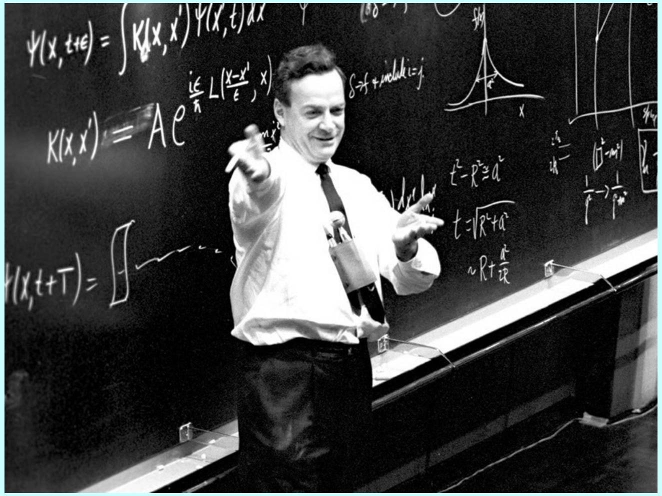
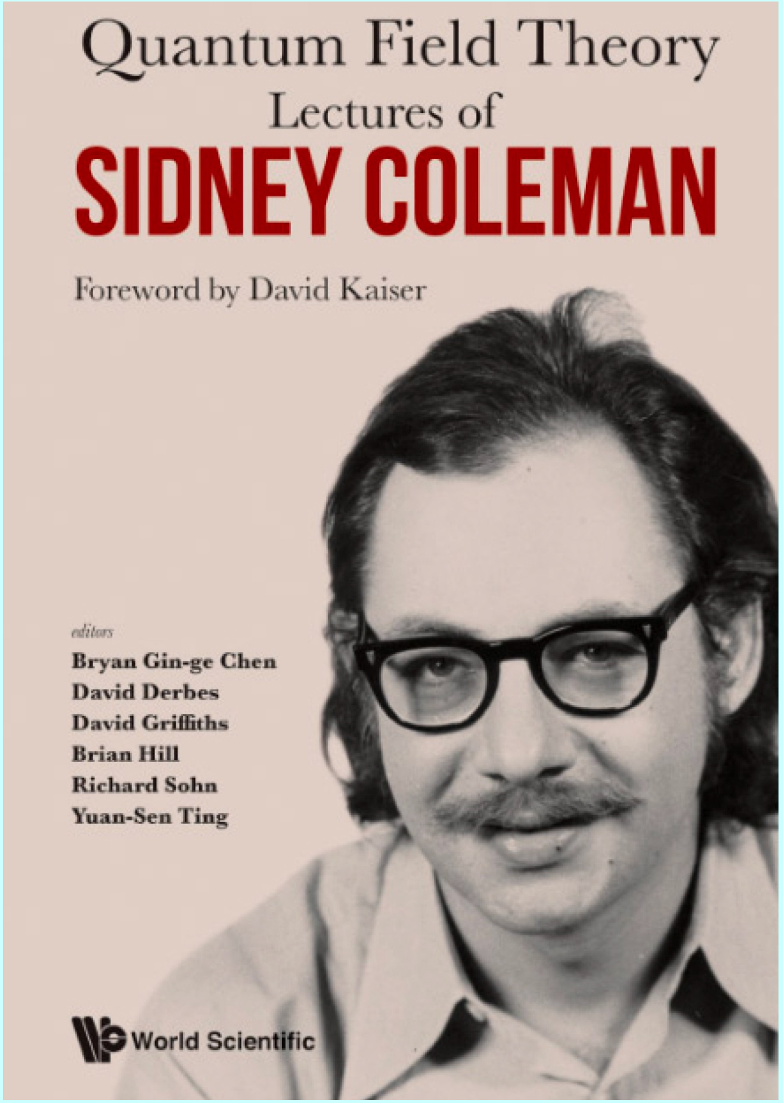
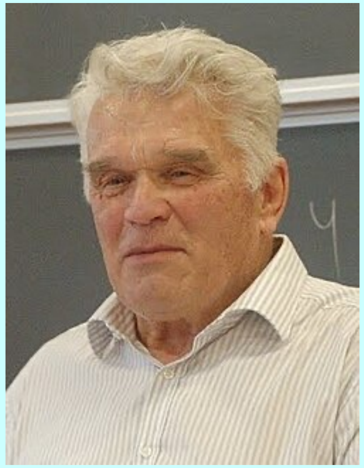
Functional Integration







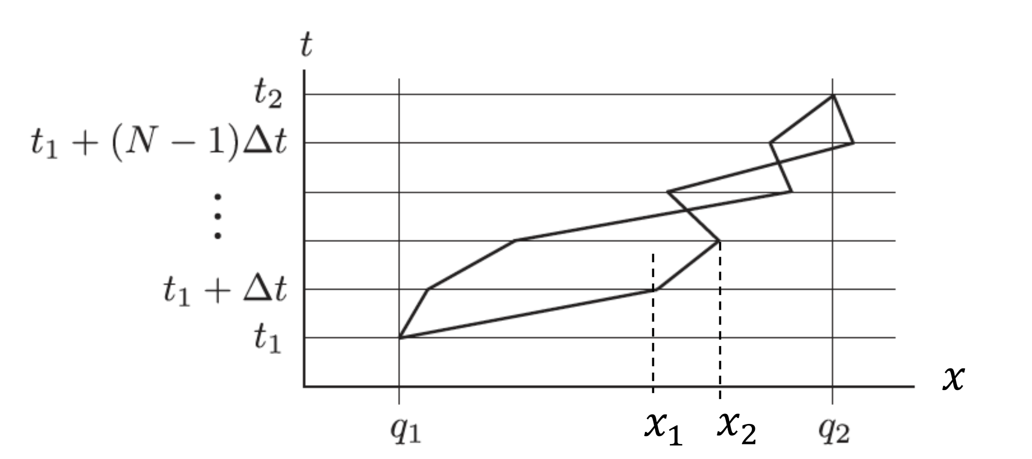
Ludvig Faddeev 1934-2017

Victor Popov 1937-1994

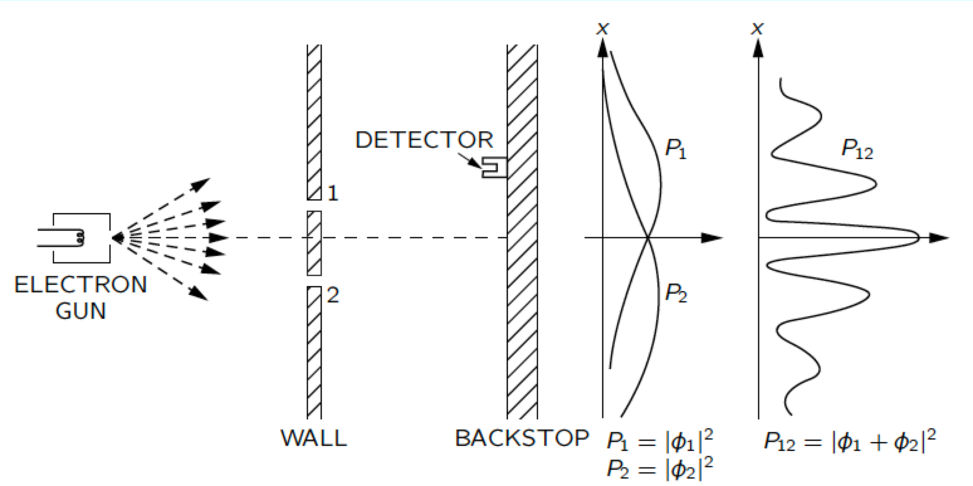
Path integral is an integral about a system's history.

But the concept of path is classical and contradictory to QM.

If we insist, at least we need to say we don't know which path it chooses.

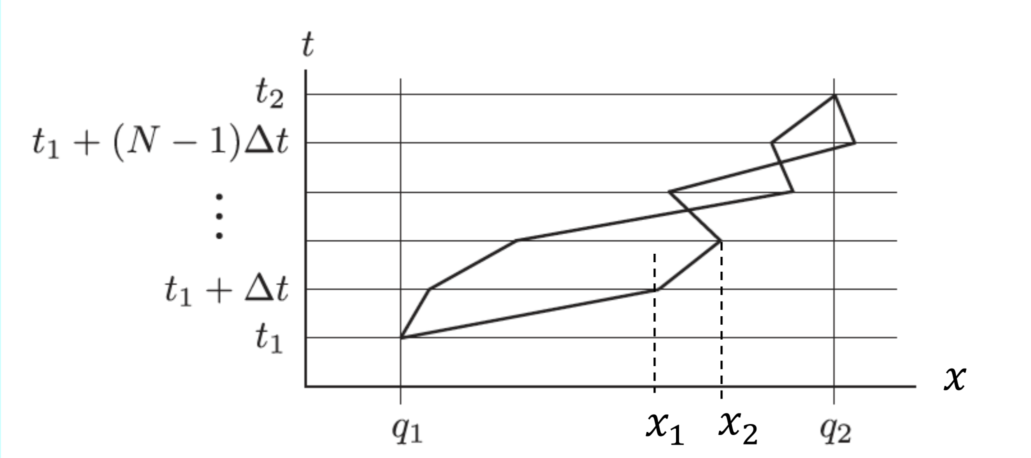


This is similar to the double-slit experiment.



