

傅立葉級數 Fourier Series

Solving Wave Equation:

求解策略 I :

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

Search for separable solutions, products of a function of x and a function of t .
They correspond to oscillation modes in coupled spring system.

$$\phi(x, t) = X(x) \cdot T(t)$$

$$\frac{d^2 X}{dx^2} = \lambda \cdot X$$

$$\frac{d^2 T}{dt^2} = v^2 \lambda \cdot T$$

$T(t)$ is solved right away as SHM.

With boundary condition, allowed $X_n(x)$, $n = 1, \dots, \infty$ would be countable. ◦

We found a series of separable solutions:

$$u_n(x, t) = X_n(x) \cdot T_n(t) = \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

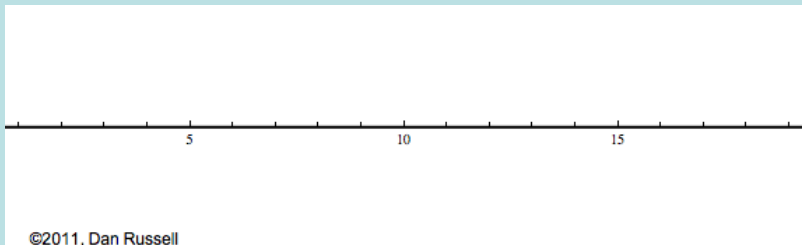
Separable solutions are really special!

$$u^{(n)}(x, t) = \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

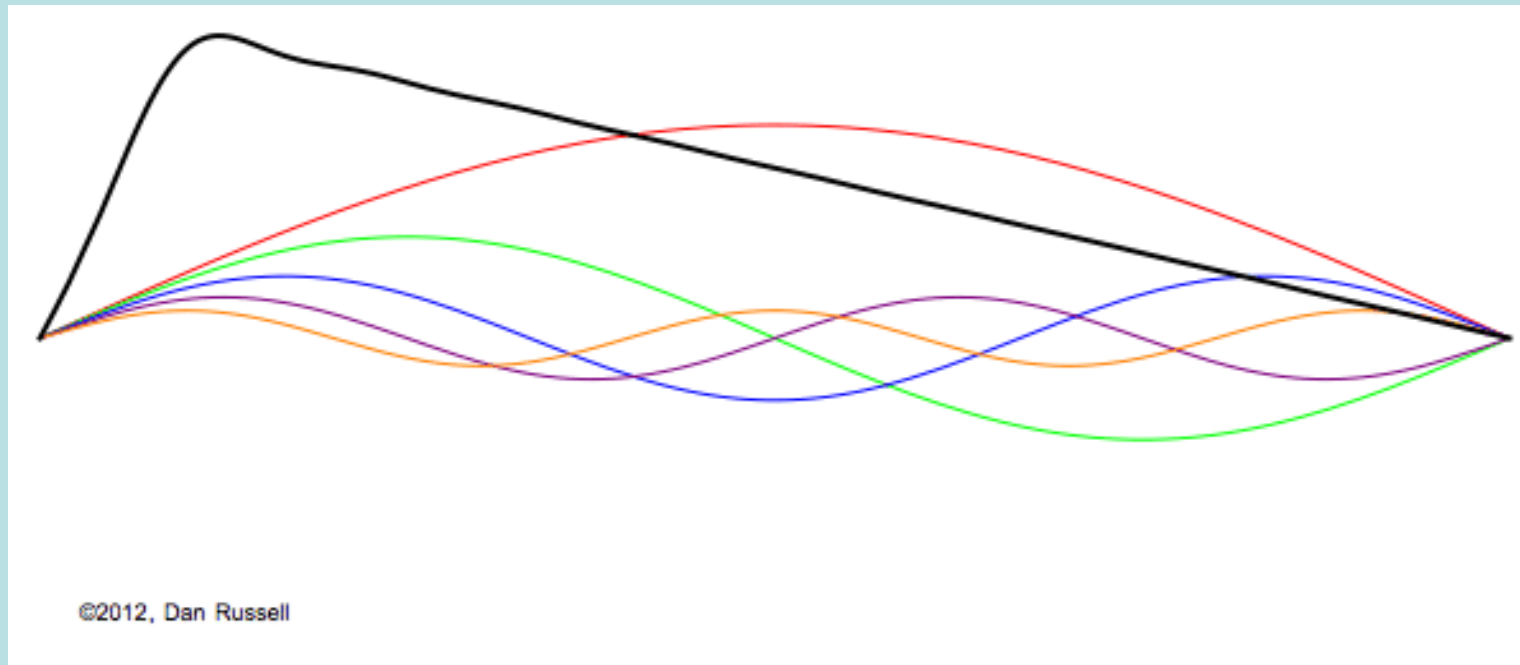
All particles oscillate as SHM by identical ω_n and phase, amplitudes $\propto \sin\left(\frac{n\pi}{L}x\right)$.

It could also be seen as a Sine function of x with time varying amplitude.

Not all solution acts like that.



Blackline is a solution when we pull string, with fixed ends, at one point and release.



D. Bernoulli, 1700-1782



求解策略II：

Since wave equation is linear, linear combinations of $X_n(x)T_n(t)$ are solutions, too.

$$\phi(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

This is an infinite series of functions, summing up lots of separables with coefficients!

These are all solutions of wave equation.

That solutions of Wave E. can be written in this form was proposed by Bernoulli.

They are called **Bernoulli Solutions**. With A_n, B_n we can form lots of Bernoulli Solutions

This also happens in coupled spring system.

Coupled Oscillator: 3 Blocks

$$x_1(t) = \frac{x_1(0) + \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2})t \right] + \frac{x_1(0) - x_3(0)}{2} \cos \left(\sqrt{\frac{2k}{m}}t \right) + \frac{x_1(0) - \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2})t \right]$$

(left)

$$x_2(t) = \frac{\sqrt{2}x_1(0) + 2x_2(0) + \sqrt{2}x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2})t \right] - \frac{\sqrt{2}x_1(0) - 2x_2(0) + \sqrt{2}x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2})t \right]$$

(center)

$$x_3(t) = \frac{x_1(0) + \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 - \sqrt{2})t \right] - \frac{x_1(0) - x_3(0)}{2} \cos \left(\sqrt{\frac{2k}{m}}t \right) + \frac{x_1(0) - \sqrt{2}x_2(0) + x_3(0)}{4} \cos \left[\sqrt{\frac{k}{m}} (2 + \sqrt{2})t \right]$$

(right)

All springs have
spring constant k ,
and all surfaces
are frictionless.

$k = 1 \text{ N/m}$
 $m = .2 \text{ kg}$
 $x_1(0) = 1 \text{ m}$
 $x_2(0) = .9 \text{ m}$
 $x_3(0) = -.3 \text{ m}$

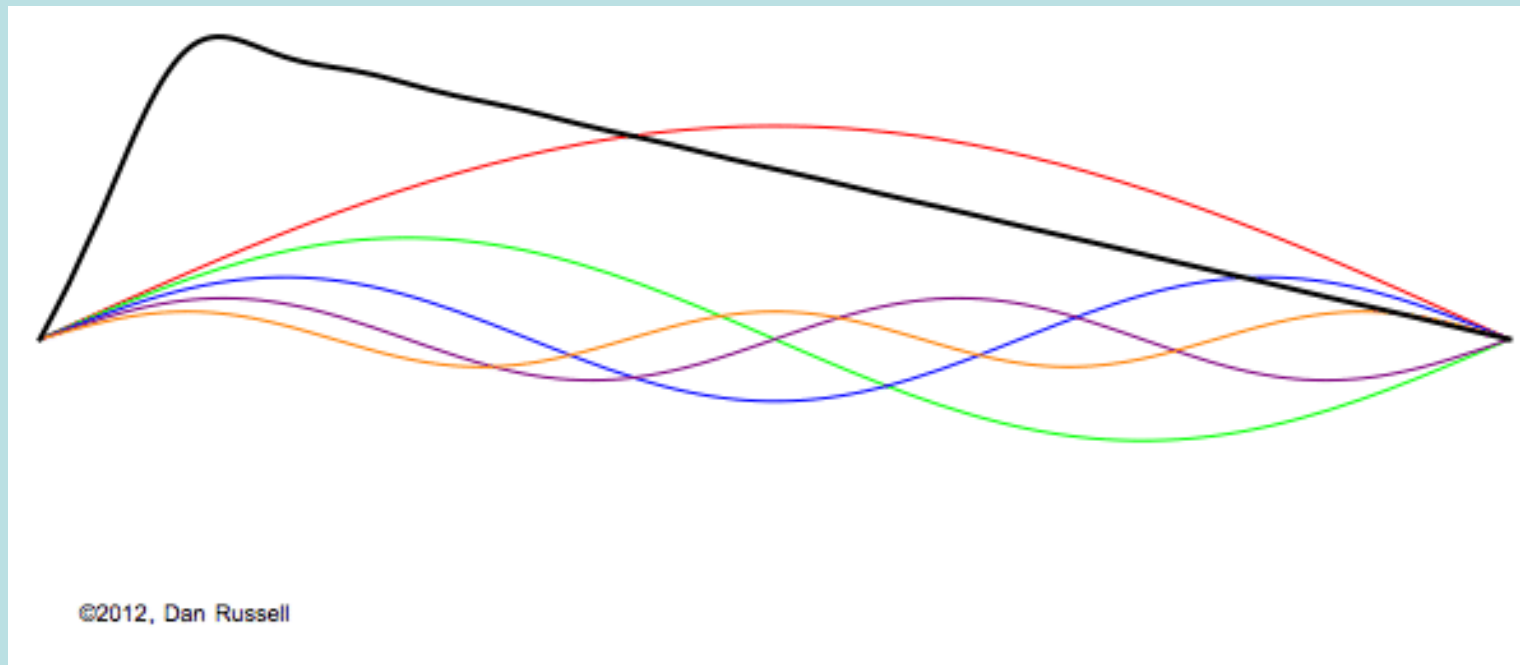


The equation of motions are linear ODE.

Linear Combinations of mode solutions are solutions, too.

For fixed end string: The following is one example of Bernoulli Solution.

The blackline solution is the sum of colored lines of separable oscillating solutions.



求解策略II：

Since wave equation is linear, linear combinations of $X_n(x)T_n(t)$ are solutions, too.

$$\phi(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

How to compute coefficients A_n, B_n for a specific problem.

Are all solutions of the form of Bernoulli Solutions?

Do Bernoulli Solutions exhaust all possible solutions of wave equation?

Equation of motion: 對照表

$$\frac{d^2 \mathbf{X}}{dt^2} = -\mathbf{A} \cdot \mathbf{X}$$

x_j

Wave Equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$\phi(jd) = \phi(x)$$

We usually choose linear combination coefficients to **fit initial condition**.

Initial Condition

$$\begin{pmatrix} x_1(0) \\ \vdots \\ x_j(0) \\ \vdots \\ x_N(0) \end{pmatrix}$$

$$\begin{pmatrix} x'_1(0) \\ \vdots \\ x'_j(0) \\ \vdots \\ x'_N(0) \end{pmatrix}$$

Time derivatives with
fixed $j \rightarrow x$.

Initial Condition

$$\phi(x, 0)$$

Wavefunction at $t = 0$

$$\frac{\partial \phi}{\partial t}(x, 0)$$

Time partial of wavefunction at $t = 0$

Could we always find coefficients A_n, B_n to satisfy specific initial condition?

If yes, we are done, because we have found Bernoulli Solutions to all problems.

Every problem can have only one solution! So yes, all solutions are like Bernoulli's.

$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t) \quad \text{Bernoulli's solution}$$

The coefficients A_n, B_n are found, (or fixed) to satisfy initial conditions.

Plug in $t = 0$ to the expression above, we find the key equation:

$$\phi(x, 0) = u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$

Technically we'd say: **function $u_0(x)$ is expanded as a series of Sine functions.**

$$\frac{\partial}{\partial t} \phi(x, 0) = v_0(x) = \sum_{n=1}^{\infty} B_n \omega_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$

How to compute A_n, B_n ?



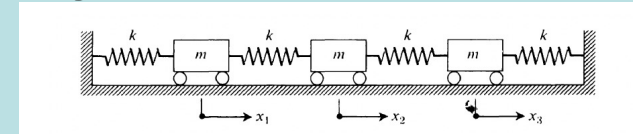
1768-1830 Jean-Baptiste Joseph Fourier

The following method is proposed by Fourier 50 years after Bernoulli.

Expansion of a function as a series of Sine functions is called **Fourier Series**.

Let's take a hint from the case of 3 body coupled springs:

Plug in $t = 0$ to sums of mode solutions.



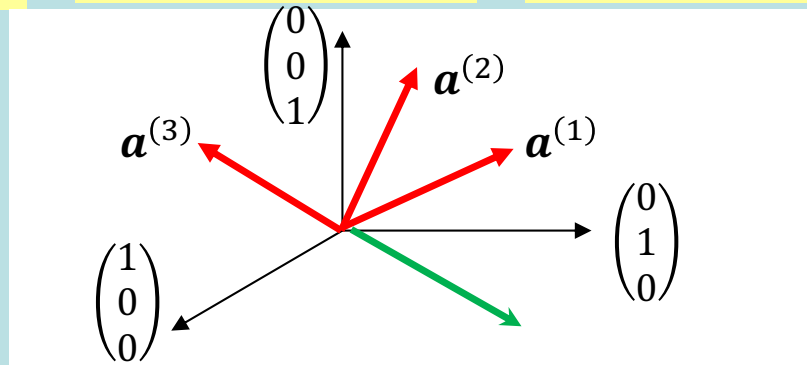
$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^3 \mathbf{a}^{(i)} f_i$$

Initial conditions determine coefficients f_i .

We choose $\mathbf{a}^{(i)}$ to be orthonormal. $\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \delta_{mn}$

In analogy, orthonormal eigenvectors look exactly like orthogonal coordinate axes!

$$\mathbf{a}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad \mathbf{a}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{a}^{(3)} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \quad \mathbf{a}^{(1)} \perp \mathbf{a}^{(2)} \perp \mathbf{a}^{(3)}$$



This is not 3D space, but its **abstract computational rules** are identical to 3D.

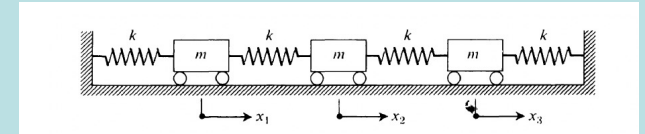
We call this kind of look-alike 3D space a (generalized) **Vector Space**.

All linear combinations of n orthonormal axes form a vector space of dimension n .

Computation rules of this vector space are identical to n D space

In the case of $N = 3$ coupled springs:

$\mathbf{a}^{(i)}$ are orthonormal.



$$\mathbf{a}^{(m)T} \mathbf{a}^{(n)} = \begin{pmatrix} a_1^{(n)} & a_2^{(n)} & a_3^{(n)} \end{pmatrix} \begin{pmatrix} a_1^{(m)} \\ a_2^{(m)} \\ a_3^{(m)} \end{pmatrix} = \sum_{j=1}^3 a_j^{(n)} a_j^{(m)} = \delta_{mn}$$

$$dj, j = 1, 2, 3$$

$$x_j(t)$$

$$a_j(t)$$

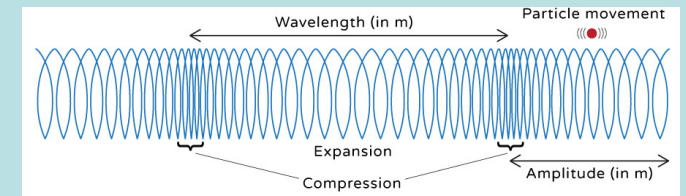


Continuous limit

$$x, 0 < x < L$$

$$\phi(x, t)$$

$$X(x, t)$$



Note that subscript j corresponds to x . Inner product in continuous limit will be multiplying $X_{m,n}(x)$ at the same x then sum up over all x : ie. integration.

Are $\sin\left(\frac{n\pi}{L}x\right)$ orthonormal?

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \stackrel{?}{=} \delta_{mn}$$

Method 0

Are $\sin\left(\frac{n\pi}{L}x\right)$ orthonormal?

If $m \neq n$

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) = \frac{1}{2} \int_0^L dx \left[\cos\frac{\pi}{L}(m-n)x - \cos\frac{\pi}{L}(m+n)x \right]$$

$$= \frac{L}{2\pi} \left[\frac{1}{m-n} \sin\frac{\pi}{L}(m-n)x + \frac{1}{m+n} \sin\frac{\pi}{L}(m+n)x \right] \Bigg|_0^L = 0$$

Sine functions are periodic.

若 $m = n$,

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) = \frac{1}{2} \int_0^L dx \left[1 - \cos\frac{2n\pi}{L}x\right]$$

$$= \frac{L}{2} + \frac{2n\pi}{L} \left[\sin\frac{2n\pi}{L}x \right] \Big|_0^L = \frac{L}{2}$$

We could redefine eigenfunctions $X_n(x)$ such that vector length equal to 1:

$$X_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L dx X_n^2 = 1$$

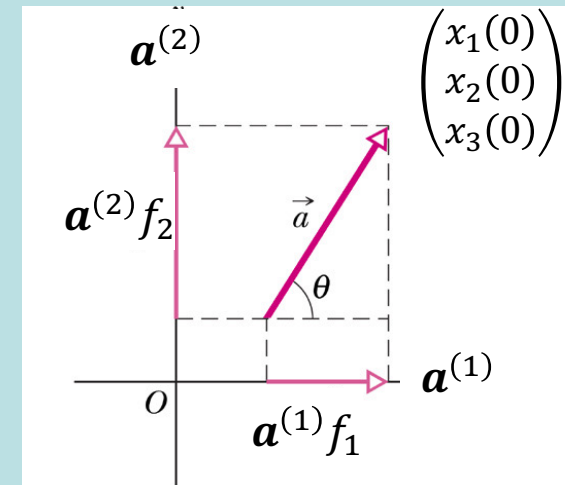
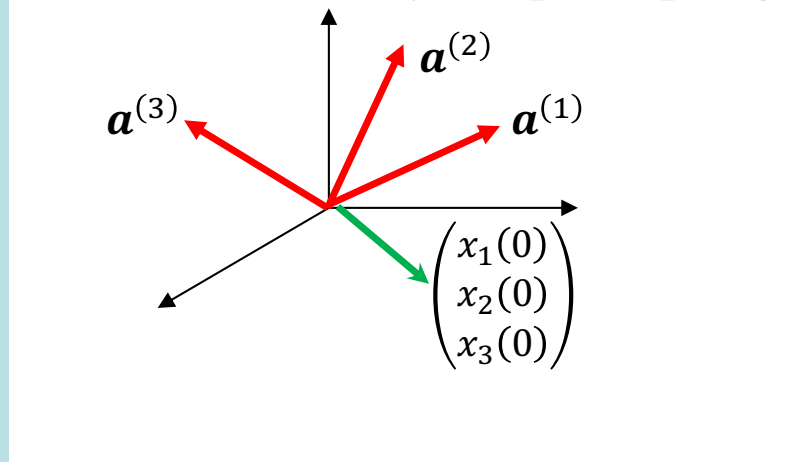
Yes, $\sin\left(\frac{n\pi}{L}x\right)$ is orthonormal if inner product is understood as integration over x .

$$\int_0^L dx X_m(x) X_n(x) = \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) = \delta_{mn}$$

Kronecker δ Function

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Back to the case of 3 body coupled springs:

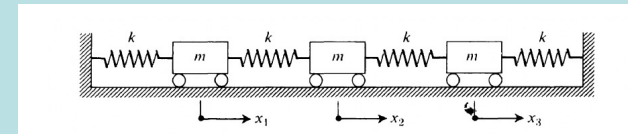


Initial conditions looks like a vector,

and is expanded in terms of axes $\mathbf{a}^{(i)}$ with f_i as components.

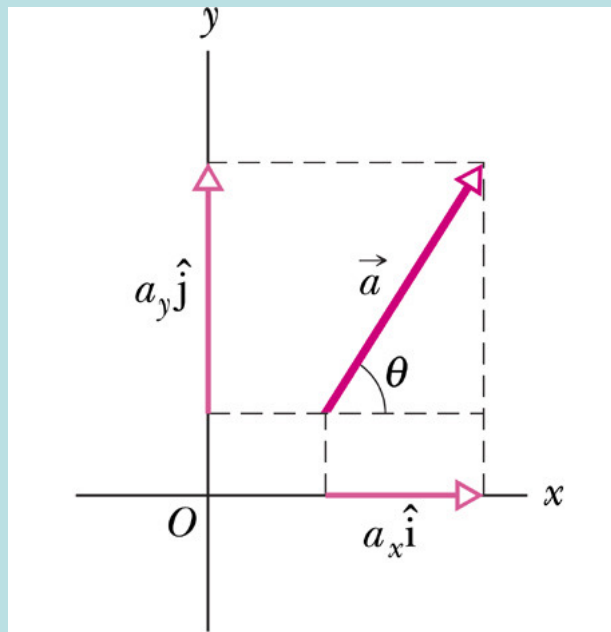
Components f_i equals the inner product of $\mathbf{a}^{(i)}$ and initial condition vector.

$$\mathbf{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$



For example,

$$f_1 = \mathbf{a}^{(1)} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \frac{1}{2} [x_1(0) + \sqrt{2}x_2(0) + x_3(0)]$$



$$\vec{a} = a_x \hat{i} + a_y \hat{j} = (a_x, a_y)$$

$$a_x = \hat{i} \cdot \vec{a}, \quad a_y = \hat{j} \cdot \vec{a}$$

For orthonormal axes $\hat{i}, \hat{j}$, components of a vector can be computed by inner products!

