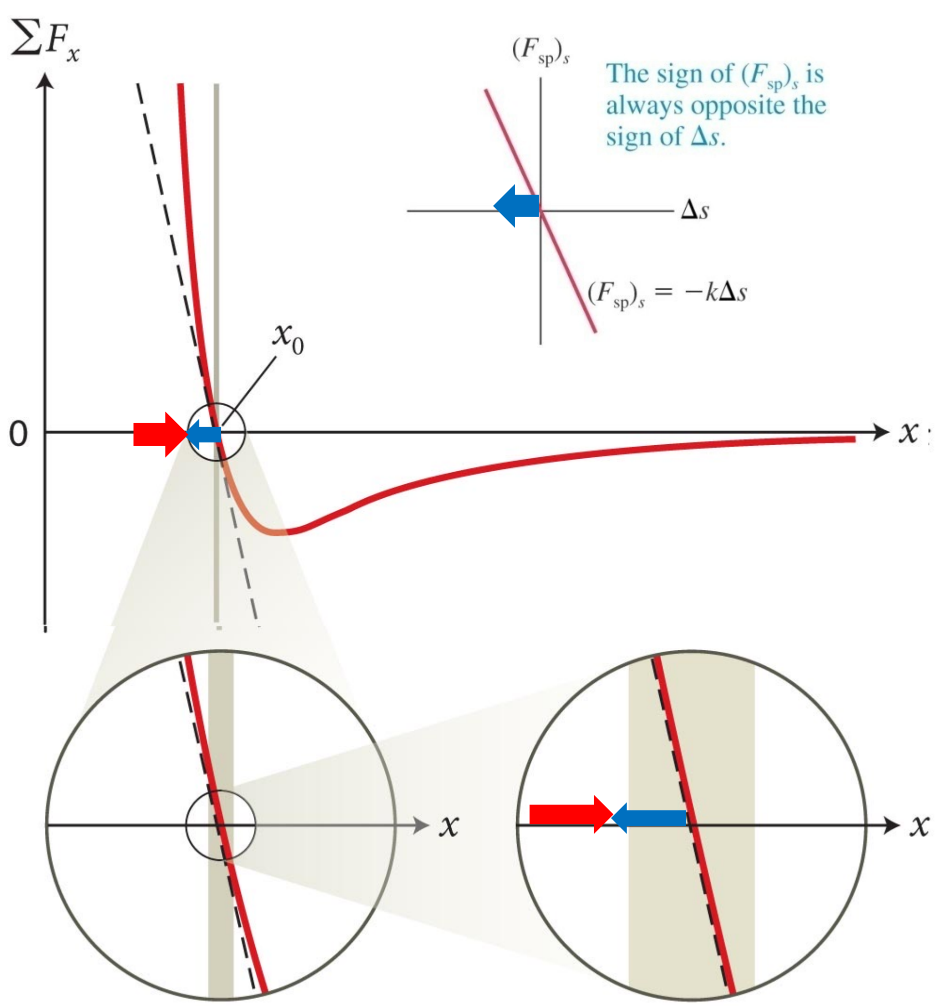
At the equilibrium point, the force is zero.

Push a particle inside the equilibrium point, a positive repulsive force will push it back.

Just like a spring.

正的原子排斥力會將原子推回平衡點。

定性上會如彈簧一樣。



The sign of $(F_{sp})_s$ is always opposite the sign of Δs.

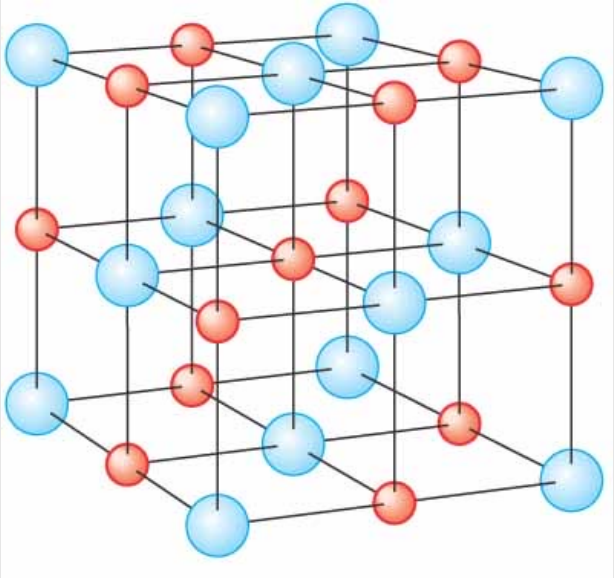
Near the equilibrium point, the force line can always be approximated by a straight line.

在相當接近平衡點的附近，原子力的曲線大致上可以用一條直線來近似。

A straight line means the force is proportional to displacement from the equilibrium point.

力正比於平衡點算起的位移，定量上也與與彈簧是相同的： $F = -k\Delta x$

如果把原子安排使得其距離恰在原子力的平衡點上，
這些原子可以組成有大規模秩序的晶格，此晶格會是平衡的，
晶格的大小與形狀處於平衡穩定狀態，這就是固態！

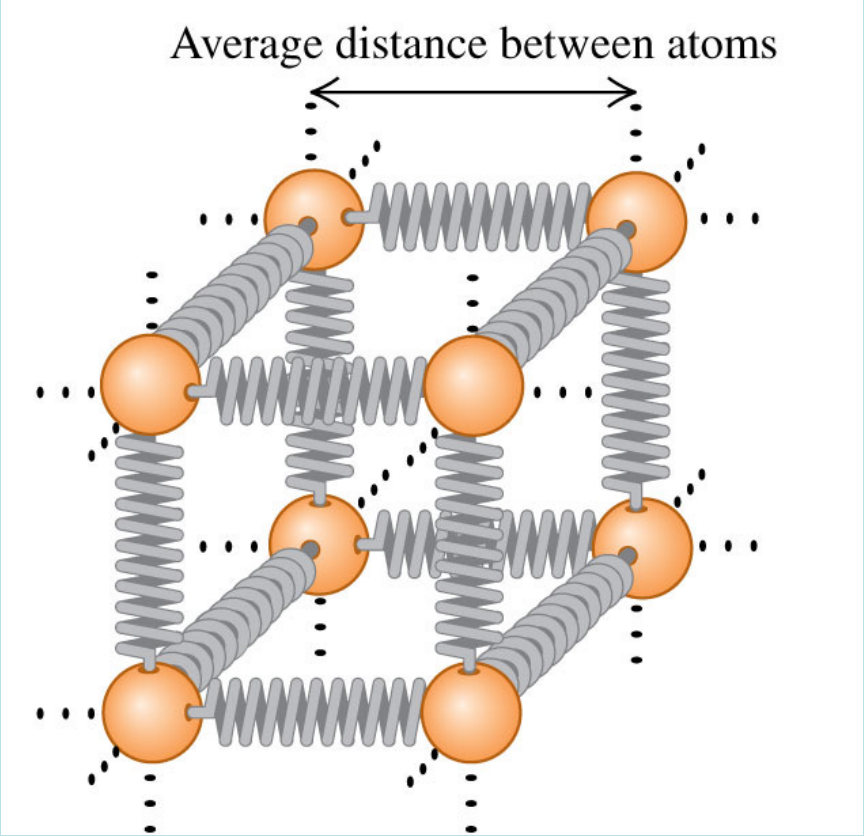


晶格會是穩定狀態的嗎？在小擾動作用下會回復原狀稱為穩定。

如果嘗試改變固體大小，固體會反彈如同彈簧！

固體中原子彼此之間，如同以彈簧連接。

固體的晶格會是穩定的！



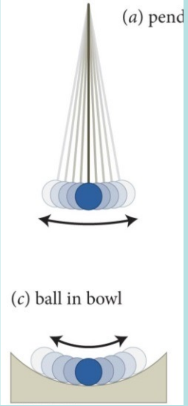
這是固體的彈性（變形回復力）的來源。

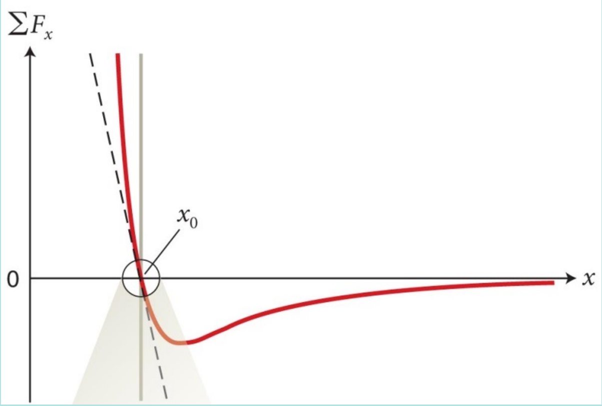
任一個有穩定平衡點的力，小範圍在平衡點附近，力的函數都可以以直線近似！

Near any stable equilibrium point, the force line can be approximated by a straight line.

因此任一個小範圍在穩定平衡點附近的運動，都可以以簡諧運動近似！

Any small motion around an equilibrium point is a SHM！

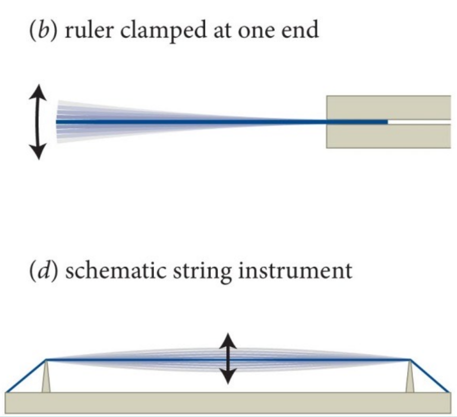




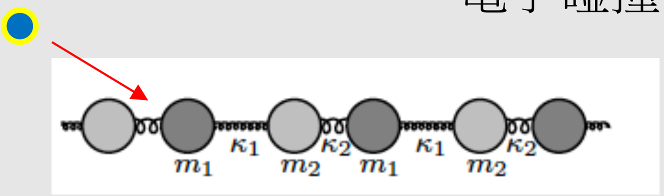
固體的原始形狀就是一平衡點，因此固體的變形運動，近似地是一個簡諧運動。

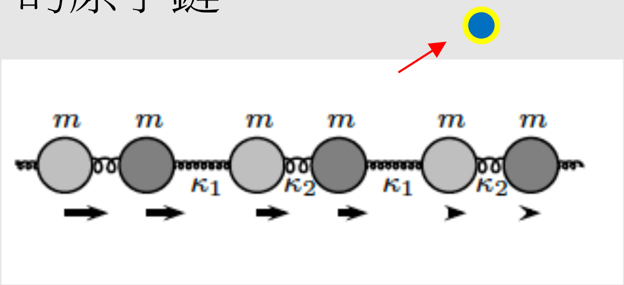
The original shape of a solid is by definition an stable equilibrium point.

Hence any small vibration or transfiguration of a structure is an SHM.



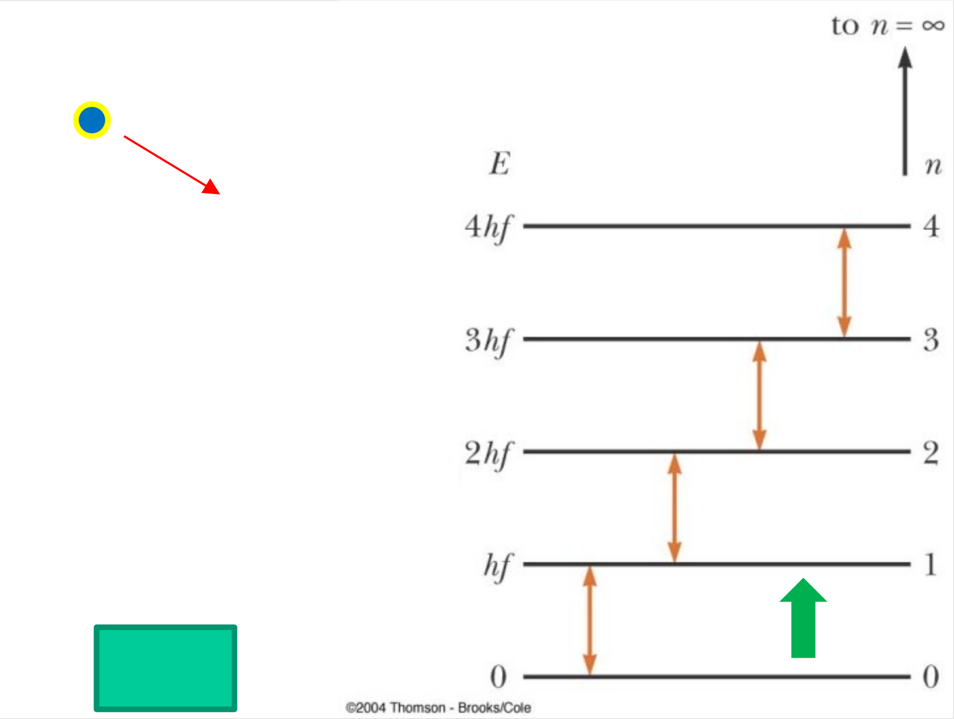
電子碰撞晶體中的原子鏈





等同

電子碰撞激發某個模式k由$n = 1$定態躍遷到$n = 2$定態。



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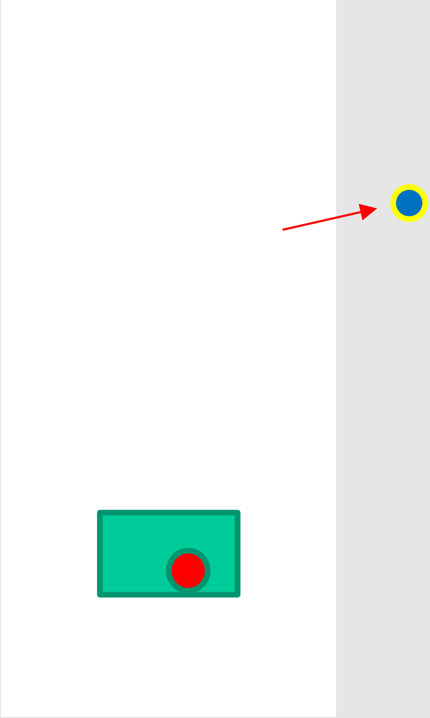
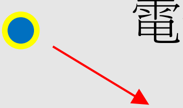
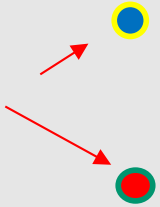


等同

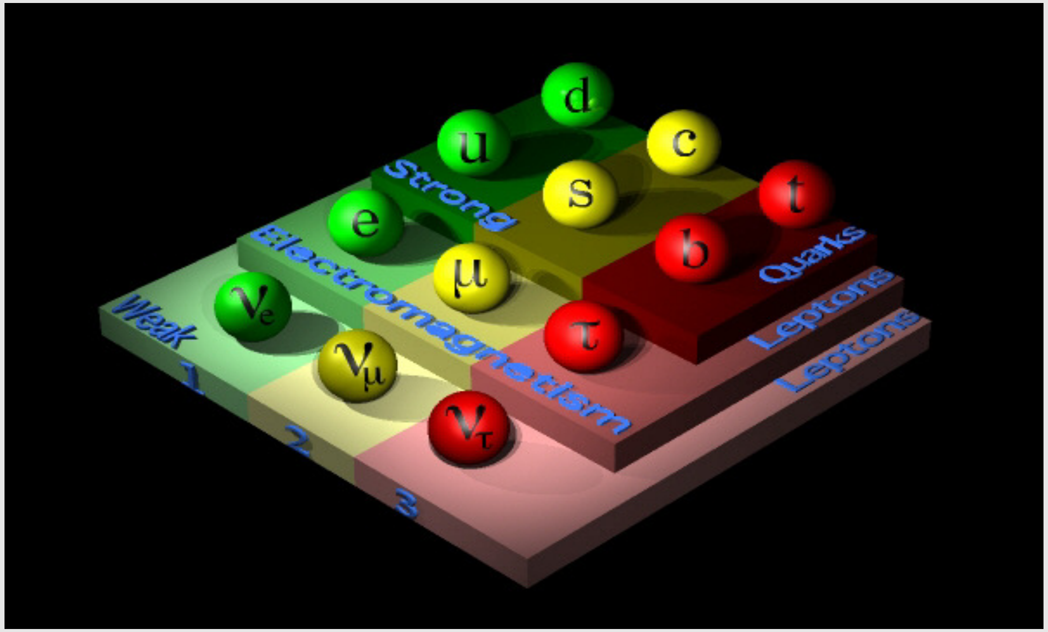


電子移動中放出一個角波數為k的聲子



聲子好像一個基本粒子！

每一個基本粒子都對應一個量子場

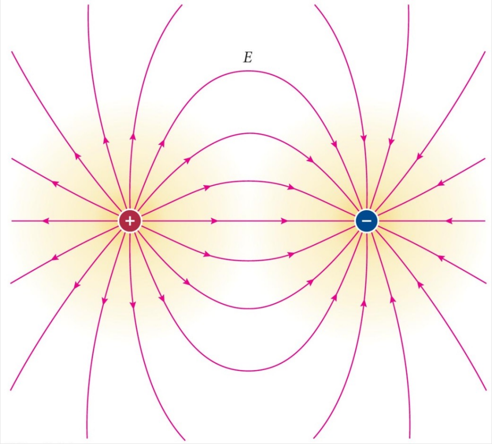


電子 ↔ 電子場

u夸克 ↔ u夸克場

描述可產生可消滅的粒子的理論：量子場論！

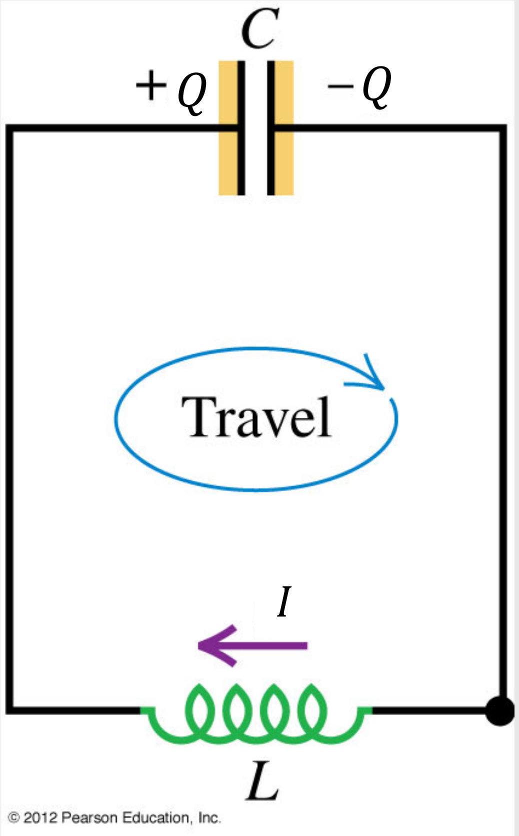
但我們早已熟悉、真空中場的擾動，也可看成一個一個的彈簧的組合！



如果場也量子化了，這就稱為量子場，激發態就會是一個個基本粒子。

我們的宇宙是由彈簧組成的

將電容器與電感器連成LC電路



電流中的電荷會停留在電容上：

導線通過的電荷就是電容上的電荷：

電流等於單位時間通過的電荷。

$$I = \frac{dQ}{dt}$$

繞導線一圈電位差為零：

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0 \Rightarrow -L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

這個方程式與簡諧運動一模一樣！

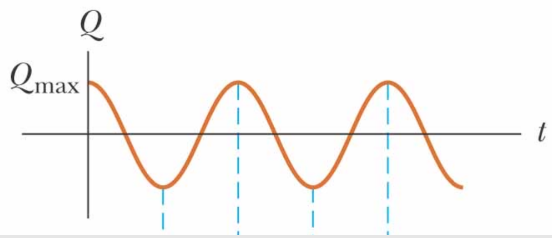
$$m\frac{d^2x}{dt^2} + kx = 0$$

$$L \leftrightarrow m$$

電感使通過它的磁場具有慣性！

$$Q \leftrightarrow x$$

$$k \leftrightarrow \frac{1}{C}$$

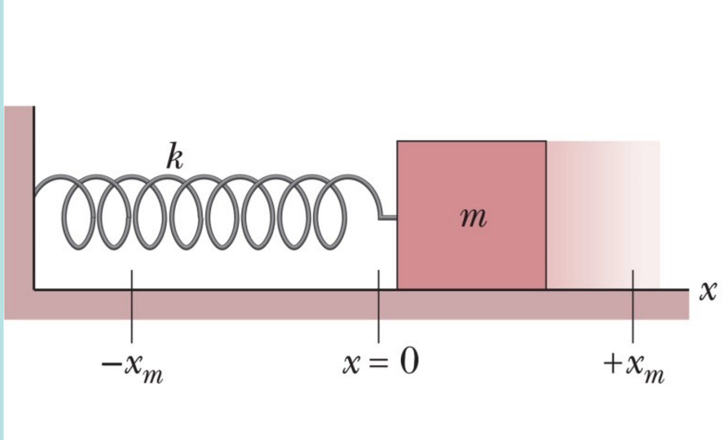


它們的解也就相同，$Q = Q_m \cos(\omega t)$

LC電路會產生電磁震盪，頻率為：

$$\omega = \sqrt{\frac{k}{m}} \rightarrow \sqrt{\frac{1}{LC}}$$

簡諧運動的運動方程式 Equation of Motion for SHM



$$F = -kx$$

$$\frac{d^2x}{dt^2} = -\omega^2 x = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

一個簡諧運動，完全由一個特徵常數ω（角頻率）決定！

A SHM is completely determined by a characteristic number ω.

具有相同的ω的簡諧運動，運動方程式的解就完全一樣。

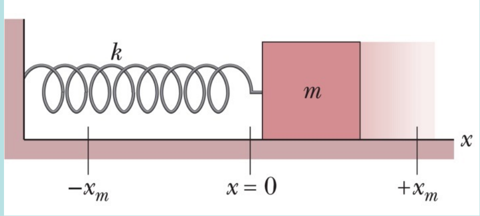
Any SHM with identical angular frequency ω has identical solution.

This is a Homogeneous 2nd order Linear ODE with constant coefficients.

$$y'' + a_1 y' + a_0 y = 0$$

簡諧運動的運動方程式求解 Solving Equation of Motion for SHM

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

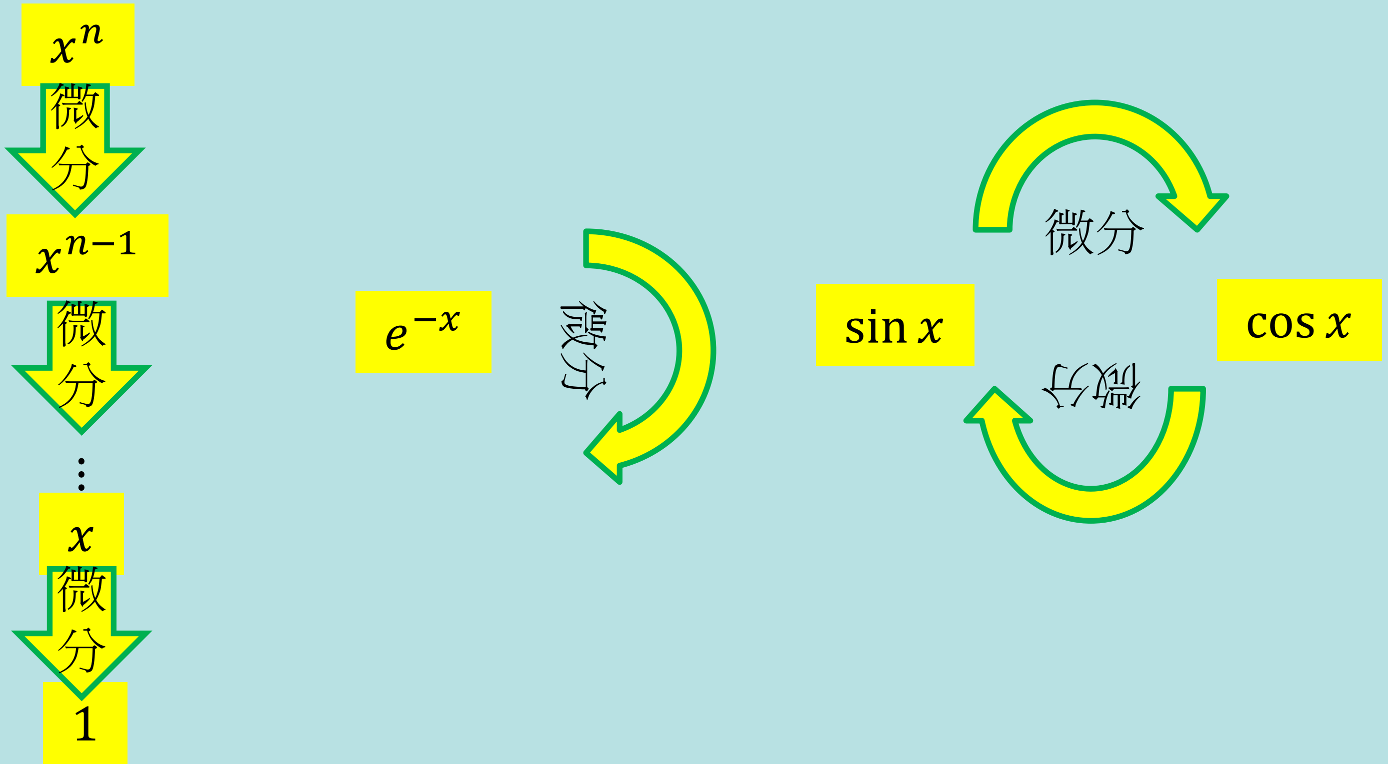


It states the second derivative of x is proportional to itself.

位置函數的兩次微分與自己成正比：多項式不符合。Polynomials could fit.

因式中的負號，指數函數也不行！Neither could Exponential due to the minus sign.

正弦函數正好具有這樣的性質！Sine function does fit.



x^n

微分

x^{n-1}

微分

$\vdots$

x

微分

1

e^{-x}

微分

$\sin x$

微分

$\cos x$

微分

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

It states the second derivative of x is proportional to itself.

正弦函數正好具有這樣的性質！Sine function does fit.

$$\frac{d(\sin\omega t)}{dt} = \frac{d\sin\omega t}{d(\omega t)} \cdot \frac{d}{dt}(\omega t) = \omega\cos\omega t$$

$$\omega\frac{d(\cos\omega t)}{dt} = -\omega^2\sin\omega t$$

$$\frac{d^2(\sin\omega t)}{dt^2} = -\omega^2\sin\omega t$$

很容易猜到餘弦函數也具有這樣的性質！Cosine function also fits.

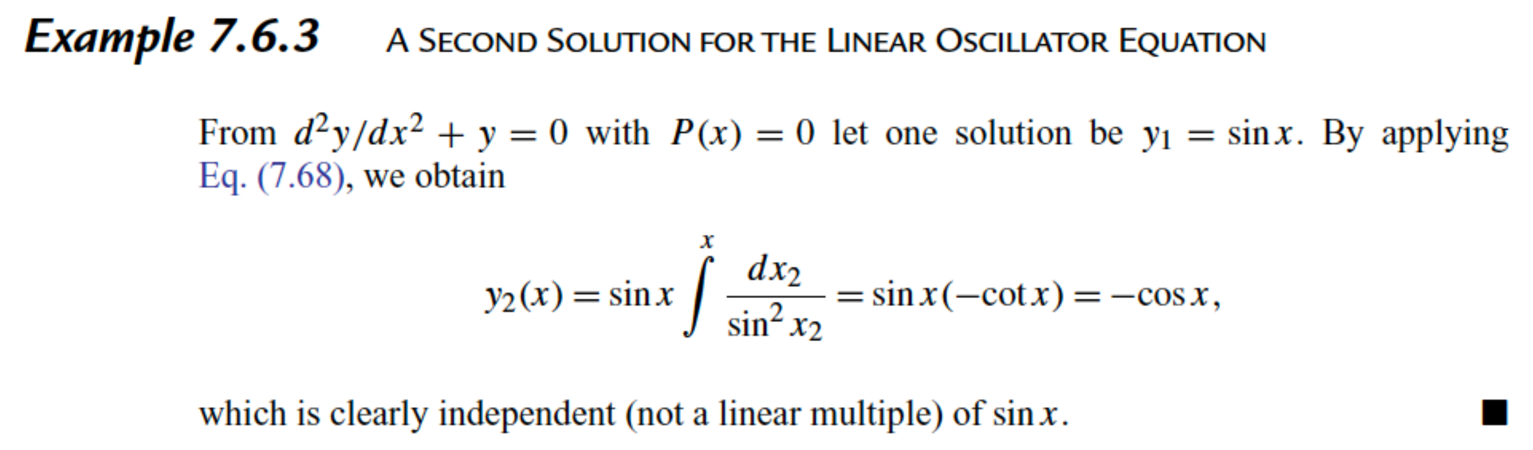
$$\frac{d^2(\cos\omega t)}{dt^2} = -\omega^2 \cos\omega t$$

We can find this simple fact using Wronskian!

If $y_1(x)$ satisfies $y'' + P(x)y' + Q(x)y = 0$

we can compute from y_1 another function $y_2(x)$ which also satisfies the ODE.

$$y_2 \sim y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$



Example 7.6.3 A SECOND SOLUTION FOR THE LINEAR OSCILLATOR EQUATION

From $d^2y/dx^2 + y = 0$ with $P(x) = 0$ let one solution be $y_1 = \sin x$. By applying Eq. (7.68), we obtain

$$y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2} = \sin x(-\cot x) = -\cos x,$$

which is clearly independent (not a linear multiple) of $\sin x$. ■

If we could find two function $y_1(x), y_2(x)$ satisfying $y'' + P(x)y' + Q(x)y = 0$

any linear combination $C_1 y_1(x) + C_2 y_2(x)$ would also satisfy the equation.

C_1 $y_1'' + P(x)y_1' + Q(x)y_1 = 0$ $+$ C_2 $y_2'' + P(x)y_2' + Q(x)y_2 = 0$

$$(C_1 y_1 + C_2 y_2)'' + P(x)(C_1 y_1 + C_2 y_2)' + Q(x)(C_1 y_1 + C_2 y_2) = 0$$

QED

If we could find two function $y_1(x), y_2(x)$, we get infinite number of solutions.

That is natural. Then we can fit the initial condition.

All solutions can then be written as： $C_1 y_1(x) + C_2 y_2(x)$

C_1, C_2 would be determined by $y'(0), y(0)$ 。

定理

如果已找到兩個函數$x_1(t), x_2(t)$都滿足齊次 Homogeneous方程式：

$$\sum_n a_n \cdot \frac{d^n x}{dt^n} = 0$$

完全由函數微分的一次項組成的微分方程式。

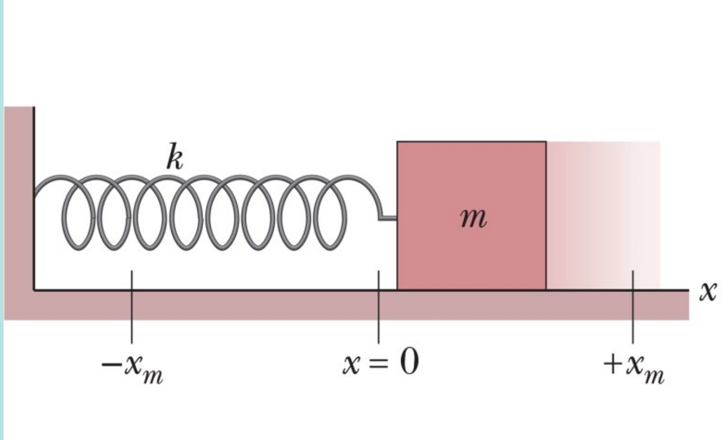
那麼任一線性組合 linear combination $ax_1(t) + bx_2(t)$ 也滿足該方程式！

$$a \quad \sum_n a_n \cdot \frac{d^n x_1}{dt^n} = 0 \quad + \quad b \quad \sum_n a_n \cdot \frac{d^n x_2}{dt^n} = 0$$

$$\sum_n a_n \cdot \frac{d^n (ax_1 + bx_2)}{dt^n} = 0$$

得證

簡諧運動的解



$$\frac{d^2x}{dt^2} = -\omega^2 x$$

很容易就找到兩個解。We found two solutions:

$$x_1 = \sin \omega t \qquad x_2 = \cos \omega t$$

那麼任一線性組合也是解！Any linear combinations would be solutions too.

$$x = a \cos \omega t + b \sin \omega t$$

我們得到無限多個解！We found infinite number of solutions.

$$x = a\cos\omega t + b\sin\omega t$$

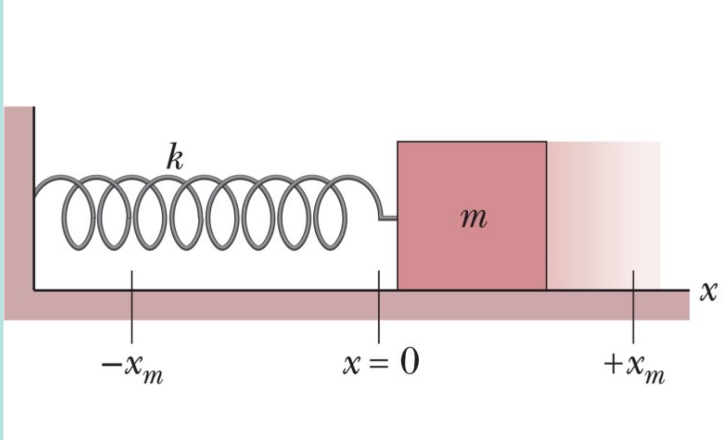
$$v = -\omega a\sin\omega t + \omega b\cos\omega t$$

a, b由起始條件決定。a, b would be determined by initial conditions$v(0), x(0)$.

$$x(0) = a = x_0$$

$$v(0) = \omega b = v_0$$

$$b = \frac{v_0}{\omega}$$



$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

這個函數同時滿足運動方程式以及兩個起始條件，因此是唯一的解！

This one solution satisfies both the Equation of motion and initial condition.