The amplitudes of each path, or each history, need be added up.

How to add?



Consider approximate path from q_1 to q_2 between t_1 to t_2.

Record the position at $t_1 + i\Delta t$ as x_i.

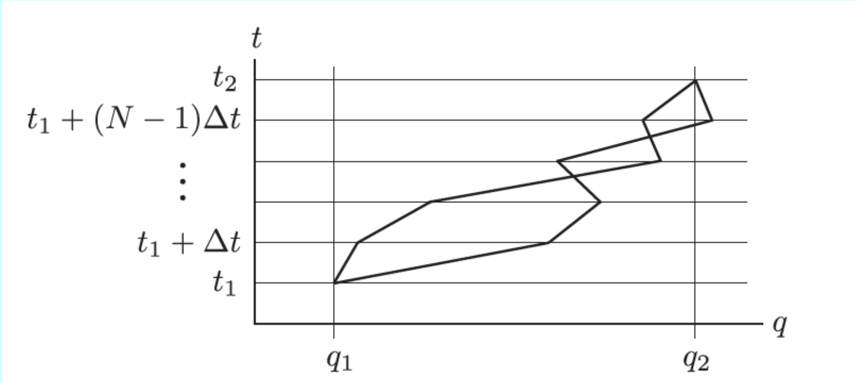
$x \equiv (q_1, x_1, x_2, \cdots x_n, q_2)$ is a history of motion or path of a particle in 1D.

Allowing for all possible path means allowing x_i to vary between $-\infty$ to ∞.

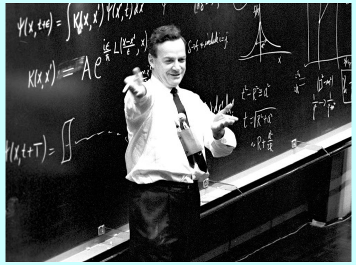
$$\sum_{\text{All possible paths}} \equiv \int d^n x \equiv \int \prod_{i=1}^{n} dx_i = \int (Dx)$$

This path integral is the sum (of amplitudes) over all possible paths.

Feynman told us how to calculate the amplitudes of each path: $e^{iS[x]}$.



The amplitude equals the sum over all possible paths.



$$\langle q_2 | q_1 \rangle = N \int (Dx)\, e^{iS[x]}$$

$S[x]$ is usually a quadratic form:

$$e^{iS[x]} = \exp -\frac{1}{2}\langle x, Ax\rangle = \exp -\frac{1}{2}\sum_{a,b=1}^{n}(x_a A_{ab} x_b)$$

the sum of amplitude from q_1 to q_2 will be

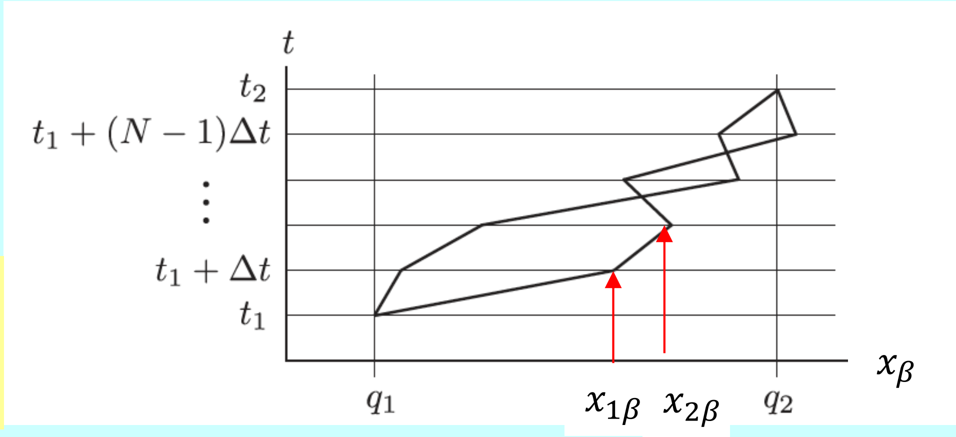
$$\langle q_2 | q_1 \rangle = \int d^n x\, e^{-\frac{1}{2}\langle x, Ax\rangle}$$

x_a can be extended to include more degrees of freedom β.

$$x_a = (x_{1\beta}, x_{2\beta}, \cdots x_{n\beta}), a = 1 \cdots mn$$

Just pile up!

For each x_β, we have a chart as:



$$\int (Dx) = \int \prod_{\beta=1}^{m} \prod_{i=1}^{n} dx_{i\beta}$$

$$\langle x, Ax \rangle = \sum_{a,b=1}^{mn} (x_a A_{ab} x_b) = \sum_{a=1}^{mn} \lambda_a x_a'^2$$

$$\int \prod_{\beta=1}^{m} \prod_{i=1}^{n} dx'_{i\beta}\, e^{-\frac{1}{2}\langle x', Ax' \rangle} = (2\pi)^{mn/2} \prod_{a=1}^{mn} (\lambda_a)^{-\frac{1}{2}} = (2\pi)^{mn/2} (\det A)^{-\frac{1}{2}}$$

$$(Dx) \equiv \frac{(Dx)}{(2\pi)^{mn/2}}$$

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

(Dx) is a sum over all possible path with multiple degrees of freedom.

$$\int (Dx)\, e^{-\frac{1}{2}\langle x, Ax \rangle} = (\det A)^{-\frac{1}{2}}$$

The quadratic form could have a linear term:

$$Q(x) = \frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle$$

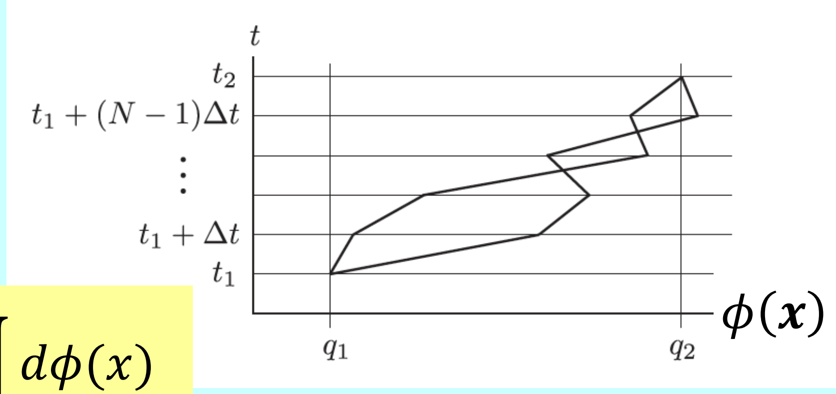
Q is maximized at $\bar{x} = -A^{-1}b$.

$$Q(x) = Q(\bar{x}) + \frac{1}{2}\langle x - \bar{x}, A(x - \bar{x})\rangle = -\frac{1}{2}\langle b, A^{-1}b\rangle + \frac{1}{2}\langle x - \bar{x}, A(x - \bar{x})\rangle$$

Infinity integration for (dx) is invariant after a shift $x \to x - \bar{x}$.

$$\int (dx)\, e^{-Q(x)} = e^{-\frac{1}{2}\langle b, A^{-1}b\rangle} \int (dx)\, e^{-\frac{1}{2}\langle x-\bar{x}, A(x-\bar{x})\rangle} = e^{-\frac{1}{2}\langle b, A^{-1}b\rangle} (\det A)^{-\frac{1}{2}}$$

We can extend the above to scalar field theory by:



$x_a \to \phi(x)$ $\quad$ $a \to t, \boldsymbol{x}$ $\quad$ $\beta \to \boldsymbol{x}$

Now space $\boldsymbol{x}$ is the index β, spacetime x is a.

$$(Dx) = \frac{1}{(2\pi)^{mn/2}} \prod_{\beta=1}^{m} \prod_{i=1}^{n} dx_{i\beta} \rightarrow (D\phi) \equiv \prod_{t,\boldsymbol{x}} d\phi(x)$$

is a sum over all possible field configuration and evolution.