Decomposing $u_0(x)$ into linear combination of $X_n(x)$ could proceed similarly.

$$u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

Take inner product of u_0 and X_n in the sense of an integration: $a_x = \hat{i} \cdot \vec{a}$

$$\sqrt{\frac{2}{L}} \int_0^L dx u_0(x) \sin\left(\frac{n\pi}{L}x\right) = \frac{2}{L} \int_0^L dx \left[\sum_{m=1}^{\infty} A_m \cdot \sin\left(\frac{m\pi}{L}x\right) \right] \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{2}{L} \sum_{m=1}^{\infty} \left[A_m \int_0^L dx \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \right] = \frac{2}{L} \sum_{m=1}^{\infty} \left[A_m \frac{L}{2} \delta_{mn} \right] = A_n$$

Jean-Baptiste Joseph Fourier

$$A_n = \sqrt{\frac{2}{L}} \int_0^L dx u_0(x) \sin\left(\frac{n\pi}{L}x\right)$$



$$a_x = \hat{i} \cdot \vec{a}$$



Plug in these coefficients to Bernoulli's solution and we solve the problem.

Bernoulli's solution

$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

Fourier Series of initial condition.

$$\phi(x, 0) = u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L}x\right) \quad \frac{\partial}{\partial t} \phi(x, 0) = v_0(x) = \sum_{n=1}^{\infty} B_n \omega_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$

Fourier Series coefficient formula.

$$A_n = \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{n\pi}{L}x\right) u_0(x)$$

$$B_n = \frac{1}{\omega_n} \sqrt{\frac{2}{L}} \int_0^L dx v_0(x) \sin\left(\frac{n\pi}{L}x\right)$$

It satisfies both the system of equations of motion and initial conditions.

There is one and only one function that satisfies both EOM and initial conditions.

This is **the only one** solution.

We prove: any solution of Wave E. is a linear combination of separable solutions.

$$\mathbf{x} = \sum_{i=1}^3 \mathbf{a}^{(i)} [f_i \cdot \cos(\omega_i t) + g_i \cdot \sin(\omega_i t)]$$

$$f_i = \mathbf{a}^{(n)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{j=1}^3 a_j^{(n)} x_j(0)$$

$$dj, j = 1, 2, 3$$

$$x_j(t)$$

$$a_j(t)$$

Continuous limit

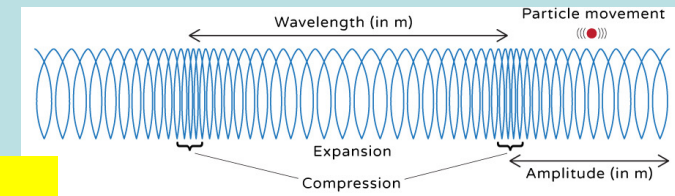
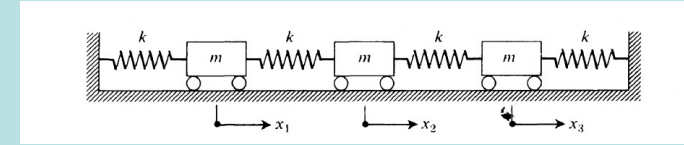
$$x$$

$$\phi(x, t)$$

$$X(x, t)$$

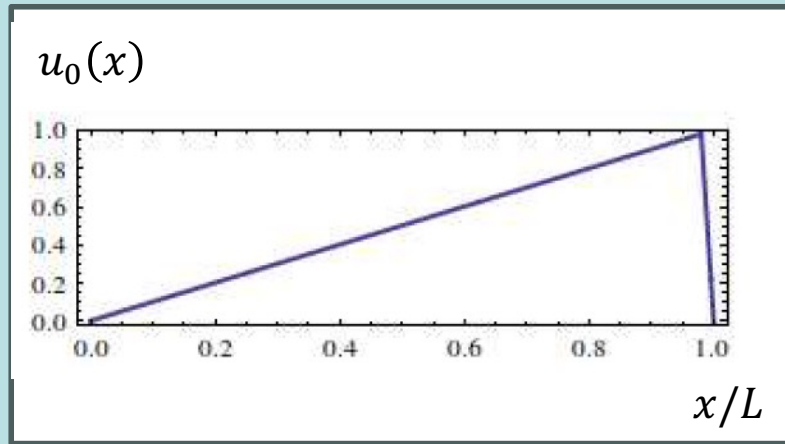
$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$A_n = \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{n\pi}{L} x\right) \phi(x, 0)$$



Inner product of coupled spring system in continuous limit
will be multiplying two functions at the same x then sum up over all x : ie. integration.

例如：考慮起始條件為一個Sawtooth鋸齒波形



$$u_0(x) = \frac{1}{L}x$$

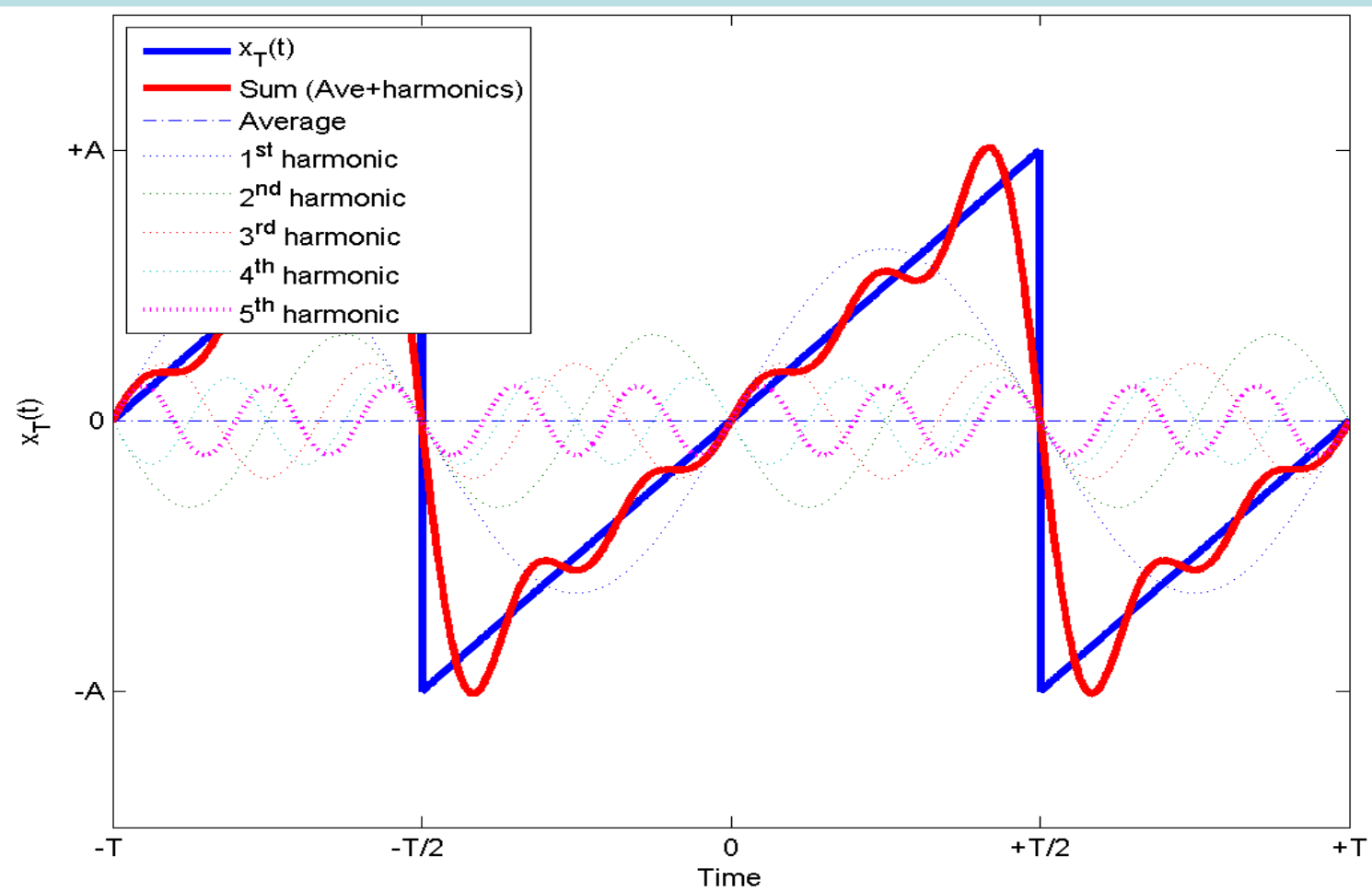
$$0 < x < L$$

$$A_n = \sqrt{\frac{2}{L}} \int_0^L dx \, u_0(x) \sin\left(\frac{n\pi}{L}x\right) = \sqrt{\frac{2}{L}} \int_0^L dx \, \frac{1}{L}x \cdot \sin\left(\frac{n\pi}{L}x\right)$$

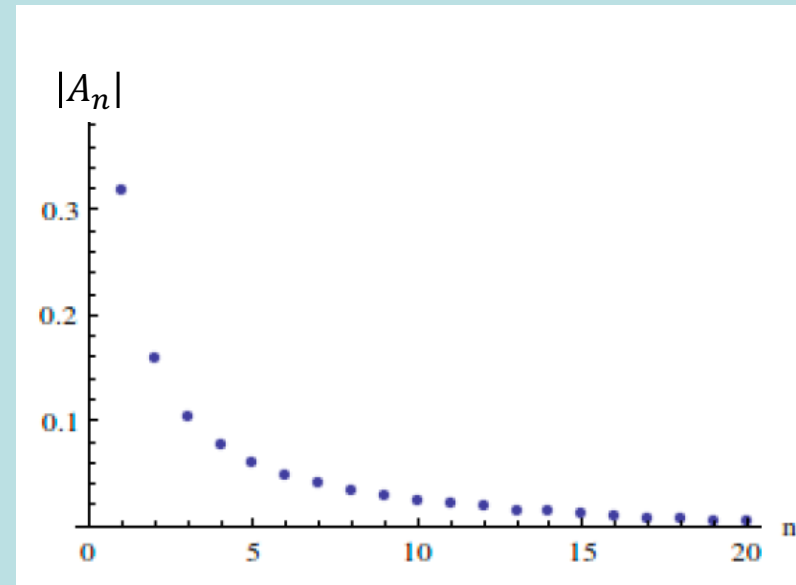
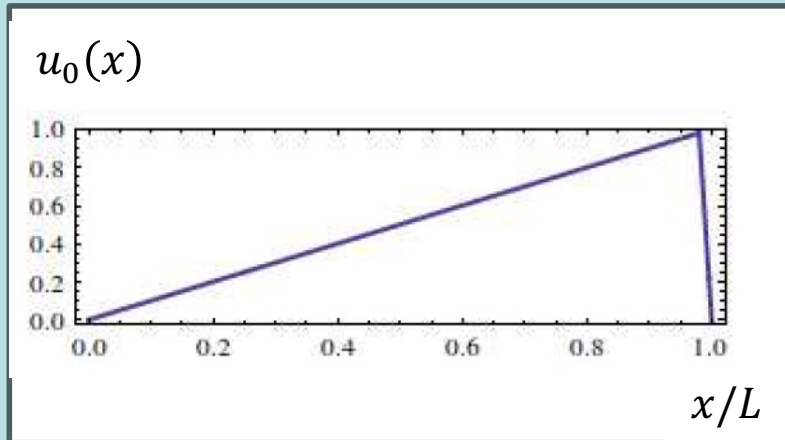
$$= \sqrt{\frac{2}{L}} \left\{ \frac{1}{n\pi} \left[x \cos\left(\frac{n\pi}{L}x\right) \right] \Big|_0^L + \frac{1}{n\pi} \int_0^L dx \, \cos\left(\frac{n\pi}{L}x\right) \right\} \quad \text{Integration by part}$$

$$= \sqrt{2L} \frac{1}{n\pi} \cos(n\pi) + \sqrt{2L} \left(\frac{1}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi}{L}x\right) \right] \Big|_0^L = \sqrt{2L} \frac{(-1)^n}{n\pi}$$

$$u_0(x) = \sum_{n=1}^{\infty} 2 \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi}{L}x\right)$$



If we start with string at rest: $v_0 = 0, B_i = 0$.



With coefficients A_n , we can write down the solution:

$$\phi(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) \cdot \cos \omega_n t = \sum_{n=1}^{\infty} 2 \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi}{L} x\right) \cdot \cos \omega_n t$$

It satisfies both the system of equations of motion and initial conditions.

This is **the only one** solution.

The end of solving wave equation with boundary conditions.



Here is an example with animation!

Acoustics and Vibration Animations

Daniel A. Russell

Graduate Program in Acoustics, The Pennsylvania State University



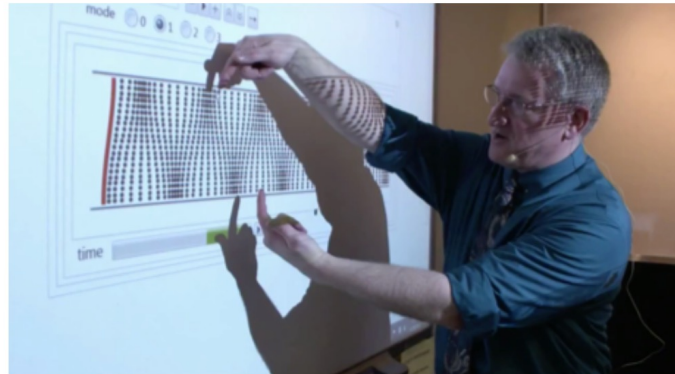
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Today is **Saturday, September 26, 2026**. The contents (and links) on this page were last updated on **August 21, 2025**.

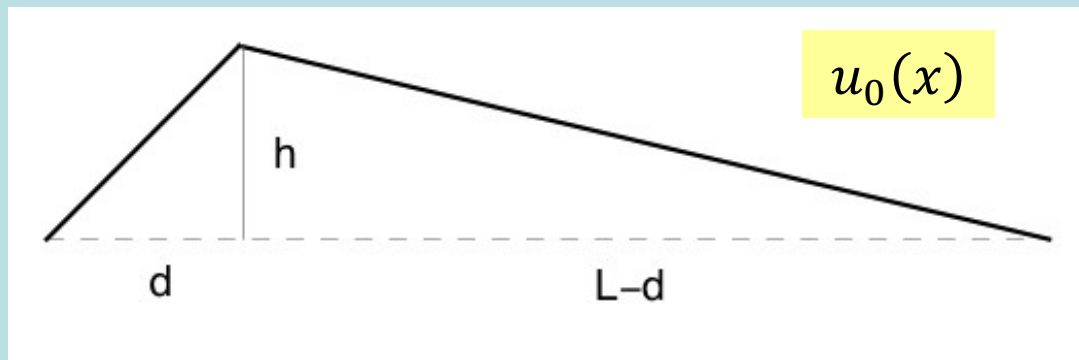
I am in the process of attempting to bring all of the pages on this site into compliance with current HTML5, CSS3, and W3C Web Accessibility standards. The bullet symbols used in the lists below identify pages as

- (filled) still needs to be updated
- (open circle) updated to HTML5 and CSS3 and W3C accessibility.

The links below contain animations illustrating acoustics and vibration, waves and oscillation concepts. I started using [Wolfram Mathematica](#) to create animations to help me understand and visualize certain acoustics and vibration phenomena in 1992 while I was a Ph.D. student in the Graduate Program in Acoustics at Penn State. For the next 16 years (1995-2011) I was a physics professor at Kettering University and continued creating animations and using them as educational tools for the courses I was teaching about waves and acoustics. Sometime around 1998 or so I began writing webpages and adding them to this online collection. Now that I'm back at Penn State, teaching graduate level acoustics, I'm continuing to add to my collection of animations. My intent has always been to create physically and mathematically correct animations, accompanied by explanatory text, that illustrate complicated phenomena involving Screenshot vibration in a manner

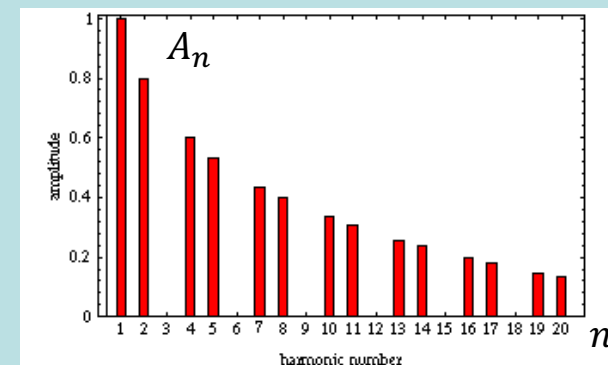
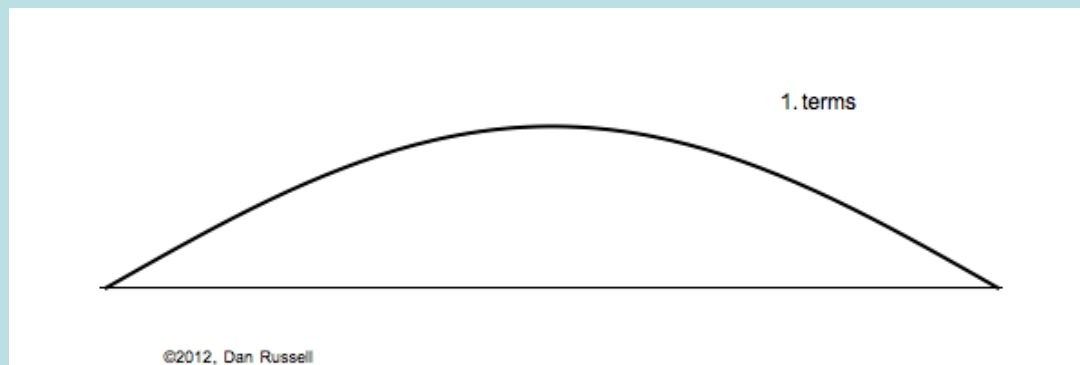


Initial condition of the string:



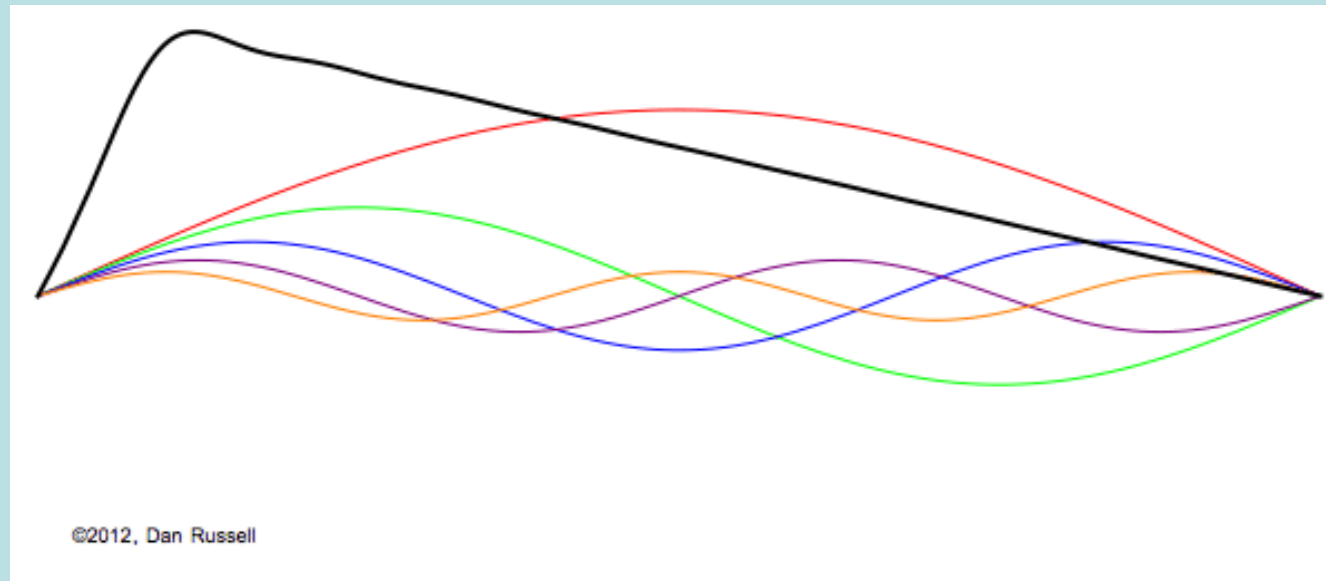
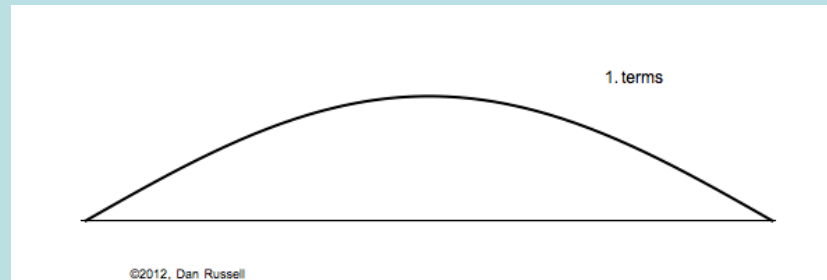
Initial condition can be decomposed into an infinite Fourier Series:

$$u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$



Note that coefficients A_n decrease rapidly with n .