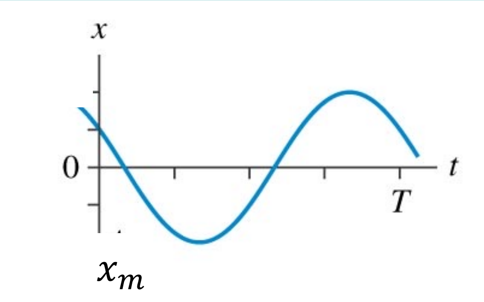
不用再找了！**It cold be proven that there is only one such function.**

$$x = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

這個式子較適合運動方程式求解。

較容易明瞭其物理意義的表示式是：

$$x = x_m \cos(\omega t + \phi)$$



這兩個數學式是一樣的，因為：

$$x = x_m \cos(\omega t + \phi) = x_m \cos \phi \cos \omega t - x_m \sin \phi \sin \omega t$$

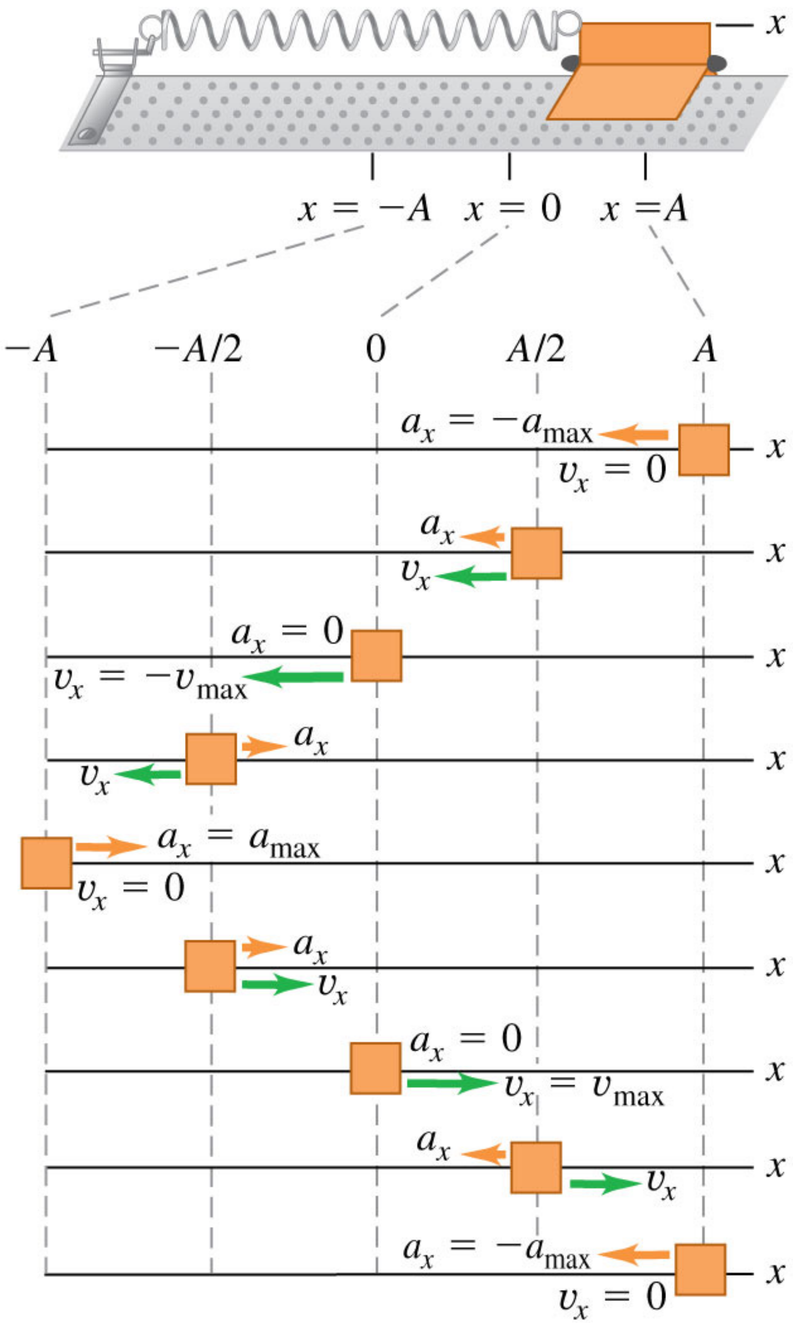
兩組常數之間的關係：

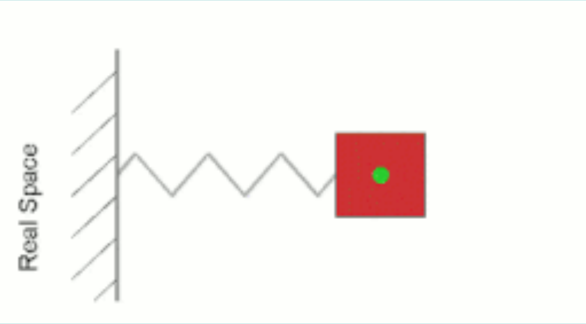
$$x_0 = x_m \cos \phi \, , \frac{v_0}{\omega} = -x_m \sin \phi$$

$$x_m = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

$$\tan \phi = -\frac{v_0}{\omega x_0}$$

簡諧運動的解就是時間平移過的一個三角函數！

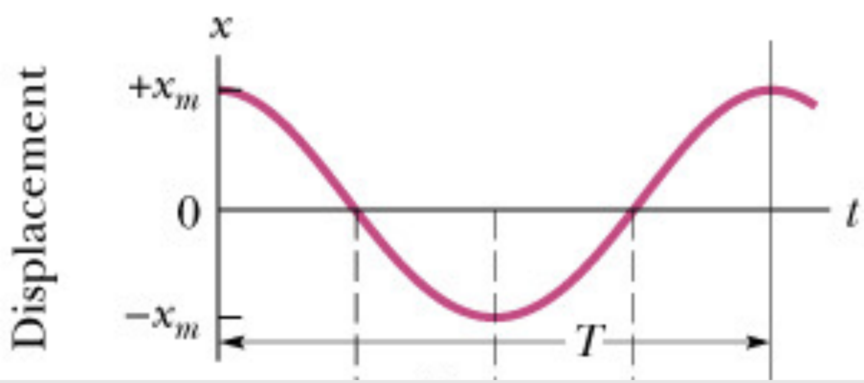




動畫 Simulation

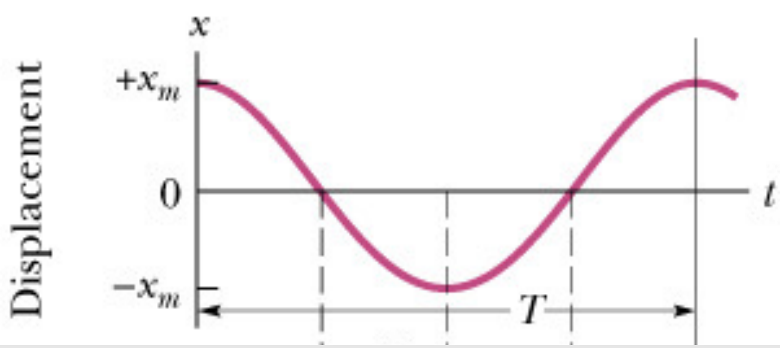
$$x = x_m \cos(\omega t + \phi)$$

常數的物理意義：



x_m振幅 Amplitude 是振動的極大值

$x = x_m \cos(\omega t + \phi)$ 三角函數是一個週期函數。 ω 決定了振盪的週期 T。



$x(t) = x(t+T)$ $\rightarrow \cos(\omega t + \phi) = \cos(\omega t + \omega T + \phi)$

三角函數一個週期後，角度增加2π。 $\omega T = 2\pi$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

ω稱為角頻率 Angular Frequency。

簡諧運動的週期 T 等於

$$T = 2\pi\sqrt{\frac{m}{k}}$$

頻率Frequency: $f = 1/T$ 每秒進行了幾個周期。單位 s^{-1} = Hz。

$$\omega = \frac{2\pi}{T} = 2\pi f$$

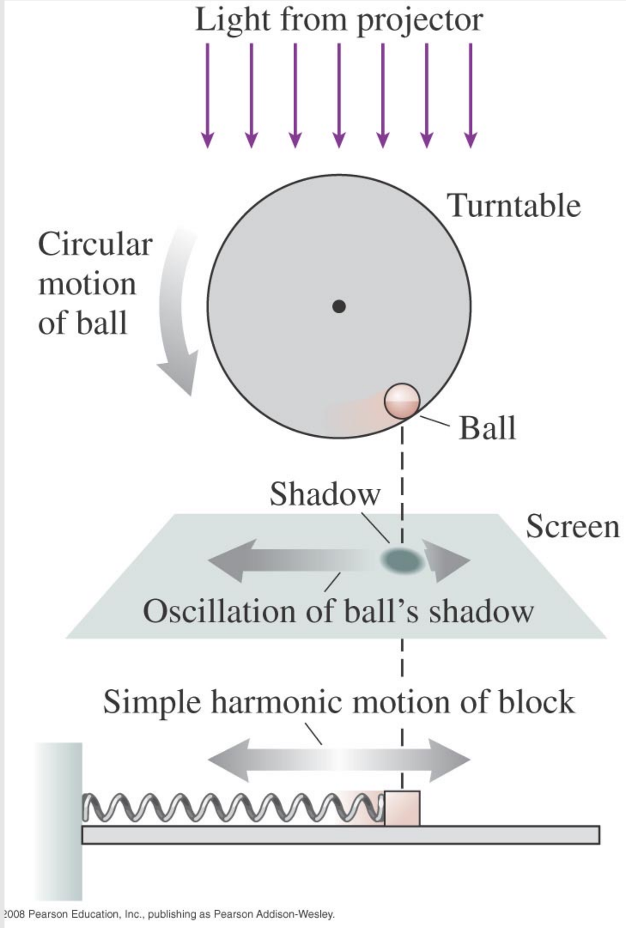
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$x = x_m \cos(\omega t + \phi)$ $\omega T = 2\pi$

正好是一個半徑為x_m的等速圓周運動的水平分量。

此等速圓周運動的速度與加速度的水平分量也好與簡諧運動相等。

一個等速圓周運動位置的水平投影就等於一個簡諧運動。



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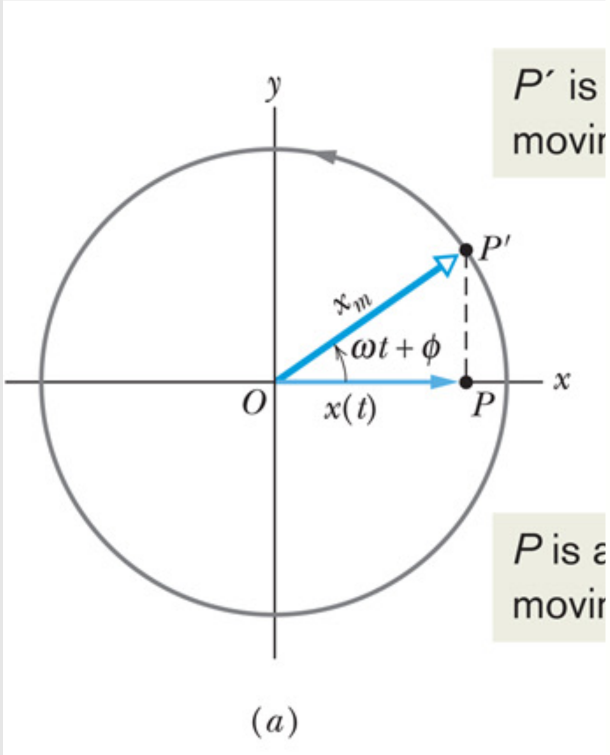
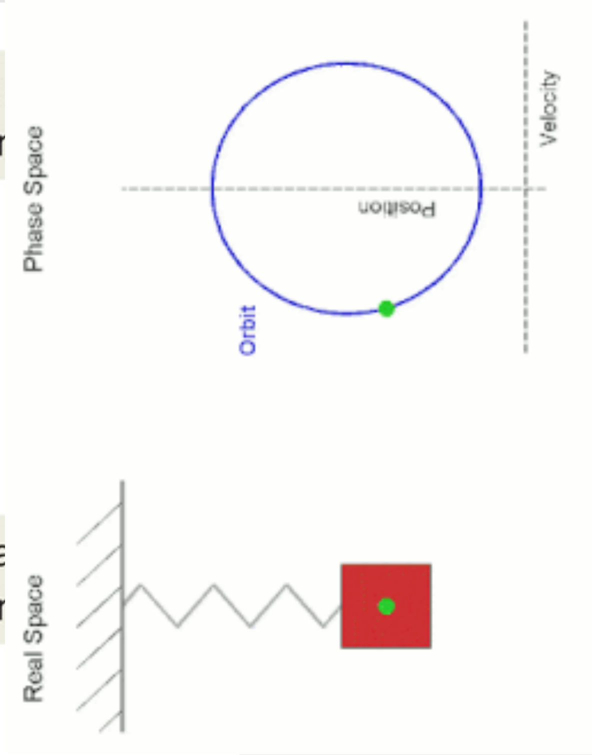
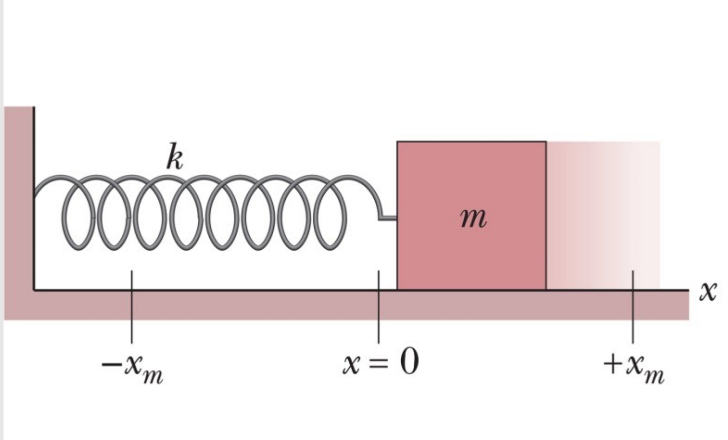




假想圓是一個很好的工具，但不是真的有一個圓！



$$x = x_m \cos(\omega t + \phi)$$

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

一個簡諧運動，有一個特有的、內在的、由ω決定的振動頻率f！

角頻率ω與振幅及相常數無關，同一個彈簧組，只有一個值。

振幅x_m及相常數ϕ由起始條件決定，

同一個彈簧組，對個別的運動，所取的值可以不同。

但自然界的振盪很少一直持續，能量總是會漸漸消耗掉。

這是來自振盪器內的阻力。

We can introduce a resistance force to account for the dissipation of energy.

近似引進一個與速率成正比的阻力：

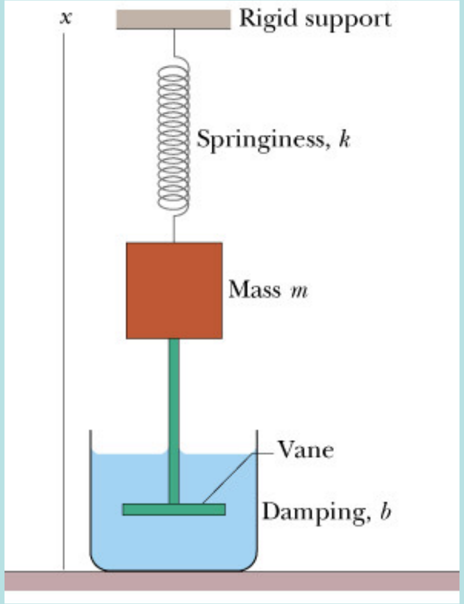
$$f_d = -bv$$

運動方程式會多一個項：

There would be one more term in the equation of motion.

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

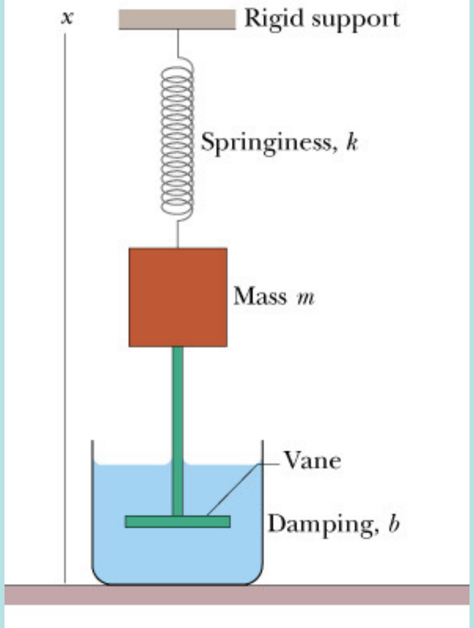
called Damped Oscillation 阻尼振盪。



Damped Oscillation 阻尼震盪

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$



This is a Homogeneous 2nd order Linear ODE with constant coefficients.

$$y'' + a_1 y' + a_0 y = 0$$

單一個正弦或餘弦函數似乎不能滿足此式。

Pure Sine (Cosine) function can not satisfy the equation since the first derivative would generate a Cosine (Sine) function.

Method 0

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2 x=0$$

Damped oscillation 應該還是會振盪，但振幅會隨時間變化。

Damped oscillation would oscillate, but with varying amplitude!

大膽的猜想：Brave Guess

$$x(t)=f(t)\cdot\cos(\omega' t+\phi)$$

$$\omega^2 f\cos(\omega' t+\phi)$$

$$\frac{b}{m}\cdot\left[f'\cdot\cos(\omega' t+\phi)-\omega' f\sin(\omega' t+\phi)\right]$$

$$(f''-\omega'^2 f)\cdot\cos(\omega' t+\phi)-2\omega' f'\sin(\omega' t+\phi)$$

Now we have both Cosine and Sine in the equation!

要求$\sin(\omega' t+\phi)$的係數為零：

Ask the coefficient of $\sin(\omega' t+\phi)$ to vanish.

$$2f'=-\frac{b}{m}f$$

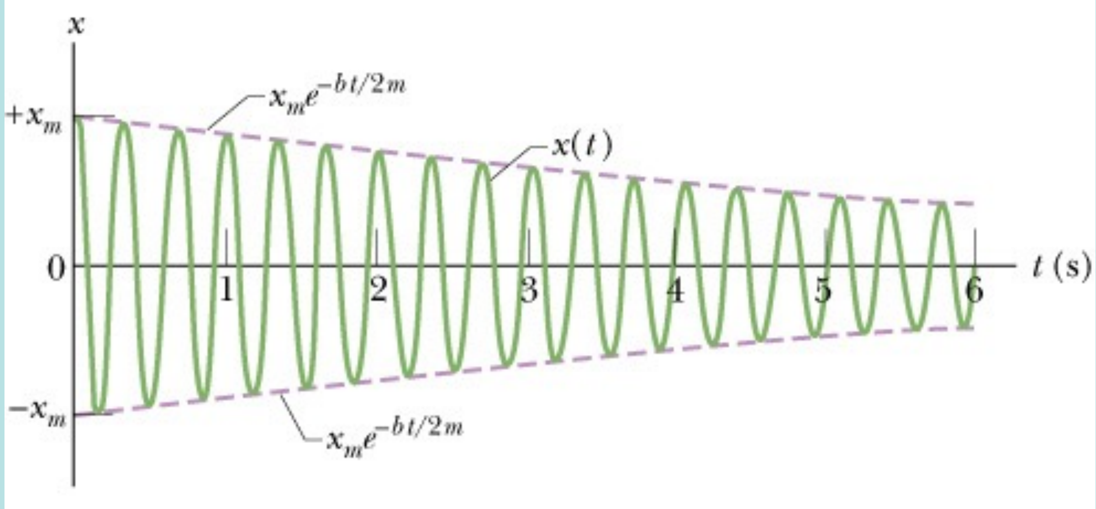
$$f=x_m\cdot e^{-\frac{b}{2m}t}$$

要求$\cos(\omega' t+\phi)$的係數為零，代入f：

Ask the coefficient of $\cos(\omega' t+\phi)$ to vanish.

$$\omega'^2-\omega^2+\frac{b^2}{4m^2}=0$$

$$\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$



$$x(t) = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

There are two unspecified constants x_m, ϕ.

We can use the two initial conditions $x(0), x'(0)$ to specify them.

振幅對時間呈現指數衰減！

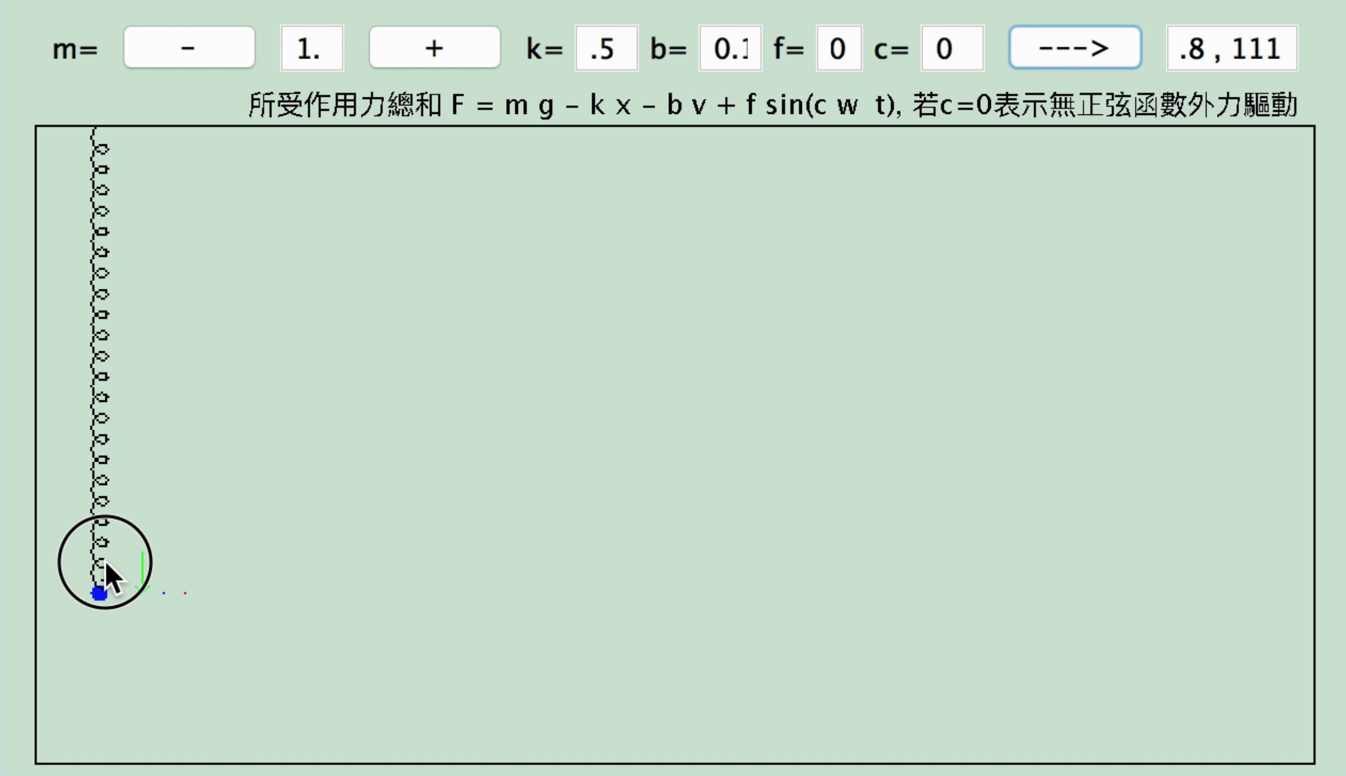
$$x_m \sim x_m e^{-\frac{b}{2m}t}$$

The amplitude decays exponentially.

角頻率會比無阻力時稍低。

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

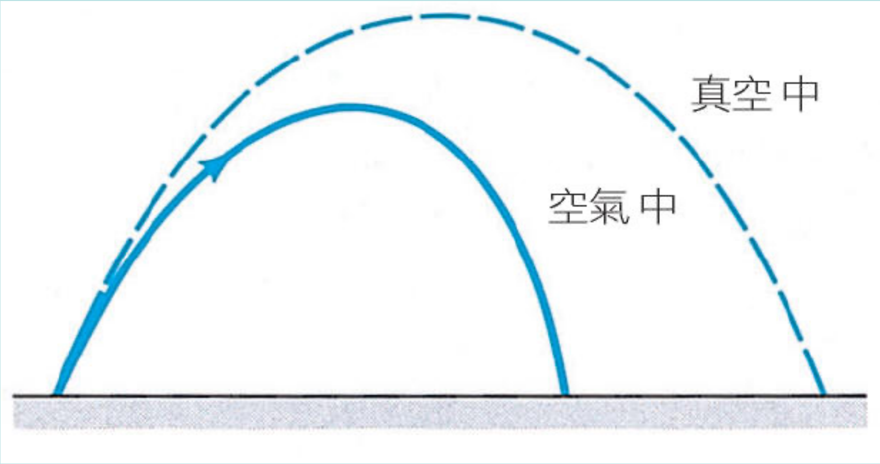
Angular Frequency is smaller than dampless SHM.

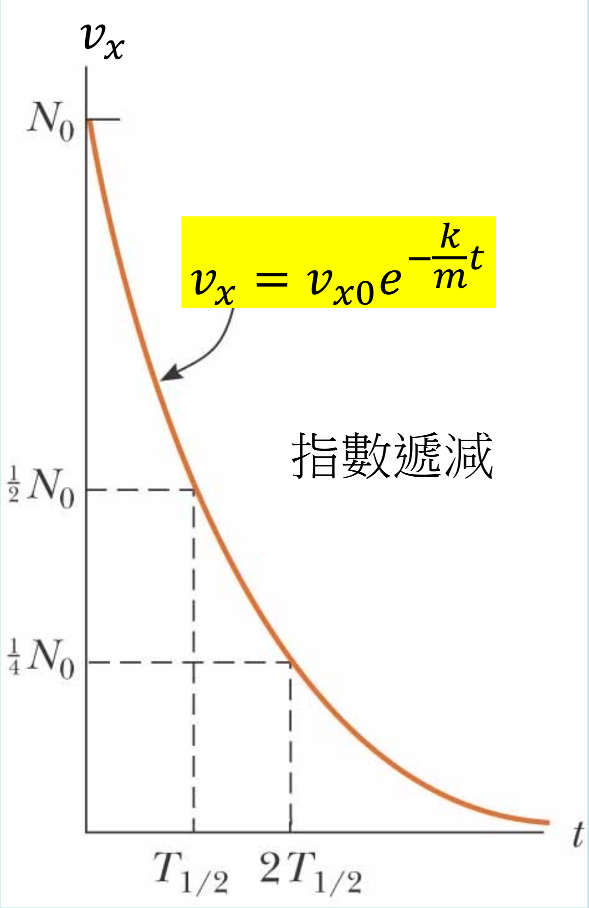


m= 1. k= .5 b= 0.1 f= 0 c= 0 .8 , 111

所受作用力總和 F = m g - k x - b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

$$x_m \sim x_m e^{-\frac{b}{2m}t}$$

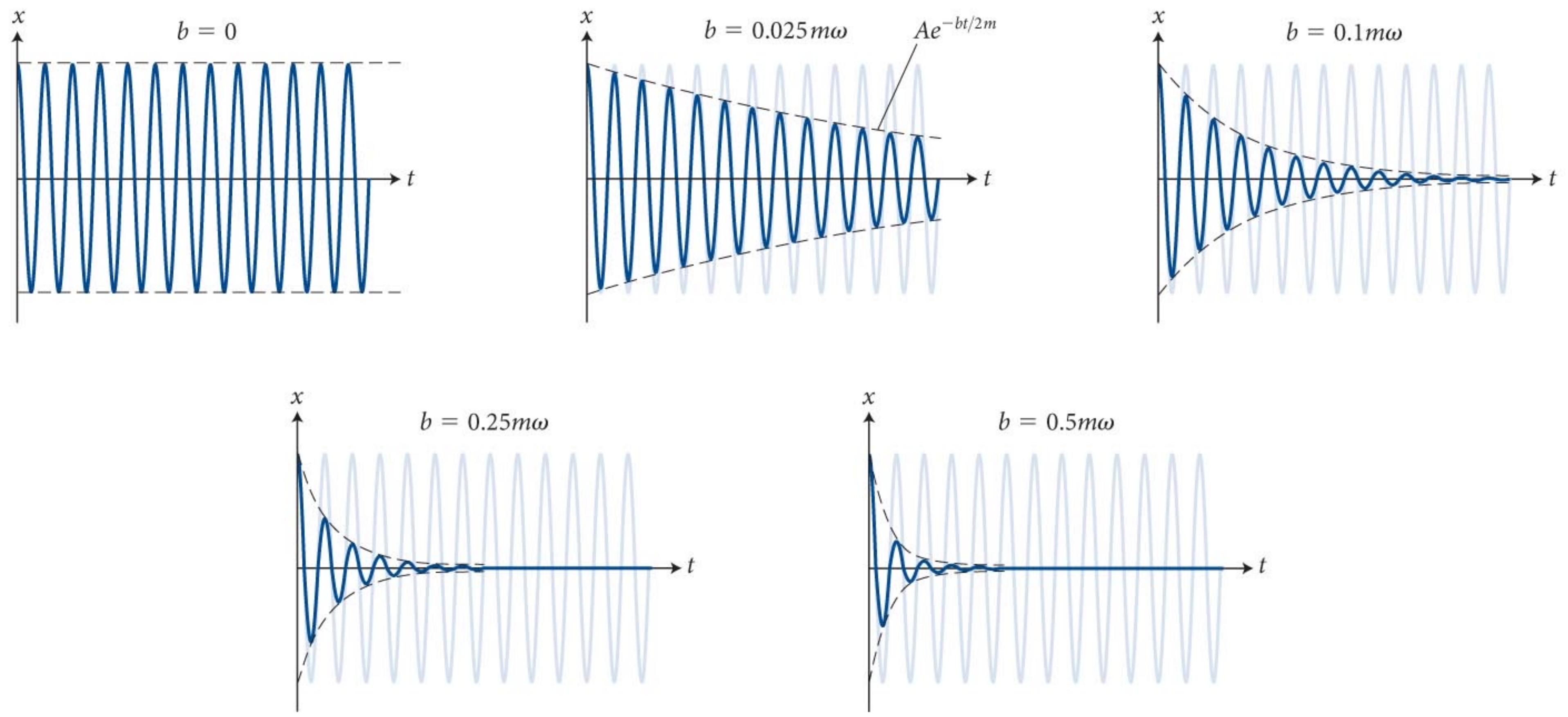




$$v_x = v_{x0} e^{-\frac{k}{m}t}$$

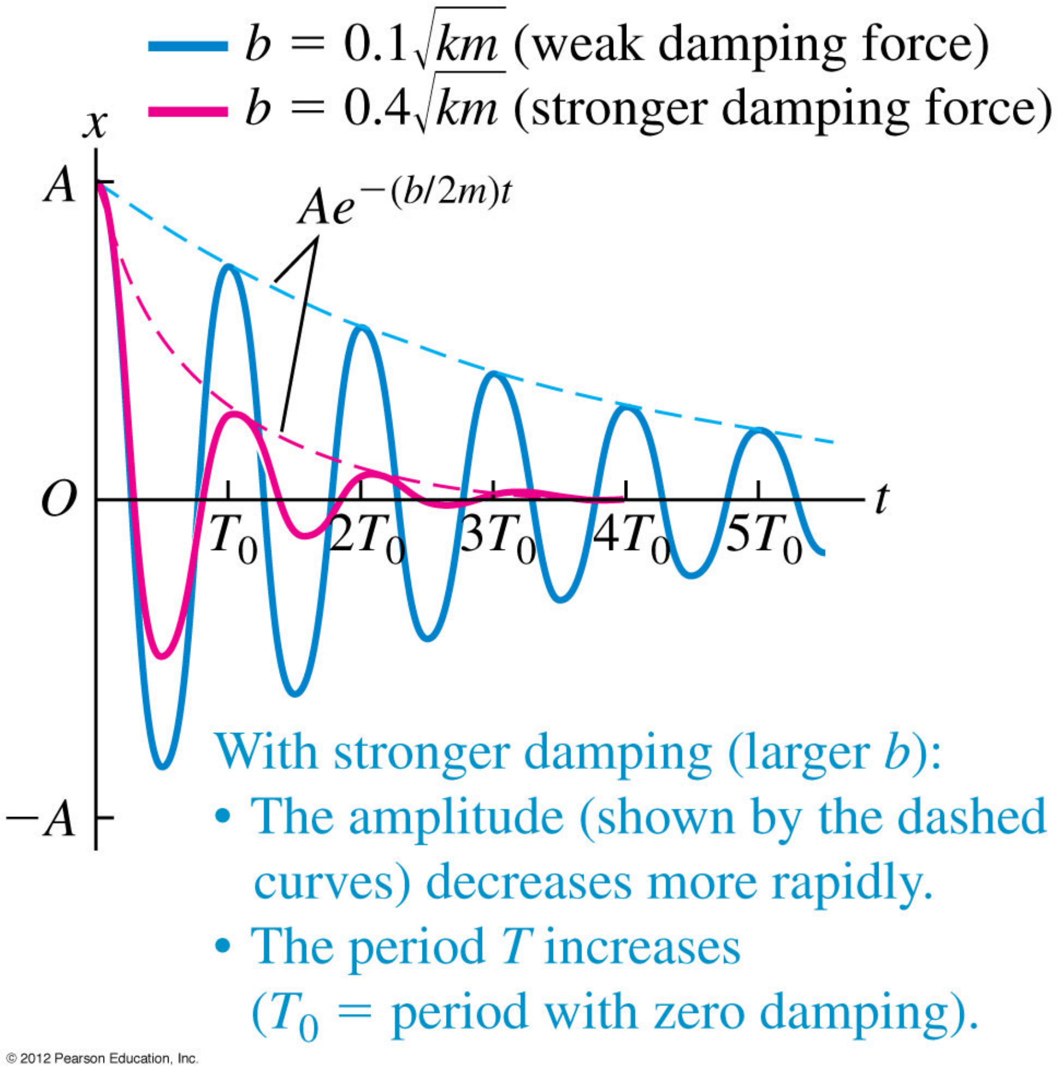
指數遞減

振幅的減小如同空氣阻力下的運動！



阻力常數*b*越大，振幅的減少也越快！

The bigger the resistance force, the faster the amplitude decays.



With stronger damping (larger b):

- The amplitude (shown by the dashed curves) decreases more rapidly.
- The period T increases (T_0 = period with zero damping).

振動角頻率減小！周期增加。

$$\omega' = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

振動頻率減小！阻力越大減少越多！

$$x(t) = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

The larger the resistance force, the smaller the Angular Frequency.