$$\langle \phi_2 | S | \phi_1 \rangle = N \int (D\phi)\, e^{iS[\phi]}$$

The amplitude or S matrix is given by path integral.

$$Q(x) = \frac{1}{2}\langle x, Ax \rangle + \langle b, x \rangle = \frac{1}{2}\sum_{a,b=1}^{n} (x_a A_{ab} x_b) + \sum_{a=1}^{n} (b_a x_a)$$

$$\Rightarrow \quad S[\phi] = \frac{1}{2}(\phi, A\phi) + (b, \phi) \equiv \frac{1}{2}\int d^4x\, d^4y \cdot \phi(x) A(x,y) \phi(y) + \int d^4x \cdot b(x)\phi(x)$$

Consider a scalar field with source $J(x)$. The action can be written as a $Q[\phi]$.

$$S = \int d^4x\, \mathcal{L} = -\int d^4x \left[\frac{1}{2}\phi(\partial_\mu \partial^\mu \phi) + \frac{1}{2} m^2 \phi^2 + J\phi\right] \qquad A = \Box + m^2$$

Consider a scalar field with source $J(x)$. The action can be written as a $Q[\phi]$.

$$S = \int d^4x\,\mathcal{L} = -\int d^4x\left[\frac{1}{2}\phi\left(\partial_\mu\partial^\mu\phi\right) + \frac{1}{2}m^2\phi^2 + J\phi\right]$$

All QFT quantities can be calculated by the vacuum-to-vacuum amplitude with source.

$$\langle\Omega|S|\Omega\rangle_J \equiv Z[J]$$

This amplitude is a functional of the function $J(x)$, ie for every $J(x)$ it gives a number.

By canonical quantization, in the interaction picture, $\mathcal{L}_I = J(x)\phi(x)$

The S matrix can be written as:

$$S = T\left[\exp\left(i\int d^4x\,[\mathcal{L}_I(x)]\right)\right] = T\left[\exp\left(i\int d^4x\,[-iJ(x)\phi(x)]\right)\right]$$

$$\langle\Omega|S|\Omega\rangle_J \equiv Z[J] = \left\langle\Omega\middle|T\left[\exp\left(i\int d^4x\,[-iJ(x)\phi(x)]\right)\right]\middle|\Omega\right\rangle$$

Its derivatives about $J(x)$ gives all the Green functions of QFT, so generating functional.

$$\frac{1}{i^n}\frac{1}{Z[0]}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} = \left\langle\Omega\middle|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\middle|\Omega\right\rangle$$

$\langle \Omega | S | \Omega \rangle_J$ can also be calculated by path or history integral.

$e^{iS[\phi,J]}$ would be the amplitudes of various histories , ie. as the weight of summation.

$$Z[J] = \langle \Omega | S | \Omega \rangle_J = N \int (D\phi)\, e^{iS[\phi,J]}$$

The advantage of doing things this way is that on the left-hand side we have an object, $Z[J]$, with all those commutators that were giving us so much trouble. But on the right-hand side, the object has *no* quantum objects, just *classical* fields which all commute with each other. They're just ordinary c-numbers, and I'm integrating over 'em. This will turn out to be an enormous advantage, and enable us to settle with a single stroke all the problems associated with derivative interactions, superfluous variables, gauge invariance and so on.

Does it make mathematical sense?

$$N\int (D\phi)\, e^{iS[\phi]}$$

It is oscillating, not exponentially decreasing as $\phi \to \infty$.

We can continue both sides to Euclidean space by a Wick Rotation.

Let x^0 becomes imaginary: $x_0 = -ix_4, x_4$ real

$$x^2 = x_0^2 - \boldsymbol{x}^2 = -x_4^2 - \boldsymbol{x}^2 = -x_E^2$$

Momenta will change accordingly so that the contour integral over half circle vanishes.

$$p_0 = ip_4, p_4 \text{ real}$$

$$p^2 = p_0^2 - \boldsymbol{p}^2 = -p_4^2 - \boldsymbol{x}^2 = -p_E^2$$

$$iS = i\int d^4x \left[\frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}\nabla\phi\nabla\phi - \frac{1}{2}m^2\phi^2 + J\phi\right]$$

$\square_E$ is d'Alembertian.

$$= -\int d^4x_E \left[\frac{1}{2}\partial_4\phi\partial_4\phi + \frac{1}{2}\nabla\phi\nabla\phi + \frac{1}{2}m^2\phi^2\right]$$

$$Z[J] = N\int (D\phi)\, e^{-\int d^4x_E\left[\frac{1}{2}\partial_4\phi\partial_4\phi+\frac{1}{2}\nabla\phi\nabla\phi+\frac{1}{2}m^2\phi^2\right]}$$

exponentially decreasing

As it stands, it doesn't look like the action $\mathcal{S}_c[\phi, J]$, with the Lagrangian (28.24) is the sort of functional integral we can do safely, even without worrying about the infinite dimensions of function space, and even for a free field theory, $\mathscr{L}' = 0$. Instead of a nice positive definite quadratic form in the exponential, or at least a form with a positive definite real part, we have an overall factor of i multiplying $\mathcal{S}_c$: the exponential *oscillates*. Put simply, to make sense of (28.27), we have to continue both sides into *Euclidean space*: to obtain four-vectors with real space components, *imaginary* time components, and a quadratic form of definite sign. We discussed this earlier (§15.5) in the context of analytic continuation of Feynman integrals. I first have to demonstrate that the Green's functions of a quantum field theory can be continued into Euclidean space. Then I will show that the functional integral is well-defined in Euclidean space, (or as well-defined as our other functional integrals have been), and then I'll be able to show that the two sides are equal.

When we were doing loop integrations, we studied the propagator in the q_0 plane and found that it had poles in a typical case as shown in Figure 15.7; either $(|\mathbf{q}|^2 - a) > 0$ or $(|\mathbf{q}|^2 - a) < 0$. In either case, we did not cross any poles if we rotated our contour of integration onto the imaginary axis:

$$q_0 \to e^{i\alpha} q_0 \quad 0 \le \alpha \le \tfrac{\pi}{2} \tag{28.29}$$

leading to the Wick-rotated values

$$\begin{aligned} q_0 &= iq_4 \quad \text{(for loop momenta)} \\ p_0 &= ip_4 \quad \text{(for other momenta)} \end{aligned} \tag{28.30}$$

With these rotated time components, the Lorentz square k^2 of a momentum four-vector k^μ turned into a *negative* Euclidean square k_E^2:

$$k^2 = k_0^2 - \mathbf{k}^2 = -k_4^2 - \mathbf{k}^2 = -k_E^2 = -\sum_{i=1}^{4} k_i^2 \tag{28.31}$$

(We made this rotation after we had performed all the momentum shifts.) If we do this simultaneously to all the external momenta in the problem, and at the same time that we rotate the external momenta in the complex plane we rotate the internal momenta in the complex plane, to preserve energy-momentum conservation at every vertex, this obviously goes through. We end up with a Feynman integral that has no zeros in its denominators anywhere. Everything is the square of a Euclidean vector plus a positive mass squared. The function is not only well-defined, it is an analytic function of the external momenta. When we rotate our external energies in the complex plane this defines an analytic continuation in k-space.

On the left-hand side of (28.27), we have an expression in x-space. It's pretty easy to see what we have to do in x-space to keep things going right in k-space; we have to rotate x in the *opposite* direction:

$$x_0 \to e^{-i\pi/2} x_0 = -ix_4 \tag{28.32}$$

The phase factor in x_0 cancels the phase factor in k_0; otherwise the Fourier transform would develop an exponential blowup. The minus sign is going to be important in making our formulas come out right. So, to all orders in perturbation theory, we can define our Green's functions for Euclidean spacetime separations, where the formal connections between the complex variable x_0 and the real variable x_4 are given above.

time coordinate $t \to -ix^0$ produces a Euclidean 4-vector product:

$$x^2 = t^2 - |\mathbf{x}|^2 \to -(x^0)^2 - |\mathbf{x}|^2 = -|x_E|^2. \tag{9.44}$$

It is possible to show, by manipulating the expression for each Feynman diagram, that the analytic continuation of the time variables in any Green's function of a quantum field theory produces a correlation function invariant under the rotational symmetry of four-dimensional Euclidean space. This Wick rotation inside the functional integral demonstrates this same conclusion in a more general way.