Usually around 10 leading terms are enough to build a satisfactory approximation.



$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

Terms $A_n \cdot \sin\left(\frac{n\pi}{L}x\right)$ in Fourier series evolve with time separately like separable solutions. Sum them up and we get the final solution.



YouTubeTM

Search



Plucked String (High Speed Video)

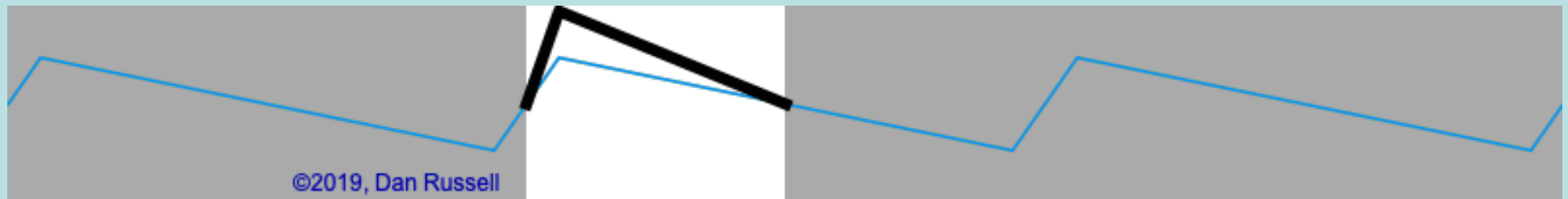


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High Speed video of a Struck Elastic String



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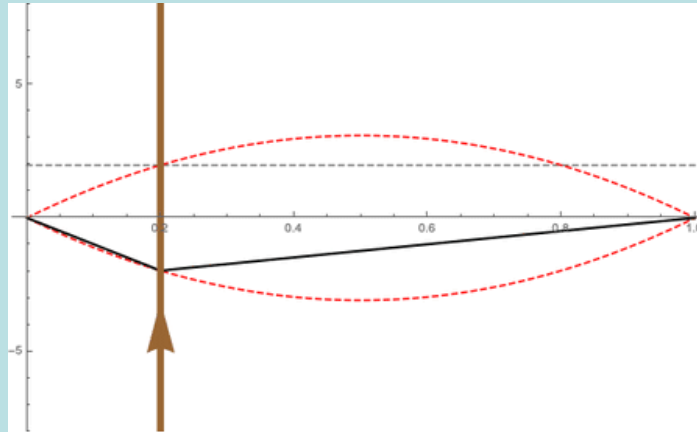
Screenshot 20



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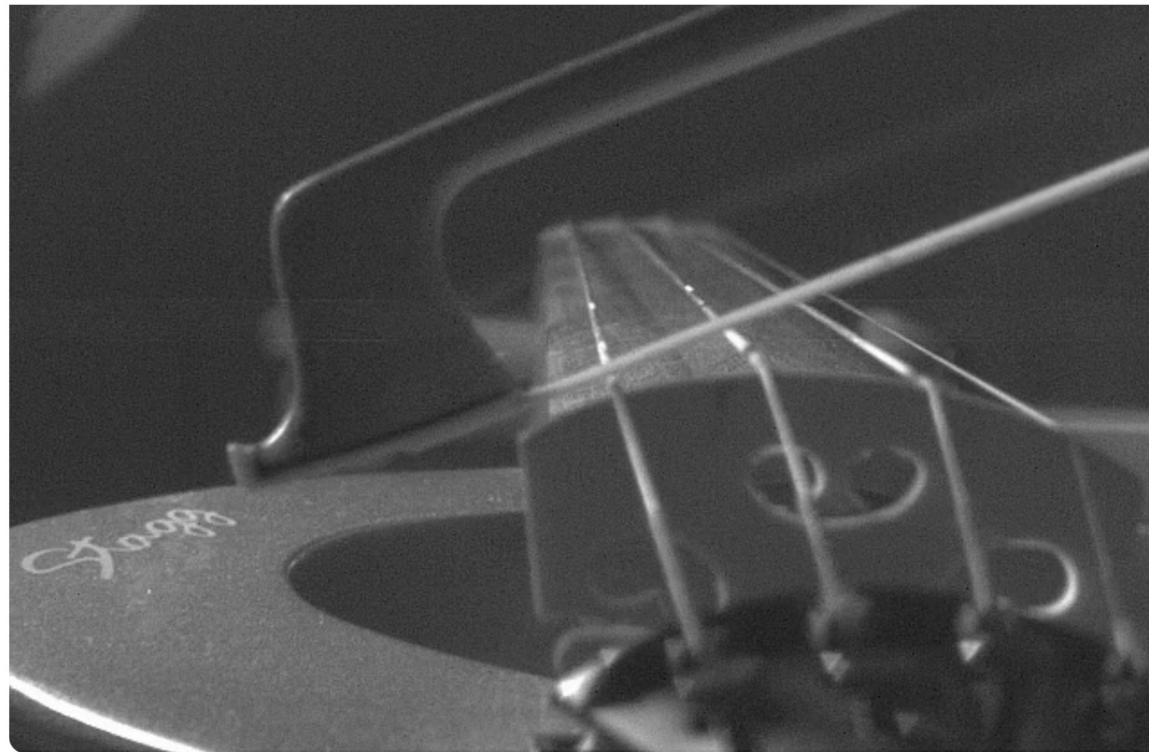
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Solving Wave Equation $\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$ 求解策略 Bernoulli Solutions
Separation of variables plus Fourier Series

Search for separable solutions, products of a function of x and a function of t .

$$\phi(x, t) = X(x) \cdot T(t)$$

Satisfying an ODE, $T(t)$ is solved right away as oscillating SHM.

With boundary condition, $X(x)$ satisfies an ODE, called **eigenvalue equation**.

Allowed $X_n(x)$, $n = 1, \dots, \infty$ would be countable, called **eigenfunctions**

We found a series of separable solutions:

$$u_n(x, t) = X_n(x) \cdot T_n(t) = \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

Since wave equation is linear, linear combinations of $X_n(x)T_n(t)$ are solutions, too.

$$\phi(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

Any initial condition $u_0(x)$ can be expanded as a linear combination of $X_n(x)$.

$$\phi(x, 0) = u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

call it a mixture if you want.

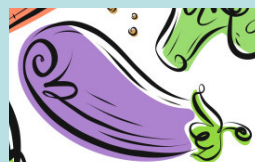
This property of Fourier Series is crucial. We say, it is complete 具有完備性.

各個配料 $X_i(x)$ 依配方 A_i 收集

Find recipe of ingredients.



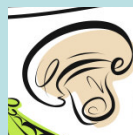
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⋮

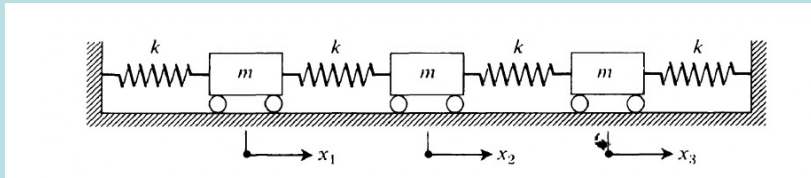
各個配料分離烹煮 $T_i(t) \sim \cos(\omega_i t)$
最後合體！

Cook and then mix cooked ingredients.

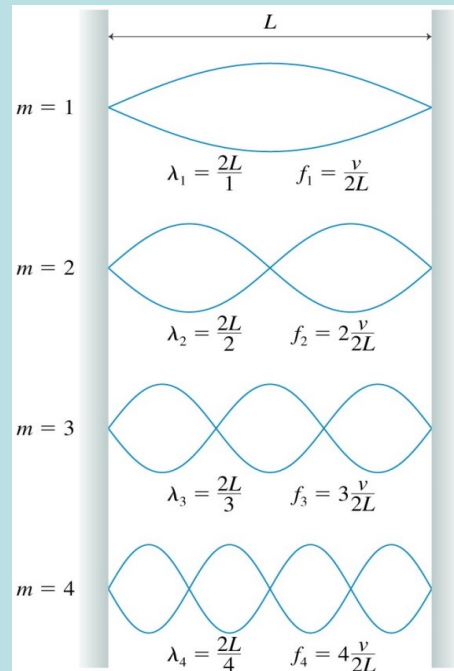


$$\phi(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

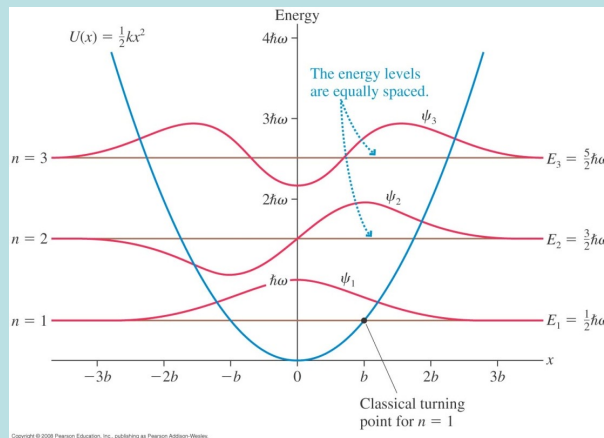
這樣的波函數顯然滿足波方程式，又滿足起始條件，就是唯一解。



Coupled springs



Classical Wave



Quantum Particle

It is striking all the 3 systems share the same mathematical technique.

$$\phi(x, 0) = u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

Any initial condition $u_0(x)$ can be expanded as a Fourier Series .

This property of Fourier Series is crucial. We say, it is complete 具有完備性.

Is this infinite Fourier Series convergent? The answer is yes.

How is this infinite Fourier series approaching $u_0(x)$? The answer is:

$$\lim_{N \rightarrow \infty} \int_0^L dx \left[u_0(x) - \sum_{n=1}^{\infty} A_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \right]^2 = 0$$

若確認任一合邊界條件的函數可以本徵函數分解，我們能計算其分解係數。

但這樣的分解真的是正確的嗎？分解所得的無限級數真的與被分解的函數相等嗎？

物理直覺：合理的起始條件下，波方程式應該都有解，因此這樣的分解一定可行。

數學上相等與否的判準：這些係數組成的 $X_n(x)$ 的線性組合，會趨近被分解的函數。

$$\lim_{N \rightarrow \infty} \int_0^L dx \left[u_0(x) - \sum_{n=1}^N A_n \cdot X_n(x) \right]^2 = 0$$

兩者差距的平方和(積分) $N \rightarrow \infty$ 時，趨近於零。

這稱為**Completeness**.完備性，必須嚴格證明的。下學期將說明。

但在波方程式的例子中，起始條件的展開其實就是**傅立葉級數**的一部分。

猜測且相信：三角函數是**Complete**完備的。傅立葉無限級數可以趨近任一週期函數！

我們可以先說明 X_n 的展開等價於傅立葉級數展開，因此一系列 X_n 也是完備的。

下學期再說明滿足某些條件的算子的本徵函數也都是完備的。

Chapter 3

Fourier Series

傅立葉級數

Just before 1800, the French mathematician/physicist/engineer Jean Baptiste Joseph Fourier made an astonishing discovery, [42]. Through his deep analytical investigations into the partial differential equations modeling heat propagation in bodies, Fourier was led to claim that “every” function could be represented as an infinite series of elementary trigonometric functions: sines and cosines. For example, consider the sound produced by a musical instrument, e.g., piano, violin, trumpet, or drum. Decomposing the signal into its trigonometric constituents reveals the fundamental frequencies (tones, overtones, etc.) that combine to produce the instrument’s distinctive timbre. This Fourier decomposition lies at the heart of modern electronic music; a synthesizer combines pure sine and cosine tones to reproduce the diverse sounds of instruments, both natural and artificial, according to Fourier’s general prescription.

Fourier’s claim was so remarkable and counterintuitive that most of the leading mathematicians of the time did not believe him. Nevertheless, it was not long before scientists came to appreciate the power and far-ranging applicability of Fourier’s method, thereby opening up vast new realms of mathematics, physics, engineering, and beyond. Indeed, Fourier’s discovery easily ranks in the “top ten” mathematical advances of all time, a list that would also include Newton’s invention of the calculus, and Gauss and Riemann’s differential geometry, which, 70 years later, became the foundation of Einstein’s general relativity. Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytic tool for solving a broad range of linear partial differential equations. Applications in physics, engineering, biology, finance, etc., are almost too numerous to catalogue: typing the word “Fourier” in the subject index of a modern science library will dramatically demonstrate just how ubiquitous these methods are. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CDs and DVDs, cell phones, movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded on the many ramifications of Fourier theory. In your career as a mathematician, scientist, or engineer, you will find that Fourier theory, like calculus and linear algebra, is one of the most basic weapons in your mathematical arsenal. Mastery of the subject is essential.

任一滿足邊界條件 $f(0) = f(L) = 0$ 的函數 $f(x)$ 都可分解為一正弦函數級數：

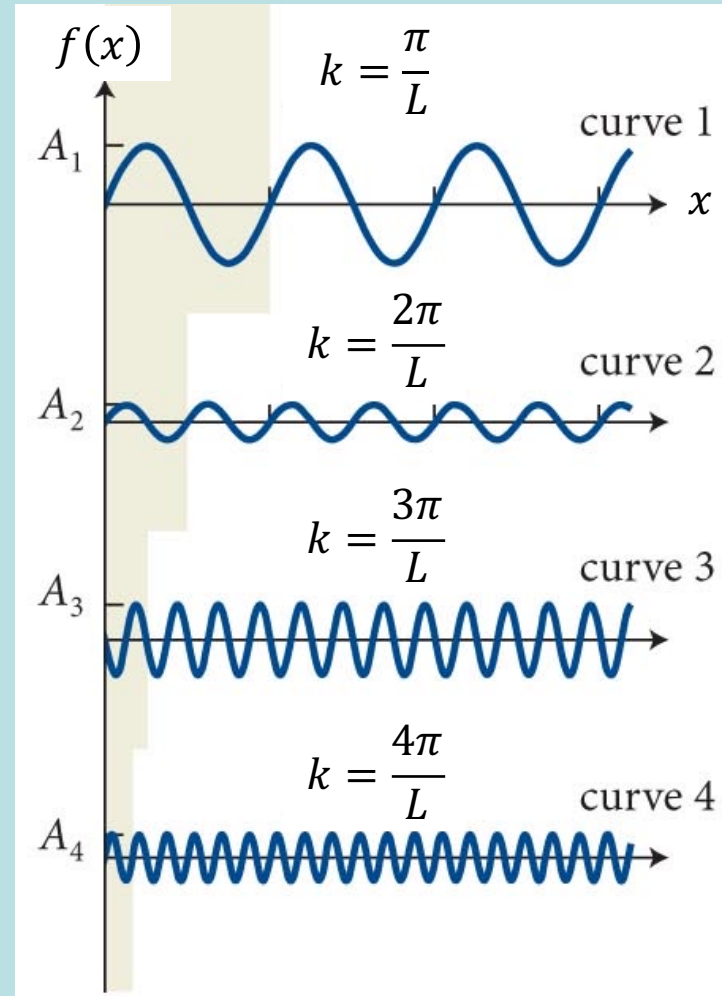
$$f(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L} x\right)$$

This is a special case of Fourier Series.



Jean-Baptiste Joseph Fourier

1768-1830



傅立葉級數 Fourier Series for periodic function

The series is usually extended to include similar Cosine functions and a constant:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cdot \sin \frac{n\pi}{L} x + b_n \cdot \cos \frac{n\pi}{L} x \right]$$

There is no doubt this expression is a periodic function of period $2L$.

It is proved **any** such periodic function can be written as such a **Fourier Series**.

Usually, the range is defined as $-L < x < L$.

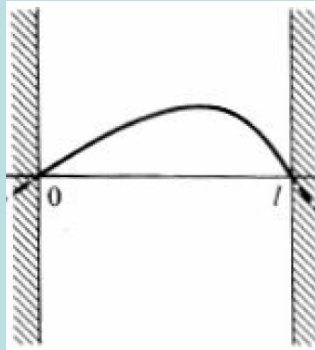
Fourier Series for periodic function and for function with fixed ends are equivalent.

Fourier Series for periodic function and for function with fixed ends:

傅立葉分析是分解週期函數為一系列正弦函數與餘弦函數的級數和。

波方程式的起始條件則是定義在一個線段中：

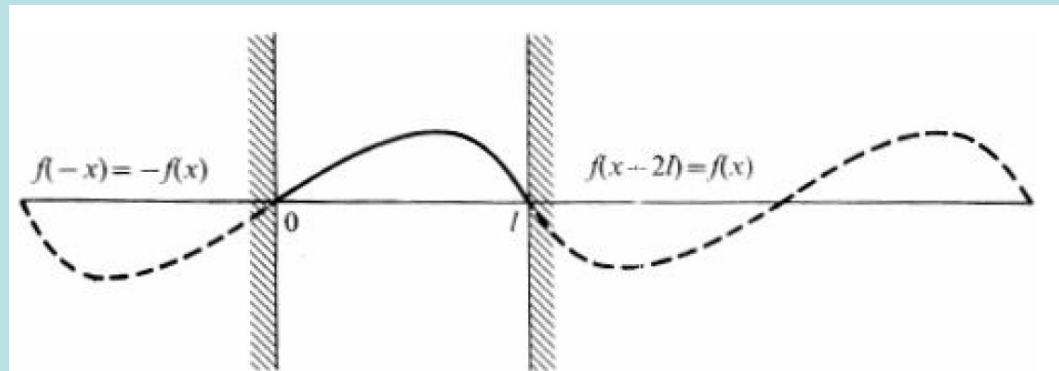
$$\phi(x, 0) = u_0(x), \quad 0 < x < L$$

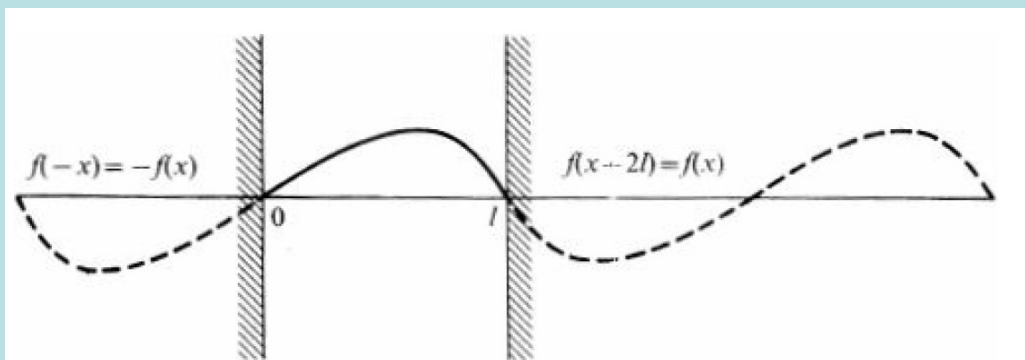


我們若將 $u_0(x)$ 延展到整個空間，成為一週期函數，傅立葉分析就能用！

定義延展後的函數為一奇函數： $u_0(-x) = -u_0(x), \quad -\infty < x < \infty$

同時是一週期為 $2L$ 的週期函數： $u_0(x + 2L) = u_0(x), \quad -\infty < x < \infty$





$$u_0(-x) = -u_0(x),$$



$$u_0(0) = -u_0(0) = 0$$

$$u_0(x + 2L) = u_0(x),$$



$$u_0(L) = u_0(-L) = -u_0(L) = 0$$

這個 $u_0(x)$ 的延展，滿足邊界條件，同時在 $0 < x < L$ 等同於原來的起始條件。
此延展為一週期函數，傅立葉分析就能用！

$$u_0(x) = \sum_{n=1}^{\infty} \left[A_n \cdot \sin \frac{n\pi}{L} x + B_n \cdot \cos \frac{n\pi}{L} x \right]$$



$$= \sum_{n=1}^{\infty} A_n \cdot \sin \frac{n\pi}{L} x$$

$u_0(x)$ 是奇函數，只有正弦函數留下。這個傅立葉數列是完備的！

這一系列算子的本徵函數 $X_n(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi}{L} x \right)$ 是完備的！

完整的傅立葉級數公式：

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cdot \cos \frac{n\pi}{L} x + b_n \cdot \sin \frac{n\pi}{L} x \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^L dx f(x)$$

$$a_n = \frac{1}{L} \int_{-L}^L dx f(x) \cos \frac{n\pi}{L} x$$

Inner product of $f(x)$ and $\cos \frac{n\pi}{L} x$.

$$b_n = \frac{1}{L} \int_{-L}^L dx f(x) \sin \frac{n\pi}{L} x$$

10.2 Fourier Series

Later in this chapter you will find that you can solve many important problems involving partial differential equations, provided that you can express a given function as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**,² they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

We begin with a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \quad (1)$$

On the set of points where the series (1) converges, it defines a function f , whose value at each point is the sum of the series for that value of x . In this case the series (1) is said to be the **Fourier series** for f . Our immediate goals are to determine what functions can be represented as a sum of a Fourier series and to find some means of computing the coefficients in the series corresponding to a given function. The first term in the series (1) is written as $a_0/2$ rather than as a_0 to simplify a formula for the coefficients that we derive below. Besides their association with the method of separation of variables and partial differential equations, Fourier series are also useful in various other ways, such as in the analysis of mechanical or electrical systems acted on by periodic external forces.