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

until it vanishes, there is no more oscillation.

$$\frac{b^2}{4m^2} = \omega^2$$

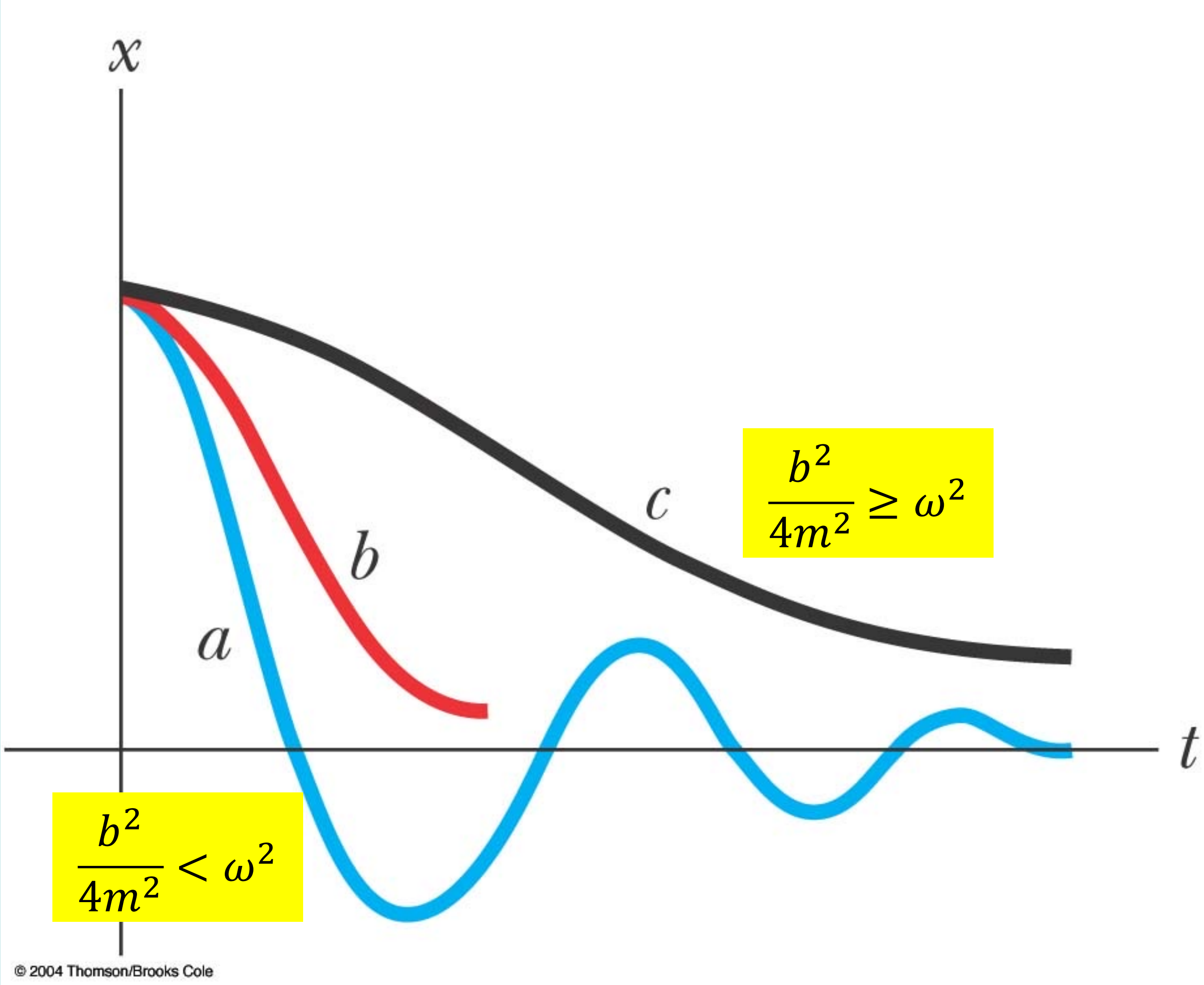
$$\omega' = 0$$

若 $\frac{b^2}{4m^2} \geq \omega^2$

那就根本沒有振動了！以上的式子就不對了。

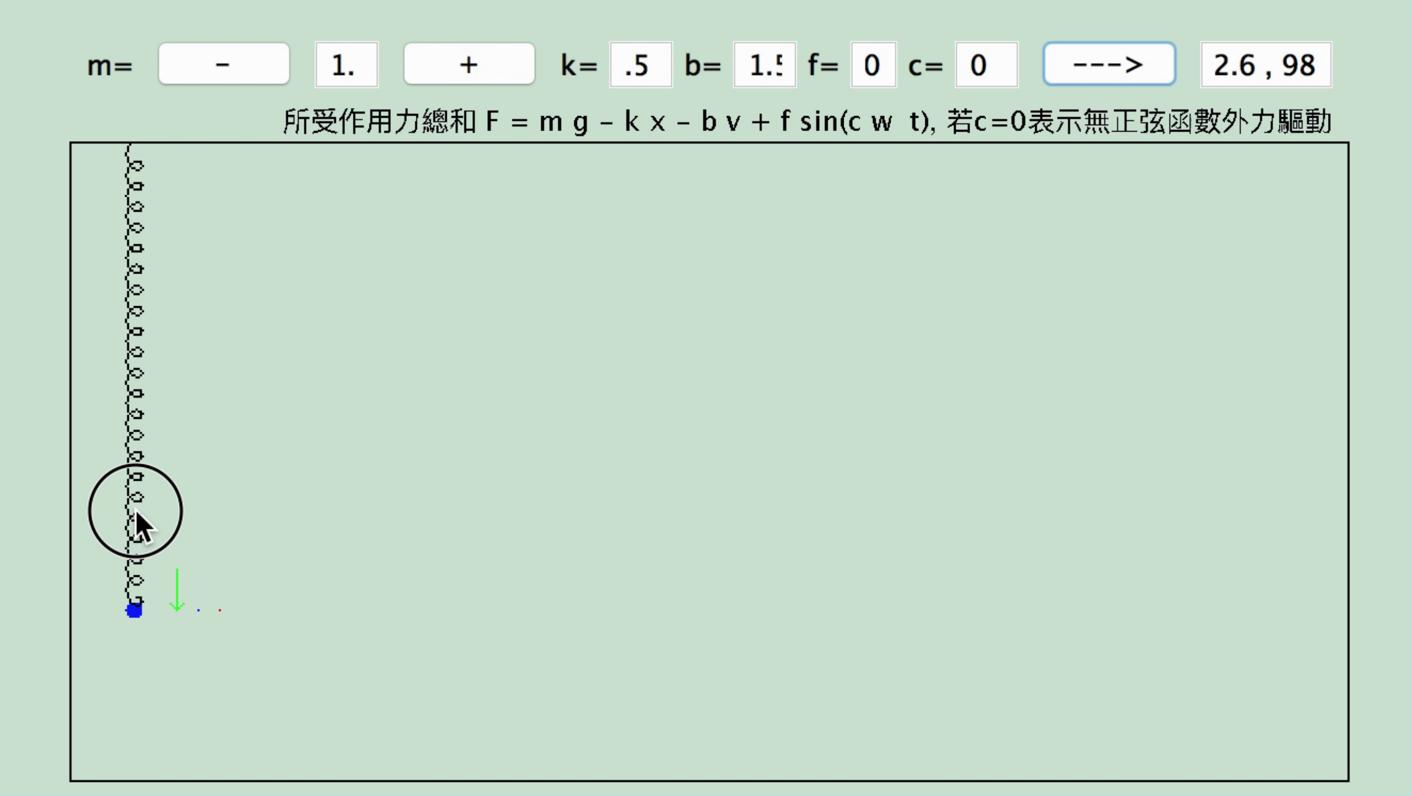
這時的解會是一個隨時間指數遞減的函數！

The solution would be a exponentially decaying function.



如果 $b/2m$ 大於 ω_0 (c)，那就根本沒有振動了！

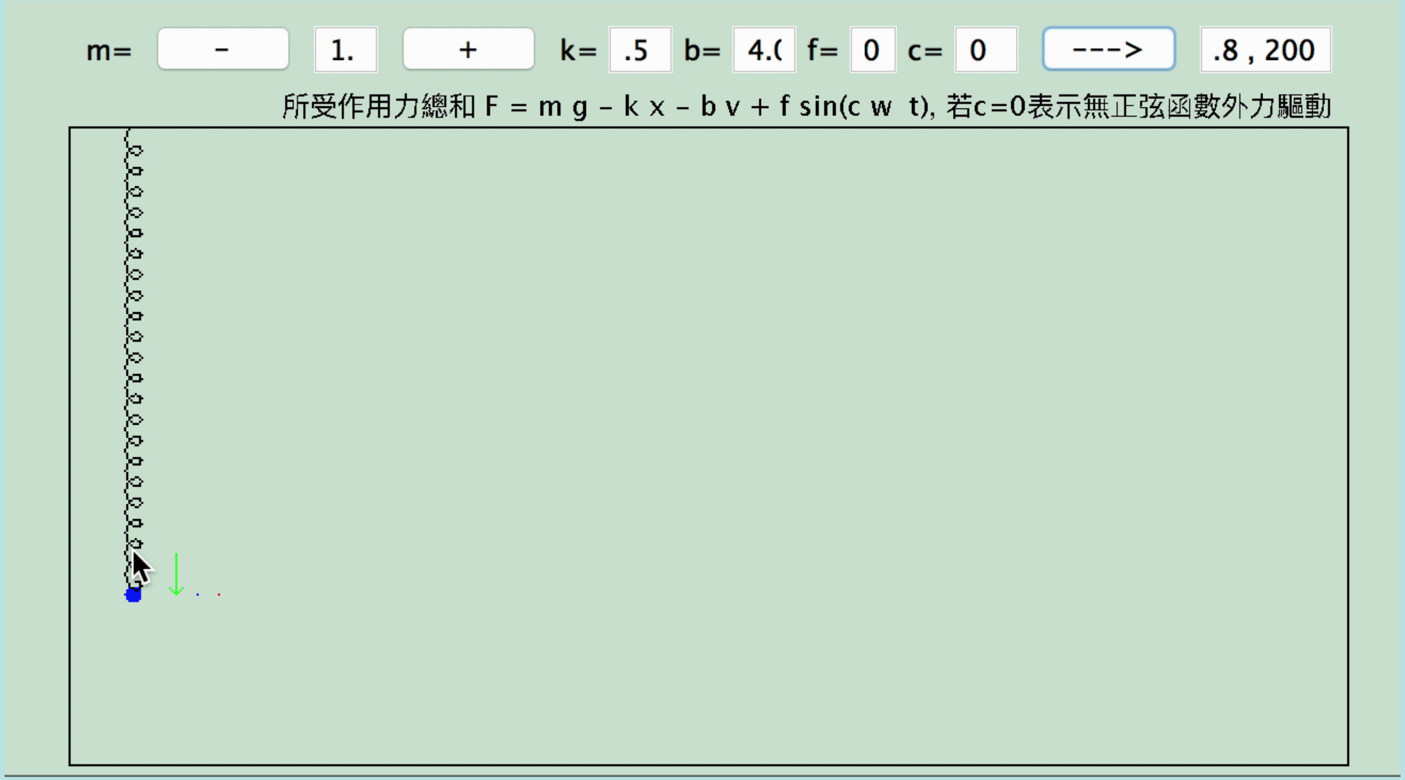
阻尼可以大到連一次震盪都未完成！



m= 1. k= .5 b= 1. f= 0 c= 0

所受作用力總和 F = m g - k x - b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

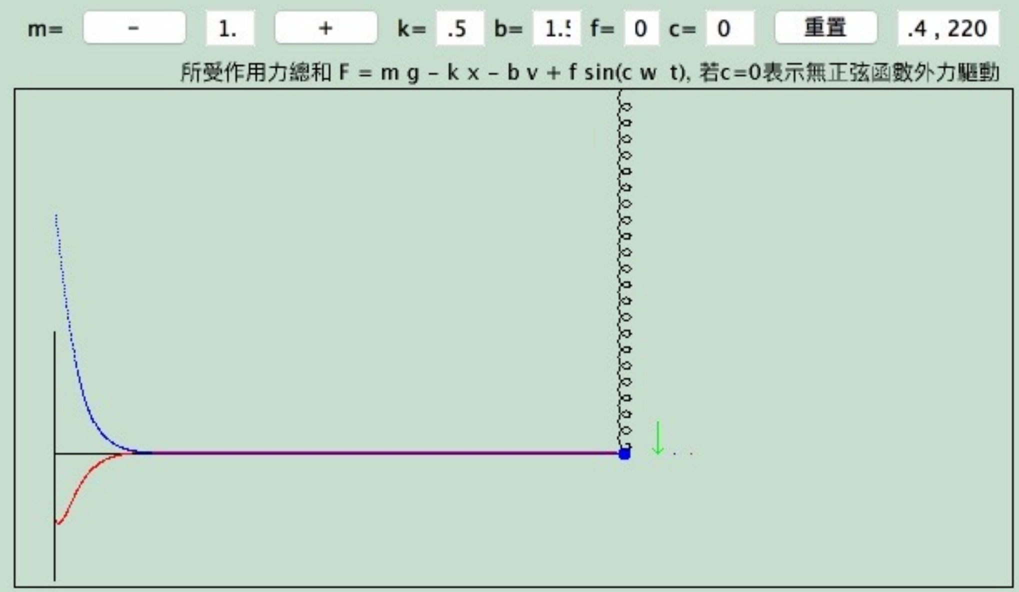
Over-damping



m= 1. k= .5 b= 4.(f= 0 c= 0 .8 , 200

所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

物體所受合力 $\mathbf{F} = m\,\mathbf{g} - k\,\mathbf{x} - b\,\mathbf{v} + \mathbf{f}_o \sin(cwt)$



m= 1. k= .5 b= 1. f= 0 c= 0 .4 , 220

所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

m= 1. k= .5 b= 4. f= 0 c= 0 .6 , 207

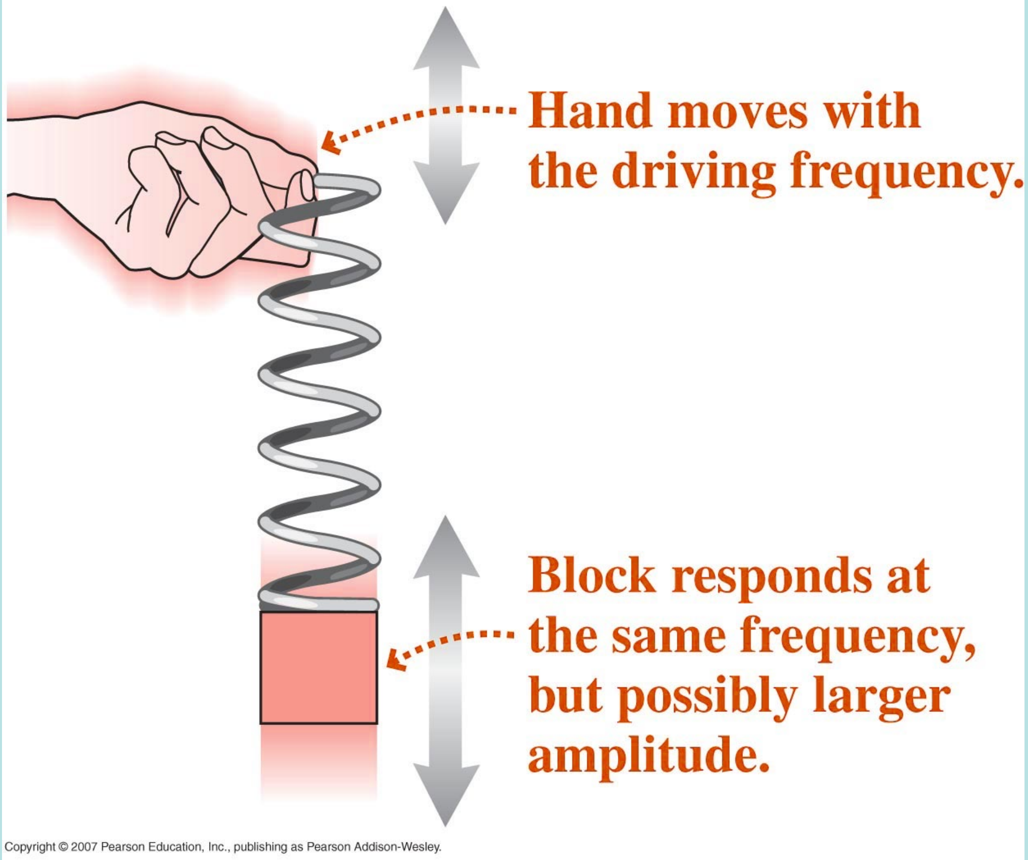
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

簡諧運動會因阻尼而使振幅減小，必須施力使它繼續振動

若施予一個常數力，所做的功在一個週期內會彼此抵消！

想使彈簧繼續振盪，必須施以一周期性的外力。

We can exert an external periodic force to keep the oscillation going.



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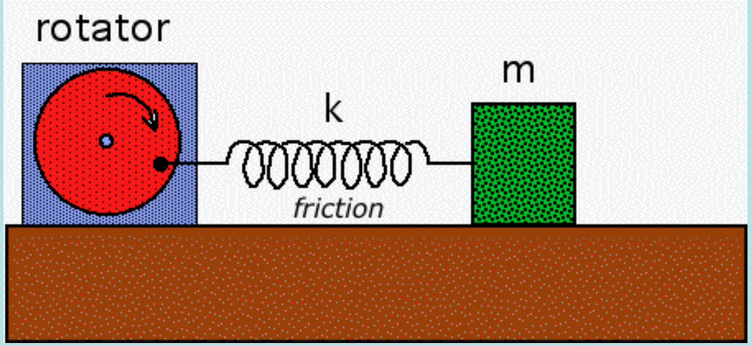

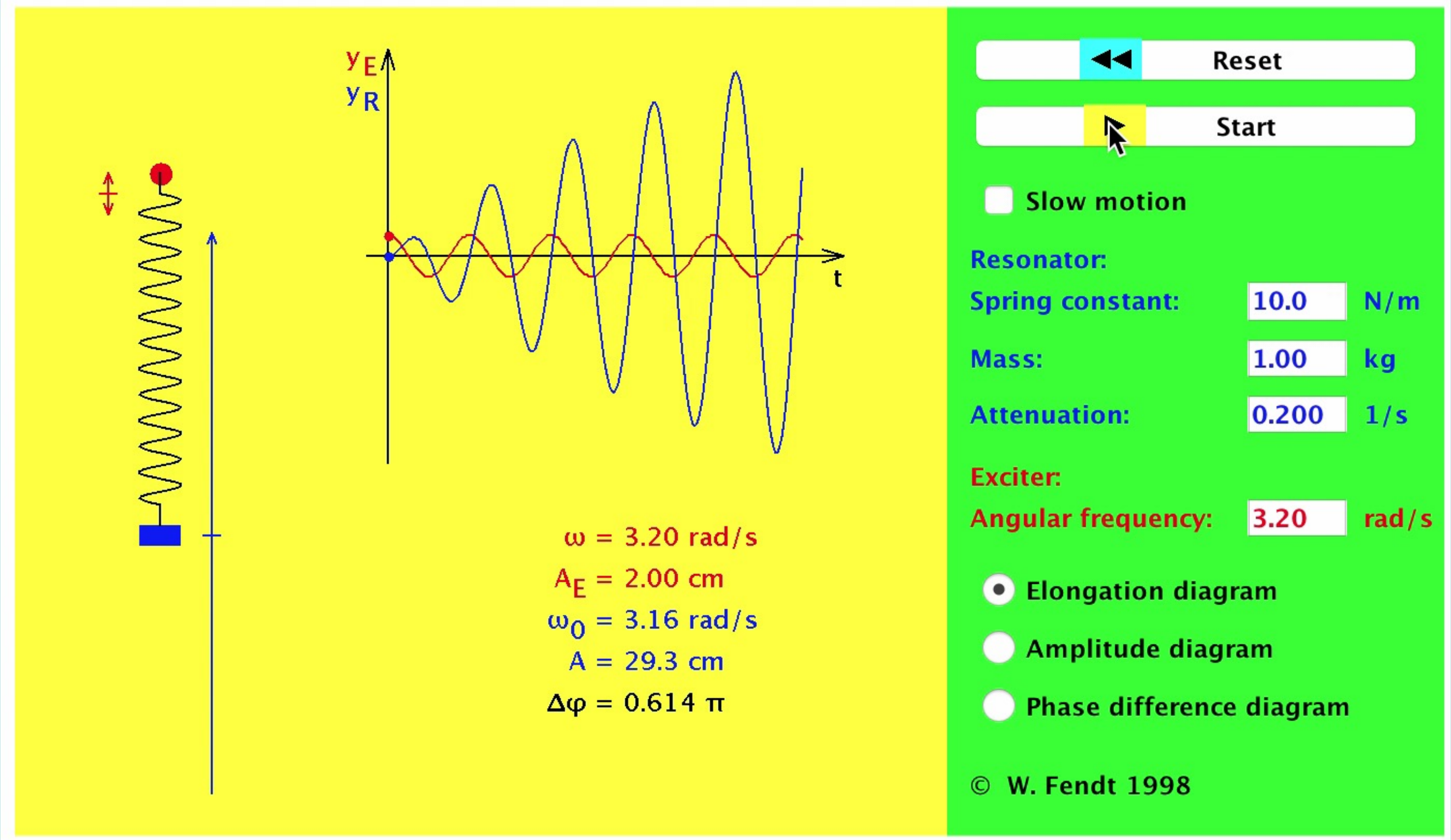

☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

Attenuation: 0.200 1/s

Exciter:

Angular frequency: 3.20 rad/s

● Elongation diagram

○ Amplitude diagram

○ Phase difference diagram

ω = 3.20 rad/s

A_E = 2.00 cm

ω_0 = 3.16 rad/s

A = 29.3 cm

$\Delta\varphi$ = 0.614 π

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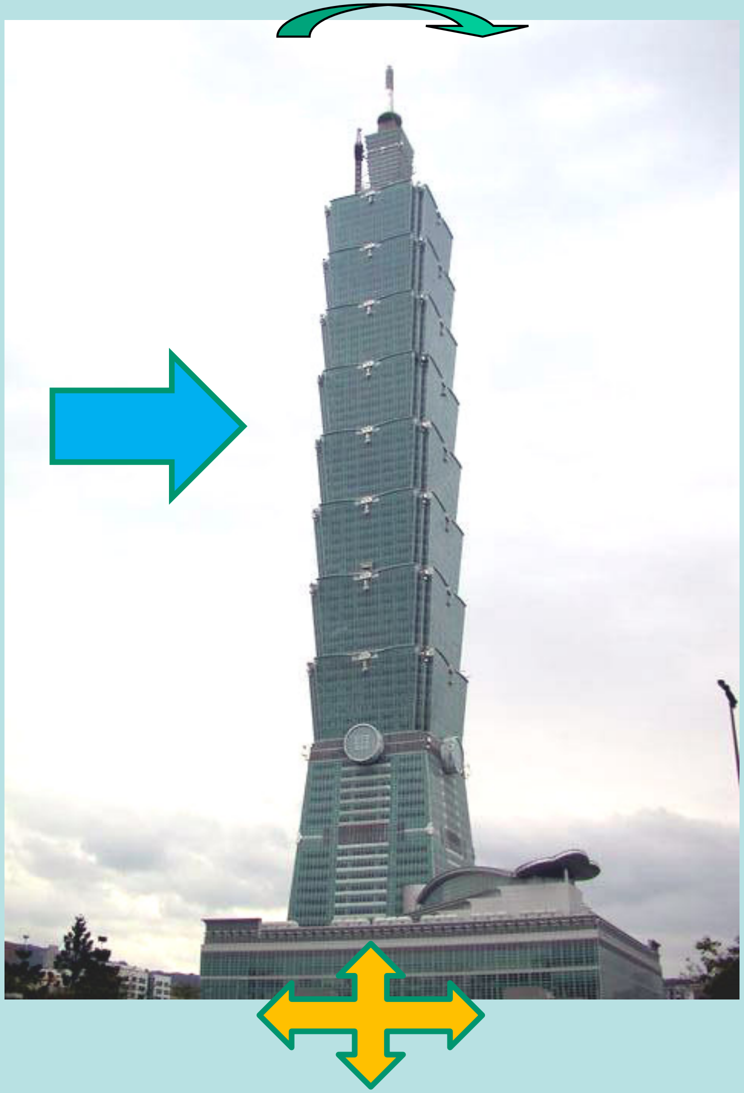
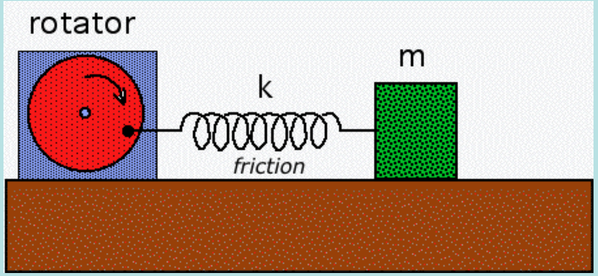
風吹或地震對101即是週期性的外力！



Method 0

外力下的震盪 Forced Oscillation



假設所施的外力可以寫成：$F_0 \cos \omega_D t$。

Assume that the additional force can be written as：$F_0 \cos \omega_D t$。

運動方程式又多了一個項：One more term for the equation of motion.

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_0 \cos \omega_D t$$

彈簧內在的角頻率

週期外力的角頻率

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos \omega_D t$$

$$y'' + a_1 y' + a_0 y = f(x)$$

This is an Inhomogeneous 2nd order Linear ODE with constant coefficients.

Maybe we call it inhomogeneous SHO just like inhomogeneous Decay E.

外力下的振盪由兩個頻率來決定：

Forced Oscillation is determined by two frequencies: ω_D, ω.

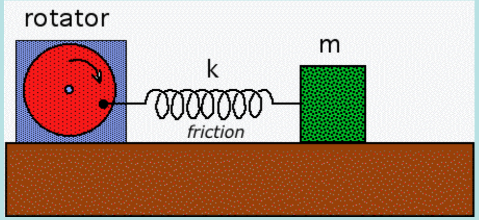
為簡單起見而專注於此二頻率的影響，先忽略阻尼：

For simplicity, ignore damping first.

$$\frac{d^2x}{dt^2} = -\omega^2 x + \frac{F_0}{m}\cos \omega_D t$$

很容易可以猜到它的解：It is easy to guess one solution!

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t$$



不是ω！

猜測 $x = A \cdot \cos\omega_D t$ 因為它的兩次微分還是正比於同樣的正弦函數！

Its derivative and the function itself are both the same function as the right-hand side.

代入 $$-A\cdot\omega_D^2\cos\omega_D t + A\cdot\omega^2\cos\omega_D t = \frac{F_0}{m}\cos\omega_D t$$

$$A = \frac{F_0}{m(\omega^2 - \omega_D^2)}$$

得到外力下振盪的一個解，姑稱為共振解 Resonance。

We get one solution of the inhomogeneous equation of forced oscillation, .

$$x_r = \frac{F_0}{m(\omega^2 - \omega_D^2)}\cos\omega_D t$$

called Resonance solution

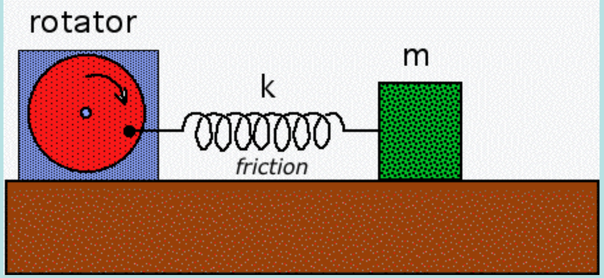
這解並不完整（起始條件還未放入），等一下會論證：長時間後，此解最重要。

This is not a complete solution yet since it can not satisfy all initial condition.

先讓我們研究一下此共振解的性質：

But let us study this Resonance solution first.

$$x_r = \frac{F_0}{m(\omega^2 - \omega_D^2)} \cdot \cos\omega_D t$$



週期外力驅動下，彈簧的反應依然是簡諧運動，

Under periodic external force, the response of the spring is still an oscillation.

但頻率是施力的頻率 ω_D ，而不是彈簧的自然頻率 ω ！

But the frequency equals that of the external force ω_D, instead of the spring ω.

此簡諧運動振幅A不再是任意，大小與施力大小成正比： $A \propto F_0$ 。

The amplitude is not arbitrary but proportional to the magnitude of external force.

振幅A的大小與施力的頻率 ω_D 密切相關：

The amplitude is sensitive to the relation between ω_D and ω.

施力的頻率 ω_D 愈接近彈簧的自然頻率 ω，反應愈強，能量的吸收愈好。

The closer ω_D is to ω, the larger the amplitude of the oscillation.

$\omega_D \to \omega$ $A \uparrow$

約會物理定律 Physical Law the dating

The dating would succeed only when the two resonate,
ie. when your frequencies are close enough.

如同每一個彈簧有一個自然頻率，每一個人也有自己的喜好！

$\omega_D \rightarrow \omega$時猶如兩人頻率一致，心意相投，所以稱之為共振現象。

$$x_r = \frac{F_0}{m(\omega^2 - \omega_D^2)} \cos \omega_D t$$

$$\omega_D \to \omega$$

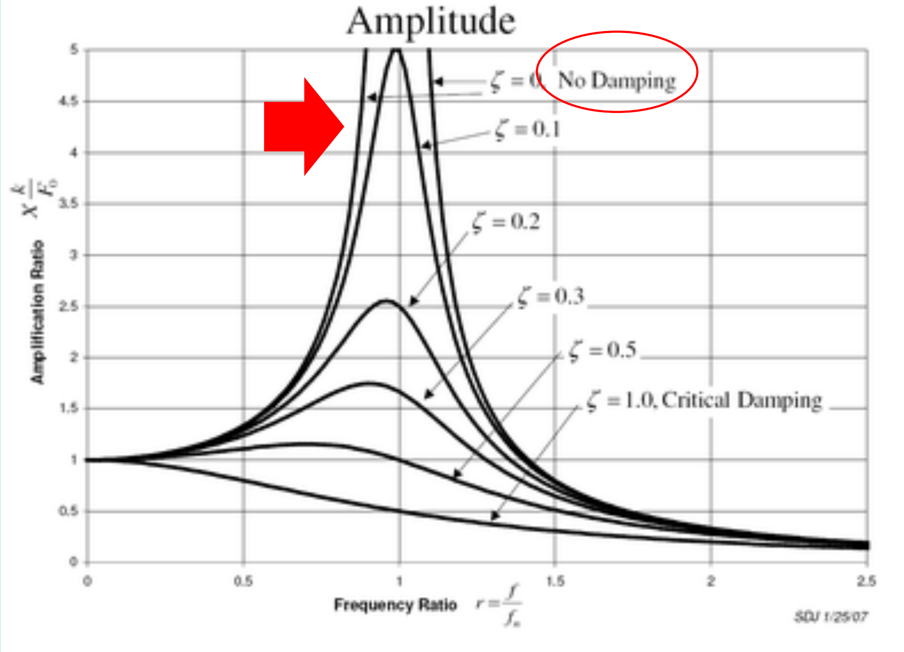
$$\frac{F_0}{m(\omega^2 - \omega_D^2)} \uparrow$$

施力頻率ω_D越接近內在頻率ω，彈簧的反應越大！

The closer ω_{D} is to ω, the larger the amplitude of the oscillation.

It falls off rapidly when ω_{D} move away from ω.

共振 Resonance



這個共振解在 $\omega_{\mathrm{D}} = \omega$ 時會趨近無限大！在自然界是不可能的。

The maximum amplitude is infinite due to ignoring damping b.

這是因為忽略阻尼：自然界一定存在阻力！

考慮阻尼後 After adding a damping:

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2 x=\frac{F_0}{m}\cos\omega_D t$$

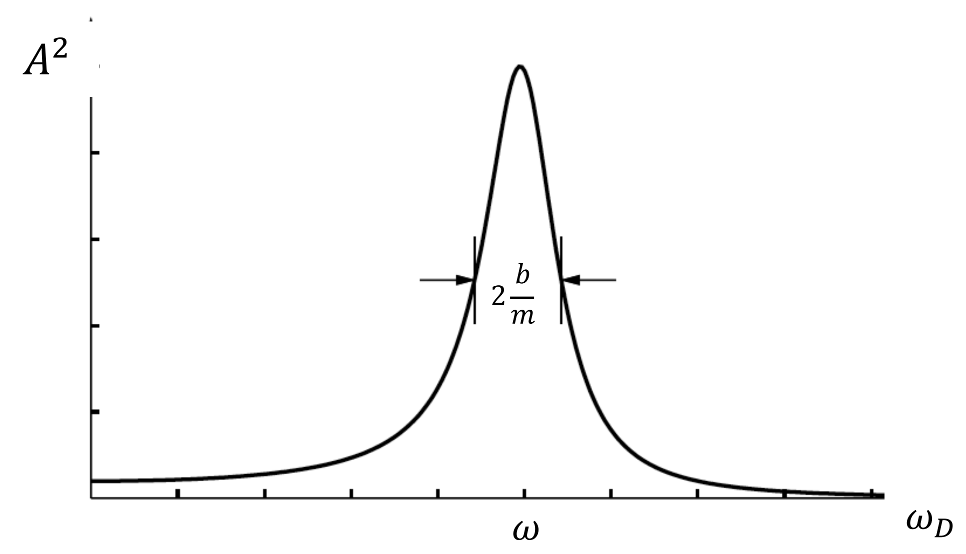
振幅的分母會多一個與阻力有關的項！
There will be one more term in the denominator.

$$A=\frac{F_0}{m\sqrt{(\omega^2-\omega_D^2)^2+\left(\frac{b}{m}\omega_D\right)^2}}$$

$$x_r=A\cos(\omega_D t+\phi)$$

振幅極大值在 $\omega_D=\omega$ 附近，但已不再是無限大！

The maximum amplitude is no longer infinite.



共振曲線的寬度現在與阻力大小成正比。

The width of the resonance curve is proportional to damping b.

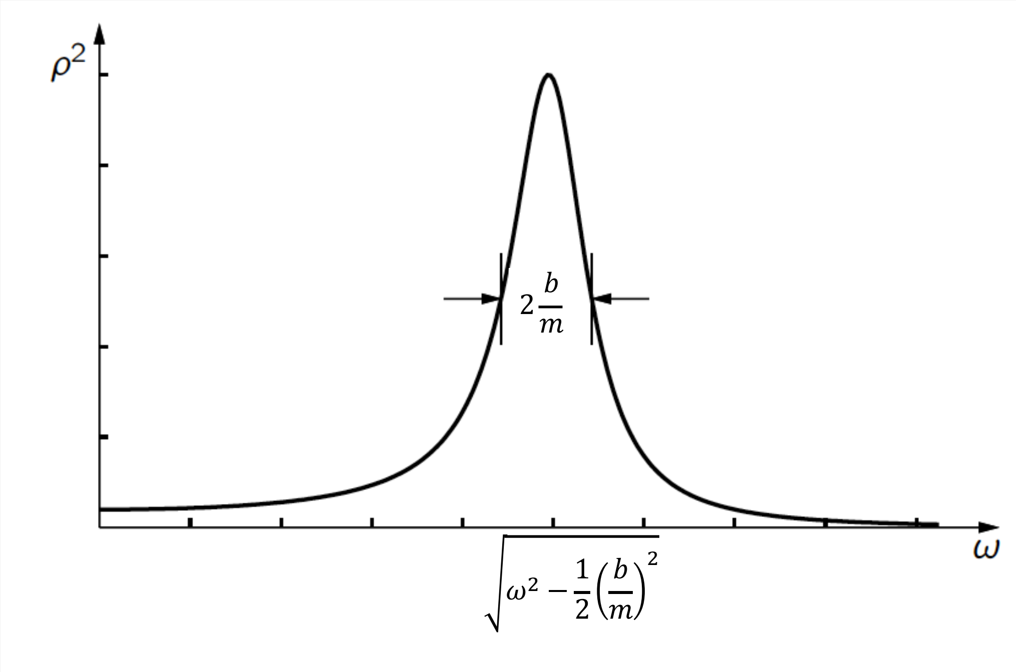
The maximum amplitude occurs where $(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega_D\right)^2$ is minimal.