To understand what we have achieved by this rotation, consider the example of ϕ^4 theory. The action of ϕ^4 theory coupled to sources is

$$\int d^4x \,(\mathcal{L} + J\phi) = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right]. \tag{9.45}$$

After the Wick rotation (9.44), this expression takes the form

$$i\int d^4x_E(\mathcal{L}_E - J\phi) = i\int d^4x_E \left[\frac{1}{2}(\partial_{E\mu}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi\right]. \tag{9.46}$$

This expression is identical in form to the expression (8.8) for the Gibbs free energy of a ferromagnet in the Landau theory. The field $\phi(x_E)$ plays the role of the fluctuating spin field $s(\mathbf{x})$, and the source $J(\mathbf{x})$ plays the role of an external magnetic field. Note that the new ferromagnet lives in four, rather than three, spatial dimensions.

The Wick-rotated generating functional $Z[J]$ becomes

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[-\int d^4x_E \,(\mathcal{L}_E - J\phi)\right]. \tag{9.47}$$

The functional $\mathcal{L}_E[\phi]$ has the form of an energy: It is bounded from below and becomes large when the field ϕ has large amplitude or large gradients. The exponential, then, is a reasonable statistical weight for the fluctuations of ϕ. In this new form, $Z[J]$ is precisely the partition function describing the statistical mechanics of a macroscopic system, described approximately by treating the fluctuating variable as a continuum field.

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In Euclidean Space, $$Z[J] = N\int (D\phi)\, e^{-\int d^4x_E[\mathcal{L}_E + J\phi]}$$

summation over history becomes summation over all possible configurations.

That is partition function in statistical mechanics with Canonical Ensemble .

The generating functional (9.42) is reminiscent of the partition function of statistical mechanics. It has the same general structure of an integral over all possible configurations of an exponential statistical weight. The source $J(x)$ plays the role of an external field. In fact, our method of computing correlation functions by differentiating with respect to $J(x)$ mimics the trick often used in statistical mechanics of computing correlation functions by differentiating with respect to such variables as the pressure or the magnetic field.

	Statistical physics (on a lattice)	Field theory (continuum)
Source	J_i	$J(x)$
Generating function	$Z(J)$	$Z[J(x)]$
Green's functions	$G^{(n)}_{i_1\ldots i_n} = \langle\hat{\phi}_{i_1}\ldots\hat{\phi}_{i_n}\rangle_{\mathrm{t}}$	$G^{(n)}(x_1,\ldots,x_n) = \langle\Omega\vert T\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\vert\Omega\rangle$
Differentiation recipe	$G^{(n)}_{i_1\ldots i_n} = \frac{1}{Z(J=0)}\left.\frac{\partial^n Z(J)}{\partial J_{i_1}\ldots\partial J_{i_n}}\right\vert_{J=0}$	$G^{(n)}(x_1,\ldots,x_n) = \frac{1}{\mathrm{i}^n}\frac{1}{Z[J=0]}\left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right\vert_{J=0}$
Order	$\langle\hat{\phi}_i\rangle_{\mathrm{t}} \neq 0$	$\langle\Omega\vert\hat{\phi}(x)\vert\Omega\rangle \neq 0$

Feynman Rule from Feynman's path integral

$Z[J]$ is called generating functional.

We define it as calculated in Euclidean space and continued back to Minkowski Space.

$$Z[J] = N \int (D\phi)\, e^{iS[\phi,J]} = N \int (D\phi)\, e^{iS[\phi,J=0]+J\phi}$$

Its derivatives with $J(x)$ can also be calculated in path integral formalism.

$$\frac{1}{i^n} \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\bigg|_{J=0} \sim \int (D\phi)\, [\phi(x_1) \cdots \phi(x_n)] e^{iS[\phi,J=0]}$$

We already know $Z[J]$'s derivatives with $J(x)$ gives the Green functions of QFT. So

$$G(x_1, x_2, \cdots x_n) = \langle \Omega | T\hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle = \int (D\phi)\, [\phi(x_1) \cdots \phi(x_n)] e^{iS[\phi,J=0]}$$

Observe that commuting $\phi(x_1) \cdots \phi(x_n)$ at R automatically arranged into time order at L.

With Green's function, we can calculate the S matrix, by LSZ reduction formula.

$$\langle p_3, \cdots p_n | S | p_1, p_2 \rangle = \left[i \int d^4 x_1 e^{-ip_1 x_1} (\Box_1 + m^2) \right] \cdots \left[i \int d^4 x_n e^{ip_n x_n} (\Box_n + m^2) \right]$$

$$\times \langle \Omega | T\hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle$$

Let’s calculate first in Euclidean Space!

$$Z[J] \sim \int (D\phi)\, e^{-\int d^4x_E \left[\frac{1}{2}\partial_4\phi\partial_4\phi + \frac{1}{2}\nabla\phi\nabla\phi + \frac{1}{2}m^2\phi^2 + j\phi\right]} \sim \int (D\phi)\, e^{-\int d^4x_E \left[\frac{1}{2}\phi(-\Box_E + m^2)\phi + J\phi\right]}$$

Using the familiar formula we just derived:

$$\int (D\phi)\, e^{-\frac{1}{2}(\phi, A\phi) + \langle b,\phi\rangle} = e^{-\frac{1}{2}\langle b, A^{-1}b\rangle} (\det A)^{-\frac{1}{2}}$$

$$Z_{\mathrm{E}}[J] = N \det(-\Box_E + m^2)^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\int d^4x_E\ J(x) \frac{1}{-\Box_E + m^2} J(x)\right\}$$

Choose N so that: $Z_{\mathrm{E}}[J = 0] = 1$

$$Z_{\mathrm{E}}[J] = \exp\left\{\frac{1}{2}\int d^4x_E\ J(x) \frac{1}{-\Box_E + m^2} J(x)\right\}$$

$$\frac{1}{-\Box_E + m^2}$$

The function connecting two sources is just the propagator!

$Z_{\mathrm{E}}[J]$ is just sum of any numbers of….

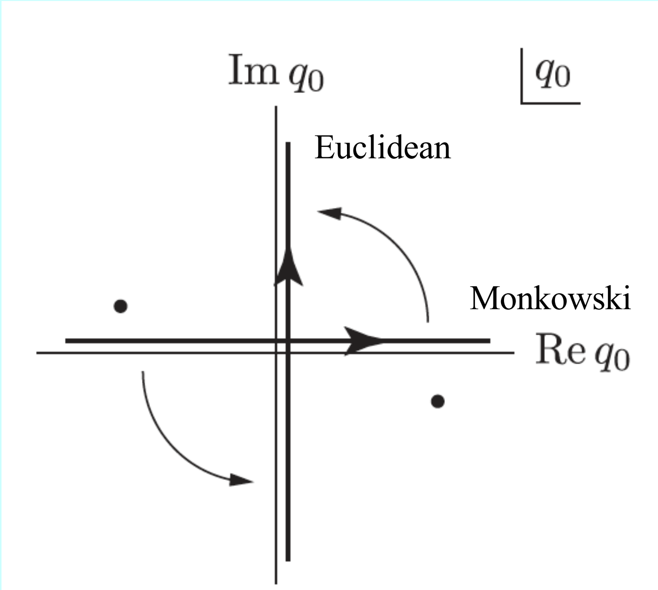
  …… 

$$\frac{1}{-\Box_E + m^2}$$ can be continued back to Monkowski space.

$$\frac{-i}{\Box + m^2 + i\varepsilon} \sim \Delta_F(x, y)$$ It would be Feynman Propagator $\Delta_F(x, y)$.

$$(\Box + m^2 + i\varepsilon)\Delta_F(x, y) = -i\delta^{(4)}(x - y)$$

$i\varepsilon$ is to ensure there is no pole in the analytic continuation.