Periodicity of the Sine and Cosine Functions. To discuss Fourier series, it is necessary to develop certain properties of the trigonometric functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer. The first property is their periodic character. A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x , and if

$$f(x + T) = f(x) \quad (2)$$

for every value of x . An example of a periodic function is shown in [Figure 10.2.1](#). It follows immediately from the definition that if T is a period of f , then $2T$ is also a period, and so indeed is any integral multiple of T . The smallest value of T for which equation (2) holds is called the **fundamental period** of f . A constant function is a periodic function with an arbitrary period but no fundamental period.

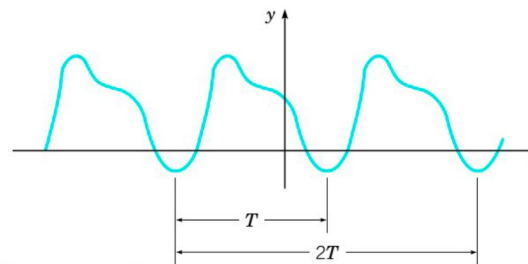


FIGURE 10.2.1 A periodic function of period T .

²Jean Baptiste Joseph Fourier (1768–1830) was twice imprisoned during the French Revolution, later served as scientific advisor in Napoleon's army in Egypt, and was prefect of the department of Isère (Grenoble) from 1801 to 1815. He made the first systematic use, although not a completely rigorous investigation, of trigonometric series in 1807 and 1811 in his papers on heat conduction. The papers were not published due to objections from the referees, principally Lagrange. Although it turned out that Fourier's claim of generality was somewhat too strong, his results inspired a flood of important research that has continued to the present day. See Grattan-Guinness or Carslaw (Historical Introduction) for a detailed history of Fourier series.

If f and g are any two periodic functions with common period T , then any linear combination $c_1f + c_2g$ is also periodic with period T . To prove this statement, begin by defining $F(x) = c_1f(x) + c_2g(x)$; then, for any x ,

$$F(x + T) = c_1f(x + T) + c_2g(x + T) = c_1f(x) + c_2g(x) = F(x). \quad (3)$$

Moreover, it can be shown that the sum of any finite number, or even the sum of a convergent infinite series, of functions of period T is also periodic with period T . In a similar way, you can show that the product fg is periodic with period T .

In particular, the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, 3, \dots$, are periodic with fundamental period $T = 2L/m$. To see this, recall that $\sin(x)$ and $\cos(x)$ have fundamental period 2π and that $\sin(\alpha x)$ and $\cos(\alpha x)$ have fundamental period $2\pi/\alpha$. If we choose $\alpha = m\pi/L$, then the period T of $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ is given by $T = 2\pi L/(m\pi) = 2L/m$.

Note also that, since every positive integral multiple of a period is also a period, each of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ has the common period $2L$.

Orthogonality of the Sine and Cosine Functions. To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, we generalize the concept of orthogonality of vectors (see Section 7.2). The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \leq x \leq \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)dx. \quad (4)$$

The functions u and v are said to be **orthogonal** on $\alpha \leq x \leq \beta$ if their inner product is zero—that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)dx = 0. \quad (5)$$

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \dots$ form a mutually orthogonal set of functions on the interval $-L \leq x \leq L$. In fact, they satisfy the following orthogonality relations:

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n; \end{cases} \quad (6)$$

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{all } m, n; \quad (7)$$

$$\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases} \quad (8)$$

These results can be obtained by direct integration. For example, to derive equation (8), note that

$$\begin{aligned} \int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= \frac{1}{2} \int_{-L}^L \left(\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right) \right) dx \\ &= \frac{1}{2} \frac{L}{\pi} \left(\frac{\sin((m-n)\pi x/L)}{m-n} - \frac{\sin((m+n)\pi x/L)}{m+n} \right) \Big|_{-L}^L \\ &= 0 \end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. Since m and n are positive, $m+n \neq 0$. On the other hand, if $m-n = 0$, then $m = n$, and the integral must be evaluated in a different way. In this case

$$\begin{aligned}
\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= \int_{-L}^L \left(\sin\left(\frac{m\pi x}{L}\right)\right)^2 dx \\
&= \frac{1}{2} \int_{-L}^L \left(1 - \cos\left(\frac{2m\pi x}{L}\right)\right) dx \\
&= \frac{1}{2} \left(x - \frac{L}{2m\pi} \sin\left(\frac{2m\pi x}{L}\right)\right) \Big|_{-L}^L \\
&= L.
\end{aligned}$$

This establishes equation (8); equations (6) and (7) can be verified by similar computations. (See Problem 12.)

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \leq x \leq L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \quad (9)$$

The coefficients a_m and b_m can be related to $f(x)$ as a consequence of the orthogonality conditions (6), (7), and (8). First multiply equation (9) by $\cos(n\pi x/L)$, where n is a fixed positive integer ($n > 0$), and integrate with respect to x from $-L$ to L . Assuming that the integration can be legitimately carried out term by term,³ we obtain

$$\begin{aligned}
\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx &= \frac{a_0}{2} \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) dx \\
&+ \sum_{m=1}^{\infty} a_m \int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\
&+ \sum_{m=1}^{\infty} b_m \int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx. \quad (10)
\end{aligned}$$

Keeping in mind that n is fixed whereas m ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only nonzero term on the right-hand side of equation (10) is the one for which $m = n$ in the first summation. Hence

$$\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = a_n \int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx = a_n L, \quad n = 1, 2, \dots \quad (11)$$

Equation (11) is valid for all $n = 1, 2, 3, \dots$. To determine a_0 , we can integrate equation (9) from $-L$ to L , obtaining

$$\begin{aligned}
\int_{-L}^L f(x) dx &= \frac{a_0}{2} \int_{-L}^L dx + \sum_{m=1}^{\infty} a_m \int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{m=1}^{\infty} b_m \int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) dx \\
&= a_0 L, \quad (12)
\end{aligned}$$

since each integral involving a trigonometric function over an interval of length $2L$ (one period) is zero. Thus

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \dots \quad (13)$$

By writing the constant term in equation (9) as $a_0/2$, it is possible to compute all the a_n from equation (13). Otherwise, a separate formula, with an extra factor of $1/2$, would have to be used for a_0 .

³This is a nontrivial assumption, because not all convergent series with variable terms can be so integrated. For the special case of Fourier series, however, term-by-term integration can always be justified.

A similar expression for b_n may be obtained by multiplying equation (9) by $\sin(n\pi x/L)$, integrating termwise from $-L$ to L , and using the orthogonality relations (7) and (8); thus

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots \quad (14)$$

Equations (13) and (14) are known as the **Euler-Fourier formulas** for the coefficients in a Fourier series. Hence, if the series (9) converges to $f(x)$, and if the series can be integrated term-by-term, then the coefficients *must be given* by equations (13) and (14).

Note that equations (13) and (14) are explicit formulas for a_n and b_n in terms of f , and that the determination of any particular coefficient is independent of all the other coefficients. Of course, the difficulty in evaluating the integrals in equations (13) and (14) depends very much on the particular function f involved.

Note also that the formulas (13) and (14) depend only on the values of $f(x)$ in the interval $-L \leq x \leq L$. Since each of the terms in the Fourier series (9) is periodic with period $2L$, the series converges for all x whenever it converges in $-L \leq x \leq L$, and its sum is also a periodic function with period $2L$. Hence $f(x)$ is determined for all x by its values in the interval $-L \leq x \leq L$.

It is possible to show (see Problem 27) that if g is periodic with period T , then every integral of g over an interval of length T has the same value. If we apply this result to the Euler-Fourier formulas (13) and (14), it follows that the interval of integration, $-L \leq x \leq L$, can be replaced, if it is more convenient to do so, by any other interval of length $2L$.

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \leq x < 0, \\ x, & 0 \leq x < 2; \end{cases} \quad L = 2$$
$$f(x + 4) = f(x).$$

Determine the coefficients in this Fourier series.

Solution

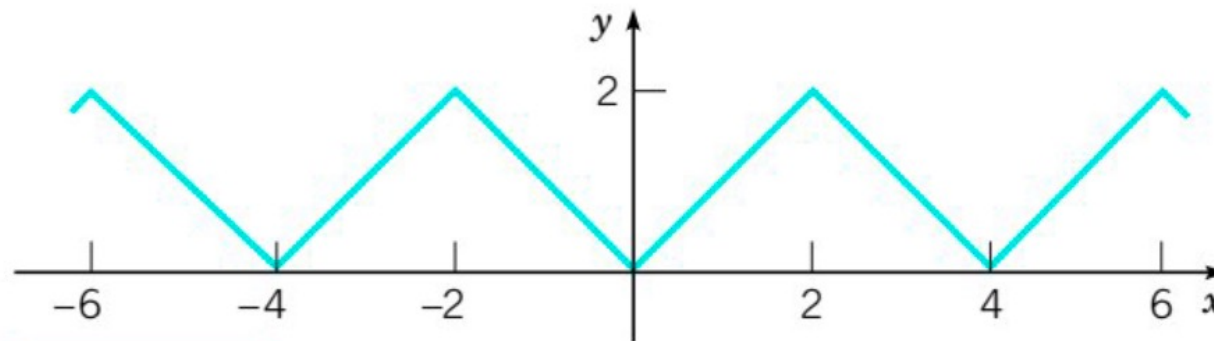


FIGURE 10.2.2 The triangular wave in Example 10.2.1.

$$\begin{aligned} f(x) &= 1 - \frac{8}{\pi^2} \left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2} \cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2} \cos\left(\frac{5\pi x}{2}\right) + \cdots \right) \\ &= 1 - \frac{8}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \cos\left(\frac{m\pi x}{2}\right) \\ &= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned}$$

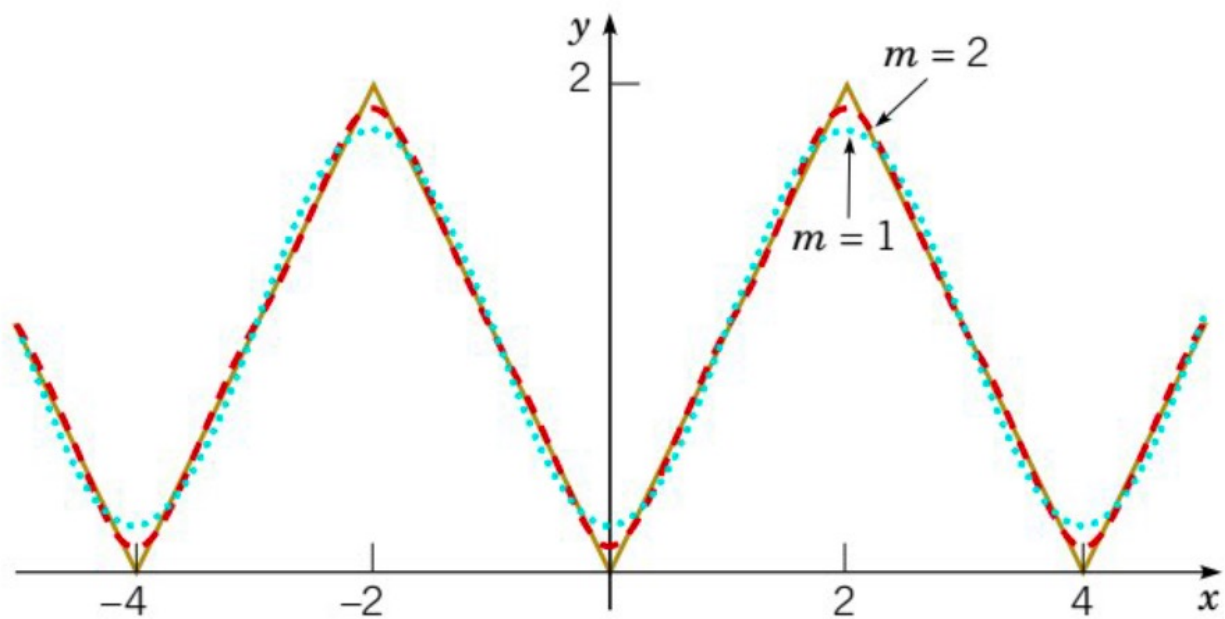


FIGURE 10.2.4 Partial sums in the Fourier series, equation (20) for $m = 1$ (dotted blue) and for $m = 2$ (dashed red), for the triangular wave (gold).

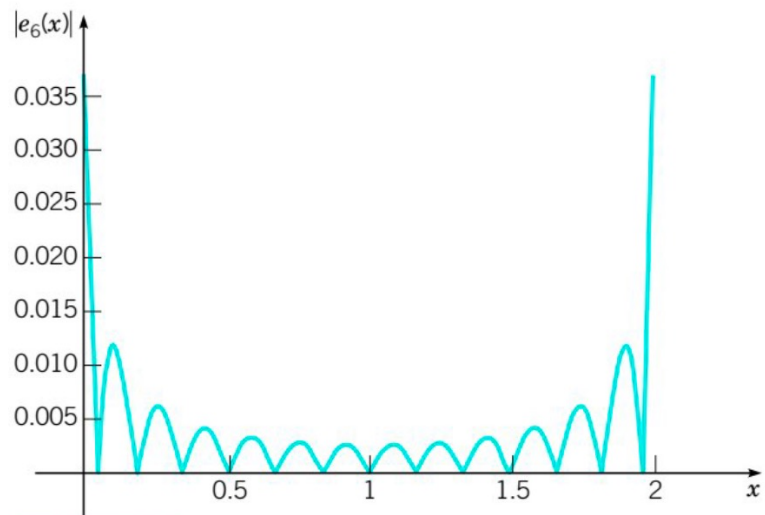


FIGURE 10.2.5 Plot of $|e_6(x)|$ versus x for the triangular wave in Example 10.2.3.

TABLE 10.2.1 Values of the Error $e_m(2)$ for the Triangular Wave

m	$e_m(2)$
2	0.09937
4	0.05040
6	0.03370
10	0.02025
15	0.01350
20	0.01013
25	0.00810

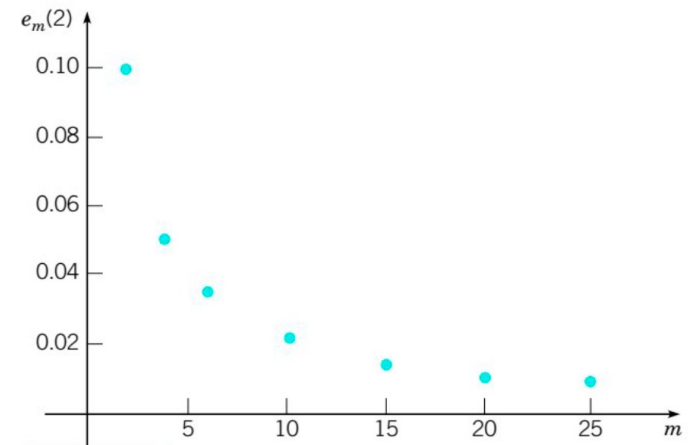


FIGURE 10.2.6 Plot of $e_m(2)$ versus m for the triangular wave.

The variable can be chosen to be time: $x \rightarrow t$

Any periodic signal of period T can be written as a **Fourier Series of Harmonics**.

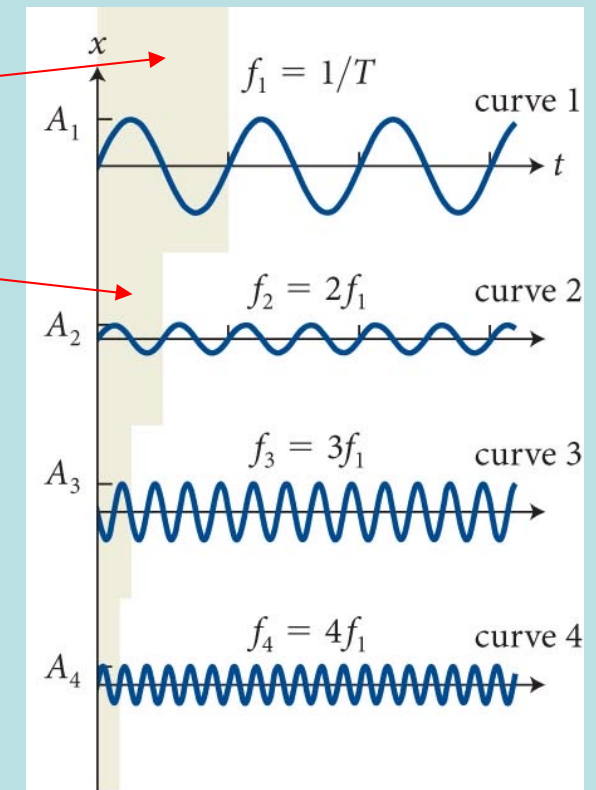
$$x(t) = \sum_{n=1}^{\infty} \left[a_n \cos \left(n \frac{2\pi}{T} t \right) + b_n \sin \left(n \frac{2\pi}{T} t \right) \right] = \sum_{n=1}^{\infty} [a_n \cos \omega_n t + b_n \sin \omega_n t]$$

各成分的週期 $T_n = \frac{T}{n}$ 是函數週期的整數分之一！

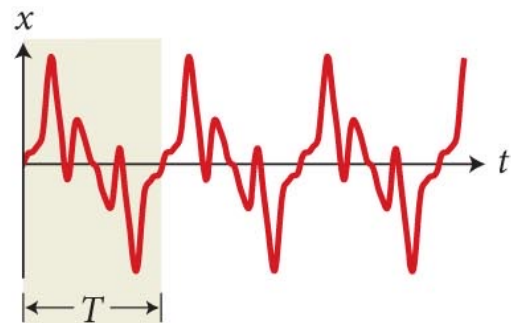
$$T_n = \frac{T}{n}$$

各成分的頻率 f_n 是訊號函數頻率 f 的整數倍！

研究三角函數就等於研究所有週期函數。



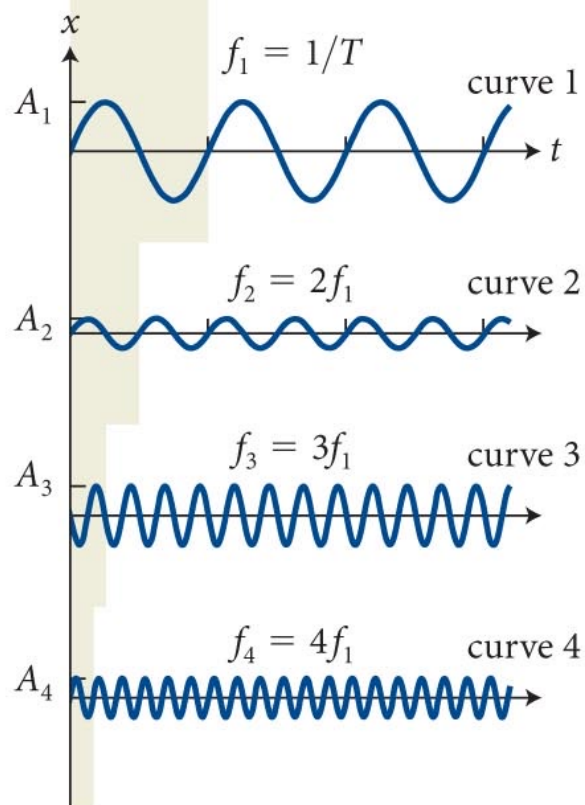
(a) Original curve



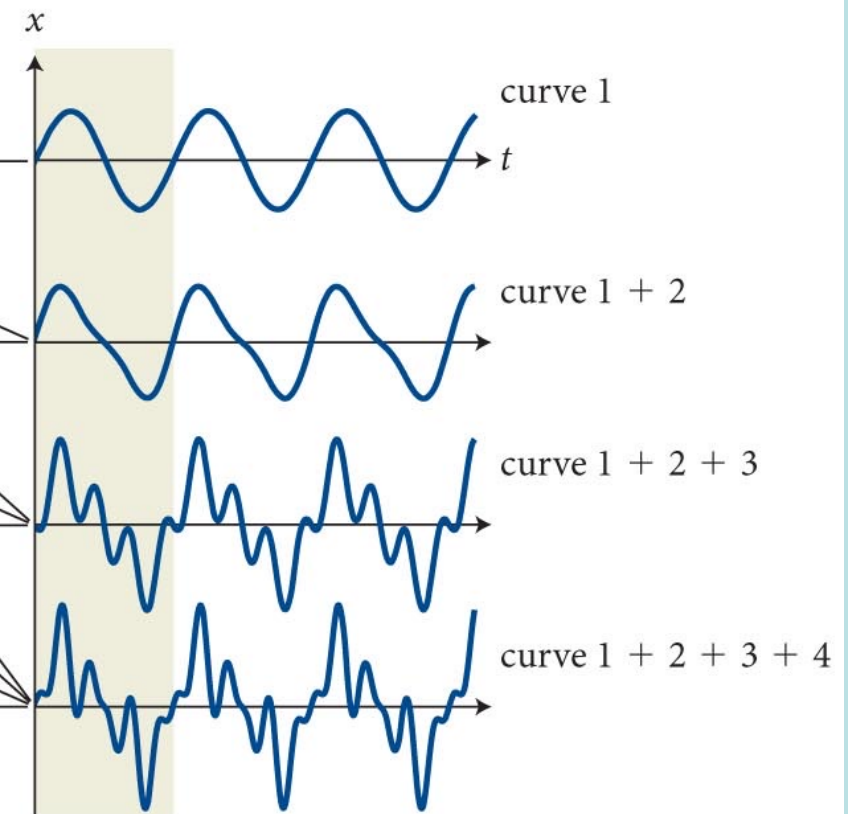
Harmonics:

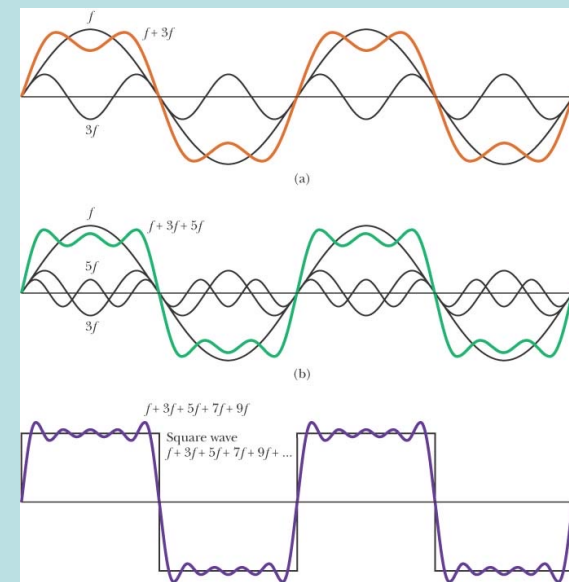
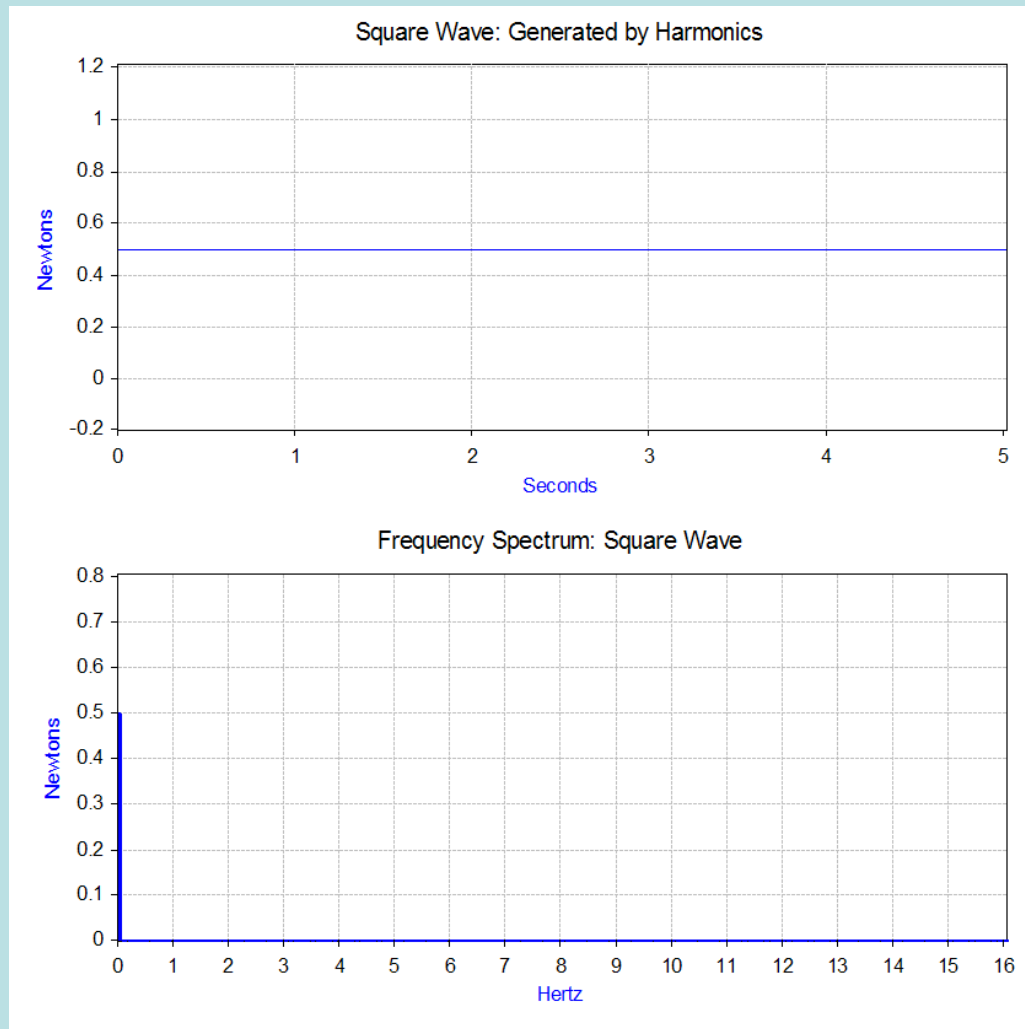
$$h_n(t) = \sin(2\pi f_n t) \quad f_n = n/T$$

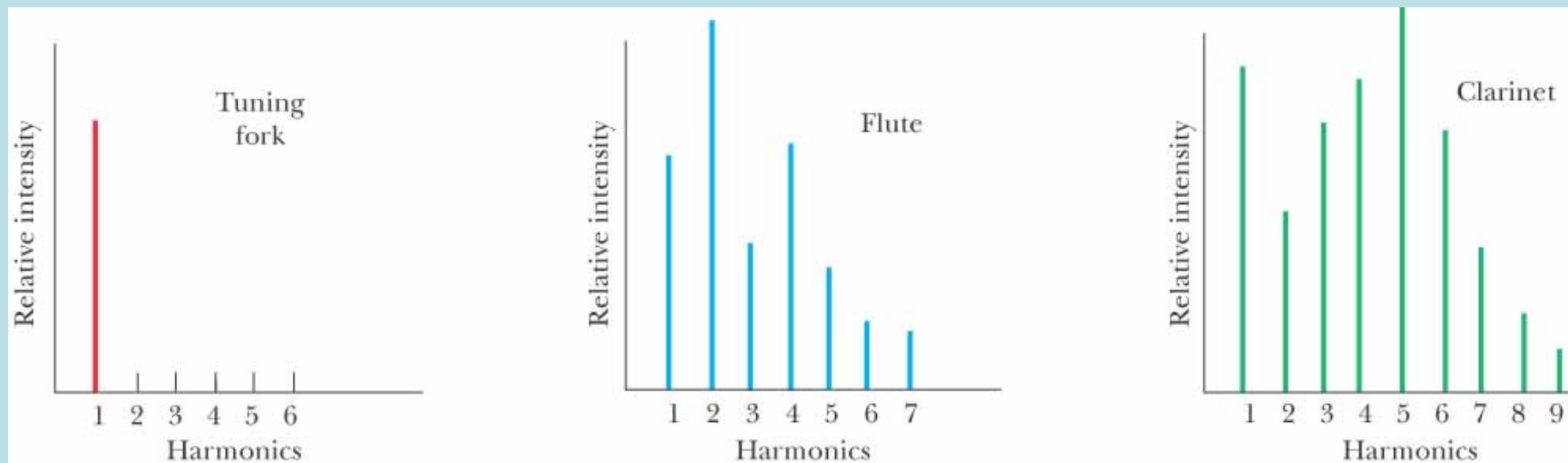
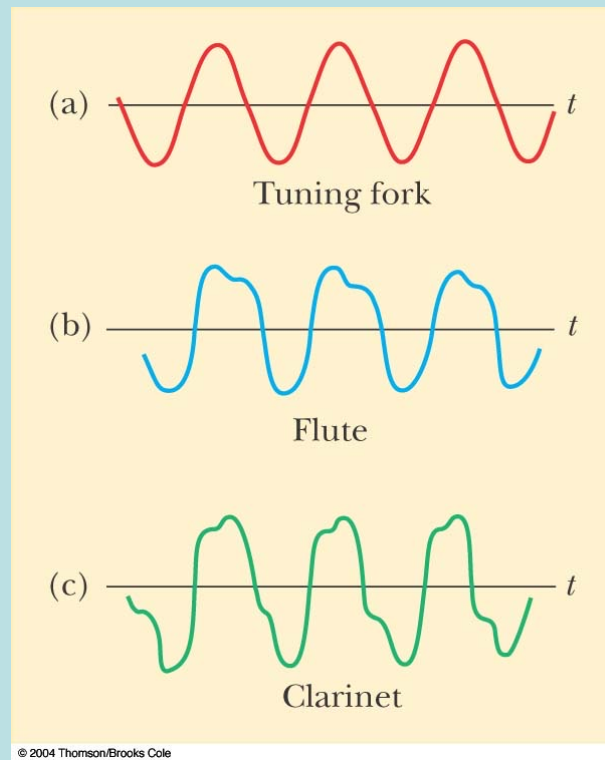
(b) Harmonics



(c) Sum of harmonics







一般來說， n 越大，強度越小，所以通常只要有限個成分就能近似一個周期運動！

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cdot \sin \frac{n\pi}{L} x + b_n \cdot \cos \frac{n\pi}{L} x \right]$$

It is sometimes useful to redefine $\frac{\pi}{L} x$ as new variable x .

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \sin nx + b_n \cdot \cos nx]$$

The period of $f(x)$ is now 2π , defined in the region: $-\pi < x < \pi$.

This is officially how Fourier Series is defined.

首先三角函數形成正交歸一的一系列函數：

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \frac{1}{2} \int_{-\pi}^{\pi} dx [\cos(m+n)x + \cos(m-n)x]$$

若 $m \neq n$

$$= \frac{1}{2} \left[\frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right] \Bigg|_{-\pi}^{\pi}$$

正弦函數是週期函數，上式為零。

若 $m = n$ ，

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \frac{1}{2} \int_{-\pi}^{\pi} dx [\cos(m+n)x + 1]$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} dx = \pi$$

綜合兩式可以表示為：

Kronecker δ Function

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \pi \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

同理：

$$\int_{-\pi}^{\pi} dx \sin mx \sin nx = \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = \frac{1}{2} \int_{-\pi}^{\pi} dx [\sin(m+n)x + \sin(m-n)x]$$

若 $m \neq n$

$$= \frac{1}{2} \left[\frac{1}{m+n} \cos(m+n)x + \frac{1}{m-n} \cos(m-n)x \right] \Big|_{-\pi}^{\pi} = 0$$

若 $m = n$ ，

$$= \frac{1}{2} \left[\frac{1}{m+n} \cos(2m)x \right] \Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = 0$$

三角函數形成正交歸一的一系列函數：

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \sin mx \sin nx = \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = 0$$

The coefficients can be easily computed:

我們只要對 $f(x)$ 成三角函數後積分即可。

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx f(x) \cos nx = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \left[a_0 + \sum_{m=1}^{\infty} [a_m \cdot \cos mx + b_m \cdot \sin mx] \right] \cos nx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} dx \left[a_0 + \sum_{m=1}^{\infty} [a_m \cdot \cos mx + b_n \cdot \sin mx] \right] \cos nx = \sum_{m=1}^{\infty} [a_m \delta_{mn}] = a_n$$

完整的傅立葉級數公式：

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos nx + b_n \cdot \sin nx], -\pi < x < \pi$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx f(x) \cos nx$$

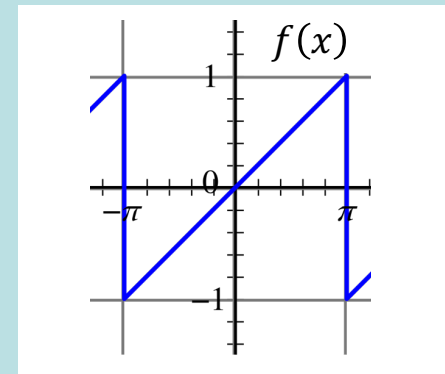
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx f(x) \sin nx$$

You observed Fourier series sometimes comes with only Sine series or Cosine series

If $f(x)$ is an Odd function, only Sine series contribute !

$$f(-x) = -f(x)$$

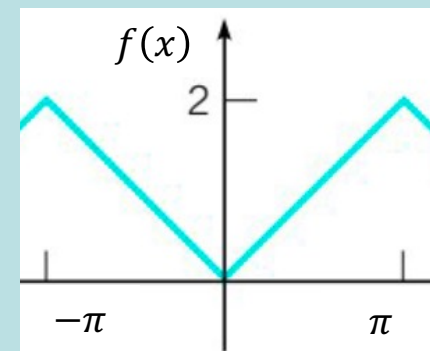
$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \sin nx$$



If $f(x)$ is an even function, only Cosine series contribute !

$$f(-x) = f(x)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \cdot \cos nx$$



例如：Sawtooth 鋸齒函數

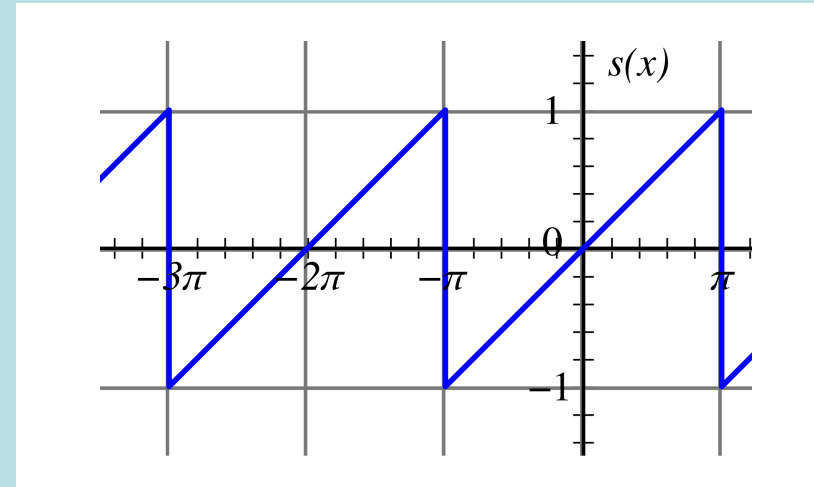
$$u_0(x) = \frac{1}{\pi} x, \quad -\pi < x < \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \frac{1}{\pi} x \sin(nx) =$$

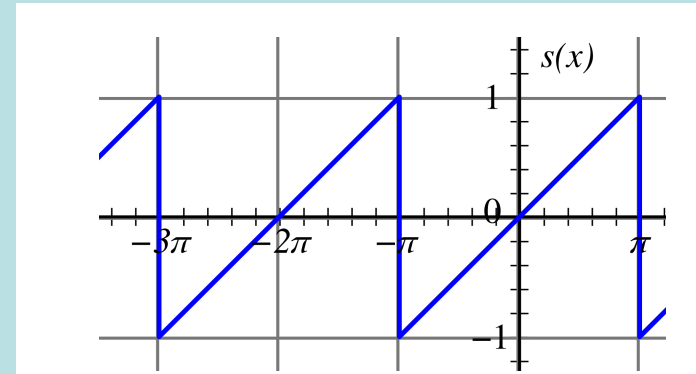
$$= \frac{1}{\pi^2} \frac{1}{n} [x \cos(nx)] \Big|_{-\pi}^{\pi} + \frac{1}{\pi^2} \frac{1}{n} \int_{-\pi}^{\pi} dx \cos(nx)$$

$$= 2 \frac{1}{n\pi} \cos(n\pi) + \left(\frac{1}{n\pi} \right)^2 \left[x \sin \left(\frac{n\pi}{L} x \right) \right] \Big|_{-\pi}^{\pi}$$

$$= 2 \frac{(-1)^n}{n\pi}$$



$$u_0(x) = \frac{1}{\pi}x, \quad -\pi < x < \pi$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \frac{1}{\pi} x \cos(nx) = \frac{1}{\pi} \int_{-\pi}^0 dx \frac{1}{\pi} x \cos(nx) + \frac{1}{\pi} \int_0^{\pi} dx \frac{1}{\pi} x \cos(nx)$$

$$= \frac{1}{\pi} \int_0^{\pi} dx \frac{1}{\pi} (-x) \cos(-nx) + \frac{1}{\pi} \int_0^{\pi} dx \frac{1}{\pi} x \cos(nx) = 0$$