The minimal condition:

$$\frac{d}{d\omega_D^2}\left[(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega_D\right)^2\right] = -2(\omega^2 - \omega_D^2) + \left(\frac{b}{m}\right)^2 = 0$$

$$\omega_D^2 = \omega^2 - \frac{1}{2}\left(\frac{b}{m}\right)^2$$

Resonance occurs now at close to but not exactly $\omega_D^2 = \omega^2$



Near Resonance $\omega \approx \omega_D$ $\omega + \omega_D \approx 2\omega_D$

$$A = \frac{F_0}{m\sqrt{(\omega-\omega_D)^2(\omega+\omega_D)^2+\left(\frac{b}{m}\omega_D\right)^2}} \sim \frac{F_0}{m\omega_D\sqrt{4(\omega-\omega_D)^2+\left(\frac{b}{m}\right)^2}}$$

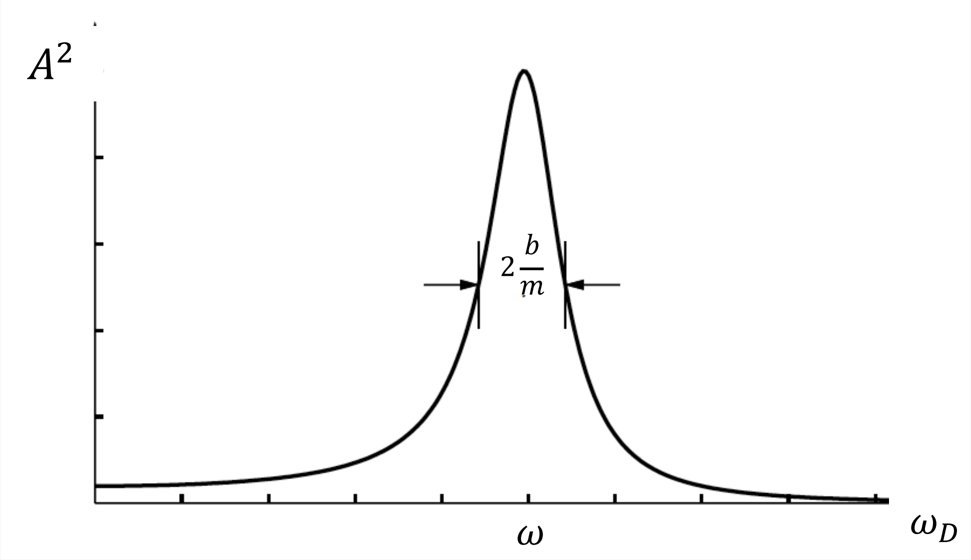
Define the width of the resonance curve as where A^2 is half its peak value:

$$\left[\frac{1}{\sqrt{4(\omega-\omega_D)^2+\left(\frac{b}{m}\right)^2}}\right]^2 = \frac{1}{2}\left[\frac{1}{\sqrt{(0)^2+\left(\frac{b}{m}\right)^2}}\right]^2$$

$$\frac{b}{m} \ll 1$$

$$(\omega-\omega_D)^2+\left(\frac{b}{m}\right)^2 = 2\left(\frac{b}{m}\right)^2$$

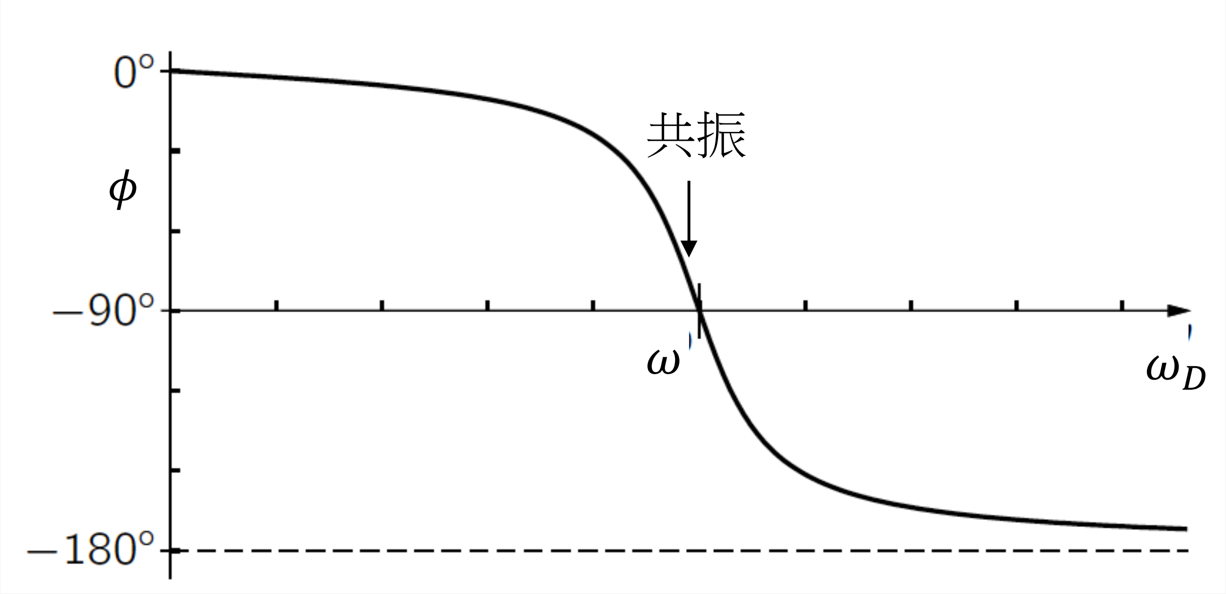
$$\omega-\omega_D \sim \frac{b}{m}$$



$x_r = A\cos(\omega_D t + \phi)$ 所施的外力：$F_0 \cos \omega_D t$。

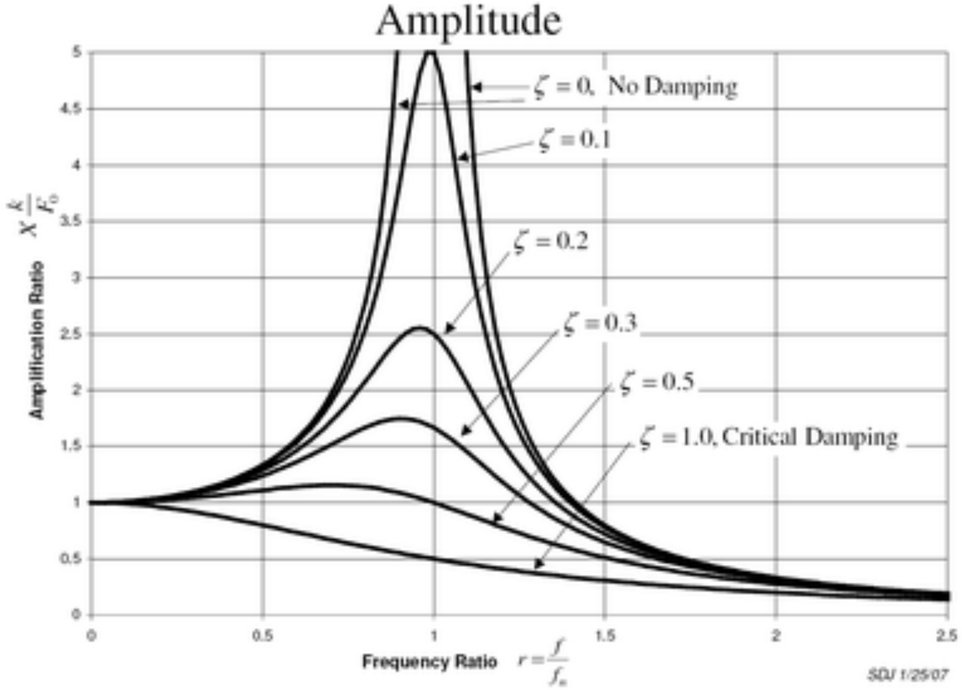
振盪與外力相較，多了一個相常數，稱相差或時間差。

$$\tan\phi = -\frac{\frac{b}{m}\omega}{\omega^2 - \omega_D^2}$$

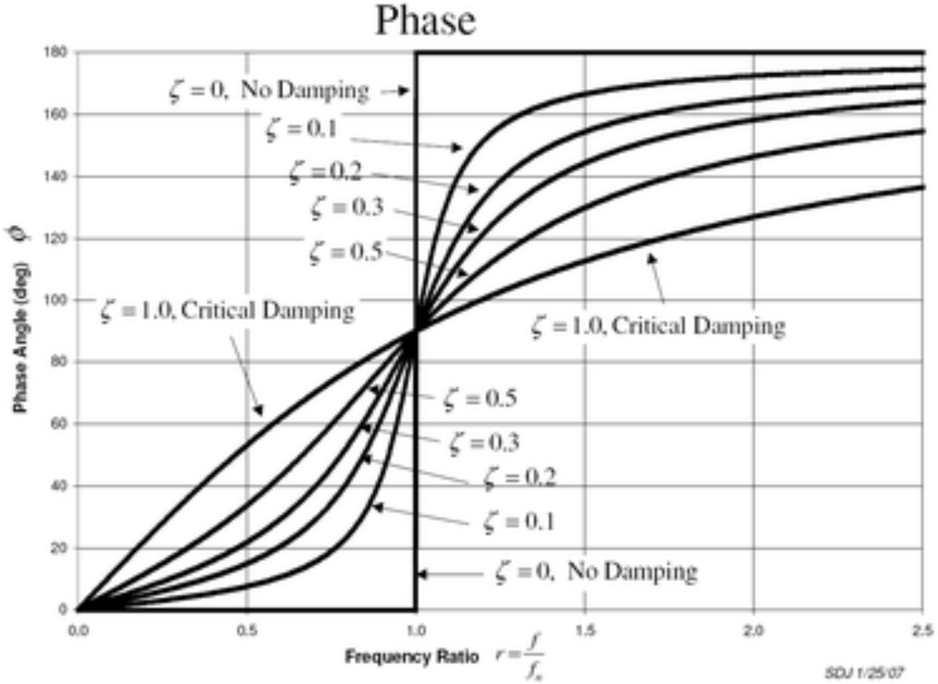


注意共振時$\phi = 90°$。也就是施力若是正弦，反應卻是餘弦。

The results depend strongly on the damping $b = \zeta$



Amplitude



Phase

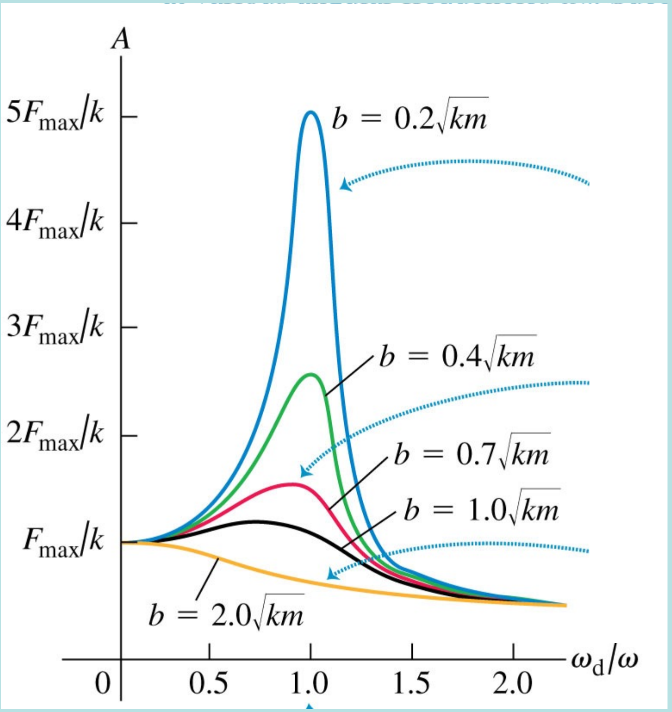
$$x_r = A\cos(\omega_D t + \phi)$$

$$A = -\frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m} b\omega_D\right)^2}}$$

$$\tan\phi = -\frac{\frac{b}{m}\omega}{\omega^2 - \omega_D^2}$$

在共振時，相差是90°。

在外力驅動下，簡諧振盪器的運動依舊是一個週期性振盪:



Resonance curves for various b.

$$x_r = A\cos(\omega_D t + \phi)$$

$$A = \frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega_D\right)^2}}$$

$$\tan\phi = -\frac{\frac{b}{m}\omega}{\omega^2 - \omega_D^2}$$

前面忽略阻尼時，得到的特性還是成立。

Basic features of the resonance solution without damping remain true here.

以外力的頻率ω_D來振盪，而不是彈簧的自然頻率ω！

It oscillates in the frequency of external force ω_D, instead of the spring ω.

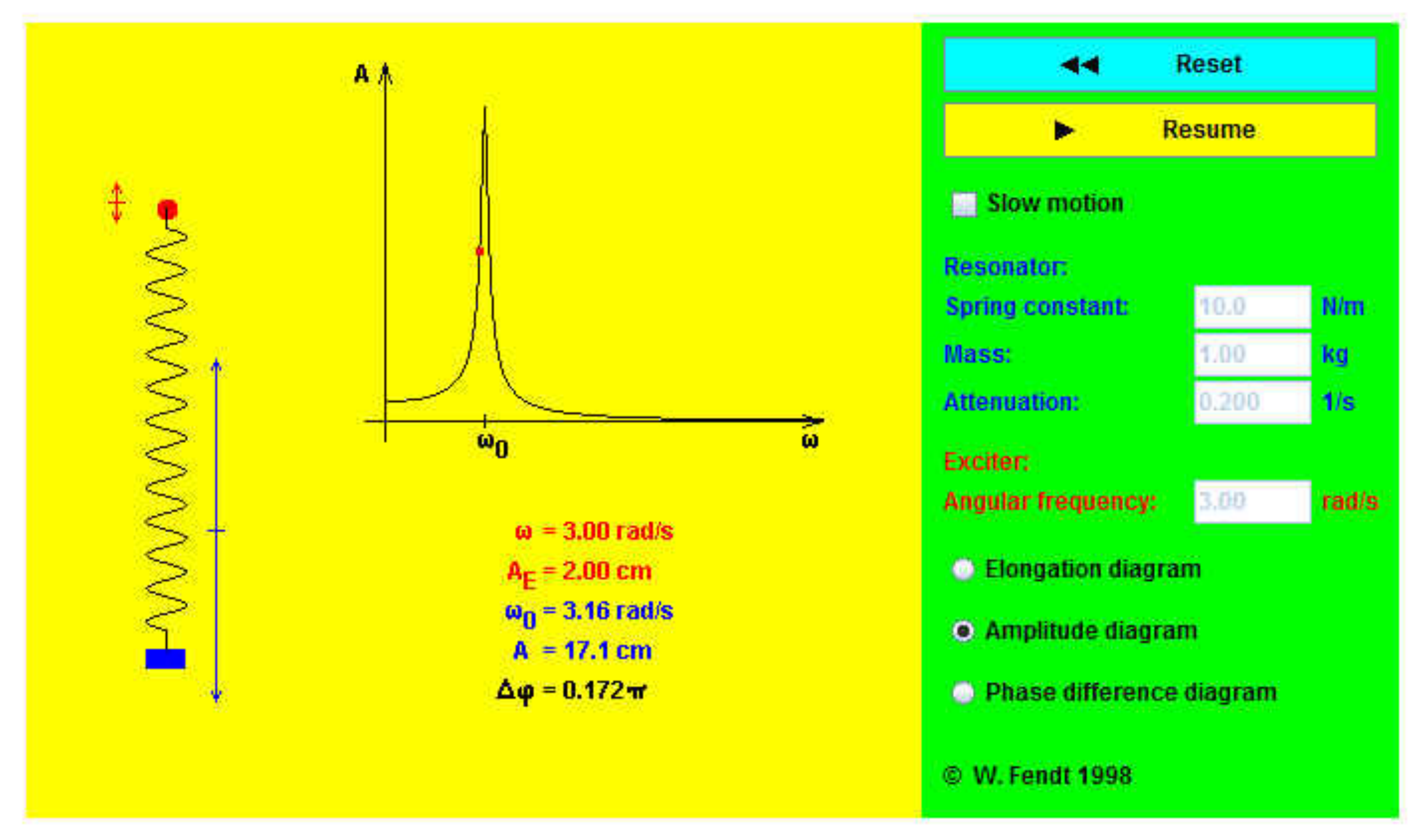
外力頻率越接近彈簧的自然頻率，振盪振幅也就越大！ $\omega_D \to \omega$ $A \uparrow$

The closer ω_D is to ω, the larger the amplitude. But it falls off rapidly away from resonance

共振曲線的寬度與阻力大小成正比，阻力會削弱共振的現象！

The width of the resonance curve is proportional to b. Damping weakens resonance.

http://www.walter-fendt.de/ph14e/resonance.htm



ω = 3.00 rad/s
A_E = 2.00 cm
ω_0 = 3.16 rad/s
A = 17.1 cm
$\Delta\varphi$ = 0.172π

☐ Slow motion

Resonator:
Spring constant: 10.0 N/m
Mass: 1.00 kg
Attenuation: 0.200 1/s

Exciter:
Angular frequency: 3.00 rad/s

○ Elongation diagram
● Amplitude diagram
○ Phase difference diagram

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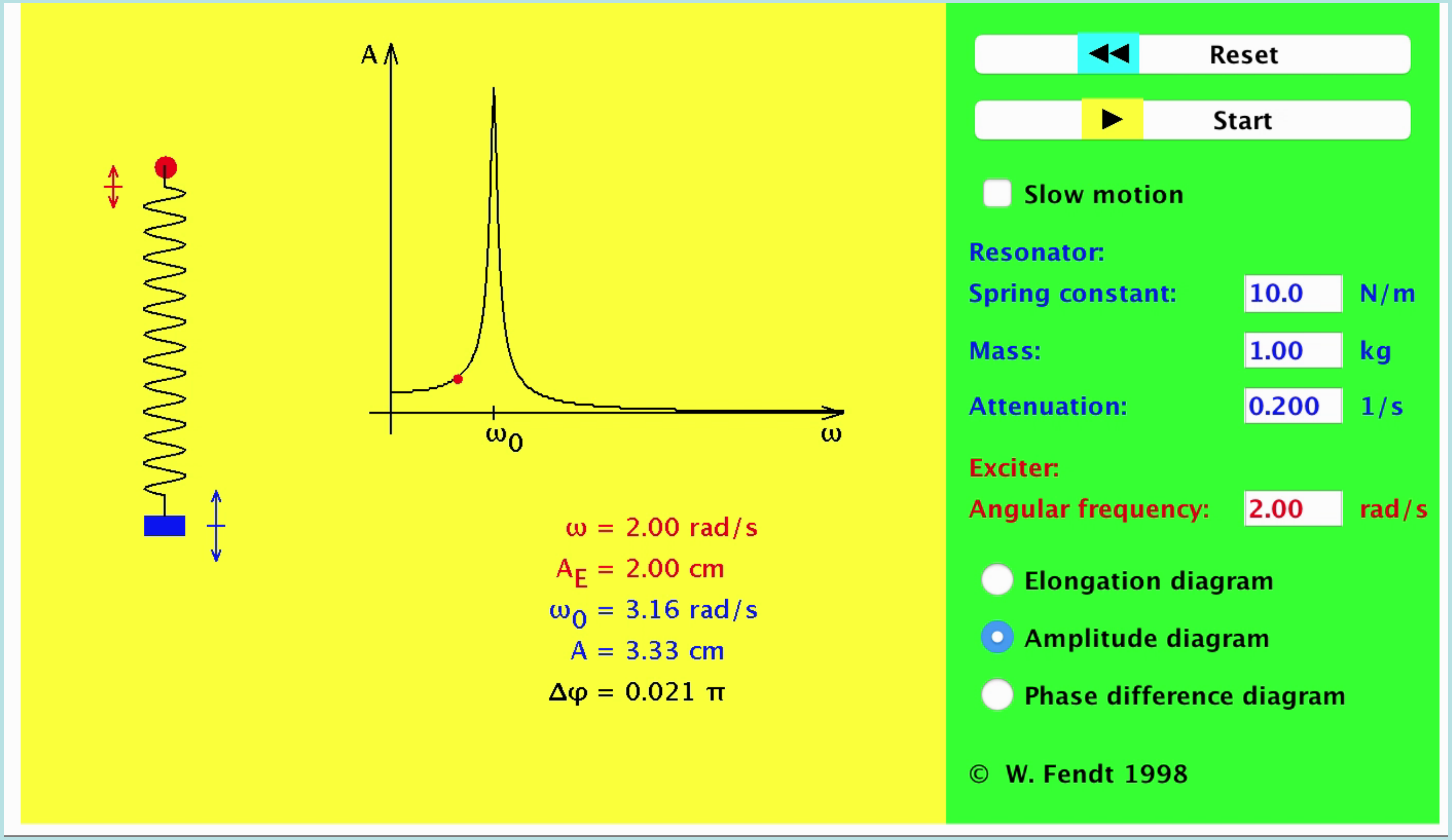
Under resonance



☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

Attenuation: 0.200 1/s

Exciter:

Angular frequency: 2.00 rad/s

○ Elongation diagram

● Amplitude diagram

○ Phase difference diagram

ω = 2.00 rad/s

A_E = 2.00 cm

ω_0 = 3.16 rad/s

A = 3.33 cm

$\Delta\varphi$ = 0.021 π

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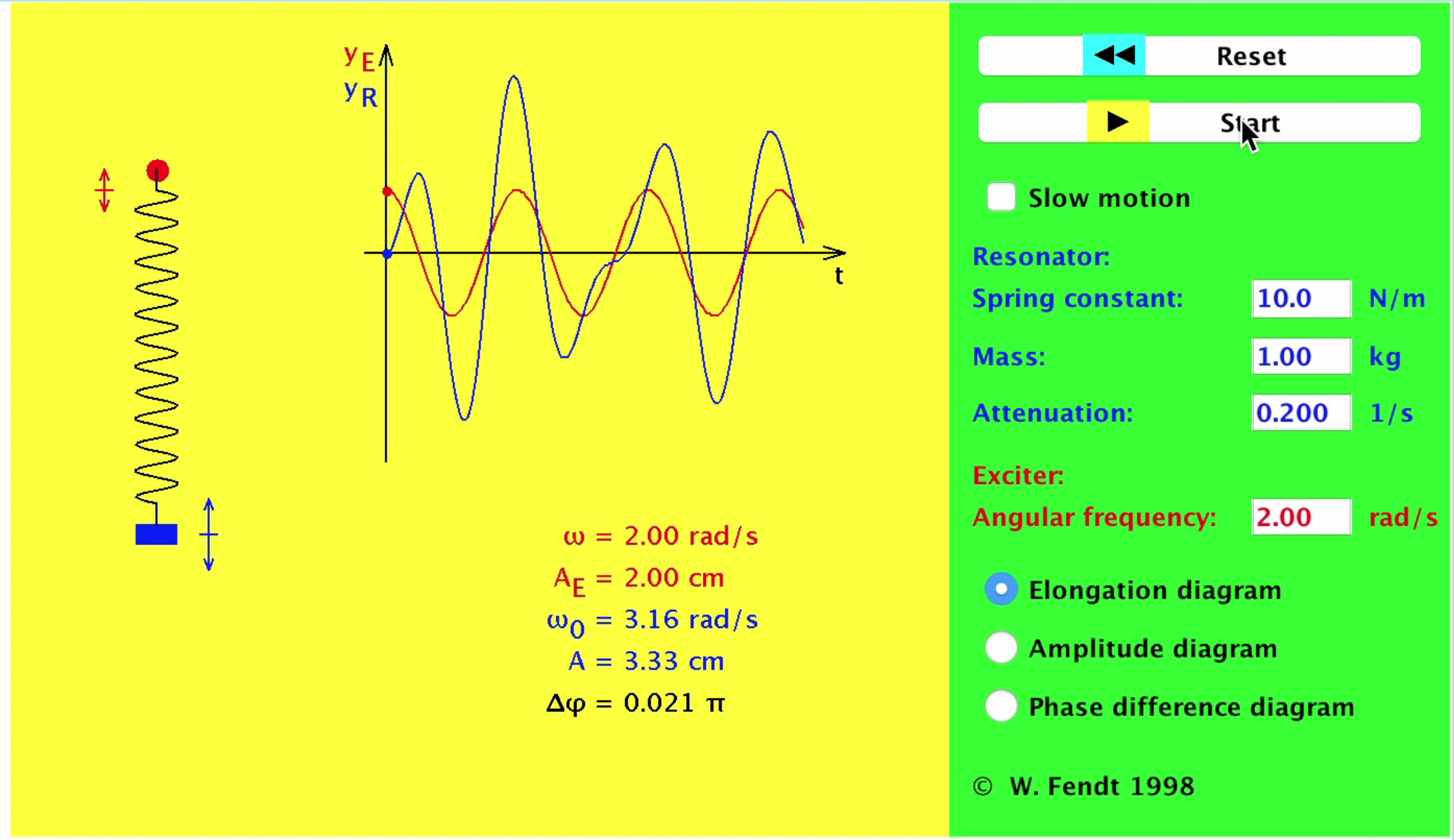


ω = 2.00 rad/s
A_E = 2.00 cm
ω_0 = 3.16 rad/s
A = 3.33 cm
$\Delta\varphi$ = 0.021 π

☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

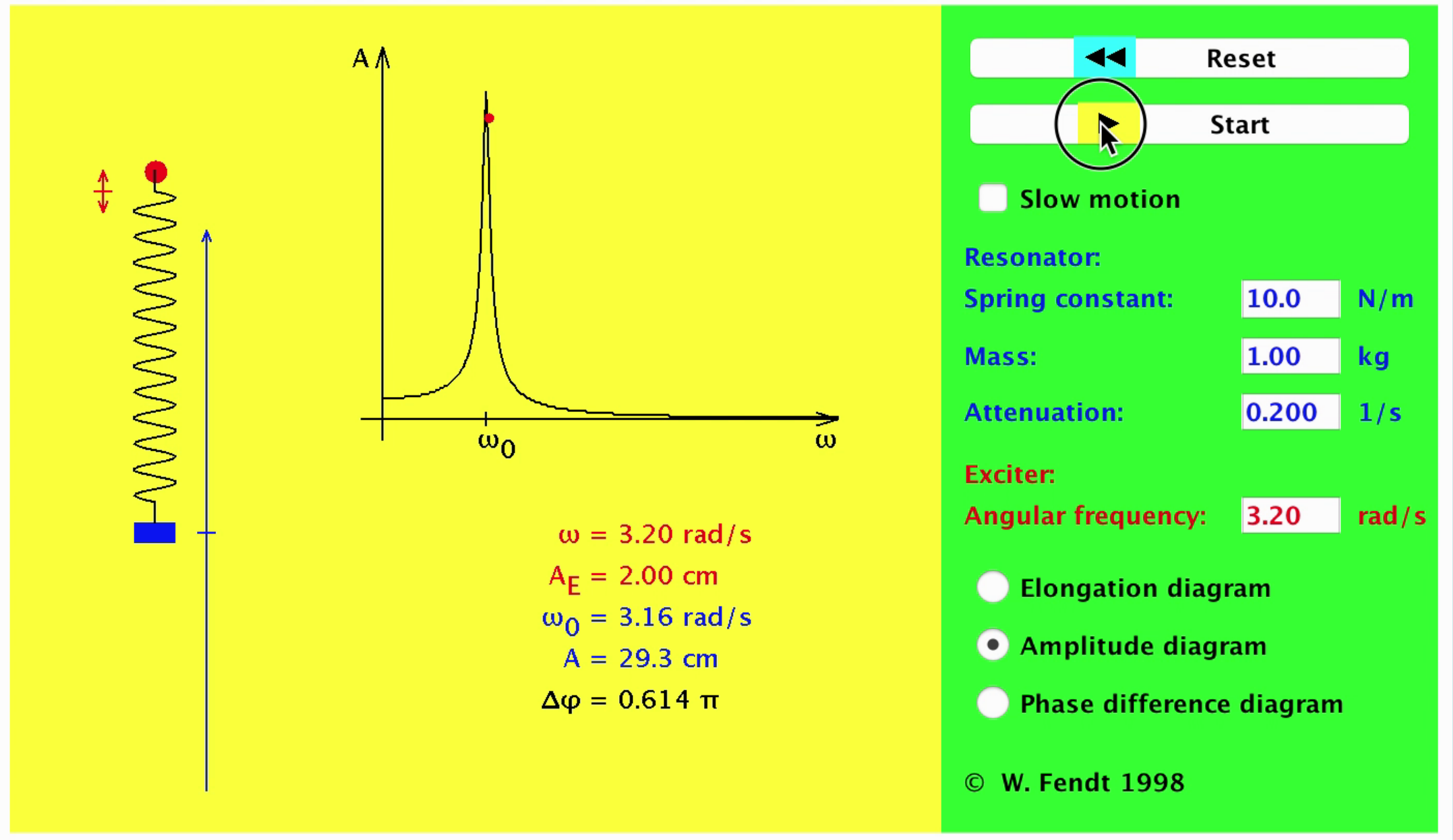
Attenuation: 0.200 1/s

Exciter:

Angular frequency: 2.00 rad/s

● Elongation diagram
○ Amplitude diagram
○ Phase difference diagram

On resonance



$\omega = 3.20$ rad/s
$A_E = 2.00$ cm
$\omega_0 = 3.16$ rad/s
$A = 29.3$ cm
$\Delta\varphi = 0.614\,\pi$

☐ Slow motion

Resonator:
Spring constant: 10.0 N/m
Mass: 1.00 kg
Attenuation: 0.200 1/s

Exciter:
Angular frequency: 3.20 rad/s

○ Elongation diagram
● Amplitude diagram
○ Phase difference diagram

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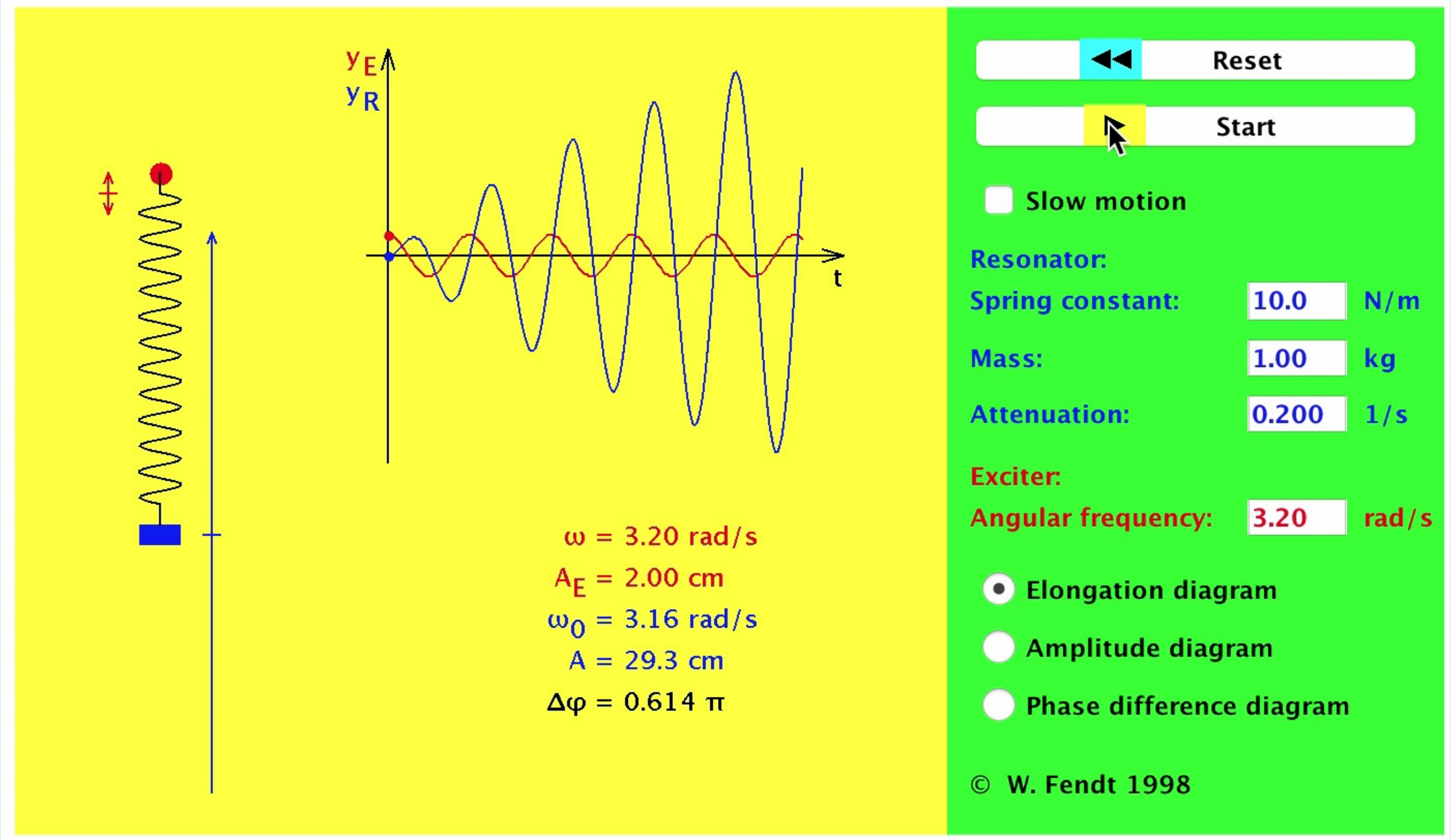

ω = 3.20 rad/s

A_E = 2.00 cm

ω_0 = 3.16 rad/s

A = 29.3 cm

$\Delta\varphi$ = 0.614 π

☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

Attenuation: 0.200 1/s

Exciter:

Angular frequency: 3.20 rad/s

● Elongation diagram

○ Amplitude diagram

○ Phase difference diagram

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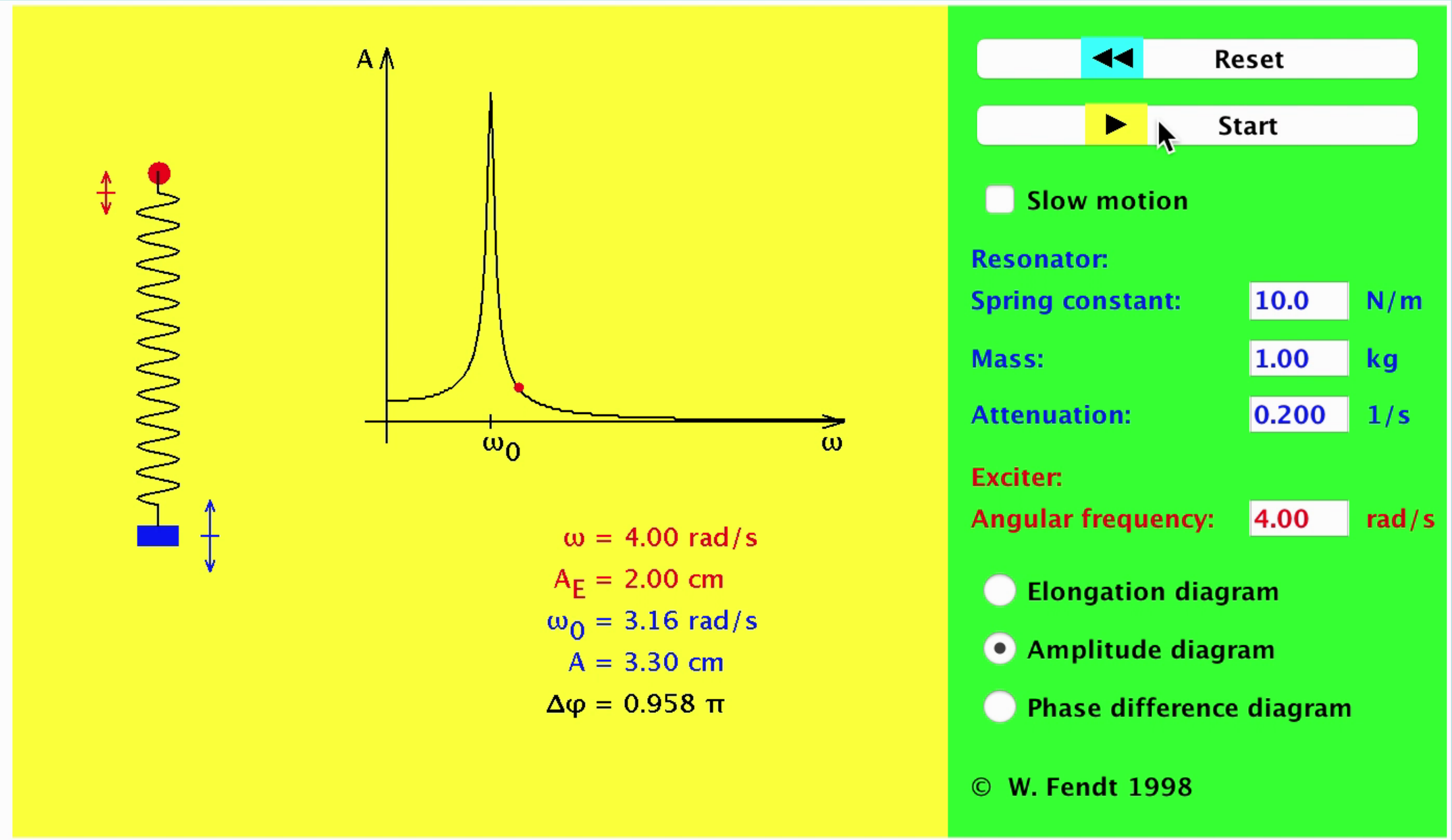
Above Resonnace



$\omega = 4.00$ rad/s

$A_E = 2.00$ cm

$\omega_0 = 3.16$ rad/s

$A = 3.30$ cm

$\Delta\varphi = 0.958\,\pi$

☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

Attenuation: 0.200 1/s

Exciter:

Angular frequency: 4.00 rad/s

○ Elongation diagram

● Amplitude diagram

○ Phase difference diagram

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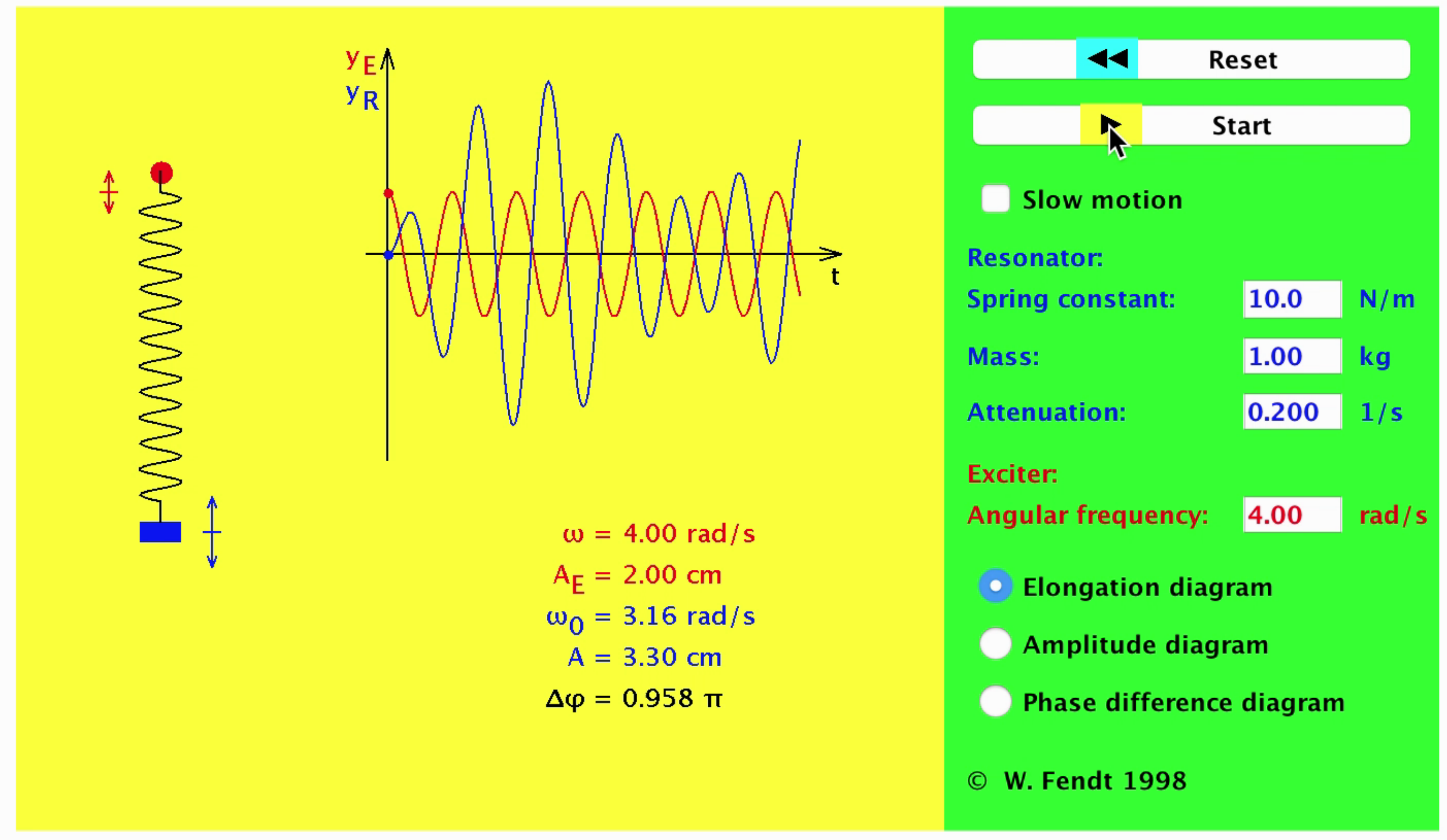

ω = 4.00 rad/s

A_E = 2.00 cm

ω_0 = 3.16 rad/s

A = 3.30 cm

$\Delta\varphi$ = 0.958 π

☐ Slow motion

Resonator:

Spring constant: 10.0 N/m

Mass: 1.00 kg

Attenuation: 0.200 1/s

Exciter:

Angular frequency: 4.00 rad/s

● Elongation diagram

○ Amplitude diagram

○ Phase difference diagram

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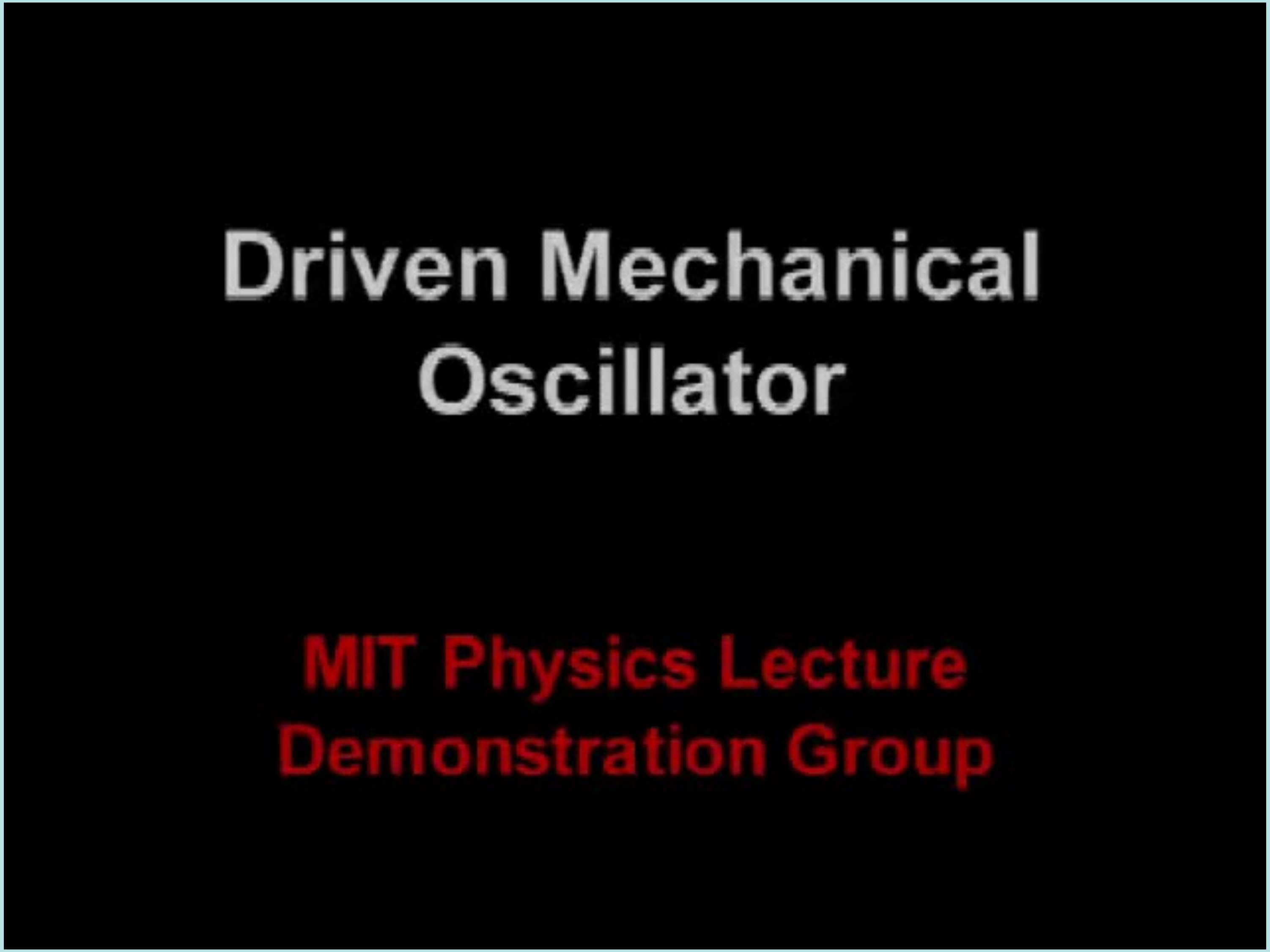
https://www.youtube.com/watch?v=aZNnwQ8HJHU

http://techtv.mit.edu/collections/physicsdemos/videos/769-mit-physics-demo----driven-mechanical-oscillator



Driven Mechanical
Oscillator

MIT Physics Lecture
Demonstration Group

Resonance solution x_r is one solution of inhomogeneous SHM.

It could not always fit the initial conditions.

The difference $x - x_r$ between any solutions of a Linear inhomogeneous ODE

and this particular one x_r equals a solution x_s of the homogeneous ODE.

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t \quad - \quad \frac{d^2x_r}{dt^2} + \frac{b}{m}\frac{dx_r}{dt} + \omega^2 x_r = \frac{F_0}{m}\cos\omega_D t$$

$$\Downarrow$$

$$\frac{d^2(x-x_r)}{dt^2} + \frac{b}{m}\frac{d(x-x_r)}{dt} + \omega^2(x-x_r) = 0 \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

$(x - x_r)$ equals a solution x_s of the homogeneous version of ODE.

Remember there are an infinite number of x_s.

Therefore, we get an infinite number of general solutions x.

$$x = x_r + x_s$$

One of them will satisfy initial conditions.

Solutions of damping SHM

$$x_s = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

Resonance solution of inhomogeneous SHM

$$x_r = \frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}} \cos(\omega_D t + \phi)$$

$x = x_r + x_s$ 滿足原來 x_r 所滿足的外力下簡諧運動的微分方程式：

x is the general solutions of inhomogeneous SHM: $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t$

While x_r is totally fixed by ODE, there are two unspecified constants x_m, ϕ in x_s.

We can choose them to satisfy the two initial conditions $x(0), x'(0)$.

這個函數同時滿足運動方程式以及兩個起始條件，因此是唯一的解！

The function we get satisfies inhomogeneous SHM ODE and initial condition simultaneously

It is the unique solution.

$$x = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi) + \frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}} \cos(\omega_D t + \phi)$$

注意非共振解x_s是以彈簧自然頻率ω震盪，而不是ω_D。

Nonresonance x_s oscillates in the damped frequency of the spring ω' instead of ω_D like x_r

但隨時間振幅會變小，長期來說可以忽略。

As time progresses, amplitude decreases exponentially. In the long term, it can be ignored

$$x = x_r + x_s \rightarrow x_r$$

長期而言，只有共振解是重要的，起始條件無關緊要

In the long term, only resonance solution survive. Initial conditions do not matter.

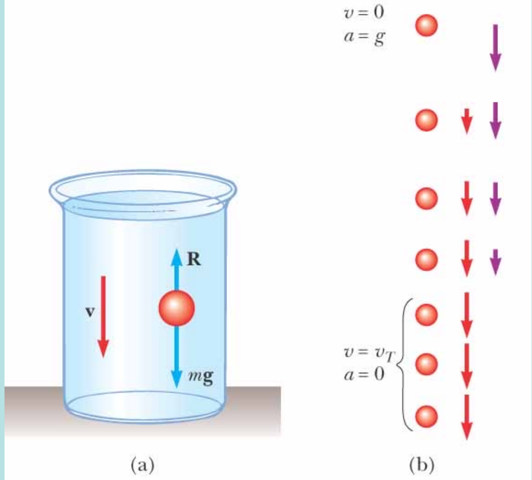
愛拼才會贏定律

It is the same as in the free fall with air resistance.

$$\frac{dv_y}{dt} + \frac{k}{m} v_y = -g$$

$$v_y(t) = \frac{C}{\alpha(t)} - \frac{1}{\alpha(t)} \int \alpha(t) g \, dt$$

$$= Ce^{-\frac{k}{m}t} - ge^{-\frac{k}{m}t} \cdot \frac{m}{k} e^{\frac{k}{m}t} = \boxed{Ce^{-\frac{k}{m}t}} - \boxed{\frac{mg}{k}}$$



The first term $Ce^{-\frac{k}{m}t}$ is the solution of the homogeneous ODE. $\frac{dv_y}{dt} + \frac{k}{m} v_y = 0$

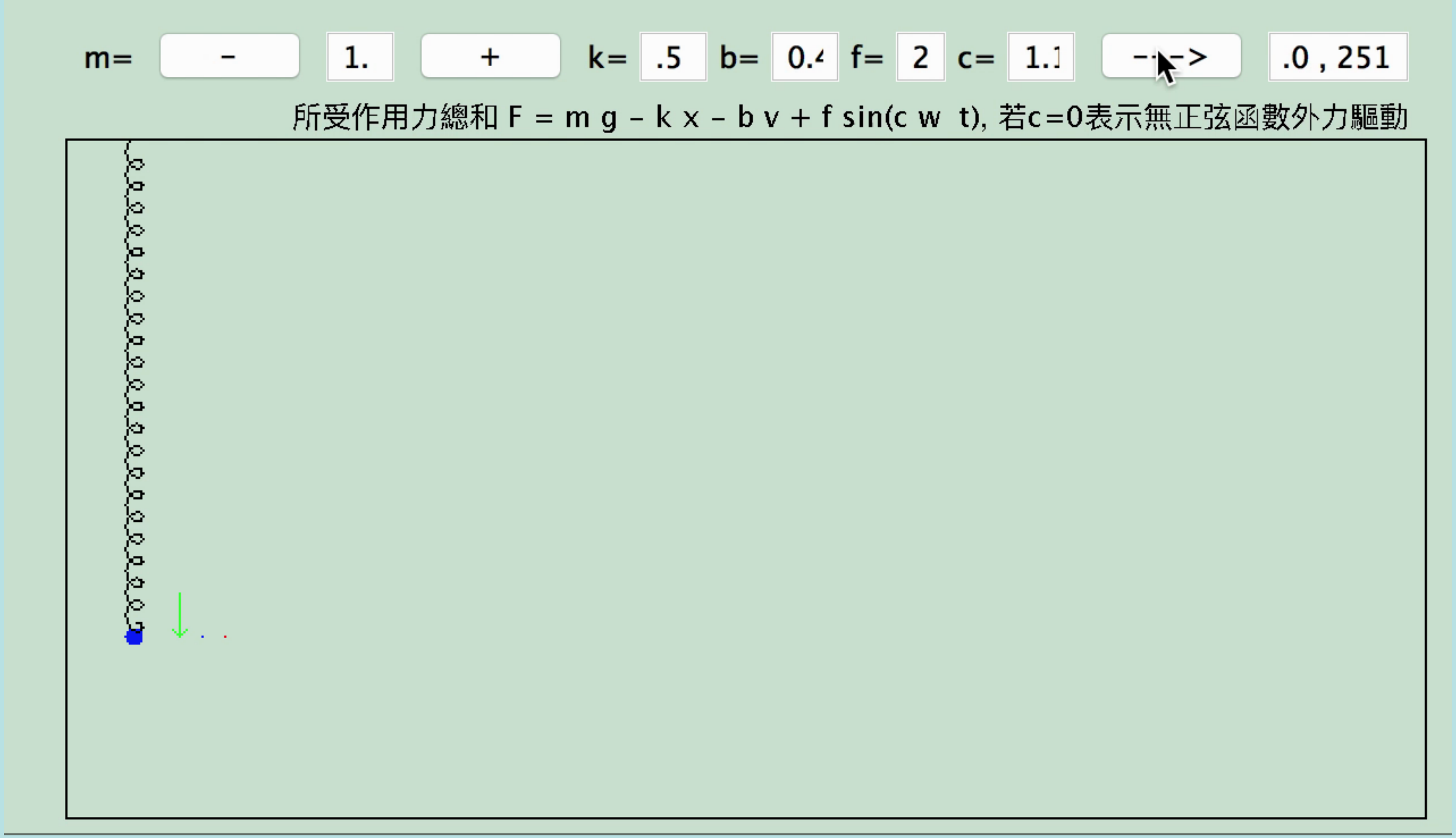
It will vanish exponentially when $t \to \infty$.

The second term is one of the solutions of the inhomogeneous ODE.

It is called particular solution and will survive as terminal speed when $t \to \infty$.

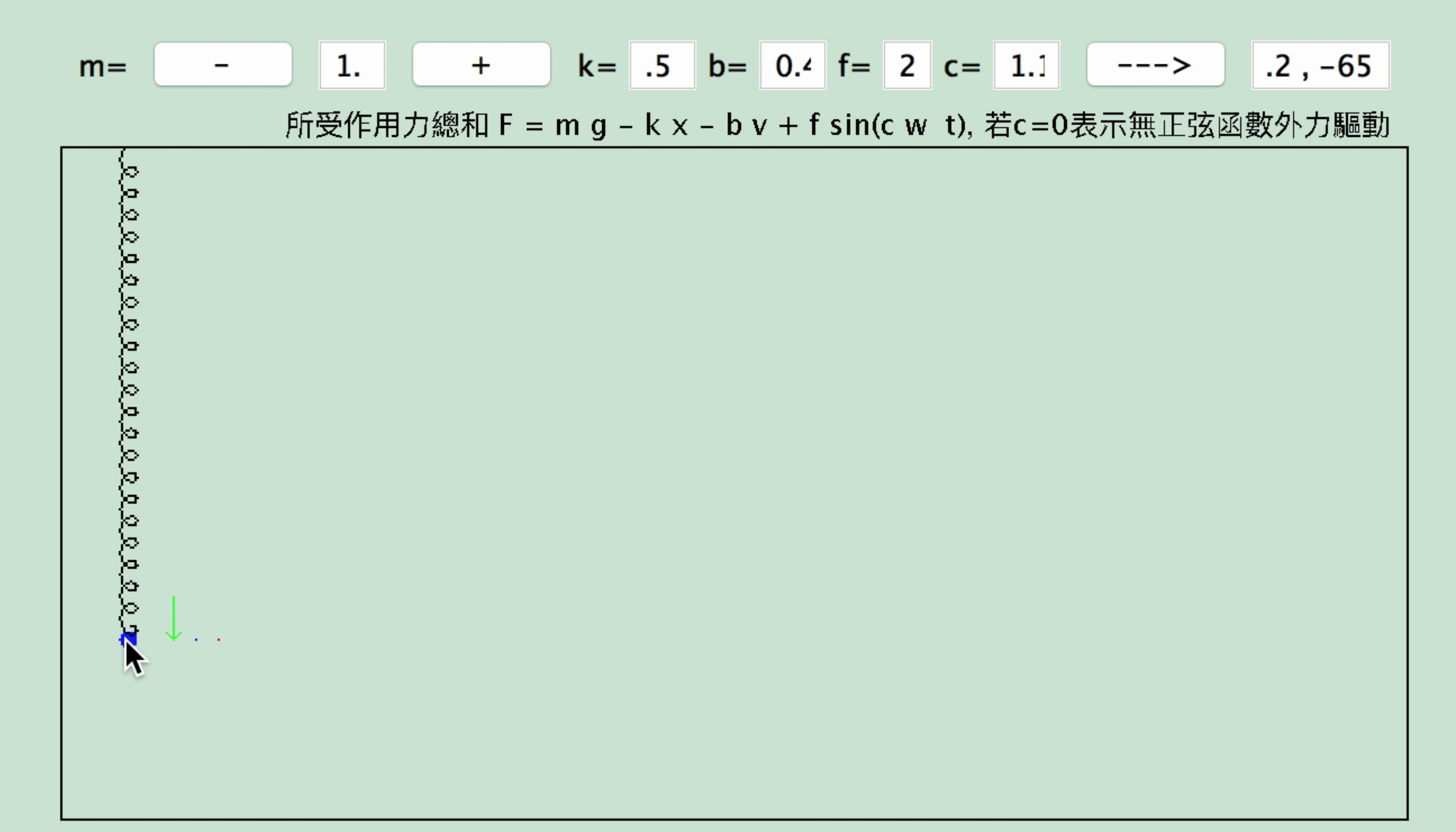
模擬

http://www.phy.ntnu.edu.tw/moodle/mod/resource/view.php?id=124



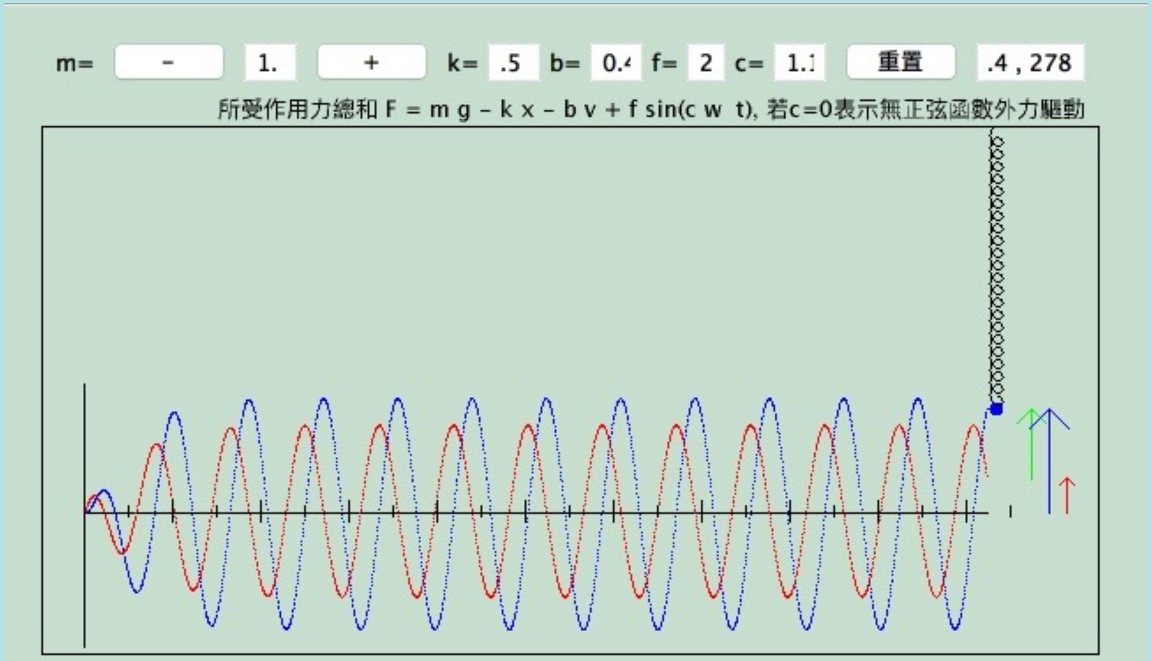
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

With a different initial condition:



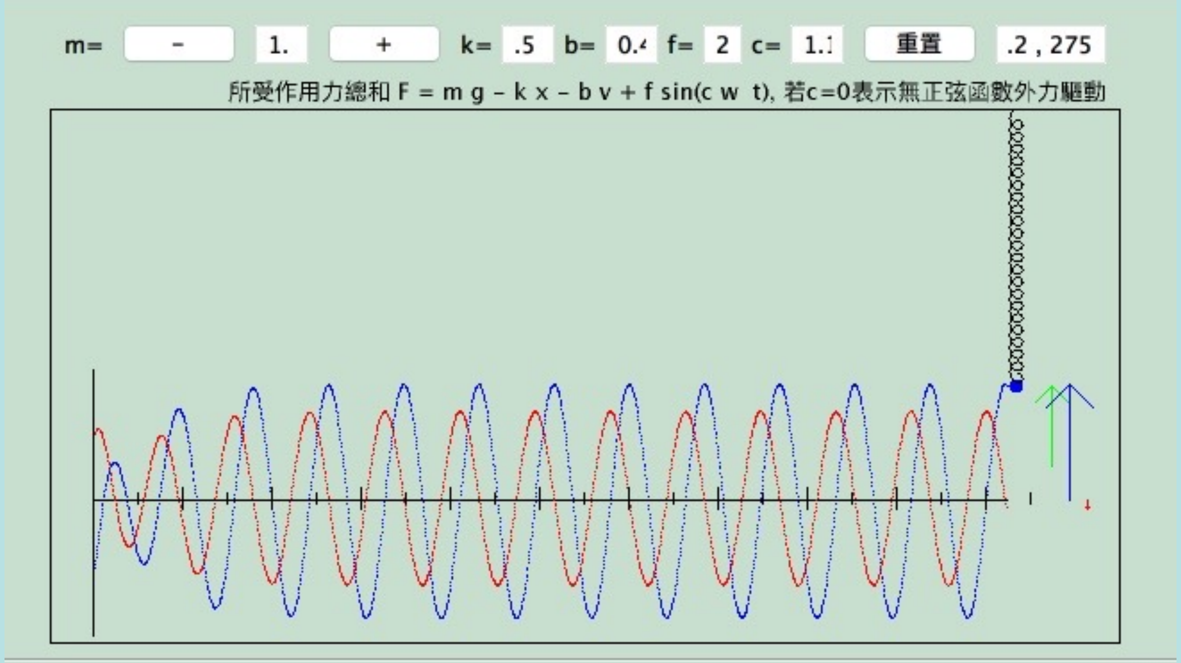
m= 1. k= .5 b= 0.4 f= 2 c= 1.1 .2 , -65

所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動



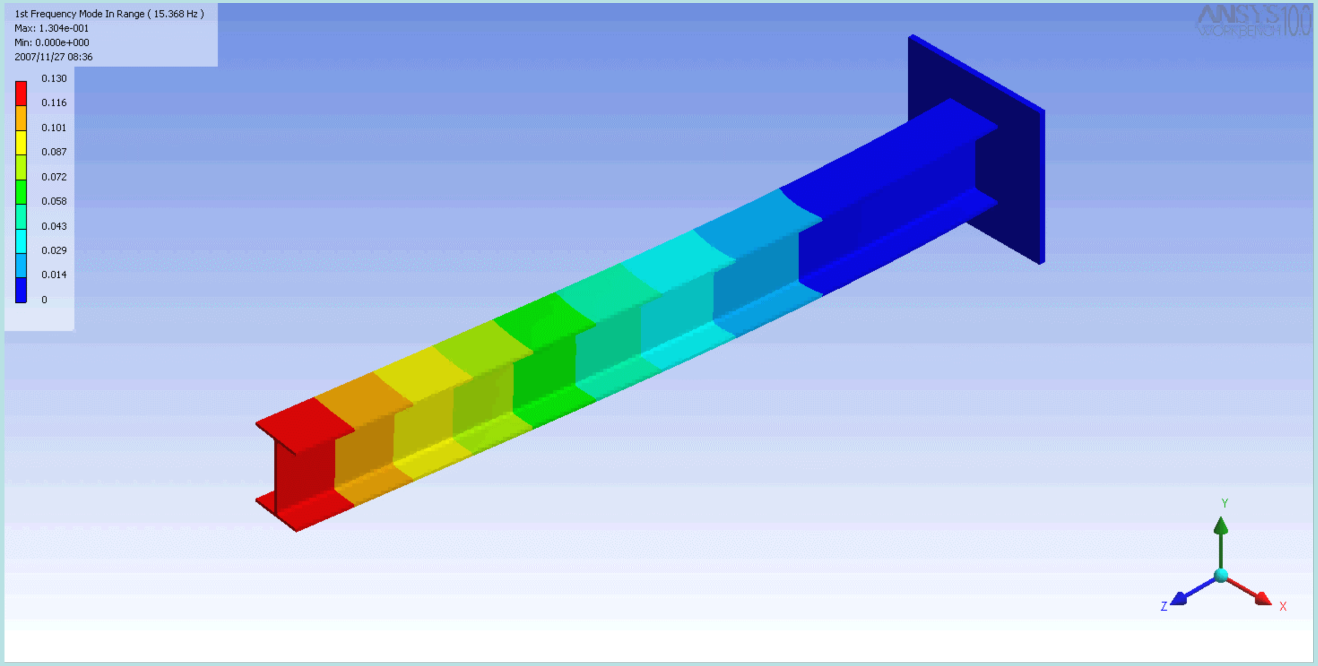
m= 1. k= .5 b= 0.4 f= 2 c= 1.1 .4 , 278

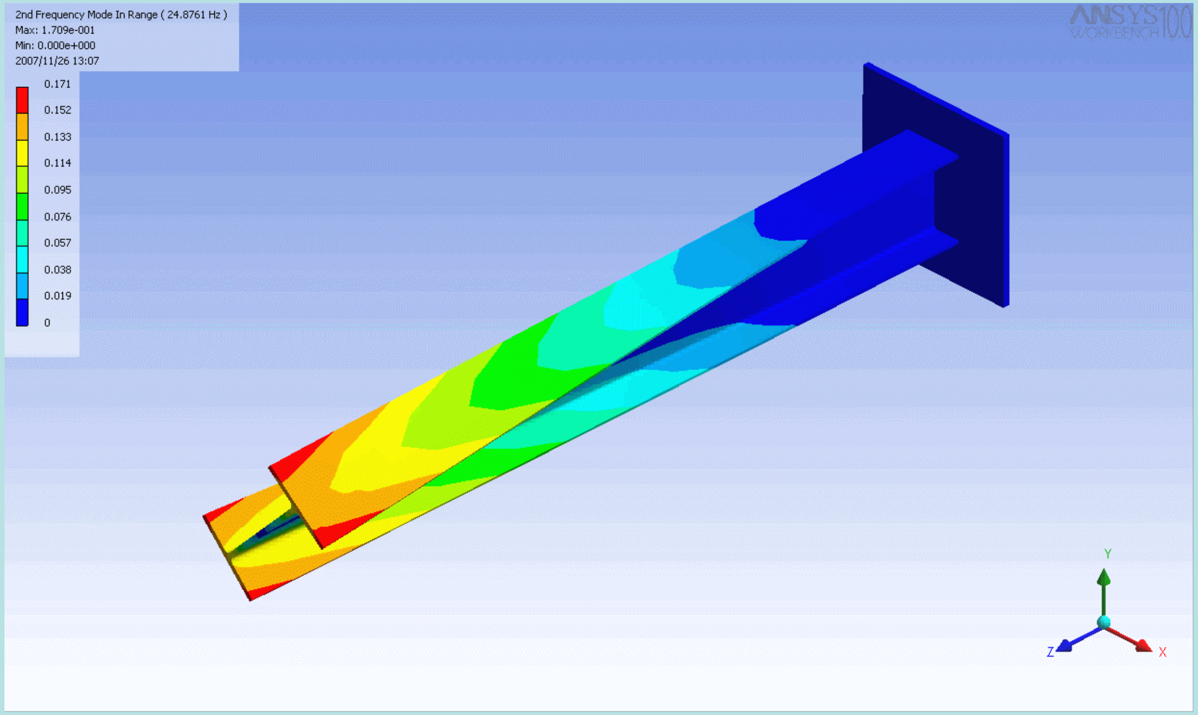
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動