By Fourier Transformation:

$$Z[J] = \exp\left\{\frac{1}{2}\int d^4x\ J(x)\frac{-i}{\Box + m^2 + i\varepsilon}J(x)\right\} = \exp\left\{\frac{1}{2}\int d^4k\ \tilde{J}(k)\frac{i}{k^2 - m^2 - i\varepsilon}\tilde{J}(k)\right\}$$

Now we can add interaction $\mathcal{L}_\mathrm{I}(\phi)$:

$$Z[J] = N \int (D\phi)\, e^{-i\int d^4x\left[\frac{1}{2}\phi(\Box+m^2+i\varepsilon)\phi+\mathcal{L}_\mathrm{I}(\phi)+J\phi\right]}$$

We can separate the interaction Lagrangian:

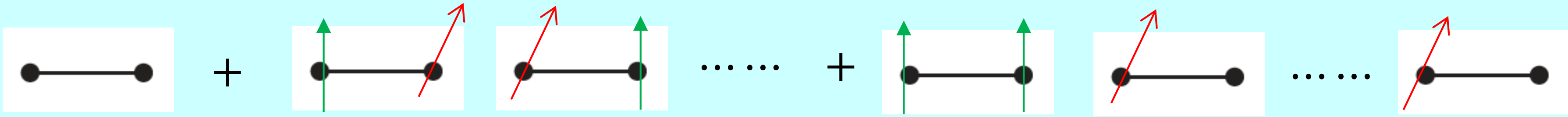
$$Z[J] = N \int (D\phi) \exp\left[i\int d^4x\, \mathcal{L}_\mathrm{I}(\phi)\right] e^{-i\int d^4x\left[\frac{1}{2}\phi(\Box+m^2+i\varepsilon)\phi-J\phi\right]}$$

ϕ can be obtained by functional derivatives of J and derivatives is taken out of integral.

$$Z[J] = N \exp\left[i\int d^4x\, \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right)\right] \int (D\phi)\, e^{-i\int d^4x\left[\frac{1}{2}\phi(\Box+m^2+i\varepsilon)\phi-J\phi\right]}$$

$$= N \exp\left[i\int d^4x\, \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right)\right] \exp\left[\frac{1}{2}\int d^4x\; J(x)\frac{-i}{\Box+m^2+i\varepsilon}J(x)\right]$$

$\sum_n$   …… 



$$\mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right) + \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right) \; \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right) \cdots\cdots + \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right) \cdots\cdots \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right)$$

$$Z[J] = N \exp\left[i\int d^4x\, \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right)\right] \exp\left[\frac{1}{2}\int d^4x\; J(x)\frac{-i}{\Box + m^2 + i\varepsilon}J(x)\right]$$

Every derivative in $\mathcal{L}_\mathrm{I}$ take out a source.

The field operators in Green's function could be written as derivatives of J, too.

Every derivative take out one source, too.

$$\langle\Omega|T\phi(x_1)\cdots\phi(x_1)|\Omega\rangle = \frac{1}{i^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} \exp\left[i\int d^4x\, \mathcal{L}_\mathrm{I}\left(-i\frac{\delta}{\delta J}\right)\right] Z_0[J]\Bigg|_{J=0}$$

All source dots in Z_0 must either end up as in-out particles or as a leg in interaction vertex.

Otherwise, it will vanish as we take $J = 0$.

$$S^{(2)} = U^{(2)}(\infty, -\infty) = -\frac{1}{2} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \; T[\mathcal{L}_\mathrm{I}(x_1) \cdot \mathcal{L}_\mathrm{I}\,(x_2)]$$

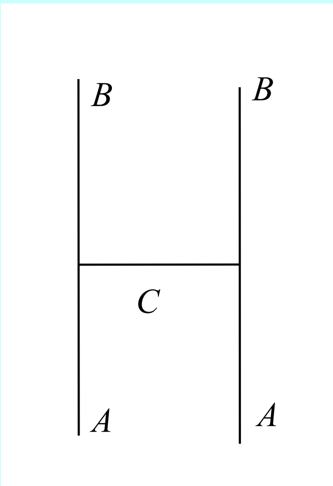
Amplitude for scattering $A + A \rightarrow B + B$

$$\langle B(\vec{p}_3)\; B(\vec{p}_4)|S|A(\vec{p}_1)A(\vec{p}_1)\rangle$$

$e^{ip_4 \cdot x_1}$ $\quad$ $e^{ip_3 \cdot x_2}$

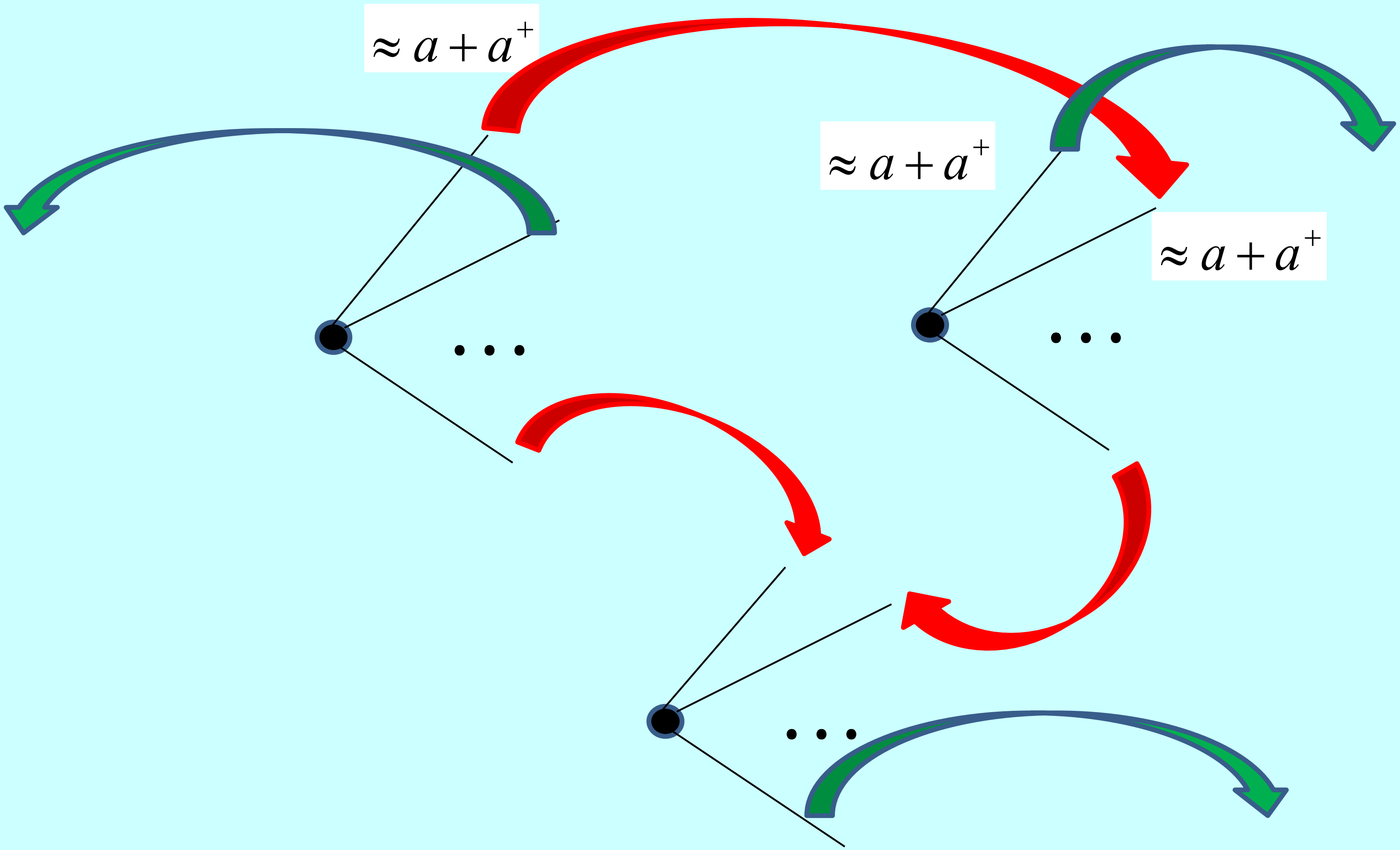
$$\Big\langle B(\vec{p}_3)\; B(\vec{p}_4)\Big| \Big[\int_{-\infty}^{\infty} d^4x_1 d^4x_1\; T\big[g\hat{A}(x_1)\hat{B}(x_1)\hat{C}(x_1) \cdot g\hat{A}(x_2)\hat{B}(x_2)\hat{C}(x_2)\big]\Big] \Big|A(\vec{p}_1)A(\vec{p}_1)\Big\rangle$$

$e^{-ip_1 \cdot x_2}$ $\quad$ $e^{-ip_2 \cdot x_1}$



$$= \int_{-\infty}^{\infty} d^4x_1 d^4x_1\; e^{-i(p_1-p_3)\cdot x_2} \cdot e^{-i(p_2-p_4)\cdot x_1} \langle 0\big|T\big[\hat{C}(x_1)\hat{C}(x_2)\big]\big|0\rangle$$

$$= \frac{i}{(p_1-p_3)^2 - m^2 + i\varepsilon}$$



Every field either couple with another field to form a propagator or annihilate (create) external particles!

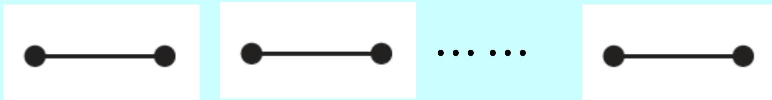
Otherwise, the amplitude will vanish when *a* operators hit vacuum!

This is the easiest way to derive the technique of Feynman diagrams.