餘弦函數是偶函數Even function

$$\cos(-nx) = \cos(nx)$$

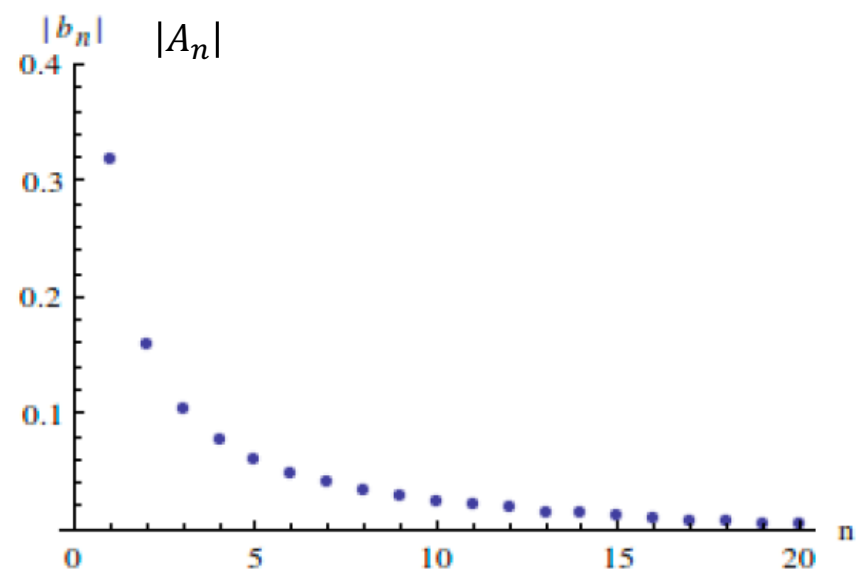
起始條件是奇函數Odd function

$$u_0(-x) = -u_0(x)$$

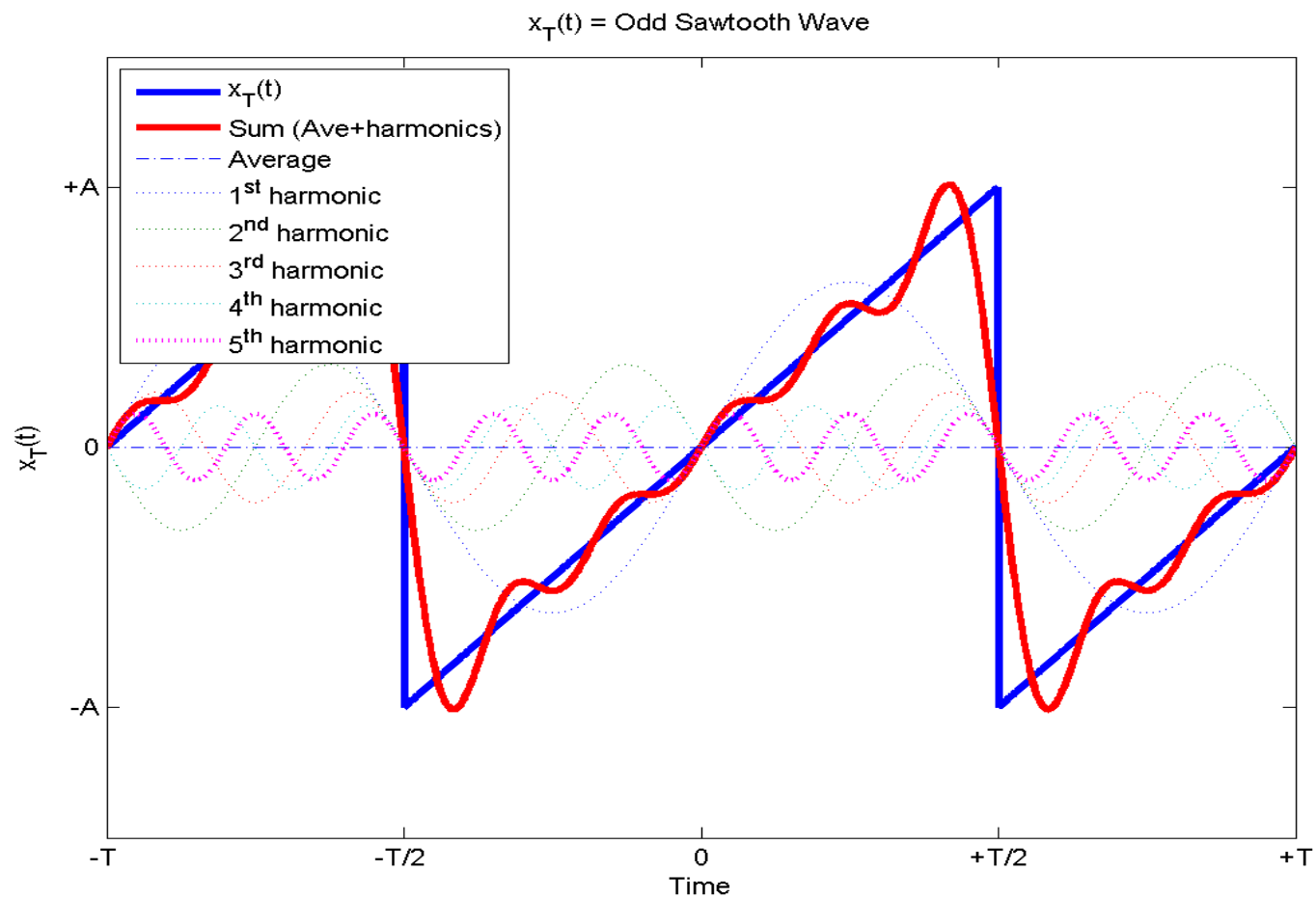
相乘後為奇函數，在原點兩邊積分將互相抵消！

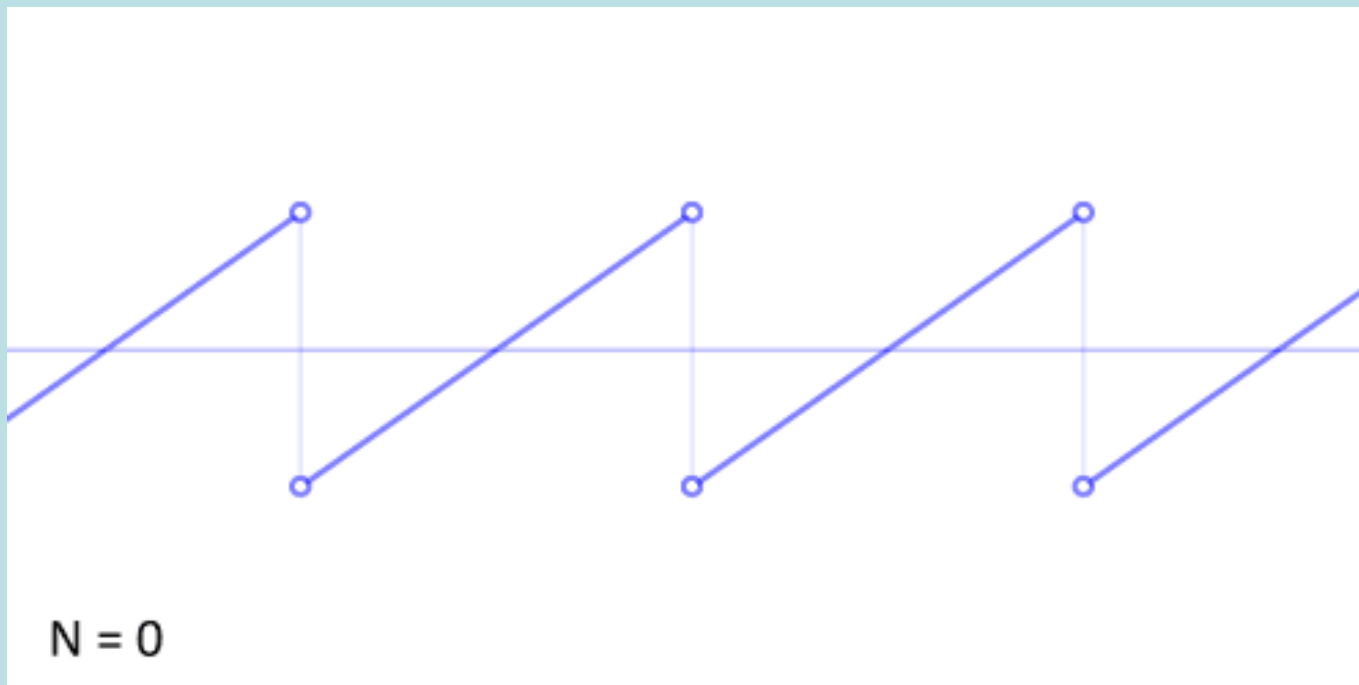
鋸齒函數的傅立葉級數展開如下：

$$\frac{1}{\pi}x = \sum_{n=1}^{\infty} \left(2 \frac{(-1)^n}{n\pi} \cdot \sin nx \right)$$



Amplitudes of the relative harmonics of a string plucked with a sawtooth plucking.





數學家早已證明三角函數是完備的。

$$\lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} dx \left[f(x) - a_0 + \sum_{n=1}^N [a_n \cdot \cos nx + b_n \cdot \sin nx] \right]^2 = 0$$



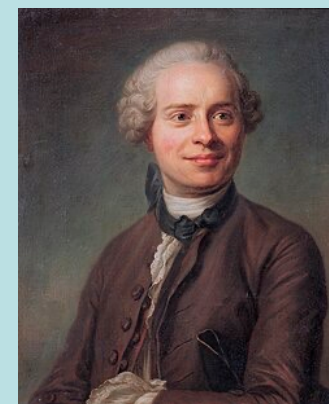
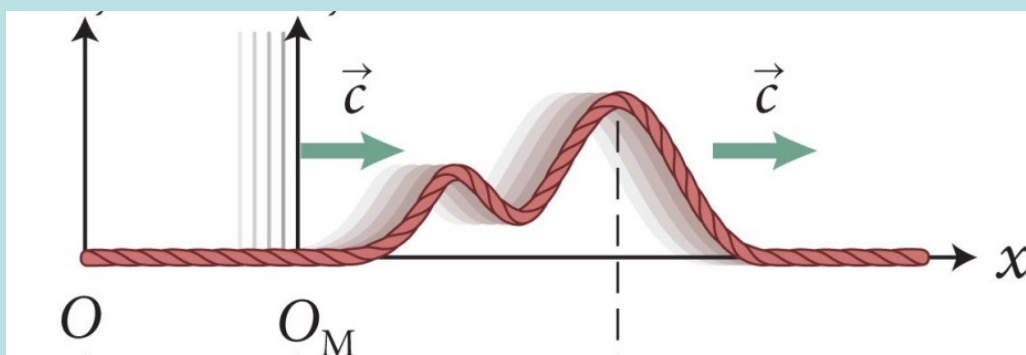
There is another easier way to solve wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

This is called d'Alembert solution.

Prove wave equation has the following solution:

$$f(x - vt)$$



Jean le Rond d'Alembert 1717-83

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{波方程式}$$



Plug in the expression. Note that it is a function of only $x' \equiv x - vt$.

$$y(x, t) = f(x - vt) = f(x') \quad x' \equiv x - vt$$

$$\frac{\partial y}{\partial x} = \frac{\partial x'}{\partial x} \frac{df}{dx'} = \frac{df}{dx'} \quad \text{fix } t \text{ and use chain rule.} \quad \frac{\partial x'}{\partial x} = 1 \quad \text{with } t \text{ fixed}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{df}{dx'} \right) = \frac{\partial x'}{\partial x} \frac{d}{dx'} \left(\frac{df}{dx'} \right) = \frac{d^2 f}{dx'^2}$$

Second partial is second derivative of f .

$$\frac{\partial y}{\partial t} = \frac{\partial x'}{\partial t} \frac{df}{dx'} = -v \frac{df}{dx'} \quad \text{fix } x \text{ and use chain rule.} \quad \frac{\partial x'}{\partial t} = -v \quad \text{with } x \text{ fixed}$$

$$\frac{\partial^2 y}{\partial t^2} = -v \frac{\partial}{\partial t} \left(\frac{df}{dx'} \right) = -v \frac{\partial x'}{\partial t} \frac{d}{dx'} \left(\frac{df}{dx'} \right) = v^2 \frac{d^2 f}{dx'^2}$$

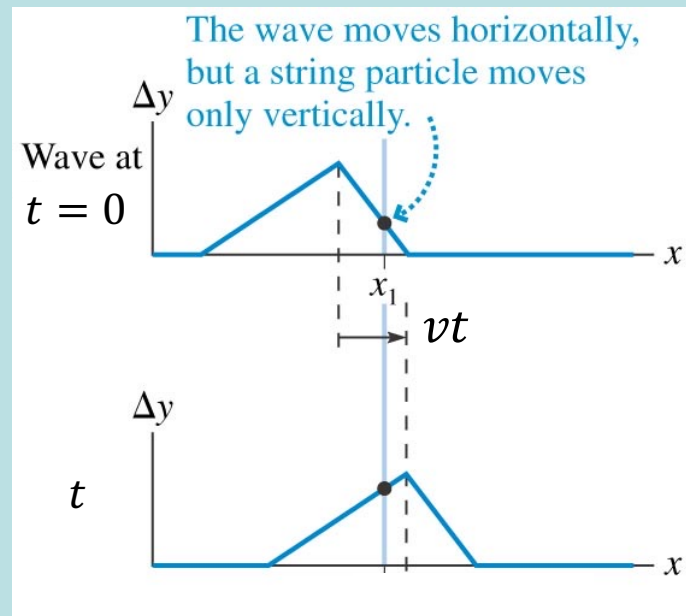
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Indeed, what we guessed is a solution of wave equation !

x' 將時間與空間連鎖在一起，於是時間微分與空間微分也是連鎖的。

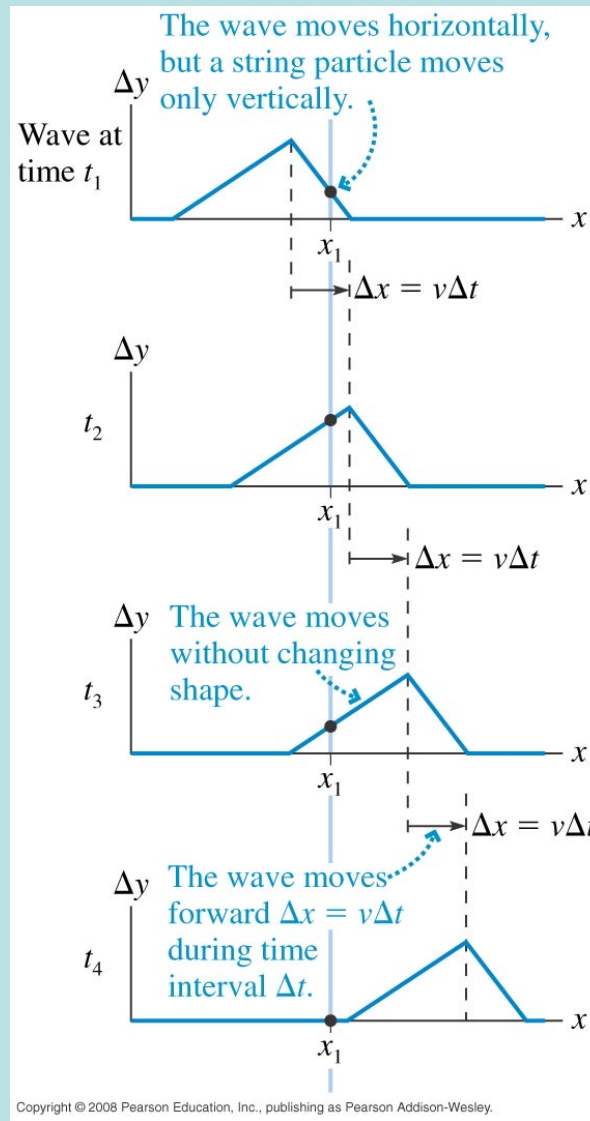
Function $f(x)$ is wave form at $t = 0$ ◦

$$y(x, t = 0) = f(x - v \cdot 0) = f(x)$$

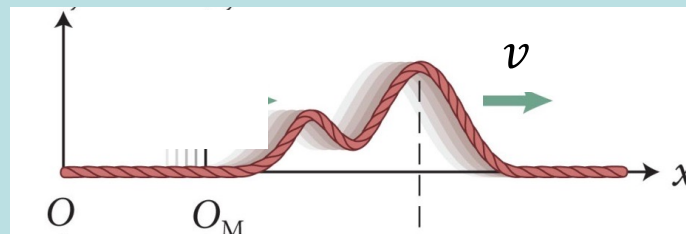


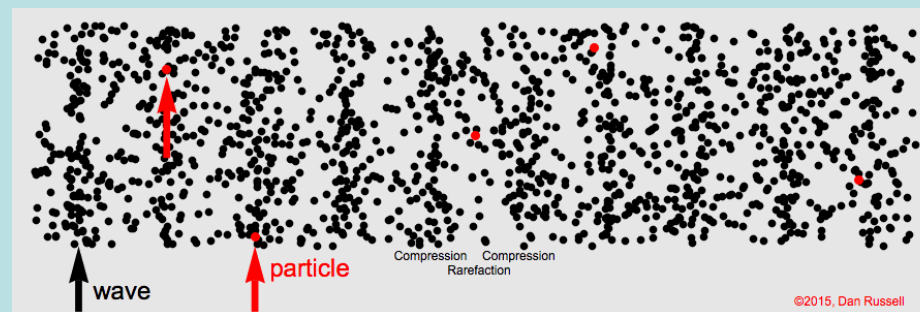
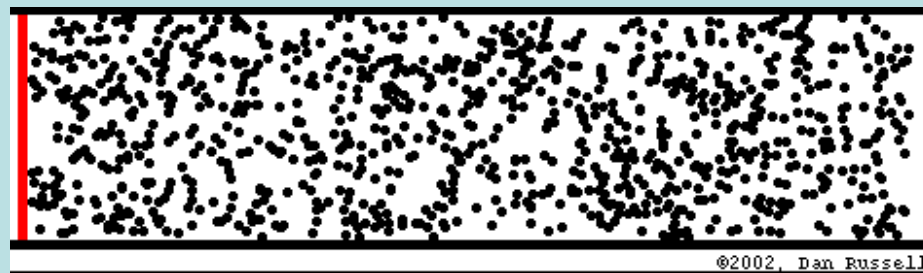
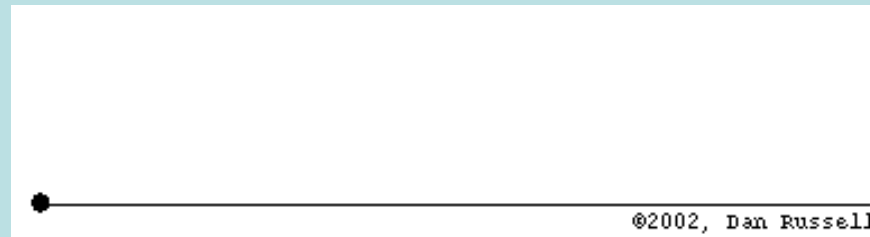
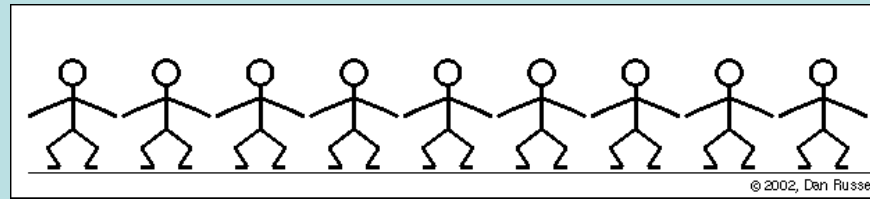
$$y(x, t) = y(x - vt, 0) = f(x - vt)$$

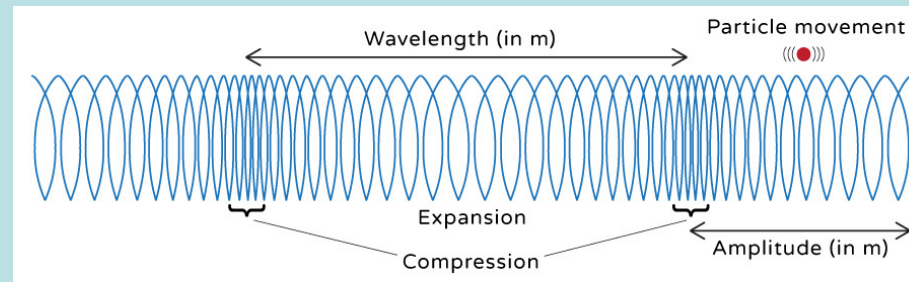
At later time t , wave at x $y(x, t)$ equals wave at $t = 0$ at location $x - vt$.



$f(x - vt)$ means that a wave form $f(x)$ is moving at speed v , as we expected.







Longitudinal and Transverse Waves (HD)
Animations for Physics and Astronomy

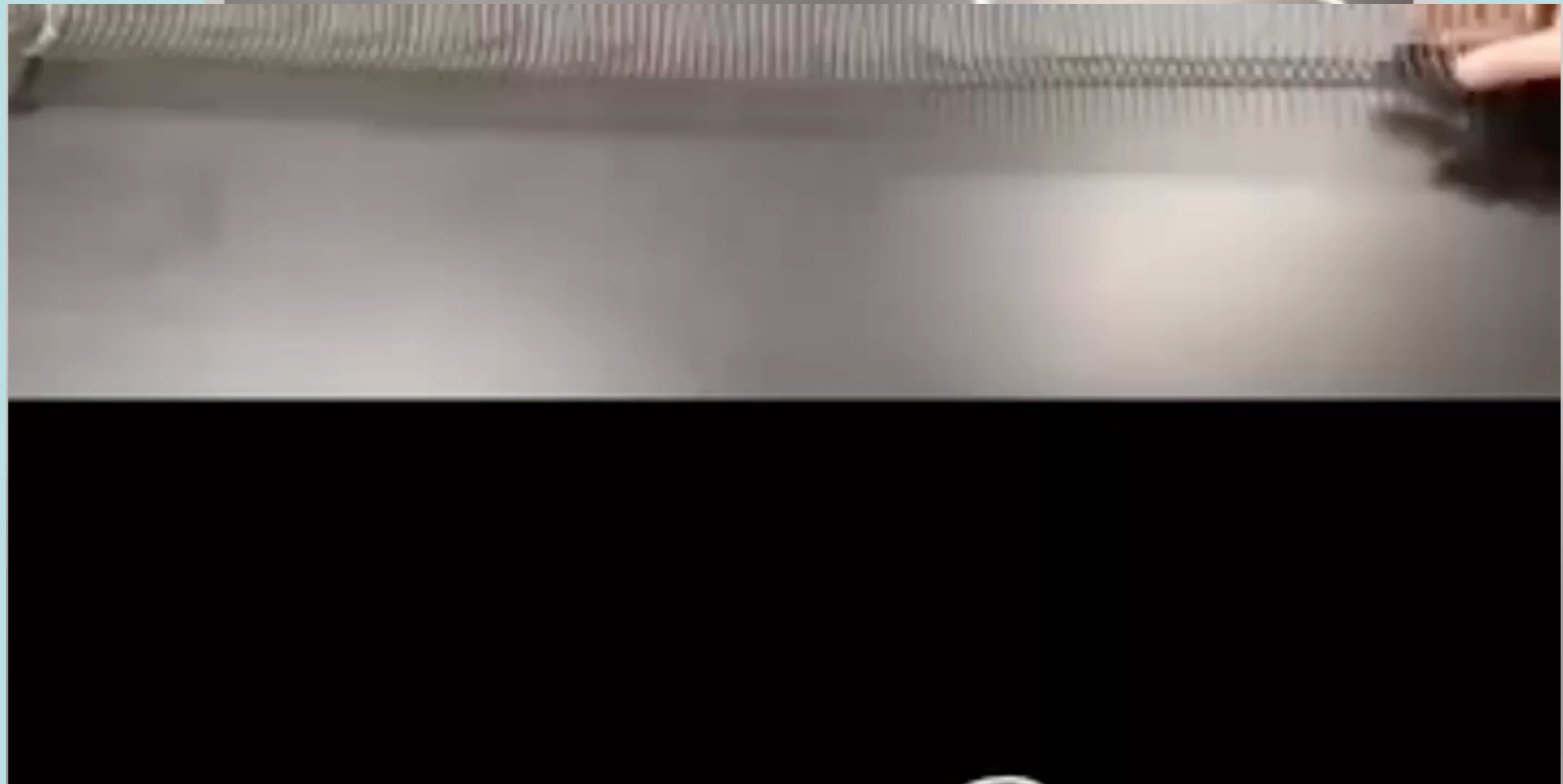
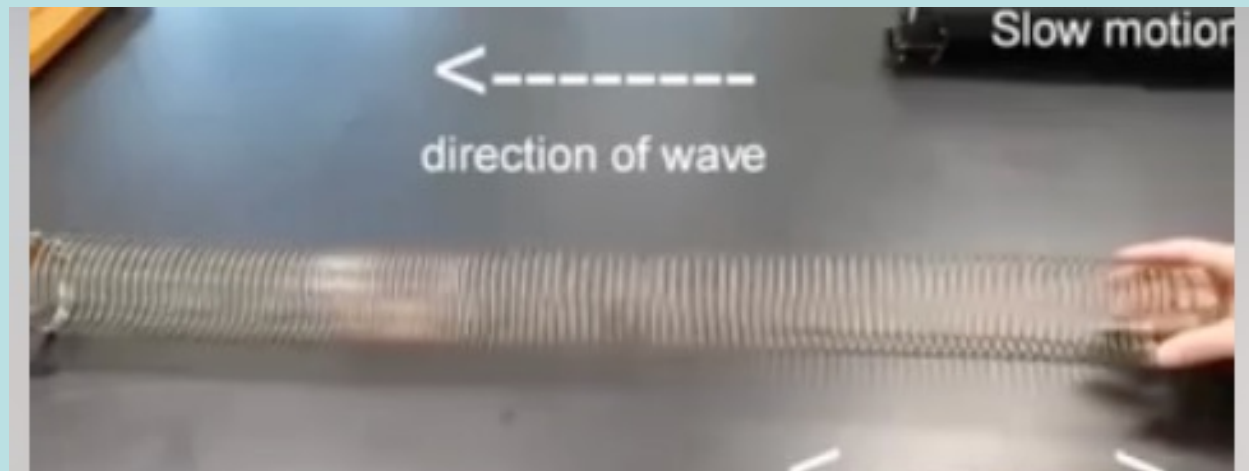
0:00 / 0:24

Longitudinal Wave

More videos 

YouTube

The video player shows a 3D animation of a longitudinal wave on a dark blue background. A yellow hand is shown moving the end of a spring up and down, indicated by a red double-headed arrow. The wave itself is a grey sinusoidal wave. A white play button is centered over the wave. The video title 'Longitudinal and Transverse Waves (HD)' and the channel name 'Animations for Physics and Astronomy' are at the top. The video progress bar shows '0:00 / 0:24'. At the bottom, there are links for 'More videos' and the 'YouTube' logo.