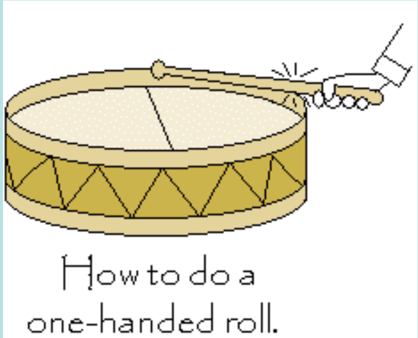
m= 1. k= .5 b= 0.4 f= 2 c= 1.1 .2 , 275

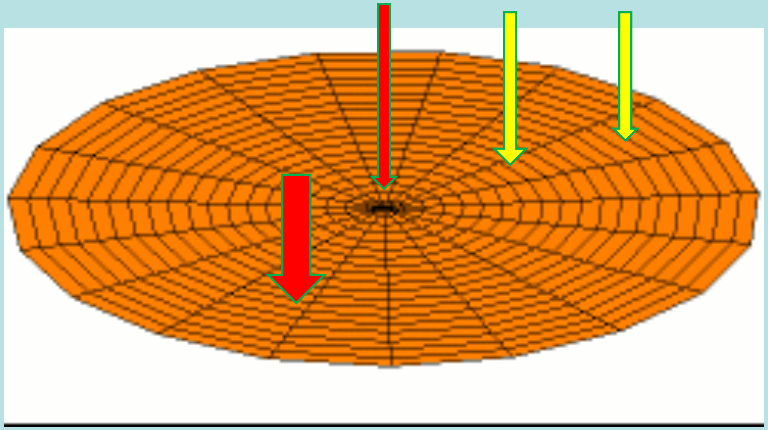
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動





物體的變形不只一種！並不是每一種變形都可以持續是簡諧運動。





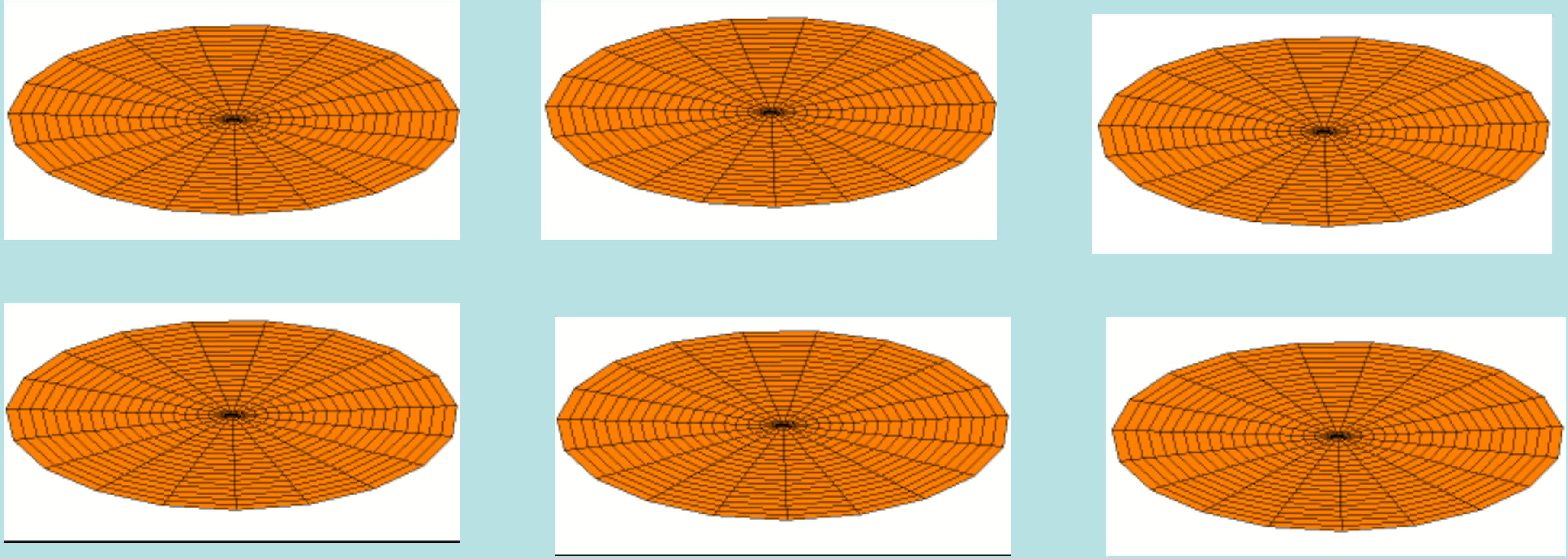
某些特定變形的模式，其隨時間的運動會如同一個彈簧，

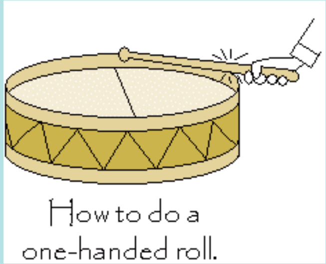
當這些模式被單獨激發時，物體中的每一點都以簡諧運動方式運動：

$$y_i(t) = y_{mi} \cdot \cos(\omega t + \phi_i)$$

每一點y_i有一特定的振幅y_{mi}與相常數ϕ_i。

可以證明：物體的所有變形就是以這些模式，或它們的疊加來進行！





物體的變形模式有無限多個，

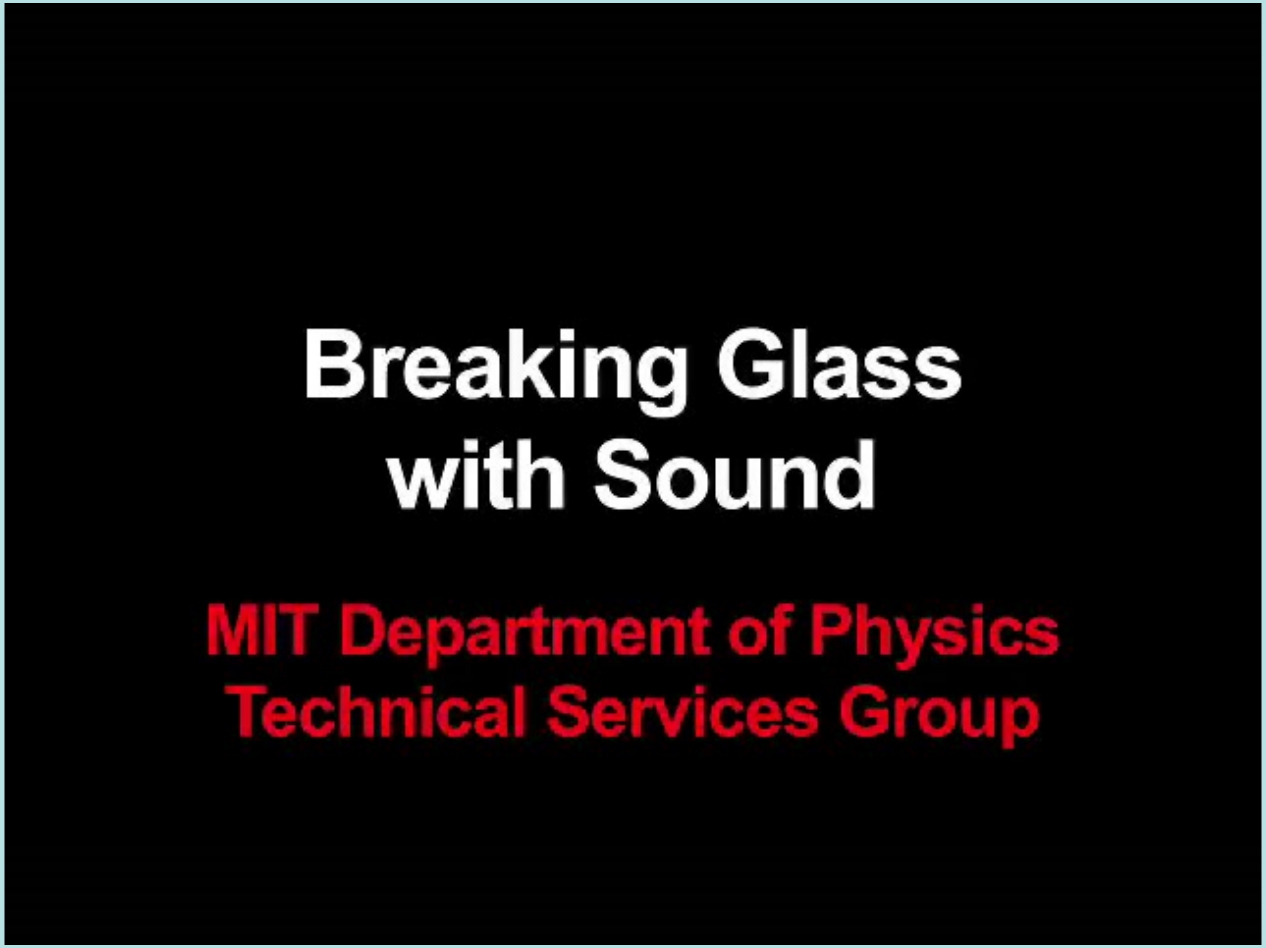
每一個模式的振動頻率不同！

一般來說，越複雜的模式，頻率越高，也越難激發。

一個物體有那些振盪模式 Norm以及對應的頻率，就是該物體的一個特徵。

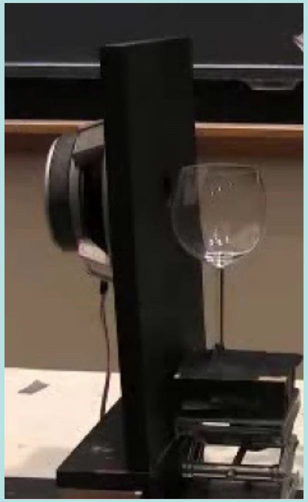
物體的振動

http://www.youtube.com/watch?v=17tqXgvCN0E&NR=1



Breaking Glass
with Sound

MIT Department of Physics
Technical Services Group

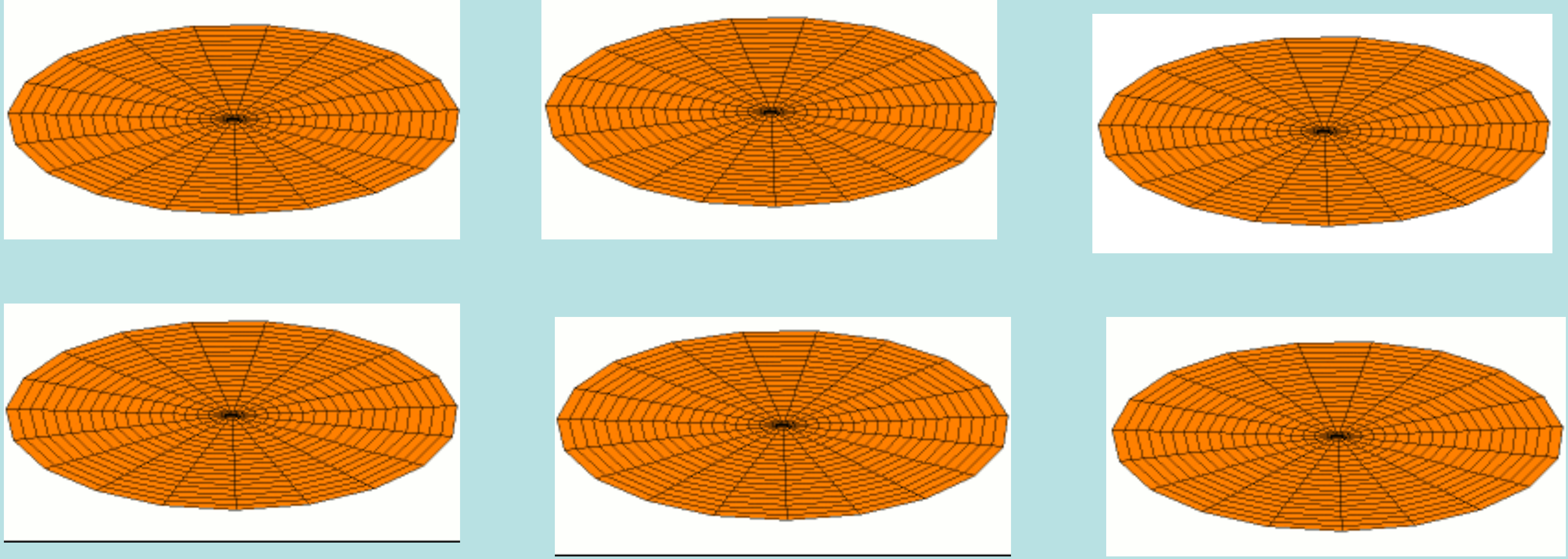


麥克風震碎玻璃杯的現象就是共振的實例。

物體的變型模式有無限多個，每一個模式的振動頻率不同。

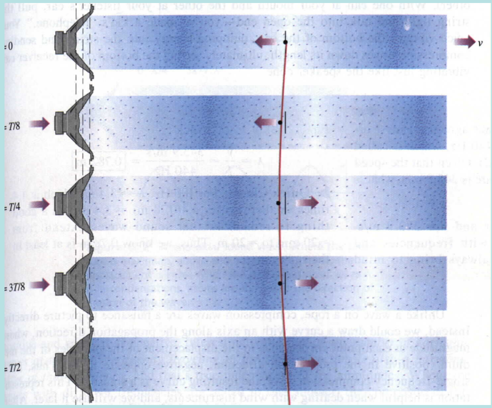
玻璃杯的變形等同於多個彈簧系統。

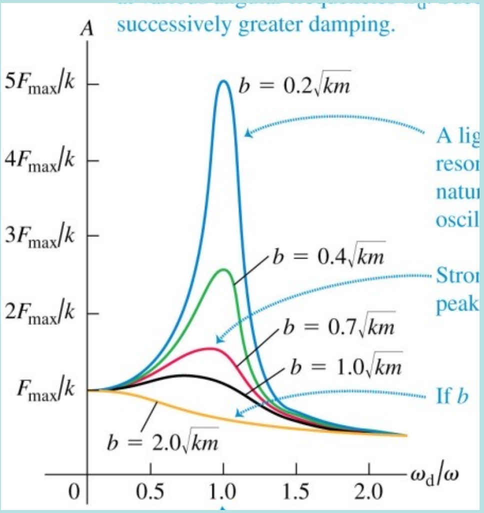
麥克風發出的聲波就是週期性外力。



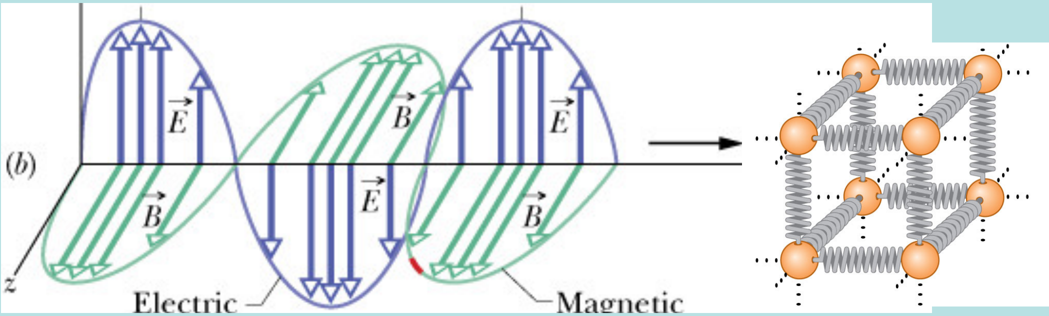
如果模式的頻率ω接近聲波頻率ω_D，週期性外力會帶動該模式開始振盪。

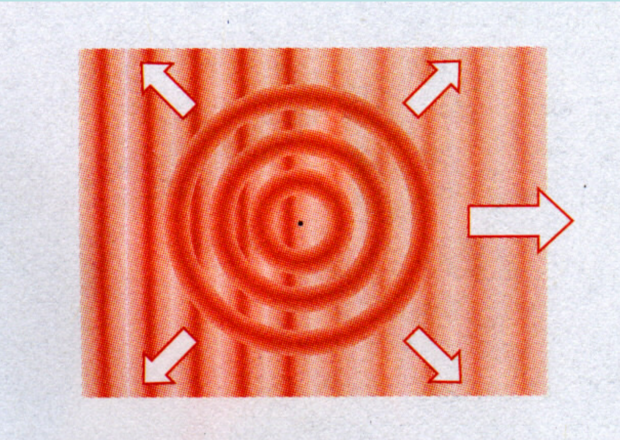
其他模式的振盪就很小。因此共振下的玻璃杯變型就如同一根彈簧！





電磁波打在一個晶體內的原子上的散射現象，也是如此。

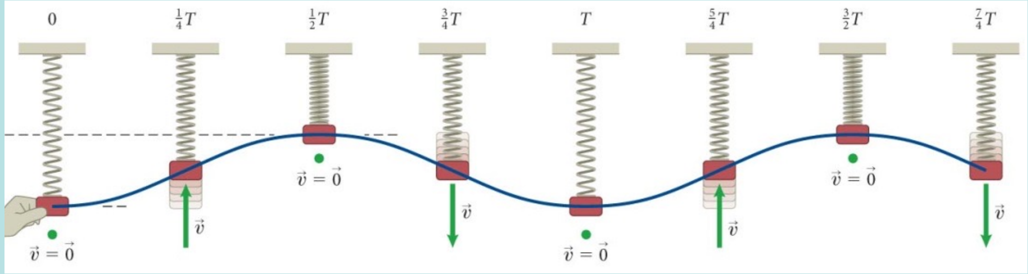




$$F = \frac{qE_0}{m}\cos\omega t$$

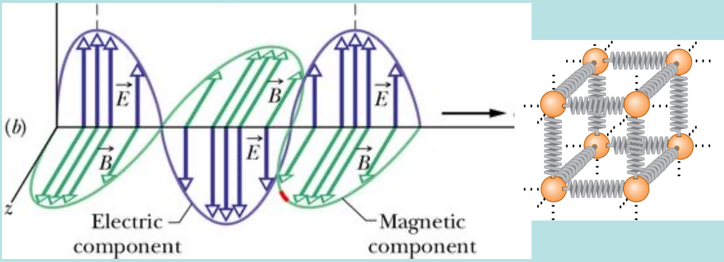
運動方程式

$$\frac{d^2y}{dt^2} = -\frac{b}{m}\frac{dy}{dt} - \omega^2 y + \frac{F_0}{m}\cos\omega_D t$$



電磁波頻率ω_D越接近分子間彈簧的振動頻率ω，彈簧振動越大，

越多電磁波能量被晶體內原子吸收！



將電磁波的吸收率對其頻率ω_D(或波長)作圖：

吸收光譜的共振頻率即是分子彈簧的自然內在頻率ω！

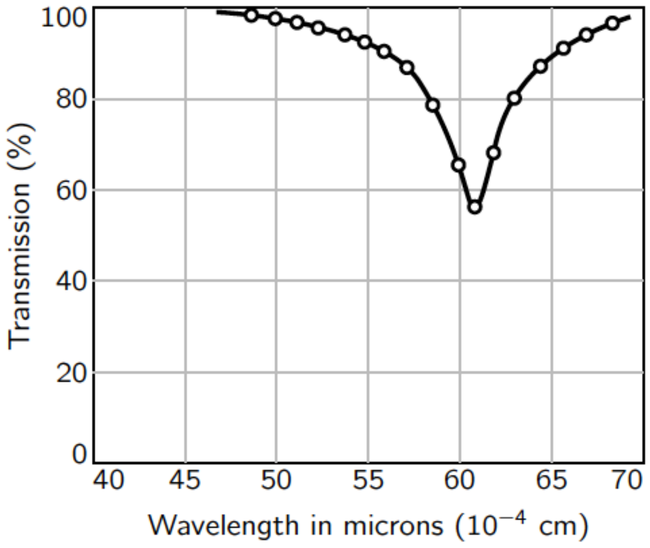


Fig. 23-7. Transmission of infrared radiation through a thin (0.17 μ) sodium chloride film. [After R. B. Barnes, *Z. Physik* **75**, 723 (1932). Kittel, *Introduction to Solid State Physics*, Wiley, 1956.]

建築必須避免其自然頻率ω，與環境外力頻率ω_D接近的狀況。

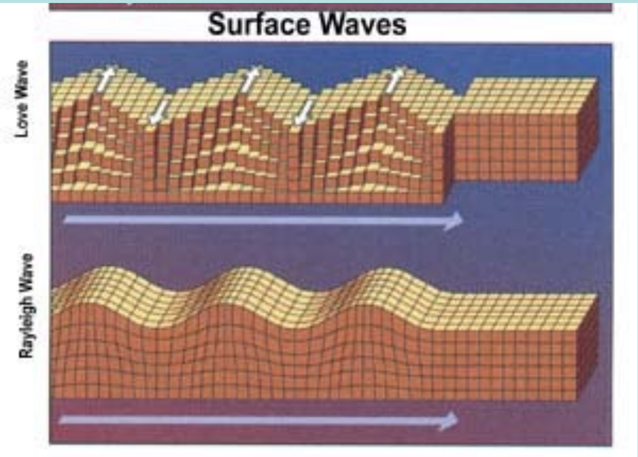
1985墨西哥市七級地震：

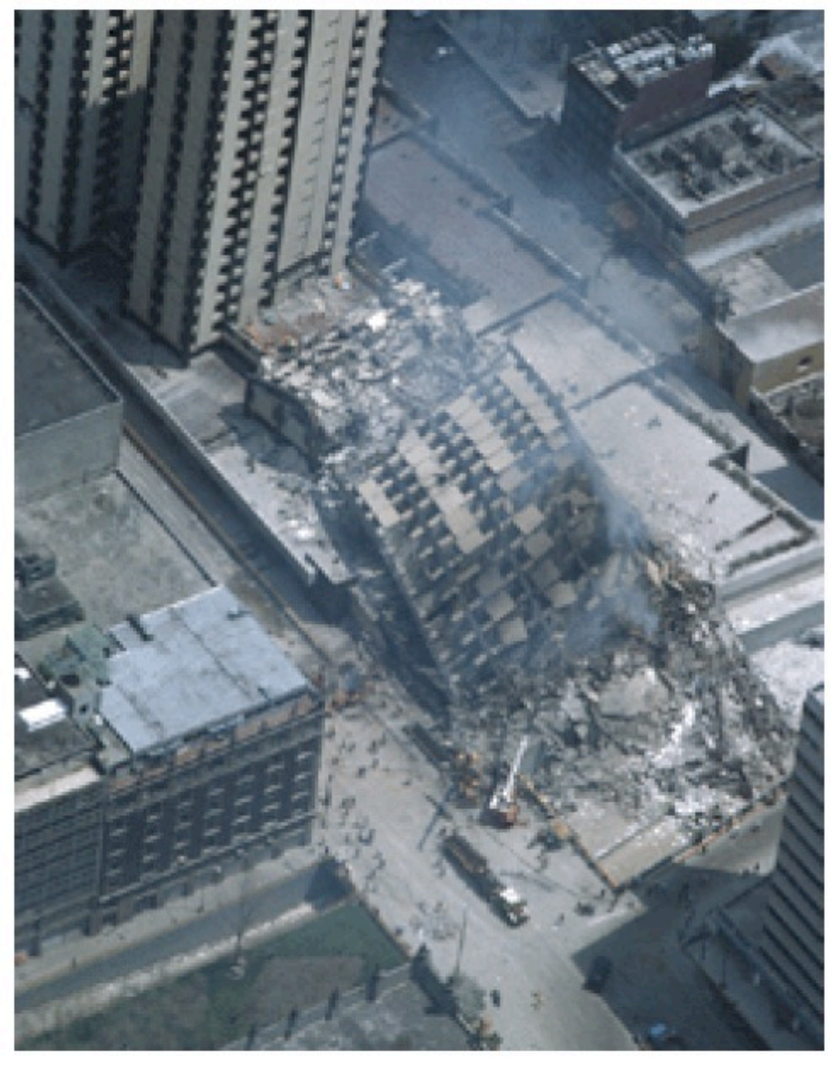
鬆軟土地面有一個自然的振盪頻率：

$f \sim 0.5\ \mathrm{Hz}$

這個頻率與淺層地震波正好接近。

共振現象使得墨西哥市地面振盪振幅大。





而建築物也有自己的自然頻率。

六至十樓中等高度建築正好有同樣自然頻率：

$f_{\text{中樓層}} \sim 0.5\ \text{Hz}$

$$f \approx \sqrt{\frac{k}{m}}$$

$h\downarrow, k\uparrow, f\uparrow$

低樓層的老建築自然頻率較高。

在軟土地面上反而沒有共振。

但很多情況下，外力施力週期會自動調整為彈簧的自然周期。

http://www.youtube.com/watch?v=gQK21572oSU

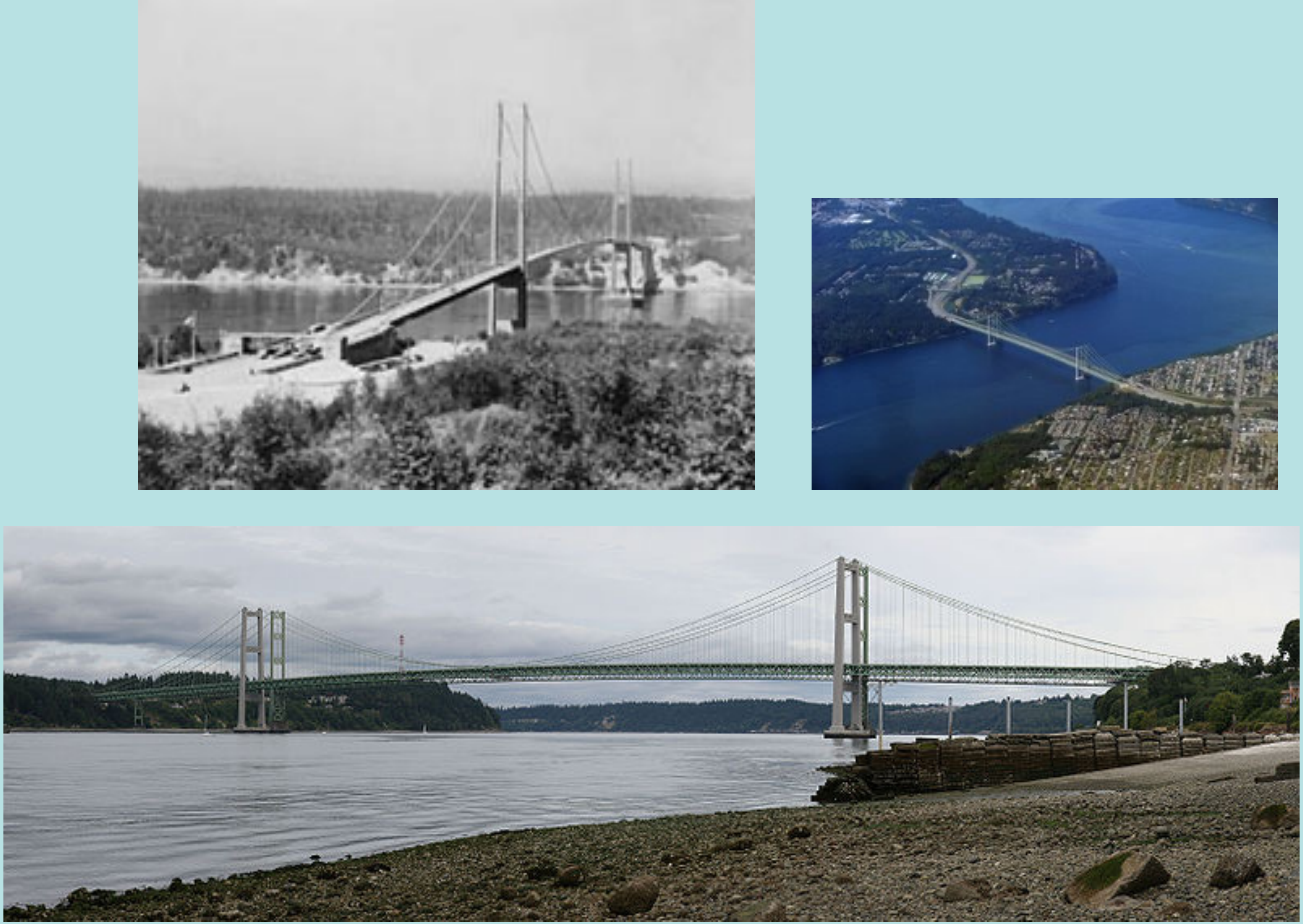
http://www.youtube.com/watch?v=eAXVa__XWZ8

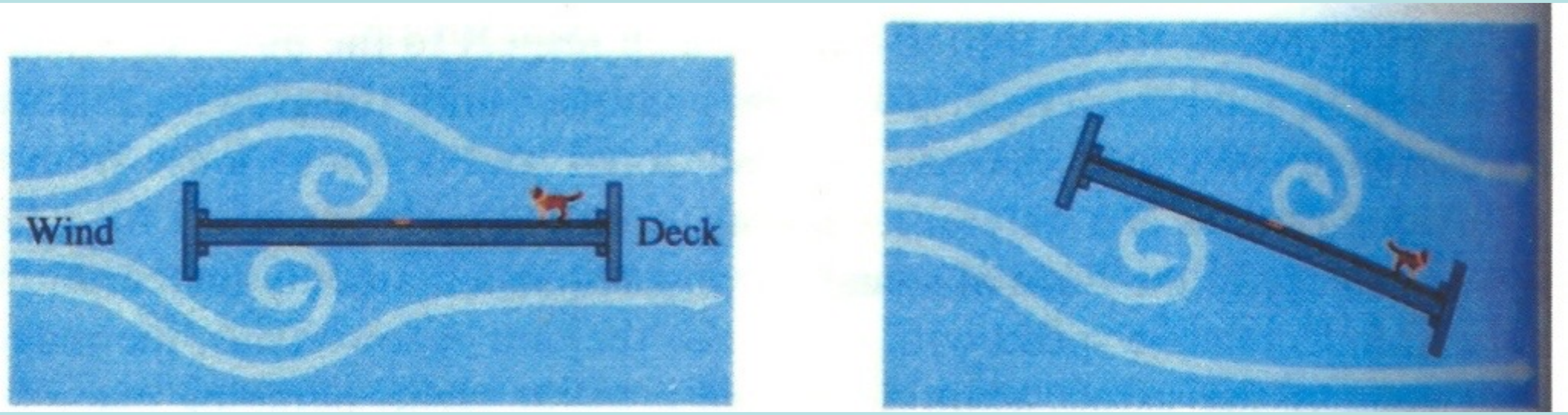

No entry
Deep water


TCR 00:10:32:18
0:00 / 1:01

但很多情況下，外力施力週期會自動調整為彈簧的自然周期。

Tacoma Narrows Bridge





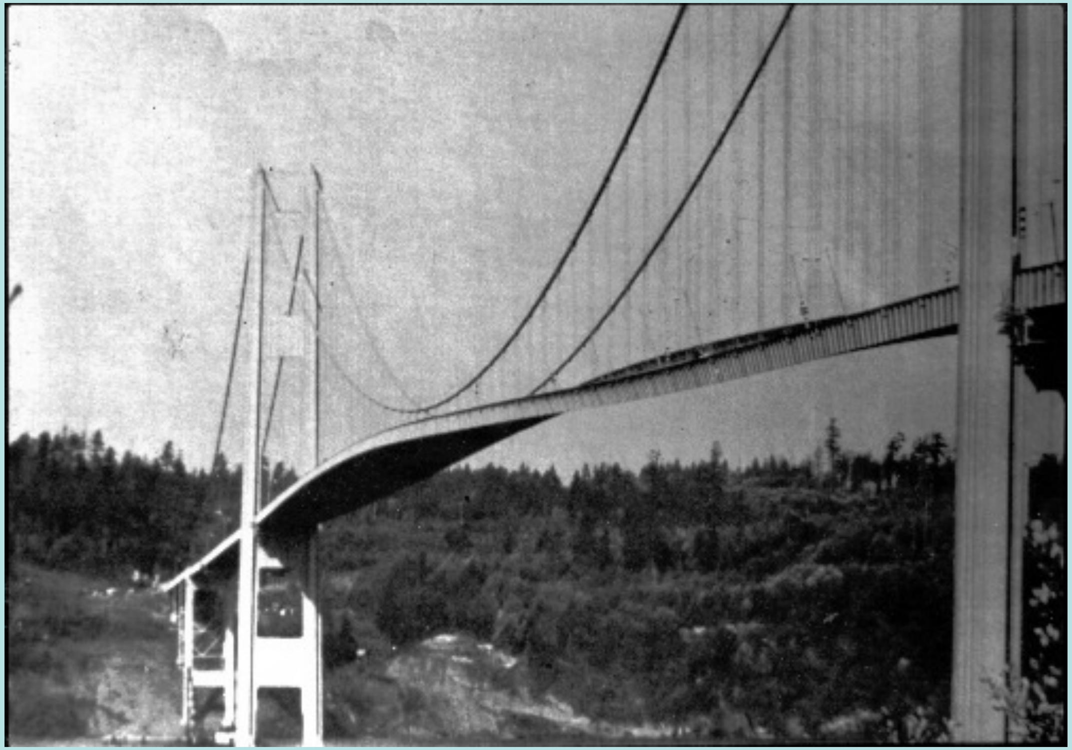
當橋未扭曲振動時，風力的力矩是平衡的，

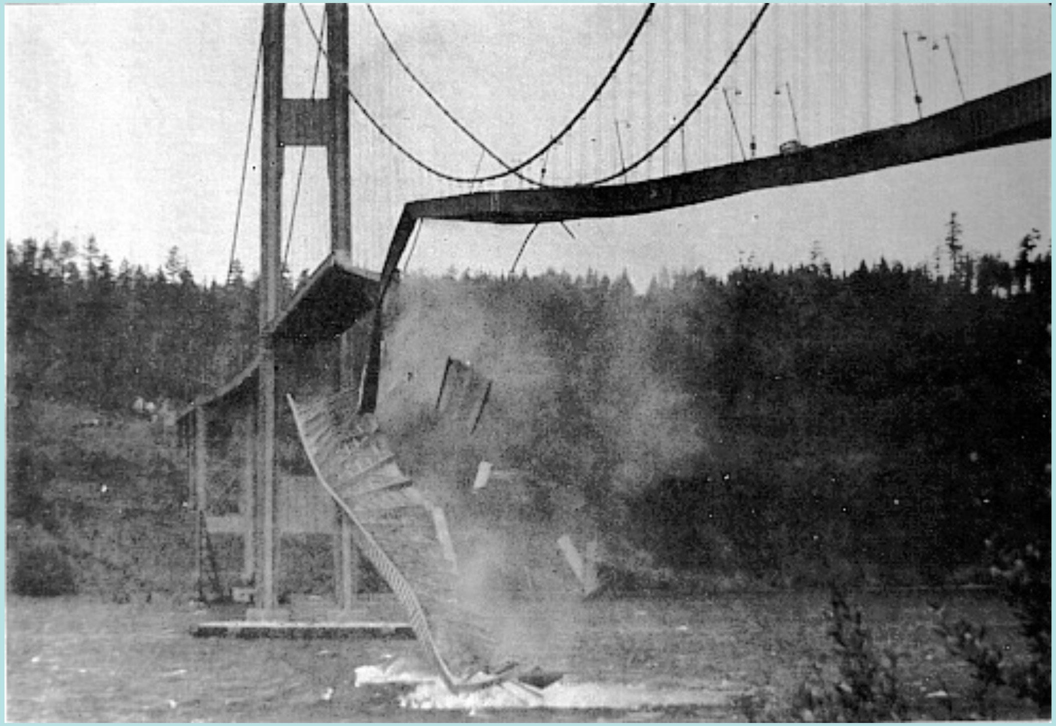
當橋開始些許某一模式的扭曲振動後，橋面傾斜，風力對橋面施予一淨力矩，

關鍵是：風力造成的力矩的頻率，自然與該扭曲振盪模式的頻率相等！

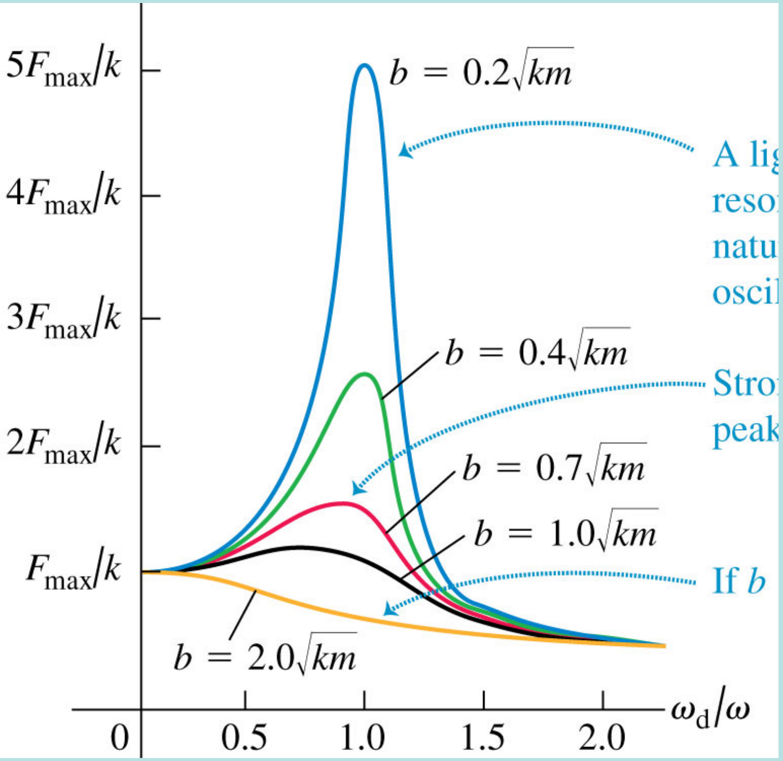
https://www.youtube.com/watch?v=lXyG68_caV4