It works even if interaction contains derivatives, a thorny case for canonical quantization.

$$\langle\Omega|T\phi(x_1)\cdots\phi(x_1)|\Omega\rangle = \frac{1}{i^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\exp\left[i\int d^4x\,\mathcal{L}_{\mathrm{I}}\left(-i\frac{\delta}{\delta J}\right)\right]Z_0[J]\Bigg|_{J=0}$$

Take as an example: $\mathcal{L}_{\mathrm{I}}(\phi) = \lambda\phi^3$

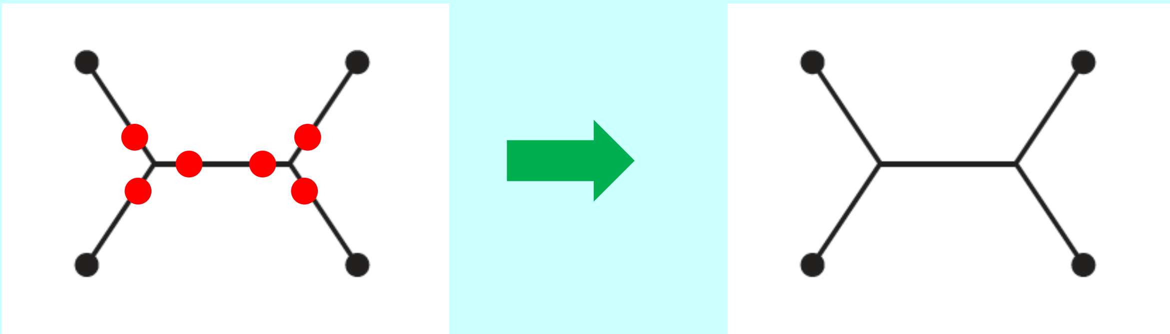


Expand the exponential to second order, ie. With two vertices.

$$\lambda\frac{\delta^3}{\delta J^3(x)}\cdot\lambda\frac{\delta^3}{\delta J^3(y)}$$

Green Function with four external lines: $\dfrac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}$

Contribution will come from five propagators with 10 J:



The source dots either end up in an interaction vertex or as in or out particles.

路徑積分運用於規範場論

You make a gauge transformation :

$V \to V + \partial_t \theta$ $\boldsymbol{A} \to \boldsymbol{A} + \nabla\theta$ or $A^\mu \to A^\mu + \partial^\mu \theta$

The observable electromagnetic fields $\boldsymbol{E}, \boldsymbol{B} = F^{\mu\nu}$ do not change!

You represent the same physical state by two different set of physical quantities A^μ.

Some information in A^μ is redundant.

This is different from symmetry.

Symmetry states two different physical states have identical properties, such as energy.

We usually choose a Gauge condition to fix the gauge.

Lorentz gauge:

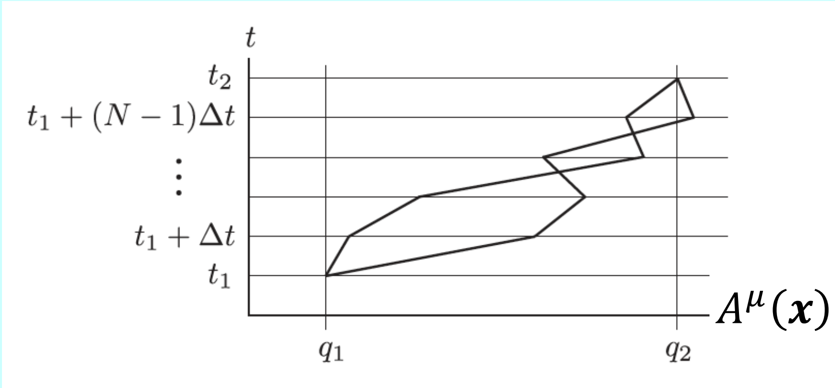
$\partial_\mu A^\mu = 0$ $\partial_\mu A^\mu = f(x)$

Landau gauge:

$\nabla \cdot \boldsymbol{A} = 0$

Axial gauge:

$A^3 = 0$

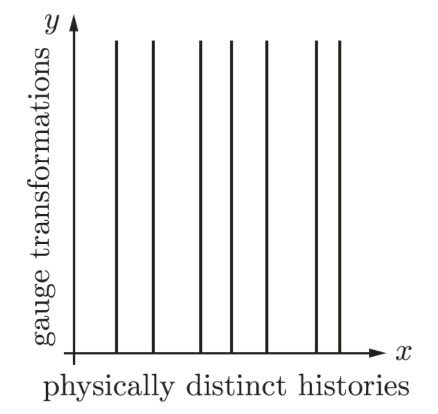


Different field evolution may represent the same history.

if we do arbitrary gauge transformation along the way.

(DA^μ), as summation over histories, is overcounting.

If the x's of these vertical lines represent physical states at successive times, and y represents the arbitrary gauge transformations you can make.
Note that different y do not mean different states. One vertical line is one state.



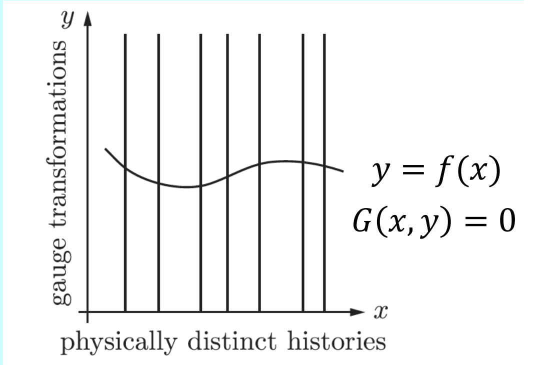


Figure 31.2: Fixing the values of the y's

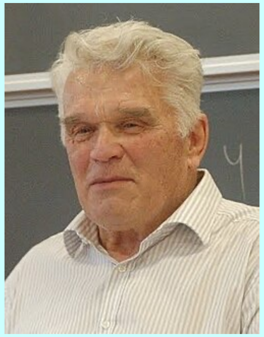
$\int(DA^\mu)$ is like summing over x and y: $\int dxdy$, which is redundant.
Choosing one y for every x ie. A line across $x - y$ plane would be enough.
We can impose a constraint or condition when we integrate: $y = f(x)$

$$\int dxdy\ \delta(y - f(x)) = \int dx \underbrace{\boxed{\int dy\, \delta(y - f(x))}}_{=1} = \int dx$$

$y - f(x) = 0$ is better written as a condition for x, y: $G(x, y) = 0$

$$\int dx = \int dx \boxed{\int dy \frac{\partial G}{\partial y} \delta(G(x, y))} = \int dxdy \frac{\partial G}{\partial y} \delta(G(x, y))$$

Note the $\frac{\partial G}{\partial y}$ that compensate the difference of gauge condition G and variable y.

Faddeev and Popov

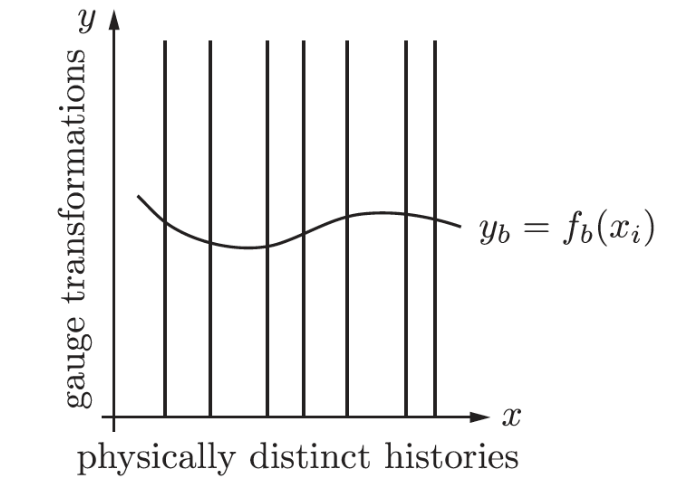


Figure 31.2: Fixing the values of the y's

This is the proper way to sum over history with redundant variables!

It can be easily extended to the case where there are more than one x and y:

There would be m conditions $G^{k=1\cdots m}$ between $x_{i=1\cdots n}$ and $y_{j=1\cdots m}$:

$$\int d^n x = \int d^n x \underbrace{\boxed{\int d^m y \ \det\left(\frac{\partial G^k}{\partial y_j}\right) \delta\left(G^k(x_i, y_j)\right)}}_{\boxed{=1}} = \int d^n x d^m y \det\left(\frac{\partial G}{\partial y}\right) \delta\big(G(x,y)\big)$$

The derivative now needs to be Jacobian $\det\left(\frac{\partial G}{\partial y}\right)$ of multivariable integration.