Sinusoidal wave

$$f(x) = y_m \sin(kx + \phi)$$

Wave form is a periodic Sine function of period λ .

$$\sin(kx + \phi) = \sin(kx + k\lambda + \phi)$$

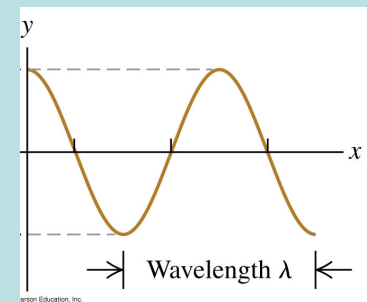
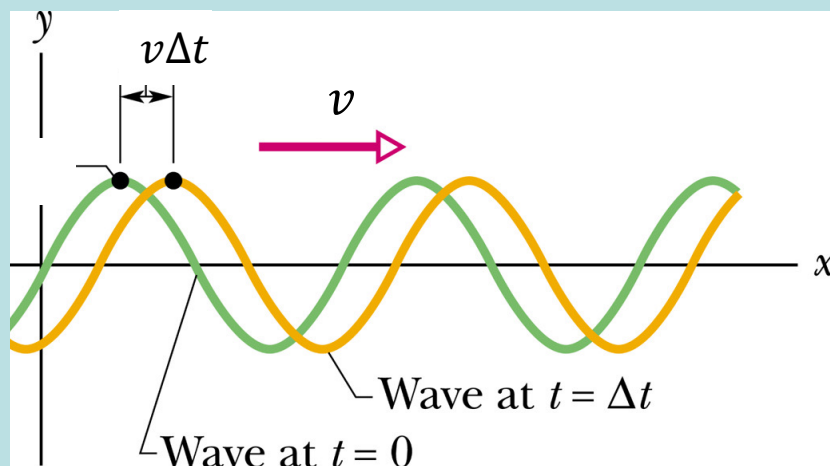
$$k\lambda = 2\pi$$

$$k \equiv \frac{2\pi}{\lambda}$$

Sinusoidal wave function:

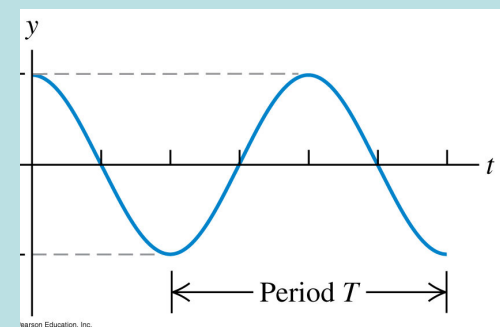
$$y(x, t) = f(x - vt) = y_m \sin(kx - kvt + \phi)$$

$$y(x, t) = y_m \sin(kx - \omega t + \phi)$$

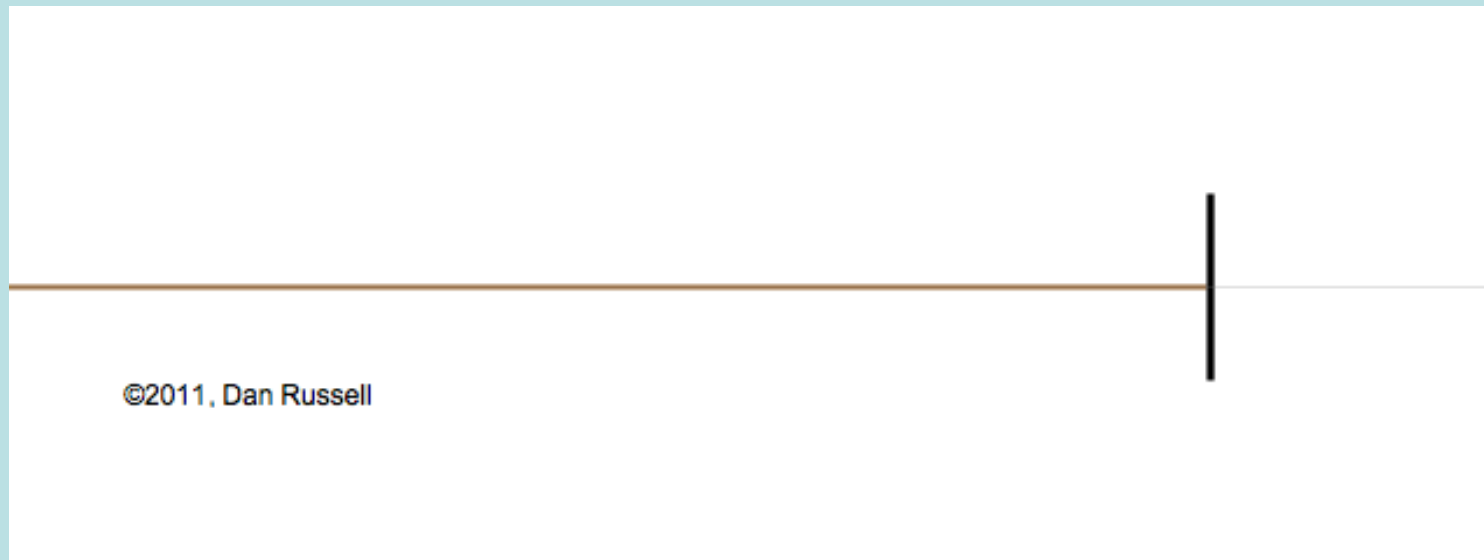


$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$f = \frac{1}{T}$$



This is periodic wave with specified wavelength and frequency in uniform medium!

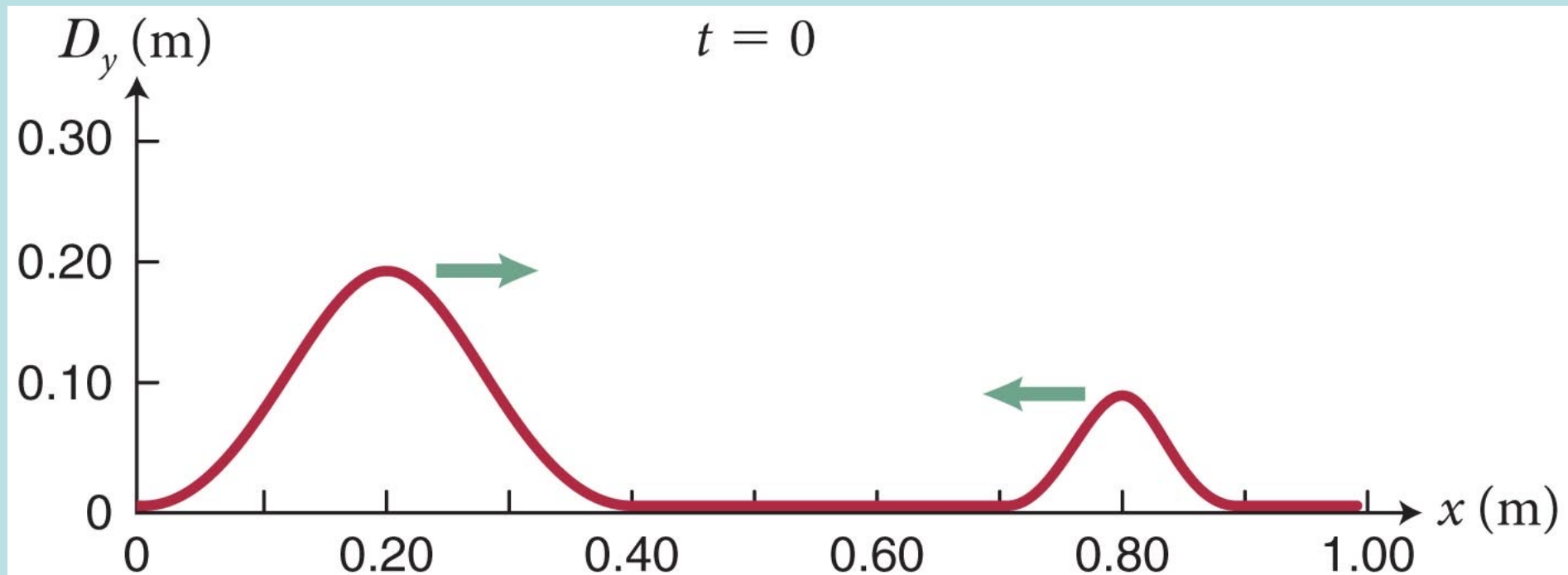


$$y(x, t) = y_m \sin(kx - \omega t + \phi)$$

$f(x - vt)$ means that a wave form $f(x)$ is moving at speed v to the right.



Replacing v with $-v$, we find a wave function $g(x + vt)$ moving to the left.



$$y(x, t) = g(x + vt)$$

疊加定理 Superposition Principle

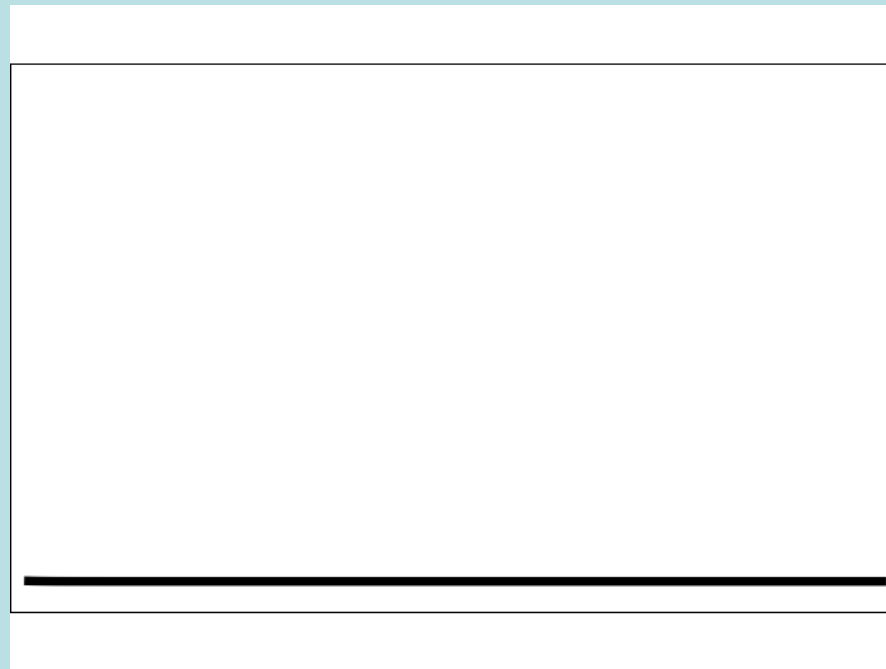
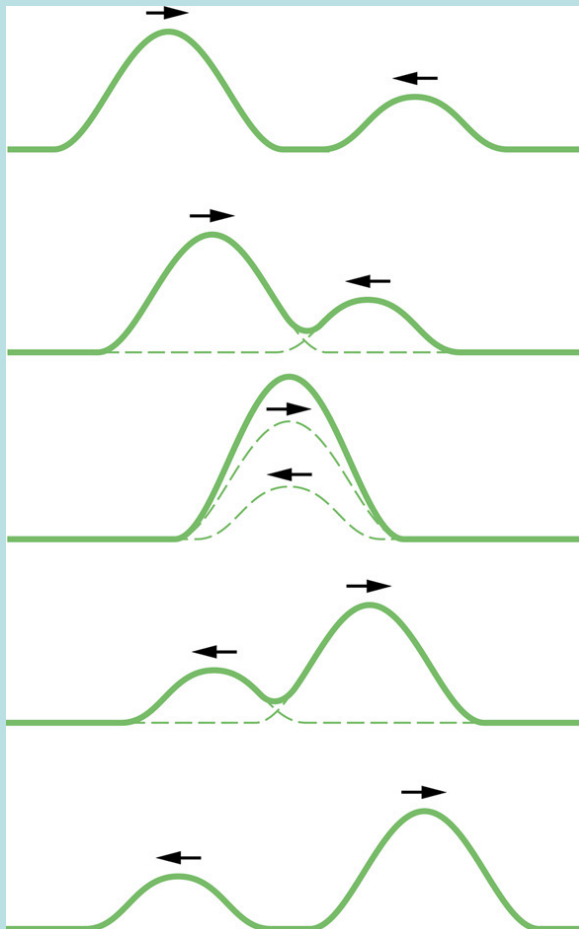
Since wave Equation is linear, linear combinations of solutions are also solutions.

$$\frac{\partial^2 y_1}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y_1}{\partial t^2}$$

$$\frac{\partial^2 y_2}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y_2}{\partial t^2}$$



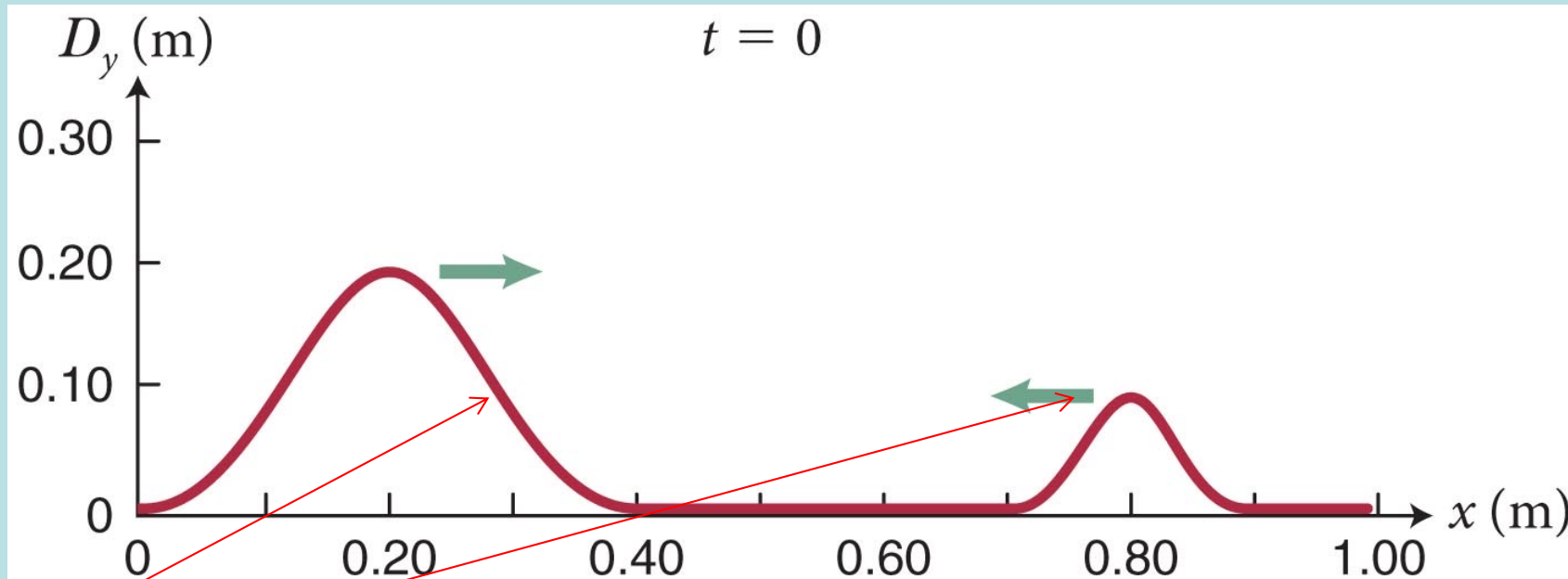
$$\frac{\partial^2 (y_1 + y_2)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 (y_1 + y_2)}{\partial t^2}$$





A general solution: d'Alembert solution

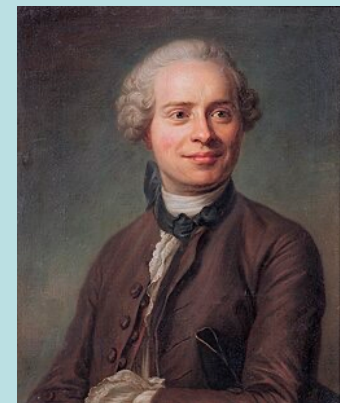
$$y(x, t) = f(x - vt) + g(x + vt)$$



$f(x), g(x)$ are waveforms propagating to the right and the left respectively.

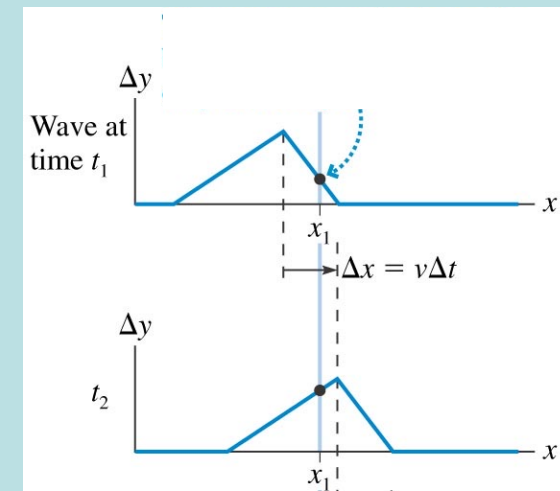
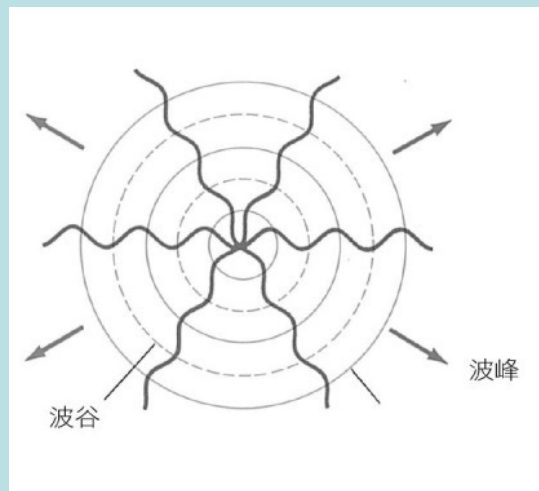
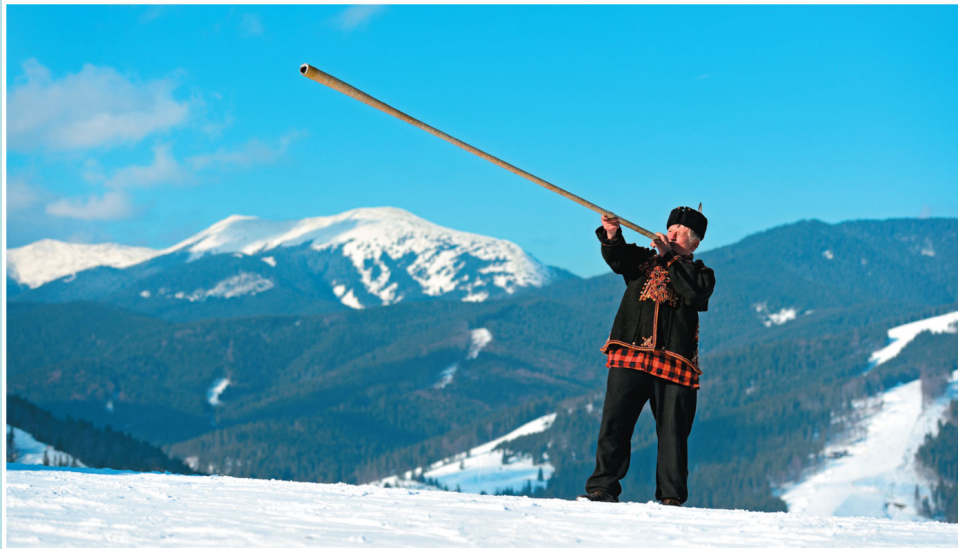
The wave forms propagates at constant wave speed.