http://www.youtube.com/watch?v=3mclp9QmCGs





要降低共振的大小，可以增加 阻尼damping *b*。



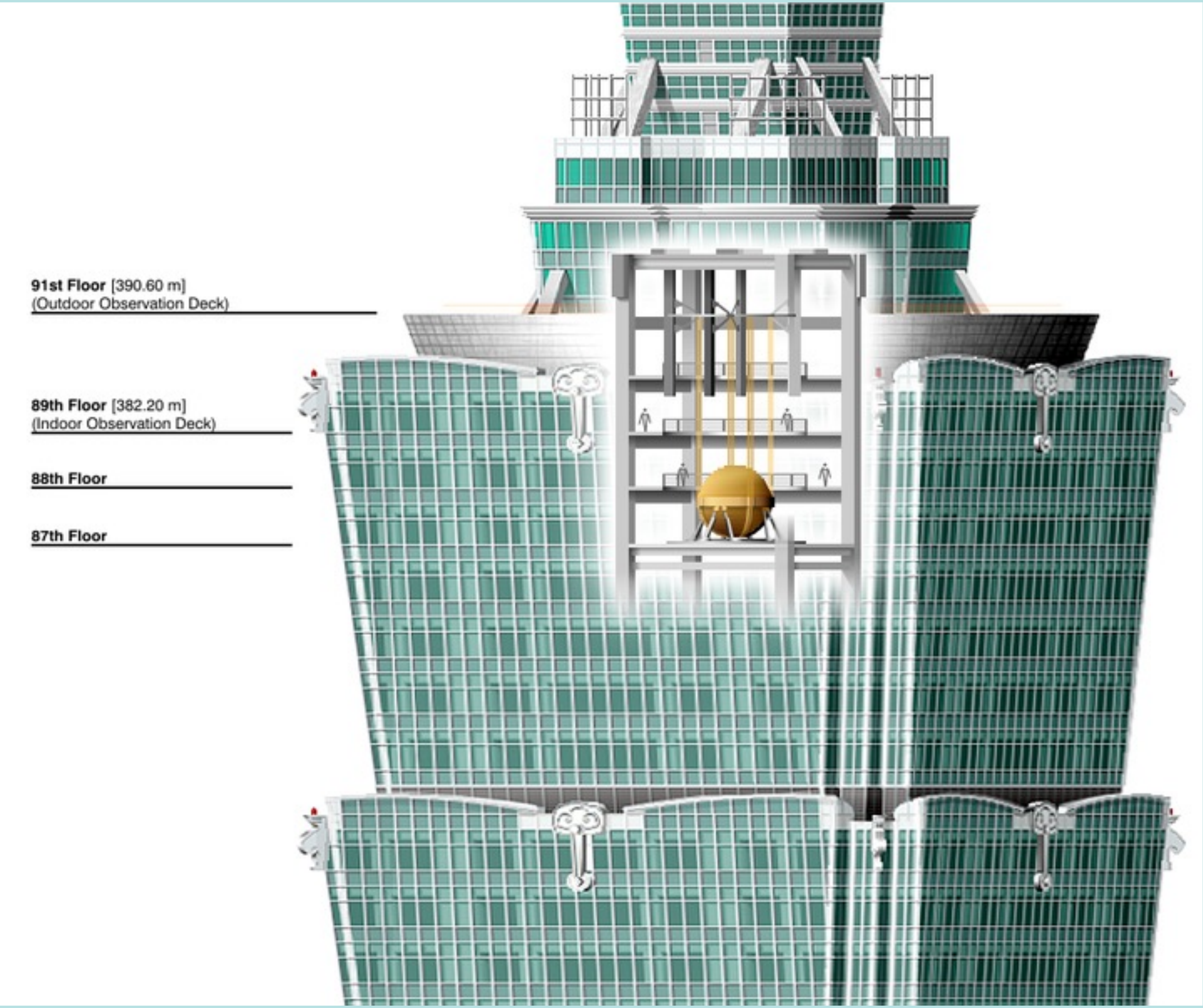
Fluid Damper

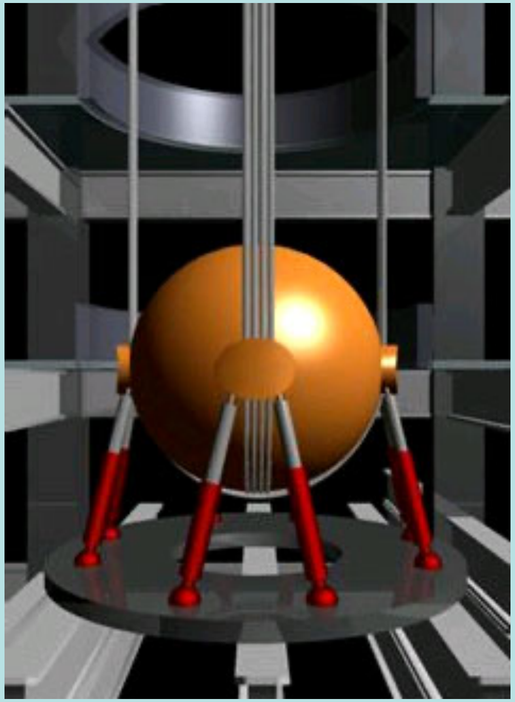
Tune Mass Damper

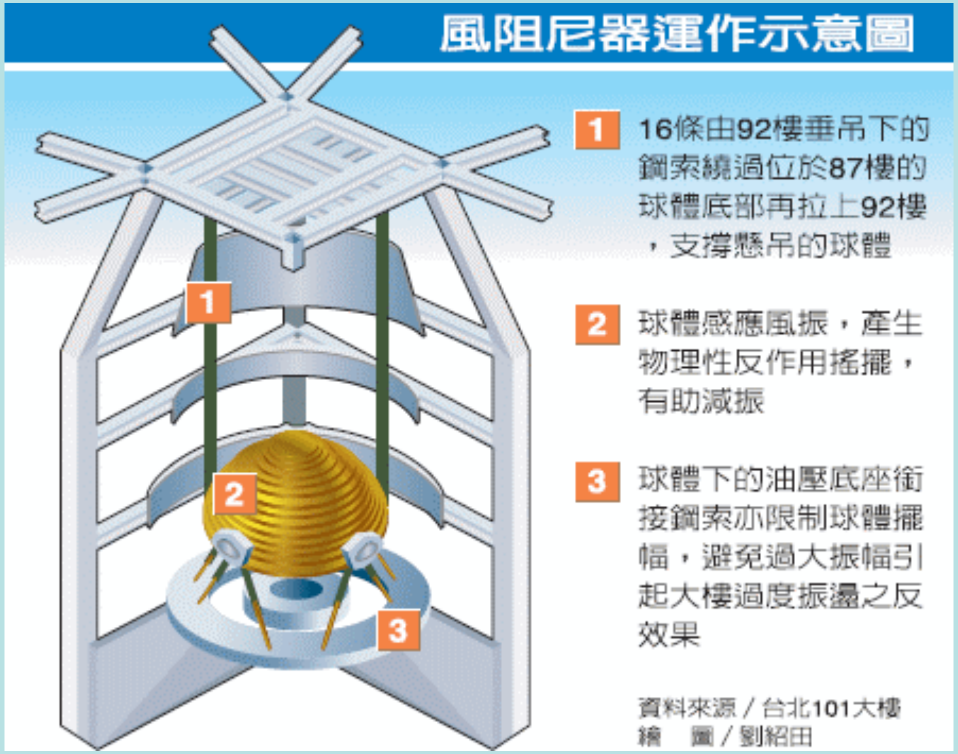
101五十噸的阻尼球

15 Oscillations

If a tall building sways slowly in a wind, the occupants may not even notice the motion, but if the swaying repeats more than 10 times per second, it becomes annoying and may even cause motion sickness. One reason is that when a person is standing, the head tends to sway even more than the feet, setting off motion sensors in the balancing region of the inner ear. Various mechanisms are employed to decrease a building's sway. For example, the large ball (5.4×10^5 kg) seen in this photograph hangs on the 92nd floor of one of the world's tallest buildings.


91st Floor [390.60 m]
(Outdoor Observation Deck)
89th Floor [382.20 m]
(Indoor Observation Deck)
88th Floor
87th Floor





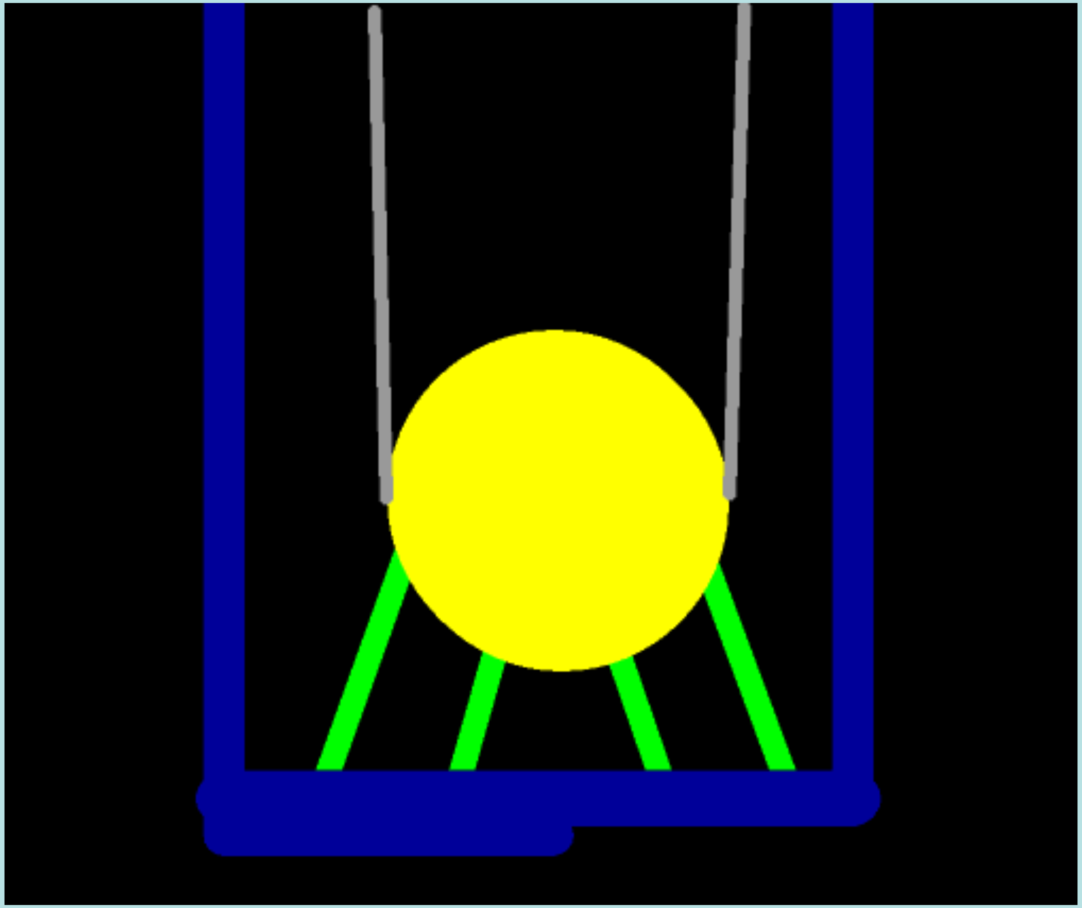
風阻尼器運作示意圖

1 16條由92樓垂吊下的鋼索繞過位於87樓的球體底部再拉上92樓，支撐懸吊的球體

2 球體感應風振，產生物理性反作用搖擺，有助減振

3 球體下的油壓底座銜接鋼索亦限制球體擺幅，避免過大振幅引起大樓過度振盪之反效果

資料來源／台北101大樓
繪　圖／劉紹田



Linear second order ODE 解法

$$y'' + P(x)y' + Q(x)y = f(x)$$

Variation of parameters
Wronskian

級數法

複變函數法

Linear ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

If we find one solution of the homogeneous ODE, we can compute the general solution.

Linear ODE with constant coefficients

$$y'' + a_1 y' + a_0 y = f(x)$$

We can find general solutions using complex number exponential function $e^{\alpha t}$.

$$y'' + P(x)y' + Q(x)y = f(x)$$

Linear ODE

Variation of Parameters
Wronskian

↓

If we find one solution of the homogeneous ODE, we can compute the second solution.

$$y'' + P(x)y' + Q(x)y = 0$$

我們猜測，inhomogeneous 有一個解可以寫成： $y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$

Assume that the solution of the inhomogeneous Eq can be written as above.

這是模仿homogeneous方程式的一般解，將常數係數改寫成函數：

$$C_1y_1(x) + C_2y_2(x)$$

Variation of Parameters

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

它要滿足的方程式：The equation $y'' + P(x)y' + Q(x)y = f(x)$

將$y(x)$微分：differentiate $y(x)$

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

選擇$v_{1,2}$滿足以下條件：Choose $v_{1,2}$ to satisfy the following condition:

$$v_1'y_1 + v_2'y_2 = 0$$

$y'(x)$可簡化：could be simplified:

$$y' = v_1y_1' + v_2y_2'$$

將$y'(x)$再微分： differentiate $y'(x)$ and plug y, y', y'' in the equation:

$$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$$

代入要滿足的方程式： $y'' + P(x)y' + Q(x)y = f(x)$

$$v_1'y_1' + v_2'y_2' + v_1\cancel{[y_1'' + Py_1' + Qy_1]} + v_2\cancel{[y_2'' + Py_2' + Qy_2]} = f(x)$$

$y_{1,2}$滿足homogeneous方程式。

$$v_1'y_1' + v_2'y_2' = f(x)$$

$v_1' y_1 + v_2' y_2 = 0$ $v_1' y_1' + v_2' y_2' = f(x)$

於是我們有兩個條件，兩個未知函數$v_{1,2}'$。正好都能解出來。

We have two equations for two unknown $v_{1,2}'$. Both can be solved.

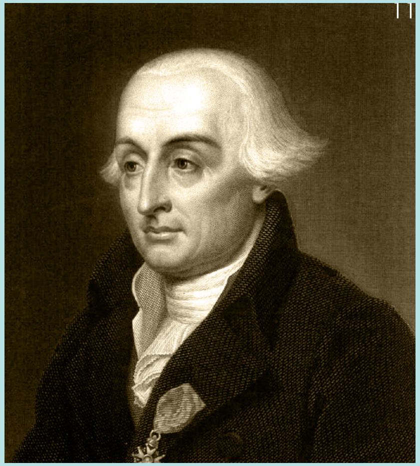
解的時候用的行列式，就是找y_2時用的Wronskian $W(x)$。

$$v_1' = -\frac{y_2 f}{W} \qquad v_2' = \frac{y_1 f}{W} \qquad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

簡單積分：

$$v_1 = -\int dx \frac{y_2 f}{W} \qquad v_2 = \int dx \frac{y_1 f}{W}$$

If we know one solution of the homogeneous ODE, we can compute the general solution of the inhomogeneous ODE.



Joseph-Louis Lagrange 1736-1813

$$y'' + P(x)y' + Q(x)y = f(x)$$

The difference $y - y_2$ between any two solutions of a Linear inhomogeneous ODE equals a solution of the homogeneous ODE.

$$y'' + P(x)y' + Q(x)y = f(x) \quad - \quad y_2'' + P(x)y_2' + Q(x)y_2 = f(x)$$

⬇

$$(y - y_2)'' + P(x)(y - y_2)' + Q(x)(y - y_2) = 0$$ homogeneous ODE

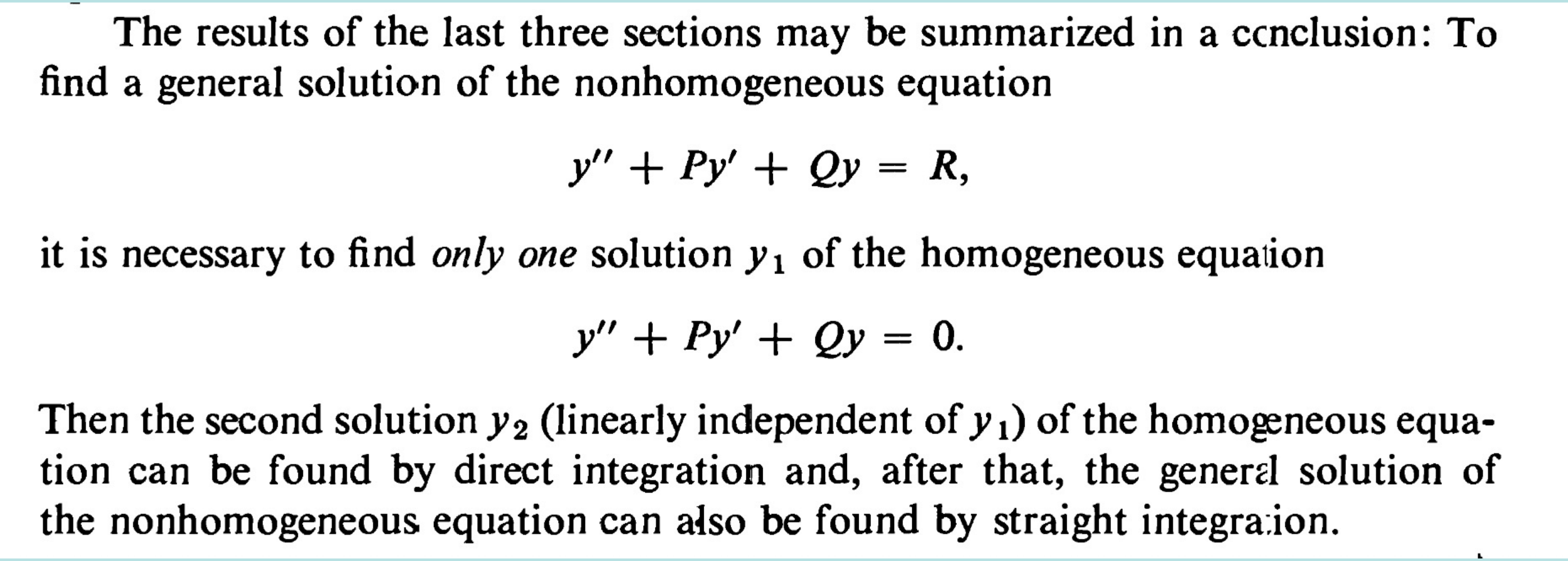
$y - y_2$ equals a solution y_1 of the homogeneous ODE. Hence:

$$y = y_1 + y_2$$

Any solutions y of Linear inhomogeneous ODE equals one solution y_2 of the inhomogeneous ODE plus the solution y_1 of the homogeneous ODE.

Linear ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$



The results of the last three sections may be summarized in a conclusion: To find a general solution of the nonhomogeneous equation

$$y'' + Py' + Qy = R,$$

it is necessary to find *only one* solution y_1 of the homogeneous equation

$$y'' + Py' + Qy = 0.$$

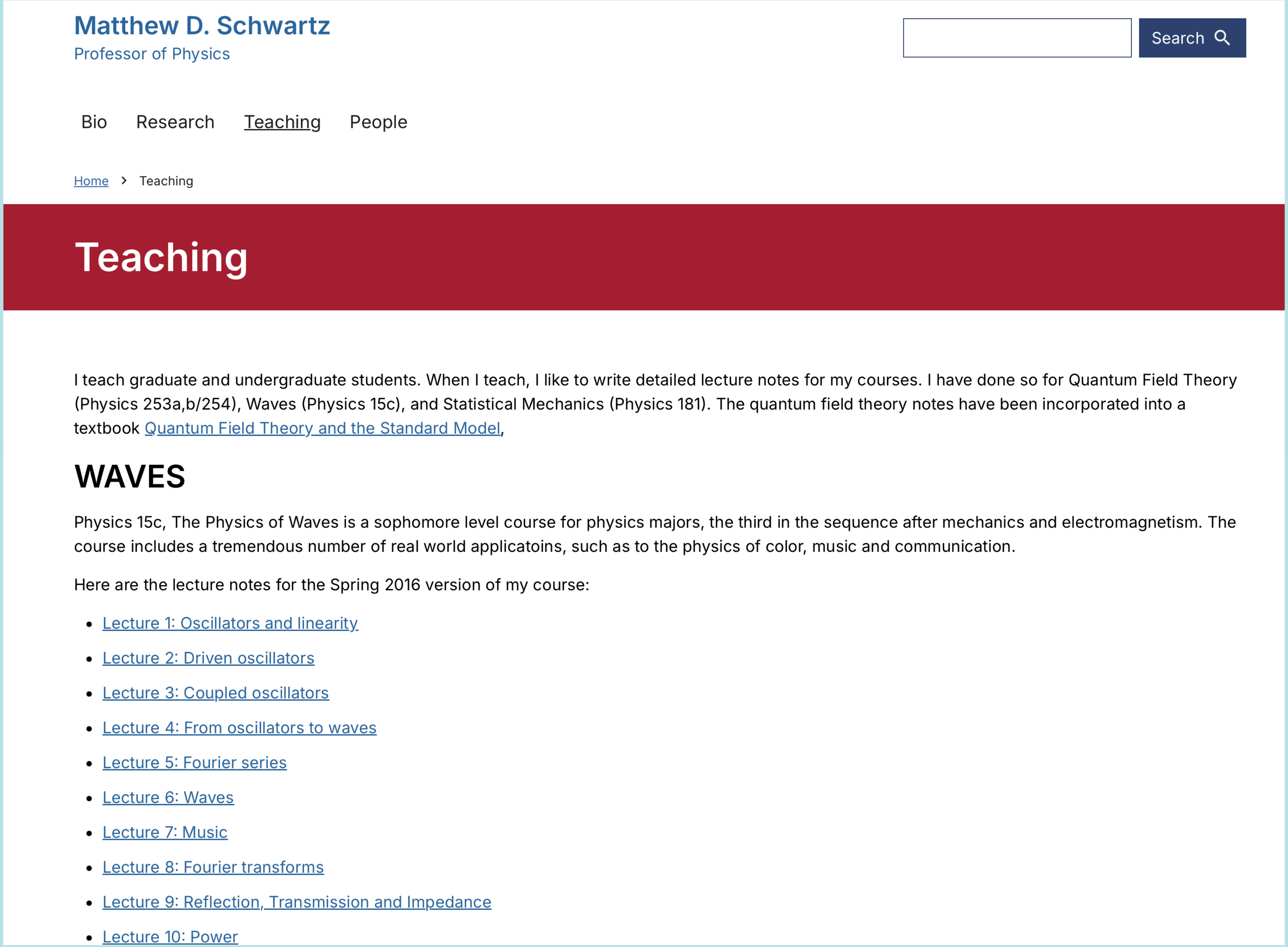
Then the second solution y_2 (linearly independent of y_1) of the homogeneous equation can be found by direct integration and, after that, the general solution of the nonhomogeneous equation can also be found by straight integration.

以複數解簡諧運動的運動方程式

Ordinary Differential Equation with constant coefficients

$$\sum_n a_n \cdot \frac{d^n x}{dt^n} = f(t)$$

Arfken 7.3



Teaching

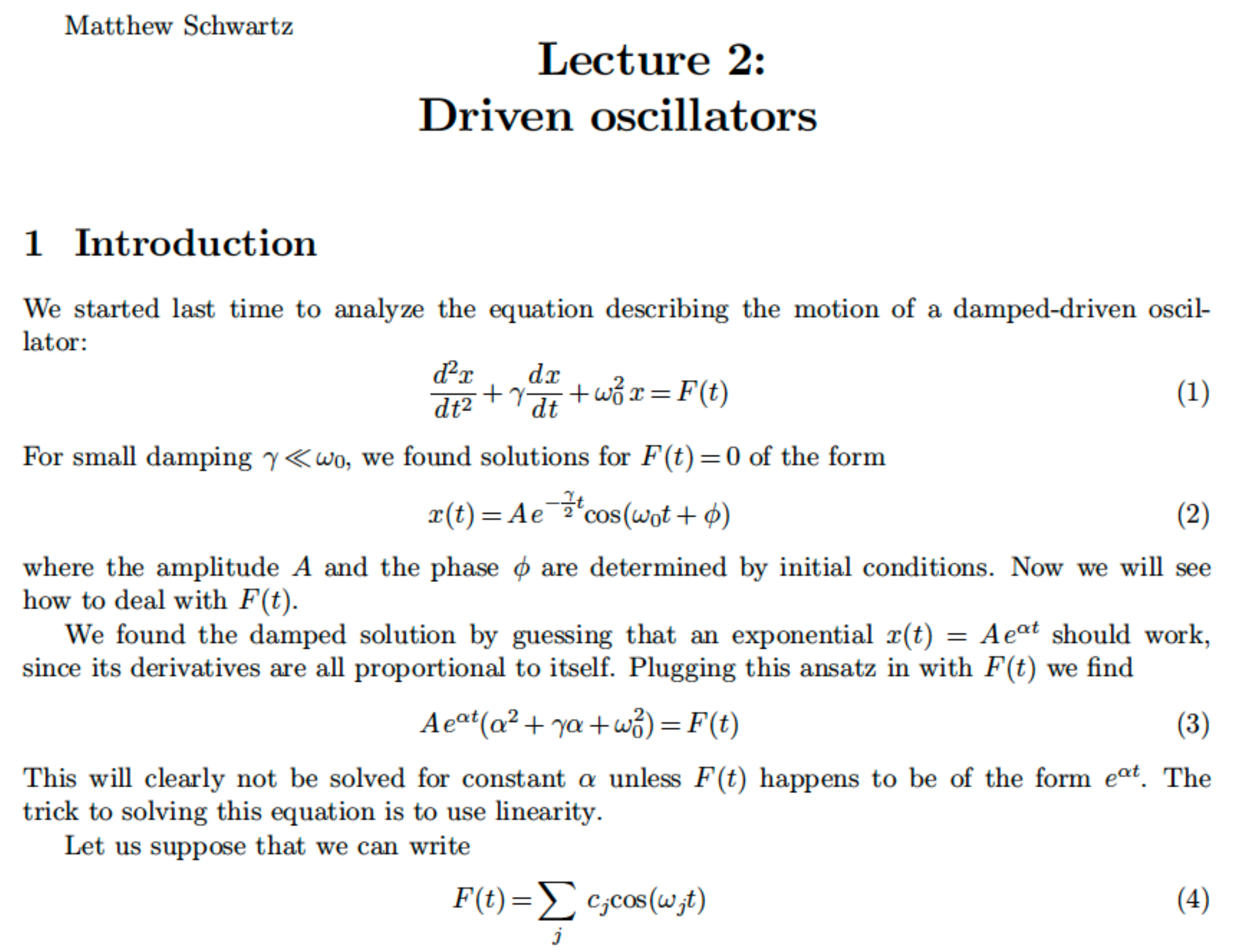
I teach graduate and undergraduate students. When I teach, I like to write detailed lecture notes for my courses. I have done so for Quantum Field Theory (Physics 253a,b/254), Waves (Physics 15c), and Statistical Mechanics (Physics 181). The quantum field theory notes have been incorporated into a textbook Quantum Field Theory and the Standard Model,

WAVES

Physics 15c, The Physics of Waves is a sophomore level course for physics majors, the third in the sequence after mechanics and electromagnetism. The course includes a tremendous number of real world applicatoins, such as to the physics of color, music and communication.

Here are the lecture notes for the Spring 2016 version of my course:

- Lecture 1: Oscillators and linearity
- Lecture 2: Driven oscillators
- Lecture 3: Coupled oscillators
- Lecture 4: From oscillators to waves
- Lecture 5: Fourier series
- Lecture 6: Waves
- Lecture 7: Music
- Lecture 8: Fourier transforms
- Lecture 9: Reflection, Transmission and Impedance
- Lecture 10: Power



Matthew Schwartz

Lecture 2:
Driven oscillators

1 Introduction

We started last time to analyze the equation describing the motion of a damped-driven oscillator:

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=F(t) \tag{1}$$

For small damping $\gamma\ll\omega_0$, we found solutions for $F(t)=0$ of the form

$$x(t)=Ae^{-\frac{\gamma}{2}t}\cos(\omega_0t+\phi) \tag{2}$$

where the amplitude A and the phase ϕ are determined by initial conditions. Now we will see how to deal with $F(t)$.

We found the damped solution by guessing that an exponential $x(t) = Ae^{\alpha t}$ should work, since its derivatives are all proportional to itself. Plugging this ansatz in with $F(t)$ we find

$$Ae^{\alpha t}(\alpha^2+\gamma\alpha+\omega_0^2)=F(t) \tag{3}$$

This will clearly not be solved for constant α unless $F(t)$ happens to be of the form $e^{\alpha t}$. The trick to solving this equation is to use linearity.

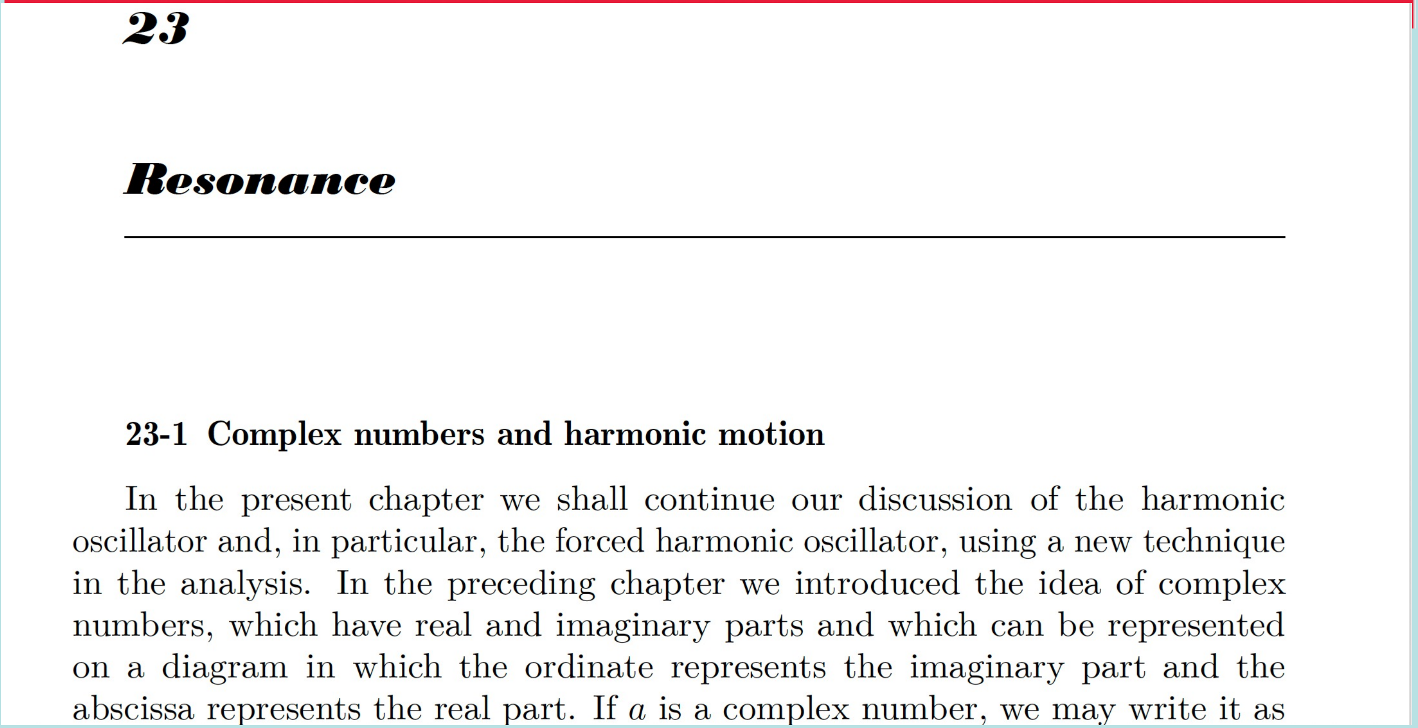
Let us suppose that we can write

$$F(t)=\sum_j c_j\cos(\omega_jt) \tag{4}$$

THE

Feynman

LECTURES ON PHYSICS



23

Resonance

23-1 Complex numbers and harmonic motion

In the present chapter we shall continue our discussion of the harmonic oscillator and, in particular, the forced harmonic oscillator, using a new technique in the analysis. In the preceding chapter we introduced the idea of complex numbers, which have real and imaginary parts and which can be represented on a diagram in which the ordinate represents the imaginary part and the abscissa represents the real part. If a is a complex number, we may write it as

我們討論一般的齊次微分方程式，原子核衰變、SHM與阻尼振盪都是特例！

We discussed the homogeneous ODE with constant coefficients, including decay, SHO etc.

$$\sum_{n=0}^{N} a_n \cdot \frac{d^n x}{dt^n} = 0$$

方程式的解有三角函數，也有指數函數，

The solutions could be Trigonometric Trig functions or Exponential functions.