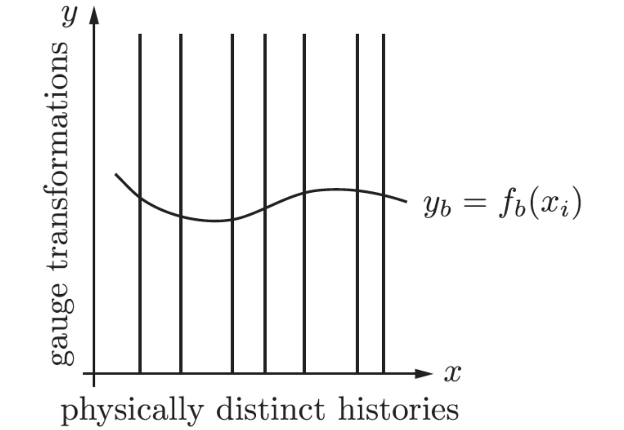


Figure 31.2: Fixing the values of the y's

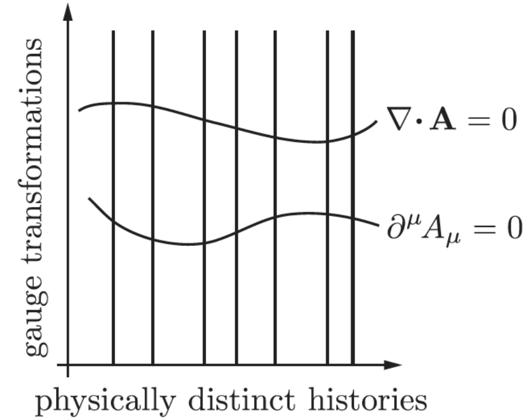


Figure 31.3: Fixing the gauge

$$\int d^n x d^m y$$

$$\int (DA^\mu)$$

$$G(x, y) = 0$$

$$G(A^\mu(x)) = 0$$

Gauge conditions

$$(x_i, 0) + (0, y_j)$$

$$A^\mu(x) + \partial^\mu \alpha(x)$$

$$y_{j=1\cdots m} \longleftrightarrow \alpha(x)$$

$$\int d^n x d^m y \det\left(\frac{\partial G}{\partial y}\right) \delta(G(x, y))$$

$$\int (DA^\mu) \det\left(\frac{\delta G(x)}{\delta \alpha(x)}\right) \delta(G(A^\mu(x))) e^{iS[A^\mu]}$$

Functional derivative

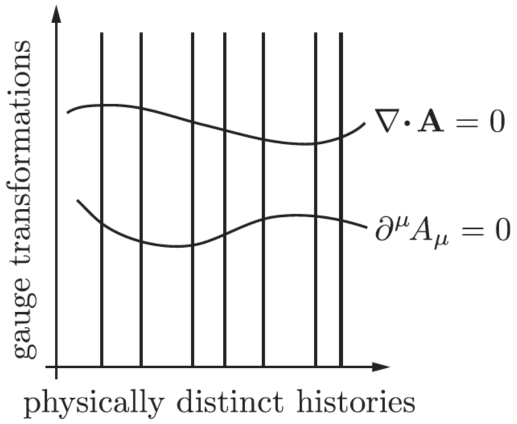


Figure 31.3: Fixing the gauge

$$G(A^\mu(x)) = 0$$ Gauge conditions

$$A^\mu(x) + \partial^\mu \alpha(x)$$

$$\int (DA^\mu) \det\left(\frac{\delta G(x)}{\delta \alpha(x)}\right) \delta\big(G(A^\mu(x))\big) e^{iS[A^\mu]}$$

This is an ansatz ie. an educated guess.

We can prove it is gauge invariance, which is just a displacement in α: $\alpha + \alpha'$.

An infinite integration $\int_{-\infty}^{\infty} d\alpha$ is always invariant under $\alpha + \alpha'$.

We can also prove it is identical to canonical quantization in axial gauge.

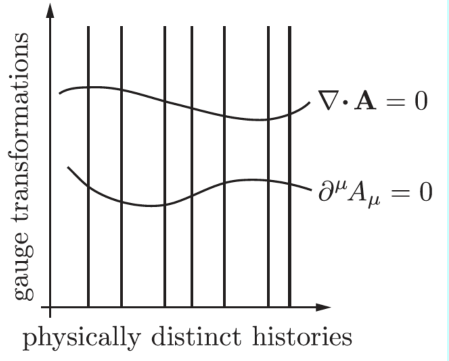


Figure 31.3: Fixing the gauge

$$\int (DA^\mu)\,\det\left(\frac{\delta G(x)}{\delta\alpha(x)}\right)\,\delta\big(G(A^\mu(x))\big)e^{iS[A^\mu]}$$

Even better, for most gauge conditions in QED, $\det\left(\frac{\delta G}{\delta\alpha}\right)$ is a constant.

Take Lorentz gauge as an example:

$$G[A^\mu] = \partial_\mu A^\mu = 0$$

The condition is transformed into:

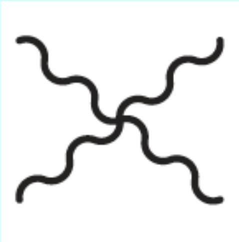
$$G[A^\mu(x) + \partial^\mu\alpha] = \partial_\mu A^\mu + \partial_\mu\partial^\mu\alpha$$

Note that the difference is independent of any physical fields.

$$\det\left(\frac{\delta G}{\delta\alpha}\right) = \det\left(\frac{\delta\Box\alpha}{\delta\alpha}\right) = \det\big(\Box\delta^{(4)}(x-y)\big)$$

This is a constant to be absorbed into normalization of Z. We ignore det in QED.

You can no longer do that in non-abelian gauge theory.



You can no longer ignore det in non-abelian gauge theory.

$$\delta \boldsymbol{W}_\mu = \frac{1}{g}\partial_\mu \boldsymbol{\alpha} + \boldsymbol{\alpha}\times\boldsymbol{W}_\mu$$

or in components:

$$\delta W_\mu^k = \frac{1}{g}\partial_\mu \alpha^k + \varepsilon^{ijk}\alpha^i W_\mu^j$$

Take Lorentz gauge: $G^k = \partial^\mu W_\mu^k = 0$

$$\int (DW_\mu^k)\det\left(\frac{\delta G^k}{\delta \alpha^i}\right)\delta(\partial^\mu W_\mu^k)e^{iS}$$

The condition is transformed into:

$$G^k\left[W_\mu^k + \frac{1}{g}\partial_\mu\alpha^k + \varepsilon^{ijk}\alpha^i W_\mu^j\right] = \partial^\mu W_\mu^k + \frac{1}{g}\partial_\mu\partial^\mu\alpha^k + \partial_\mu\left(\varepsilon^{ijk}\alpha^i W_\mu^j\right)$$

$$= \partial^\mu W_\mu^k + \frac{1}{g}\partial_\mu\left[\partial^\mu\alpha^k - g\partial_\mu\left(\varepsilon^{jik}W_\mu^j\alpha^i\right)\right]$$

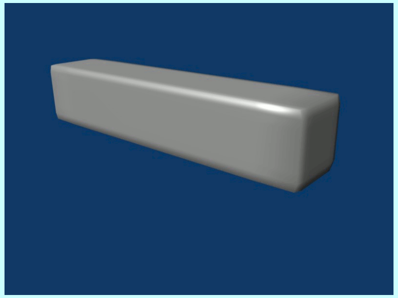
Note that the difference depends on gauge field and can't be ignored.

$$\det\left(\frac{\delta G^k}{\delta\alpha^i}\right) = \det\left(\frac{\delta\Box\alpha}{\delta\alpha}\right) \sim \det\left[\partial_\mu\partial^\mu\delta^{(4)}(x-y)\delta^{ki} - g\delta^{(4)}(x-y)\varepsilon^{jik}W_\mu^j\right]$$

Det term involves gauge bosons.

Another way to twist Lorentz gauge. Consider conditions with a variable function f.

$$G[A^\mu] = \partial_\mu A^\mu - f(x) = 0$$



Gauge transformation will change $f(x)$. So it is a surrogate of $\alpha(x)$.

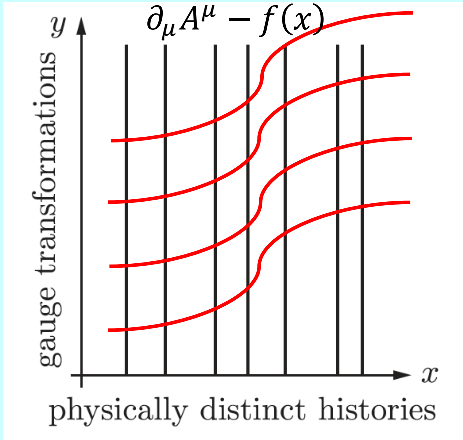
Gauge invariance suggest the following will not depend on $f(x)$.

$$\int (DA^\mu)\, \delta\big(\partial_\mu A^\mu - f\big) e^{iS[A^\mu]}$$

We ignore the det.