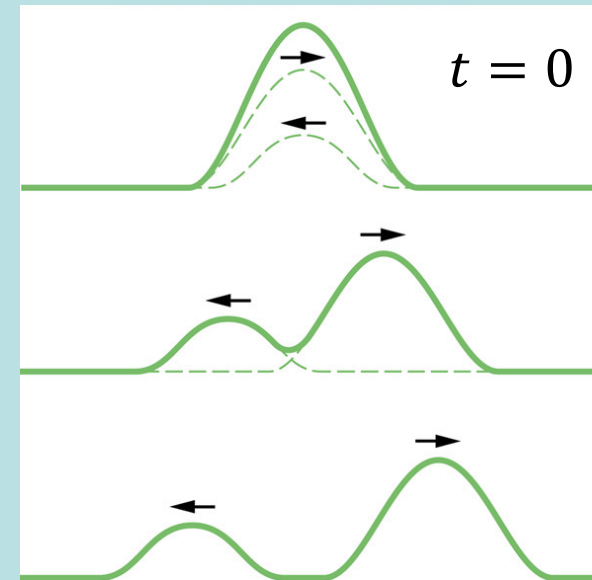
The wave forms do not change during propagation.



Jean le Rond d'Alembert 1717-83

These are properties we expect from a Propagating Wave





$$y(x, t) = f(x - vt) + g(x + vt)$$

Functions $f(x)$, $g(x)$ can be solved from initial conditions: $y(x, 0)$, $\frac{\partial y}{\partial t}(x, 0)$.

$$y(x, 0), \frac{\partial y}{\partial t}(x, 0) \leftrightarrow f(x), g(x)$$

$$y(x, 0) = u_0(x) = f(x) + g(x)$$

$$\frac{\partial y}{\partial t}(x, 0) = v_0(x) = v[f'(x) - g'(x)]$$

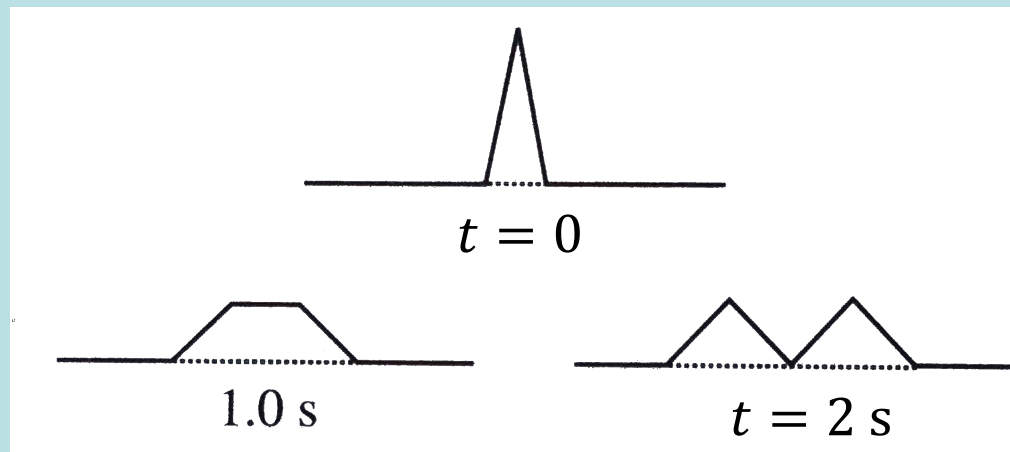
$$y(x, t) = \frac{1}{2}[u_0(x - vt) + u_0(x + vt)] + \frac{1}{2v} \int_{x-vt}^{x+vt} dx' v_0(x')$$

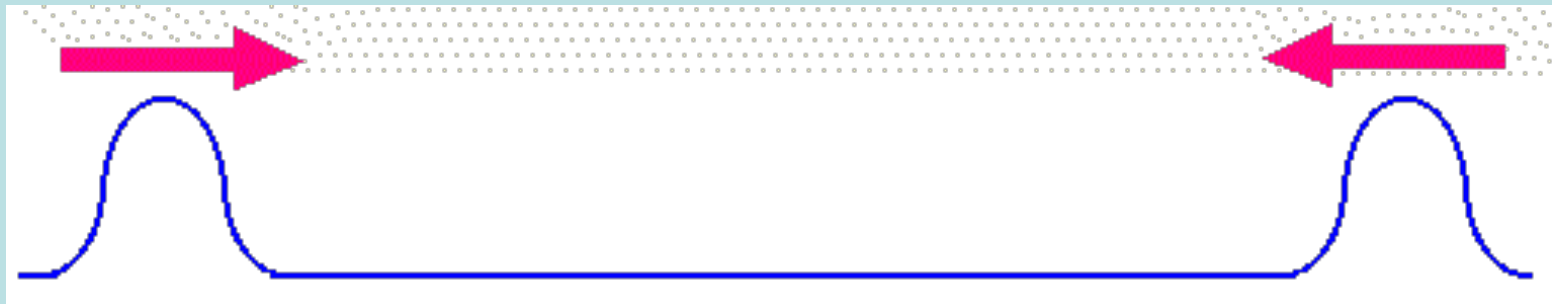
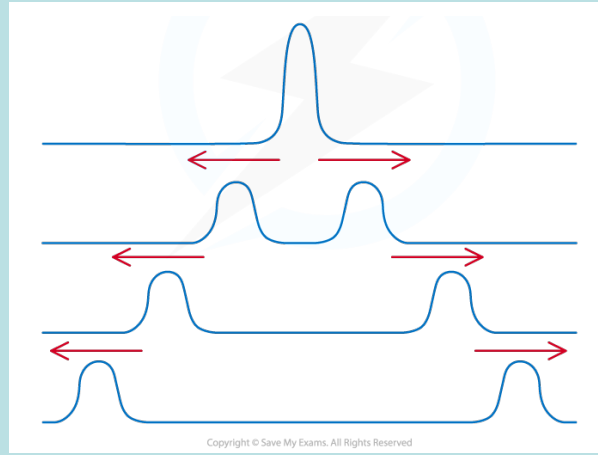
Take the case $v_0 = 0$ as examples. $u_0(x)$ is the initial wave form.

$$y(x, t) = \frac{1}{2} [u_0(x - vt) + u_0(x + vt)]$$

Right moving wave form and left are exactly one half the initial wave form at rest.

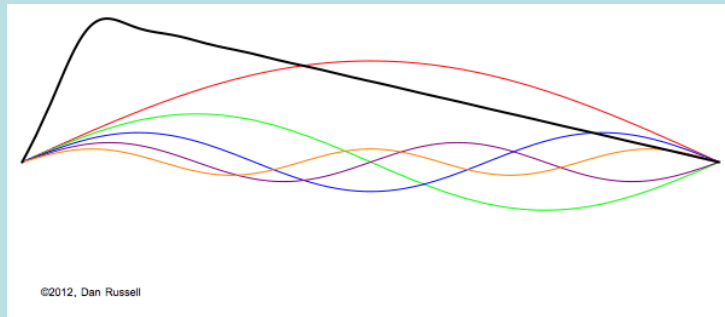
The wave form will slip into two, one propagating to the right, one to the left.





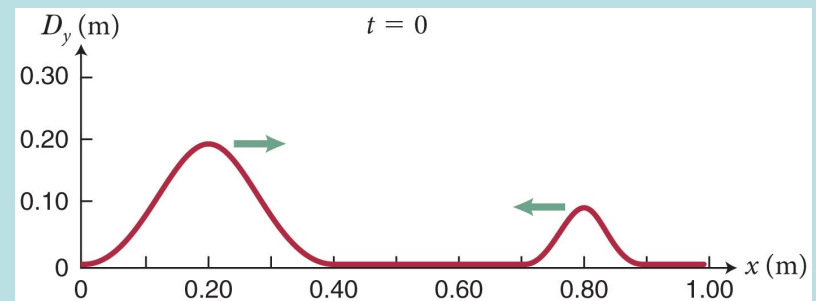
Bernoulli's solution

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$



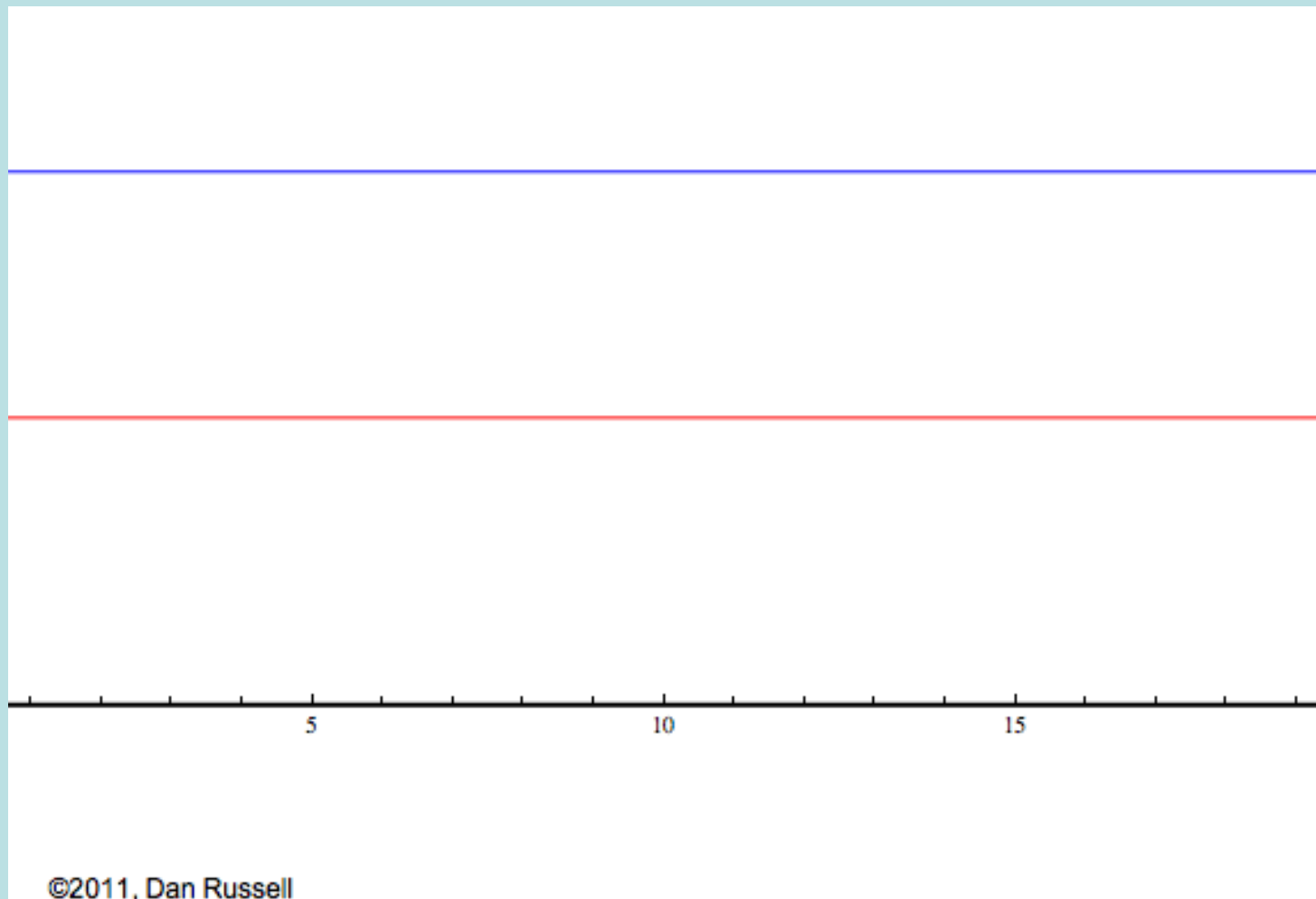
d'Alembert solution:

$$f(x - vt) + g(x + vt)$$

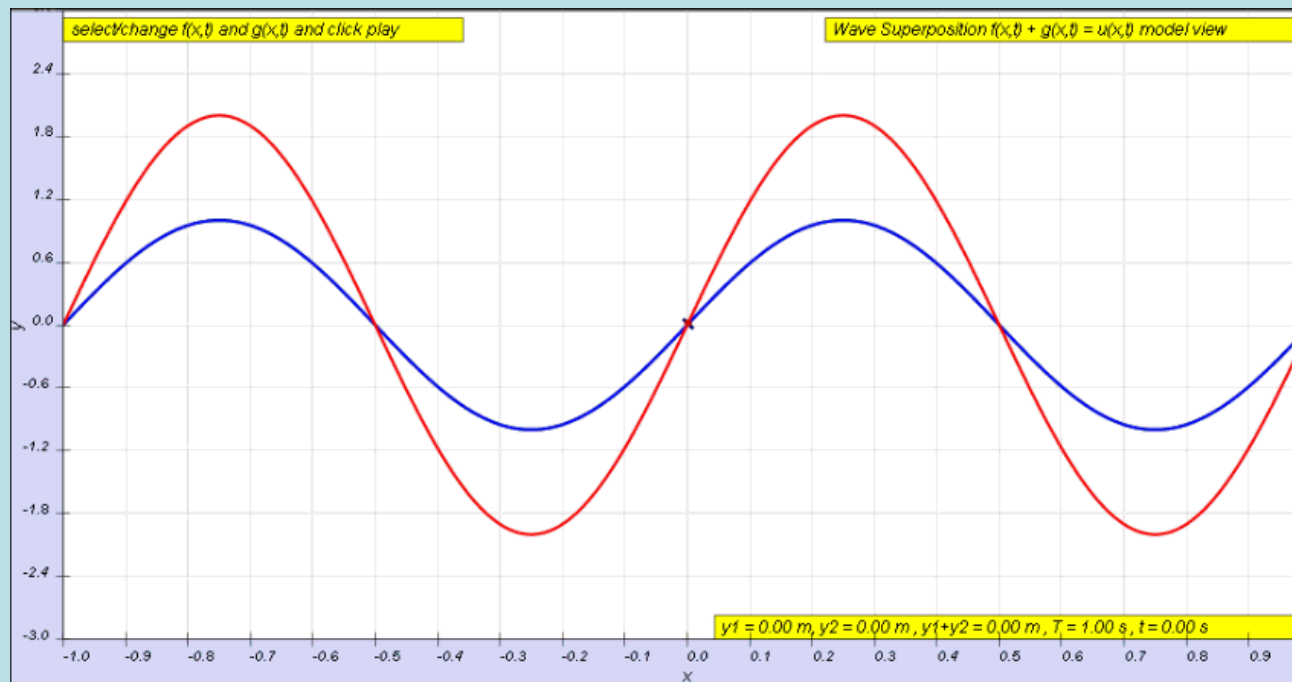


Two types of solutions are equivalent.

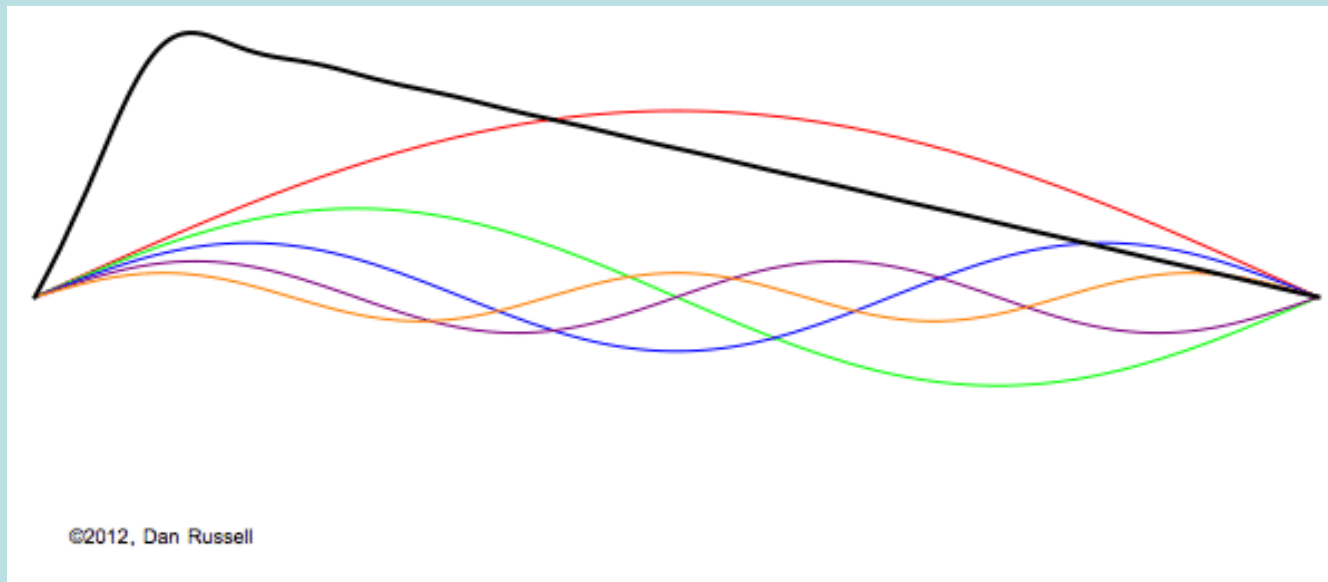
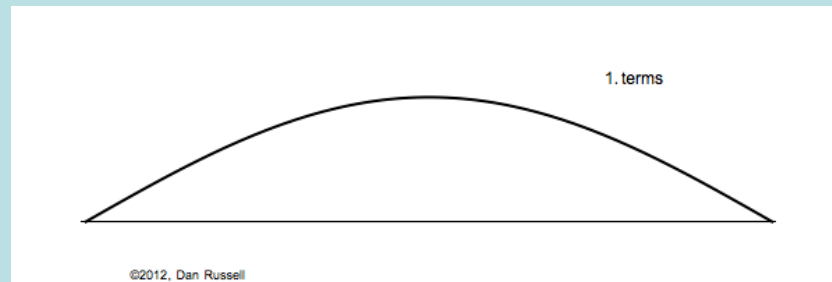
Standing waves (separable solutions) are d'Alembert solutions.



Standing waves (separable solutions) are d'Alembert solutions.

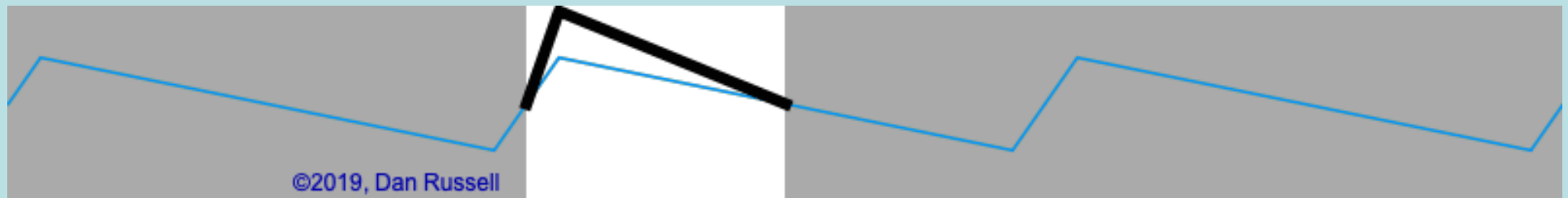


Sawtooth waves (as a Bernoulli solution-coloured lines)



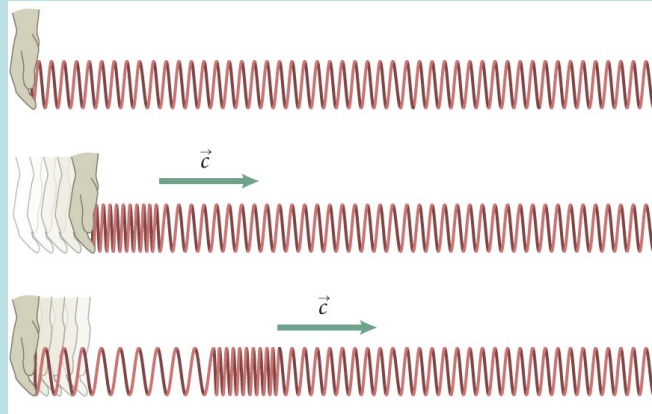
$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

Sawtooth waves (as a Bernoulli solution-coloured lines) are d'Alembert solutions.



Wave Equation

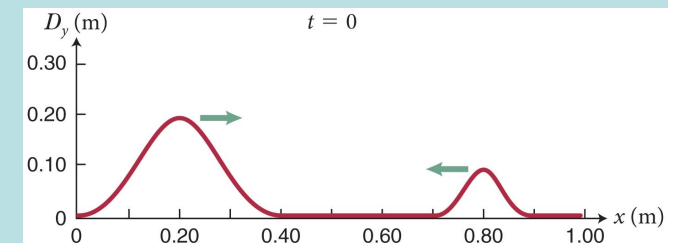
$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$



If the size of the medium is infinite or its boundaries are far away, we have d'Alembert solution of a **travelling or propagating** wave from :

$$\phi(x, t) = f(x - vt) + g(x + vt)$$

行進傳播





If the size of the medium is finite, its boundaries are nearby.

振盪

Reflection from boundaries would make wave behave more like **vibrations**!

It resembles a system of coupled springs! Bernoulli's solution works better.