而阻尼振盪的解是三角函數與指數函數的乘積。

The solutions of damped SHO is even a product of both.

有可能將三角函數與指數函數統合在一起嗎？

To have a generalized approach, is it possible to unify the trig function with exponential function?

The answer is yes.

指數函數與三角函數有許多類似之處：They are similar.

$$\frac{d^2e^{b\theta}}{d\theta^2} = b^2e^{b\theta} \quad \Longleftrightarrow \quad \frac{d^2\sin\theta}{d\theta^2} = -\sin\theta$$

如果只看二次微分，大膽假設：If we assume: $b^2 = -1, b = \sqrt{-1} = i$

正弦函數似乎可以是虛數的指數函數。

It seems Sine function could be the exponential of imaginary variable:

$$\sin\theta \sim e^{i\theta}$$

但此定義，對一次微分不成立： $\sin\theta$的微分不是$\sin\theta$。

But this does not work for first derivative.

$$\frac{de^{i\theta}}{d\theta} = ie^{i\theta} \qquad \frac{d(\sin\theta)}{d\theta} = \cos\theta$$

第一式顯示：$e^{i\theta}$必須是複數，有實數部及虛數部！

The first formula indicates that $e^{i\theta}$ must be complex with a real and a imaginary part.

$$\frac{de^{i\theta}}{d\theta} = ie^{i\theta}$$

$$i(a+ib) = -b+ia$$

$e^{i\theta}$ 的一次微分是自己乘上 i ，也就是將實數部及虛數部互換。

The derivative of $e^{i\theta}$ is $e^{i\theta}$ multiplied by i, exchanging the real and imaginary part.

三角函數的一次微分將 cos 與 sin 互換

The derivative of Trig functions will exchange Cosine and Sine.

$$\frac{d(\cos\theta)}{dt} = -\sin\theta$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta$$

何不就假設 $e^{i\theta}$ 的實數部與虛數部分別是正弦與餘弦？

With this correspondence, why don’t we just assume that:

the real and imaginary parts of $e^{i\theta}$ are Cosine and Sine respectively.

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

Euler's Formula　虛數的指數函數，如果這樣定義：

Carl Friedrich Gauss (1777–1855)

$$\frac{d}{d\theta}e^{i\theta} = -\sin\theta + i\cos\theta = ie^{i\theta}$$

$$\frac{d^2}{d\theta^2}e^{i\theta} = -\cos\theta - i\sin\theta = -e^{i\theta} = i^2e^{i\theta}$$

正好是我們期待指數函數必須滿足的微分關係。

$$\frac{d^n}{dx^n}e^{i\alpha x} = (i\alpha)^n \cdot e^{i\alpha x}$$

$$e^{i\alpha} \cdot e^{i\beta} = (\cos\alpha + i\sin\alpha) \cdot (\cos\beta + i\sin\beta) =$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) =$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$$

正好是我們期待指數函數必須滿足的乘積關係。

此定義滿足指數函數所有重要性質！

另一個推導：

三角函數的泰勒展開：

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots$$

指數函數的泰勒展開：

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

要求指數函數的泰勒展開對虛變數還是對的：

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \cdots$$

$$= 1 - \frac{1}{2!}\theta^2 + + \frac{1}{4!}\theta^4 + \cdots + i\left(\theta - \frac{1}{3!}\theta^3 + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

我們可以更進一步定義複數$\alpha = a + i\theta$的指數函數：

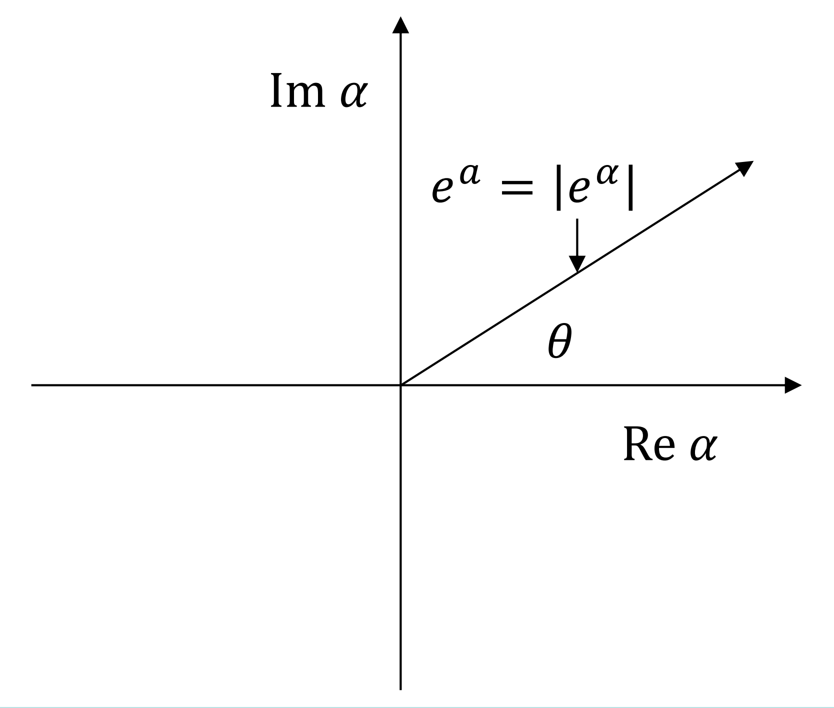
$$e^{\alpha} = e^{a+i\theta} = e^{a}e^{i\theta} = e^{a}(\cos\theta + i\sin\theta)$$

$$\left|e^{i\theta}\right| = 1$$

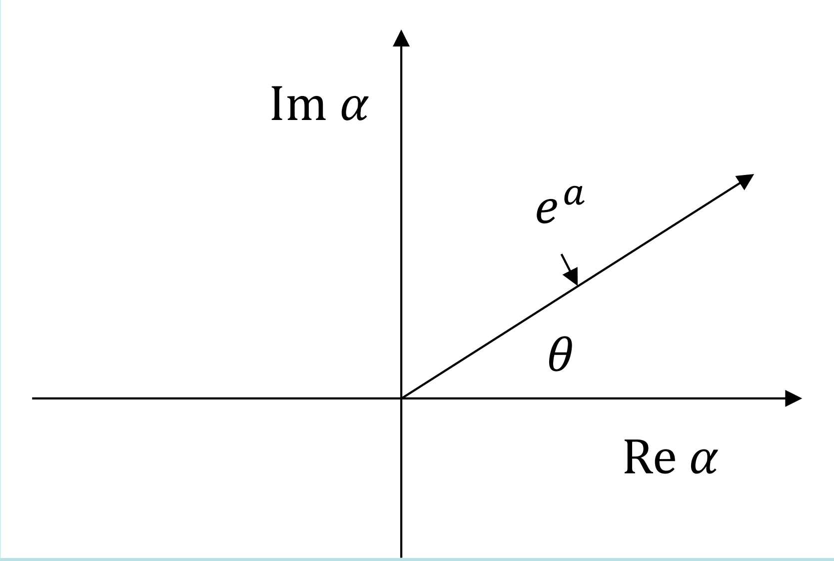
e^{α}在複數平面上表示，e^{a}就是絕對值$|e^{\alpha}|$，θ就是幅角

$$\left|e^{a+i\theta}\right| = e^{a}$$

Any complex number z can be expressed this way: $z = |z| \cdot e^{i\theta}$



$$|mn| = |m| \cdot |n|$$



$$\frac{d}{dx}e^{\alpha x} = \frac{d}{dx}\left(e^{ax}e^{i\theta x}\right)$$

$$= (ae^{ax})e^{i\theta x} + e^{ax}\left(i\theta e^{i\theta x}\right)$$

$$= (a + i\theta)e^{a+i\theta} = \alpha e^{\alpha x}$$

The derivative of a complex exponential equals the coefficient times the exponential.

可以證明： $$\frac{d^n}{dx^n}e^{\alpha x} = (\alpha)^n \cdot e^{\alpha x}$$

所以，複數的指數函數，所有的微分都與自己成正比！

The derivative of nth order equals the coefficient to the nth power times the exponential.

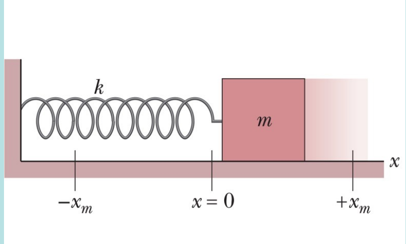
簡諧振盪器，以複數方法求解 SHO by complex number method：

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

首先將這個式子裡的 x 推廣為一個複數 z。

First elevate the real function x into a complex number function $z(t)$。

$$\frac{d^2z}{dt^2}+\omega^2z=0$$



複數 z 包含實數部Re z與虛數部Im z。

Complex z consist of real part Re z and imaginary part Im z。

$$\frac{d^2(\mathrm{Re}\,z+i\mathrm{Im\ z})}{dt^2}+\omega^2(\mathrm{Re}\,z+i\mathrm{Im\ z})=0+i0$$

因為方程式是線性的，z的實數部與虛數部也同時滿足原來x滿足的方程式！

Since the equation is linear, both the real and imaginary part satisfy the Eq.

$$\frac{d^2(\mathrm{Im}\,z)}{dt^2}+\omega^2(\mathrm{Im}\,z)=0 \qquad \frac{d^2(\mathrm{Re}\,z)}{dt^2}+\omega^2(\mathrm{Re}\,z)=0$$

如果能解出複數z，再取其實數部或虛數部，即可得到原方程式的實數解 x。

If we solve complex z, Re z and Im z would be solutions x of the real Eq.

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$

我們大膽地猜，解正比於一個複數指數函數：

Guess: $z = z_0 e^{\alpha t}$

$$\alpha^2 z + \omega^2 z = 0$$

上式所有項都正比於z：

Both terms are proportional to z.

微分的次數對應α的冪次。

$$\frac{d^n}{dt^n} z = \alpha^n z$$

Orders of derivatives corresponds to powers of α.

$$\alpha^2 + \omega^2 = 0$$

原來的微分方程式現在一項對一項地轉化為未知的α滿足之代數方程式。

The original differential Equation can be transformed into an Algebraic Equation.

α 可被解出： α can solved

$$\alpha_\pm = \pm i\omega$$

解出複數解z：The complex solutions are:

$$z_\pm = z_0 e^{\pm i\omega t}$$

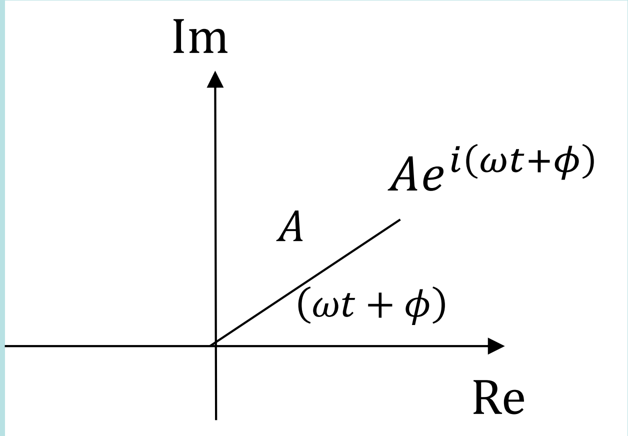
z_0是一個複數常數，因此可以寫成：

The constant z_0 can be written by its Absolute value A and argument ϕ:

$z_0 = Ae^{i\phi}$ A, ϕ是兩個實數常數。

$$z_+ = Ae^{i(\omega t+\phi)} = A\cos(\omega t+\phi) + iA\sin(\omega t+\phi)$$

這就是簡諧運動的複數解！The is the complex solution.

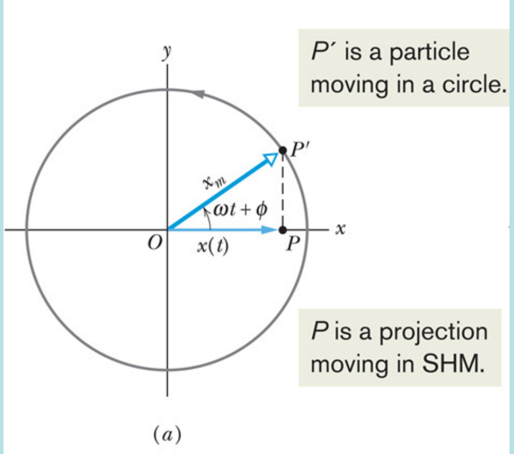


假想圓就是此複數解$Ae^{i(\omega t+\phi)}$在其複數平面上的表現！

取其實數部，就得到簡諧運動的實數解！

Pick the real part of z_+, we arrive at real solutions x.

$$A\cos(\omega t+\phi) = \mathrm{Re}\left[Ae^{i(\omega t+\phi)}\right]$$



P′ is a particle moving in a circle.

P is a projection moving in SHM.

這個解有兩個未定常數，因此就是最普遍的解了。

The general solution contains two unknown constants A, ϕ just to fit two initial conditions.

The final function will be the unique solution to both satisfy the Eq. and initial Conditions.

那z_+的虛數部及z_-呢？How about the imaginary part of z_+ and z_-?

$$\mathrm{Im}\left[Ae^{i(\omega t+\phi)}\right] = A\sin(\omega t+\phi) = A\cos\left(\omega t+\phi-\frac{\pi}{2}\right)$$

z_+的虛數部及z_+的實數部等價。

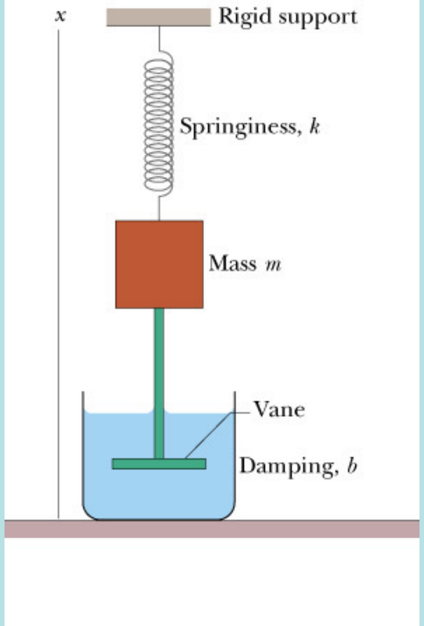
The imaginary part of z_+ is equivalent to its real part.

同理z_-的實數部及虛數部也與z_+的實數部等價。

The the real and imaginary part of z_- is equivalent to the real part of z_+.

$$A\cos(-\omega t+\phi) = A\cos(\omega t-\phi)$$

$$A\sin(-\omega t+\phi) = -A\sin(\omega t-\phi) = -A\cos\left(\omega t-\phi-\frac{\pi}{2}\right)$$



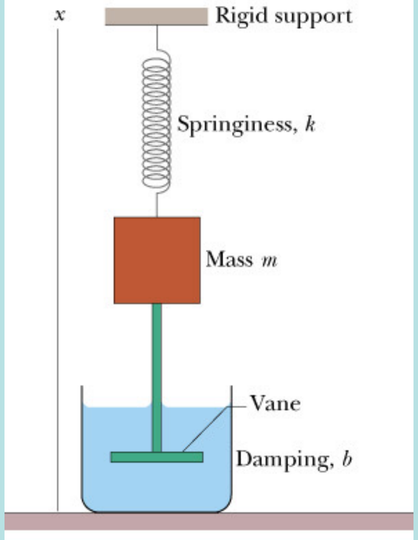
阻尼簡諧運動，也是複數指數函數的實數部！

$$x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi) = x_m e^{-\frac{b}{2m}t} \cdot \mathrm{Re}\left[e^{i(\omega' t+\phi)}\right] = \mathrm{Re}\left[x_m e^{-\frac{b}{2m}t+i(\omega t+\phi)}\right]$$

是不是所有線性微分方程式的解都是複數指數函數？

有阻力的簡諧振盪器，以複數方法求解：

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=0$$



首先將這個式子裡的 x 推廣為一個複數 z。

First elevate the real x into a complex $z(t)$。

$$\frac{d^2z}{dt^2}+\frac{b}{m}\frac{dz}{dt}+\omega^2z=0$$

注意複數 z 包含實數部Re z與虛數部Im z。

$$\frac{d^2(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})}{dt^2}+\frac{b}{m}\frac{d(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})}{dt}+\omega^2(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})=0+i0$$

因為方程式是線性的，z的實數部與虛數部也同時滿足原來x滿足的方程式！

Since the equation is linear, both the real and imaginary part satisfy the Eq.

如果能解出複數z，再取其實數部或虛數部，即可得到原方程式的實數解 x。

If we solve complex z, Re z and Im z would be solutions x of the real Eq.

$$\frac{d^2(\mathrm{Im}\,z)}{dt^2}+\frac{b}{m}\frac{d(\mathrm{Im}\,z)}{dt}+\omega^2(\mathrm{Im}\,z)=0$$

$$\frac{d^2(\mathrm{Re}\,z)}{dt^2}+\frac{b}{m}\frac{d(\mathrm{Re}\,z)}{dt}+\omega^2(\mathrm{Re}\,z)=0$$

$$\frac{d^2z}{dt^2} + \frac{b}{m}\frac{dz}{dt} + \omega^2 z = 0$$

如果我們大膽地猜，解正比於一個複數指數函數：

Guess: $z = z_0 e^{\alpha t}$

$$\alpha^2 z + \frac{b}{m}\alpha z + \omega^2 z = 0$$

上式所有項都正比於z：

all terms are proportional to z.

$$\frac{d^n}{dt^n} z = \alpha^n z$$

微分的次數對應α的冪次。

Orders of derivatives corresponds to powers of α.

$$\alpha^2 + \frac{b}{m}\alpha + \omega^2 = 0$$

This is called Characteristic Equation.

原來的微分方程式現在一項對一項地轉化為未知的α滿足之代數方程式。

The original differential Equation is transformed into an Algebraic Equation of α.

α 可被解出： α can solved

These are called Eigenvalues of the SHO.

$$\alpha_\pm = -\frac{b}{2m} \pm i\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} \equiv -\frac{b}{2m} \pm i\omega'$$

$$\omega' \equiv \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

若阻力不大 $\omega > \frac{b}{2m}$ 通常我們得到兩個複數解 $\alpha_{\pm} = -\frac{b}{2m} \pm i\omega'$

The two complex solutions are:

$$z_{\pm} \equiv z_0 e^{\alpha_{\pm} t} = z_0 e^{-\frac{b}{2m}t} \cdot e^{\pm i\omega' t}$$

The constant z_0 can be written in terms of Absolute value A and argument ϕ: $z_0 = Ae^{i\phi}$

$$= Ae^{i\phi} \cdot e^{-\frac{b}{2m}t} e^{\pm i\omega' t} = Ae^{-\frac{b}{2m}t} e^{\pm i(\omega' t \mp \phi)}$$

$$= Ae^{-\frac{b}{2m}t} \cdot [\cos(\omega' t \mp \phi) \pm i\sin(\omega' t \mp \phi)]$$

取複數解z_+的實數部，即可得到原方程式的實數解 x。

Pick the real part of the complex solution z_+, we arrive at real solutions x of the Eq.

$$x = \mathrm{Re}[z_+] = A \cdot e^{-\frac{b}{2m}t} [\cos(\omega' t + \phi)]$$

此一般解有兩個尚未決定的常數：A, ϕ。而我們正好有兩個起始條件決定A, ϕ，

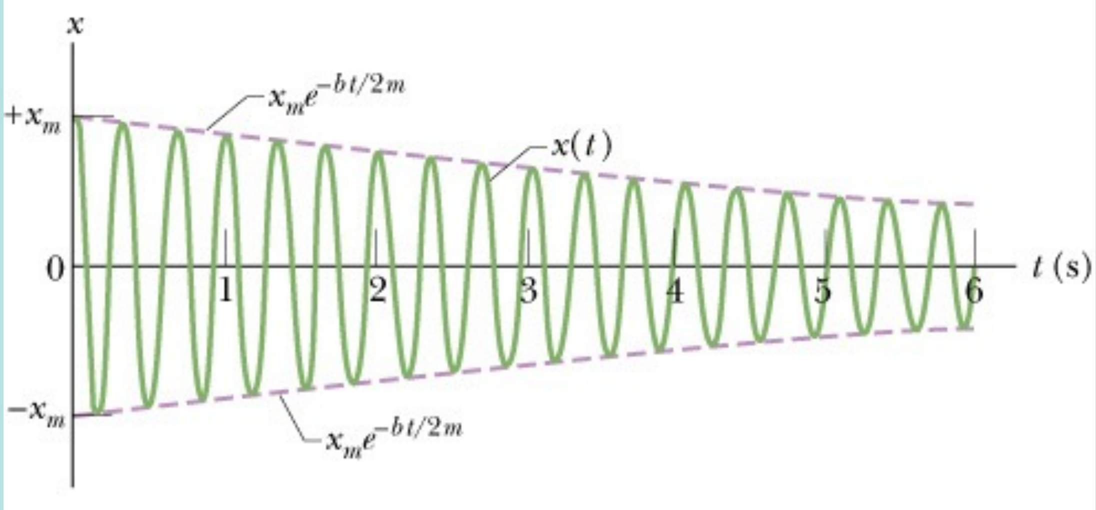
The general solution contains two unknown constants A, ϕ just to fit two initial conditions.

所得到就是方程式的唯一解。

The final function will be the unique solution to both satisfy the Eq. and initial Conditions.

$$x = x_m \cdot e^{-\frac{b}{2m}t}\,[\cos(\omega' t + \phi)]$$

$$\omega' = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$



振幅呈現指數衰減！

振動頻率減小！

若阻力很大 $\omega < \dfrac{b}{2m}$　α得到兩個實數解：α has two real solutions:

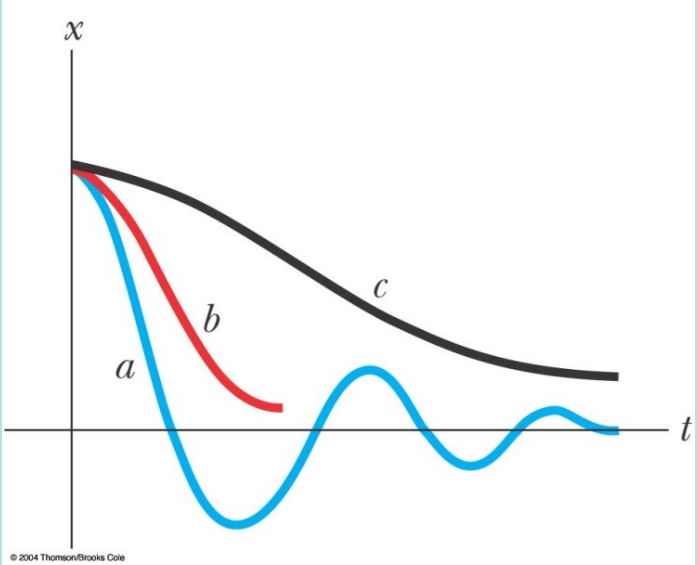
$$\alpha_{\pm} = -\frac{b}{2m} \pm i\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} \Rightarrow \alpha_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}$$

$\alpha_{\pm}$都是負的，兩個解都是隨時間指數遞減，而沒有振盪！

Both solutions $\alpha_{\pm}$ are negative, the solution $z_{1,2}$ of Eq. is exponentially decaying.

$$z_1 \equiv e^{-\left[\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}\right]t}$$

$$z_2 \equiv e^{-\left[\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}\right]t}$$



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There is no oscillation in this case.

如果 $\dfrac{b}{2m}$大於ω，那就根本沒有振動了！阻尼可以大到連一次震盪都未完成！

若阻力恰好 $\omega = \frac{b}{2m}$ α得到兩個相同實數解：

$$\alpha_\pm = -\frac{b}{2m}$$

$$\frac{d^2z}{dt^2} + \frac{b}{m}\frac{dz}{dt} + \omega^2 z = 0$$

我們只得到一組解：

$$z_1 = e^{-\frac{b}{2m}t}$$

由 y_1我們計算出另一個線性獨立的函數$y_2(x)$，也滿足此方程式。

$$y_2 \sim y_1 \int dx \frac{\exp\left(-\int_0^x P(x')\cdot dx'\right)}{y_1^2}$$

$$z_2 \sim e^{-\frac{b}{2m}t} \int dt \frac{\exp\left(-\int \frac{b}{m}\cdot dt\right)}{e^{-\frac{b}{m}t}} = e^{-\frac{b}{2m}t} \int dt \frac{e^{-\frac{b}{m}t}}{e^{-\frac{b}{m}t}} = te^{-\frac{b}{2m}t}$$

$$z_1 = e^{-\frac{b}{2m}t}$$ $$z_2 = te^{-\frac{b}{2m}t}$$ 你可以代入確認。

以上的解法很容易地就可以推廣到任意的齊次微分方程式

$$\sum_{n=0}^{N} a_n \cdot \frac{d^n x}{dt^n} = 0$$

The above can be generalized to ODE of arbitrary order N.

先推廣為複數 elevated to complex number.

$$\sum_{n=0}^{N} a_n \cdot \frac{d^n z}{dt^n} = 0$$

$$\frac{d^2 x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 z = 0$$

猜解並代入 guess and plug in $z = z_0 e^{\alpha t}$

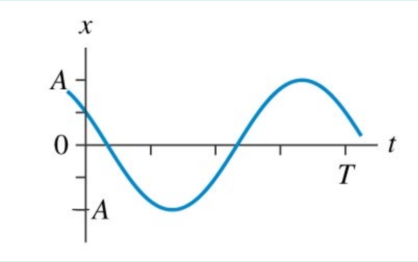
$$\sum_{n=0}^{N} a_n \cdot \alpha^n z = 0 \quad \Rightarrow \quad \sum_{n=0}^{N} a_n \cdot \alpha^n = 0$$

$$\alpha^2 + \frac{b}{m}\alpha + \omega^2 = 0$$

微分方程式被轉化為未知數α的代數方程式：ODE into Algebraic Eq.

代數方程式中的有N個解$\alpha_{1,2,3\cdots N}$： Algebraic Eq of order N has N solutions.

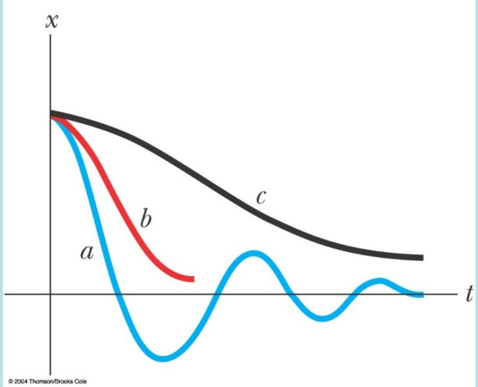
$z = z_0 e^{\alpha t}$



If α is imaginary, $z = z_0 e^{i\omega t}$

$$= Ae^{i(\omega t+\phi)} = A\cos(\omega t + \phi) + iA\sin(\omega t + \phi)$$

$\mathrm{Re}[z] = A\cos(\omega t + \phi)$ The solution is oscillating SHM.



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If α is real, $z = z_0 e^{\pm bt}$

$\mathrm{Re}[z] = Ae^{\pm bt}$ The solution is exponentially decreasing or increasing.

If α is complex, $z = z_0 e^{\pm bt + i\omega t} = Ae^{\pm bt} \cdot e^{i(\omega t+\phi)}$

$\mathrm{Re}[z] = Ae^{\pm bt} \cdot \cos(\omega t + \phi)$

The solution is a oscillation with exponentially decreasing amplitude.

General solutions are linear combinations $z = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} + \cdots c_N e^{\alpha_N t}$

最後取實數部即可得實數解 Pick the real part $x = \mathrm{Re}\, z$

有N個解就有N個未知數，因此就需要N個起始條件，fit initial conditions.

一般就是起始值及起始N次以下微分。Initial condition contains initial value and initial derivatives of order smaller than N.

周期外力下，有阻力的簡諧振盪器，也可以複數方法求解。

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=\frac{F_0}{m}\cos\omega_D t$$

非齊次微分方程式

將它先推廣為複數的微分方程式，注意右手邊的技巧：

Elevate the real x into a complex $z(t)$ with $e^{i\omega_D t}$ replacing $\cos\omega_D t$.

$$\frac{d^2z}{dt^2}+\frac{b}{m}\frac{dz}{dt}+\omega^2z=\frac{F_0}{m}e^{i\omega_D t}$$

$$e^{i\omega_D t}=\cos\omega_D t+i\sin\omega_D t$$

取此複數的微分方程式的實數部，果然$\mathrm{Re}(z)$滿足原來的方程式：

Take the real part of the whole equation and we recover the original ODE.

$$\frac{d^2\mathrm{Re}(z)}{dt^2}+\frac{b}{m}\frac{d\mathrm{Re}(z)}{dt}+\omega^2\mathrm{Re}(z)=\frac{F_0}{m}\mathrm{Re}\left(e^{i\omega_D t}\right)=\frac{F_0}{m}\cos\omega_D t$$

with solutions which are the real part of z.

$$x=\mathrm{Re}(z)$$

$$\frac{d^2z}{dt^2} + \frac{b}{m}\frac{dz}{dt} + \omega^2 z = \frac{F_0}{m}e^{i\omega_D t}$$

Again, postulate the solution is an exponential function of complex variable.

我們可以依舊猜解為 $z = z_0 e^{\alpha t}$ 代入上式

$$\left(\alpha^2 + \frac{b}{m}\alpha + \omega^2\right) z_0 e^{\alpha t} = \frac{F_0}{m}e^{i\omega_D t}$$

未知數α只有一種可能 $\alpha = i\omega_D$ 此解與外力以同樣頻率震盪！

α could only have one possibility.

z_0也只有一種可能 $\left(-\omega_D^2 + \frac{ib}{m}\omega_D + \omega^2\right) z_0 e^{-i\omega_D t} = \frac{F_0}{m}e^{-i\omega_D t}$

z_0 could only have one possibility, too.

$$z_0 = \frac{F_0}{m}\frac{1}{\left(\omega^2 - \omega_D^2 + \frac{ib}{m}\omega_D\right)}$$

於是我得到一個特別解。I get one particular solution!

以絕對值A及幅角ϕ表示z_0最為方便:

$$z_0 = \frac{F_0}{m}\frac{1}{\left(\omega^2 - \omega_D^2 + \frac{ib}{m}\omega_D\right)} = \frac{F_0}{m}\frac{(\omega^2 - \omega_D^2) - \frac{ib}{m}\omega_D}{(\omega^2 - \omega_D^2)^2 + \left(\frac{b\omega_D}{m}\right)^2}$$

$$\equiv Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$

$$A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

$$\tan\phi = -\frac{\frac{b\omega_D}{m}}{(\omega^2 - \omega_D^2)}$$

This particular solution can be written as:

$$z = z_0 e^{-i\omega_D t} = Ae^{-i(\omega_D t + \phi)}$$

Please note that A, ϕ are both fixed.

其實數部即原方程式實數解 Its real part is a real solution.

$$\mathrm{Re}\, z = \mathrm{Re}\, Ae^{-i(\omega_D t + \phi)} = A\cos(\omega_D t + \phi) \equiv x_r$$

這個就稱為共振解，並沒有任何自由度可以附合起始條件。

This is the so-called resonance solution, with no freedom to fit initial conditions..

Resonance solution x_r is one solution of inhomogeneous SHM.

It could not always fit the initial conditions.

The difference $x - x_r$ between any solutions of a Linear inhomogeneous ODE

and this particular one x_r equals a solution x_s of the homogeneous ODE.

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t \quad - \quad \frac{d^2x_r}{dt^2} + \frac{b}{m}\frac{dx_r}{dt} + \omega^2 x_r = \frac{F_0}{m}\cos\omega_D t$$

$$\Downarrow$$

$$\frac{d^2(x - x_r)}{dt^2} + \frac{b}{m}\frac{d(x - x_r)}{dt} + \omega^2(x - x_r) = 0 \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

$(x - x_r)$ equals a solution x_s of the homogeneous version of ODE.

Remember there are an infinite number of x_s.

Therefore, we get an infinite number of general solutions x.

$$x = x_r + x_s$$

One of them will satisfy initial conditions.

Solutions of damping SHM

$$x_s = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

Resonance solution of inhomogeneous SHM

$$x_r = \frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}} \cos(\omega_D t + \phi)$$

$x = x_r + x_s$ 滿足原來 x_r 所滿足的外力下簡諧運動的微分方程式：

x is the general solutions of inhomogeneous SHM: $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t$

While x_r is totally fixed by ODE, there are two unspecified constants x_m, ϕ in x_s.

We can choose them to satisfy the two initial conditions $x(0), x'(0)$.

這個函數同時滿足運動方程式以及兩個起始條件，因此是唯一的解！

The function we get satisfies inhomogeneous SHM ODE and initial condition simultaneously

It is the unique solution.

$$x = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi) + \frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}} \cos(\omega_D t + \phi)$$

注意非共振解x_s是以彈簧自然頻率ω震盪，而不是ω_D。

Nonresonance x_s oscillates in the damped frequency of the spring ω' instead of ω_D like x_r

但隨時間振幅會變小，長期來說可以忽略。

As time progresses, amplitude decreases exponentially. In the long term, it can be ignored

$$x = x_r + x_s \rightarrow x_r$$

長期而言，只有共振解是重要的，起始條件無關緊要

In the long term, only resonance solution survive. Initial conditions do not matter.

愛拼才會贏定律