We can tame infinity of integrating α by giving it an exponentially damping weight.

$$\int (Df) \int (DA^\mu)\, \delta\big(\partial_\mu A^\mu - f\big) \cdot \boxed{\exp \frac{i}{2\xi} \int d^4x\, f^2} \cdot e^{iS[A^\mu]}$$



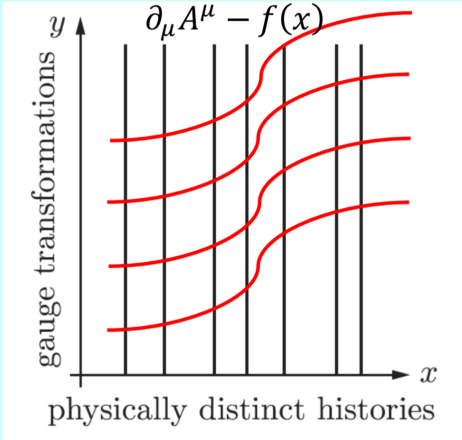
Change the order of integration:

$$\int (DA^\mu) \boxed{\int (Df)\, \delta\big(\partial_\mu A^\mu - f\big) \cdot \exp \frac{i}{2\xi} \int d^4x\, f^2} \cdot e^{iS[A^\mu]}$$

Integrate over the delta function:

$$\int (DA^\mu) \exp \frac{i}{2\xi} \int d^4x \left(\partial_\mu A^\mu\right)^2 \cdot e^{iS[A^\mu]}$$

$$\int (DA^\mu)\cdot e^{\frac{i}{2\xi}\int d^4x(\partial_\mu A^\mu)^2}\cdot e^{i\int d^4x\mathcal{L}[A^\mu]}$$



Yes, we are resuming the redundant variables.

Yes, the new gauge fixing term breaks gauge symmetry.

But it yields same Green functions as gauge symmetric theory.

That means redundant variables are carefully handled so that it doesn't cause troubles.

See also Schwartz P268

The new exponential changes the kinetic energy of A^μ and hence the propagator.

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 = \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\partial_\nu A_\mu \partial^\mu A^\nu + \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

After integration by part:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathcal{L} \supset -\frac{1}{2}A_\nu \partial_\mu \partial^\mu A^\nu - \frac{1}{2}A_\mu \partial^\mu \partial_\nu A^\nu + \frac{1}{2\xi}A_\mu \partial^\mu \partial_\nu A^\nu$$

$$= -\frac{1}{2}A_\mu\left[g^{\mu\nu}\Box - (1-\frac{1}{\xi})\partial^\mu \partial^\nu\right]A_\nu$$

$$-\frac{1}{2}A_\mu\left[g^{\mu\nu}\Box-(1-\frac{1}{\xi})\partial^\mu\partial^\nu\right]A_\nu$$

We can find photon propagator from the above kinetic energy:

It is the inverse of the operator in [ ], ie. Green function of this Differential Eq:

$$\left[g^{\mu\nu}\Box-\left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]D_{\nu\rho}(x-y)=\delta^\mu_\rho\delta^{(4)}(x-y)$$

Its Fourier transformation is easier to solve.

$$\left[-g^{\mu\nu}k^2-\left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]\widetilde{D}_{\mu\nu}(k)=i\delta^\mu_\rho$$

We can use Transversal and Longitudinal projections. Note that $P^{T,L}P^{T,L}=P^{T,L}$.

$P^L_{\mu\nu}$ project out 4-vectors $\perp$ to k_ν, while $P^T_{\mu\nu}$ picks these transverse components.

$$P^T_{\mu\nu}=g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\qquad\qquad P^L_{\mu\nu}=\frac{k_\mu k_\nu}{k^2}$$

$$\left[-g^{\mu\nu}k^2-\left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]=k^2P^T_{\mu\nu}+\frac{k^2}{\xi}P^L_{\mu\nu}$$

Its inverse is just inverting the Transversal and Longitudinal part.

$$\widetilde{D}_{\mu\nu}(k)=\frac{i}{k^2+i\varepsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\frac{i\xi}{k^2+i\varepsilon}\frac{k_\mu k_\nu}{k^2}$$

$$\widetilde{D}_{\mu\nu}(k) = \frac{i}{k^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{i\xi}{k^2 + i\varepsilon}\frac{k_\mu k_\nu}{k^2}$$

The parameter ξ could be anything we choose for our convenience.

Physical results should be independent of ξ.

Two popular choices:

$\xi = 1$, Feynman Gauge:

$$\widetilde{D}_{\mu\nu}(k) = \frac{i g_{\mu\nu}}{k^2 + i\varepsilon}$$

$\xi = 0$, Landau Gauge:

$$\widetilde{D}_{\mu\nu}(k) = \frac{i}{k^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

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How about fermions? Could I use the same master equation of functional integration?

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

$$Z[J] = N \int (D\psi)(D\bar{\psi})\, e^{i \int d^4x [\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + +\bar{J}\psi + \bar{\psi}J]}$$

Of course we cannot. But the same almost applies with modification.

The key problem is that for fermion ψ could not be ordinary number.

As an operator, ψ anti-commute. But do we have numbers that anti-commute?

The same is yes, called Grassmann numbers.

Hermann Grassmann
1809-1877

But it is shocking we know how to calculate $Z[J]$ without using Grassmann numbers.

From earlier pages, now do a complex version:

$$Q(x) = \frac{1}{2}\langle x^*, Ax\rangle + \langle b^*, x\rangle + \langle x^*, b\rangle$$

Q is maximized at $\bar{x} = -A^{-1}b$.

$$Q(x) = Q(\bar{x}) + \frac{1}{2}\langle (x-\bar{x})^*, A(x-\bar{x})\rangle = \frac{1}{2}\langle b^*, A^{-1}b\rangle + \frac{1}{2}\langle (x-\bar{x})^*, A(x-\bar{x})\rangle$$

The functional integration:

$$\int (dxdx^*)\, e^{-Q(x)} = \int (dxdx^*)\, e^{-\frac{1}{2}\langle b^*, A^{-1}b\rangle - \frac{1}{2}\langle (x-\bar{x})^*, A(x-\bar{x})\rangle}$$

$$= e^{-\frac{1}{2}\langle b^*, A^{-1}b\rangle} \int (dxdx^*)\, e^{-\frac{1}{2}\langle (x-\bar{x})^*, A(x-\bar{x})\rangle}$$

Then we said the infinite integration is invariant as x shift by a number.

$$\int (dxdx^*)\, e^{-\frac{1}{2}\langle x^*, Ax\rangle} = \int (dxdx^*)\, e^{-\frac{1}{2}\langle (x-\bar{x})^*, A(x-\bar{x})\rangle}$$

Note that this Gaussian integration is independent of $\bar{x}$ or b.

Note that the integration is now independent of $\bar{x}$ or b.

$$\int (dx dx^*)\, e^{-Q(x)} = e^{-\frac{1}{2}\langle b^*, A^{-1} b\rangle} \int (dx dx^*)\, e^{-\frac{1}{2}\langle x^*, Ax\rangle}$$

Apply this formula to our Naïve fermion generating functional:

$$Z[J] = N \int (D\psi)(D\bar{\psi})\, e^{i \int d^4x [\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \bar{J}\psi + \bar{\psi}J]}$$

$$Z[J = 0] = 1$$

$$= e^{i \int d^4x \left[\bar{J}(i\gamma^\mu \partial_\mu - m)^{-1} J\right]} \cdot N \int (D\psi)(D\bar{\psi})\, e^{i \int d^4x [\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi]} = e^{i \int d^4x \left[\bar{J}\boxed{(i\gamma^\mu \partial_\mu - m)^{-1}} J\right]}$$

We are done and it is correct! But the key we need shift symmetry in Grassman number.

All diagrammatic program of QFT of fermions (QED) follows as before from here.

Grassman number and its integration with shift invariance:

θ, ξ anticommute $\{\theta, \xi\} = \theta\xi + \xi\theta = 0$

θ, ξ anticommute with themselves: $\{\theta, \theta\} = \theta\theta + \theta\theta = 0$

Any larger than one power of θ, ξ vanish $\theta\theta = 0, \theta^{n>1} = 0$

Any function of θ can expanded like: $f(\theta) = a + b\theta$ $e^{a\theta} = 1 + a\theta$

θ integration must satisfy shift invariance:

$$\int d\theta\, \theta = \int d\theta (a + \theta) = a \int d\theta + \int d\theta\, \theta \Rightarrow \int d\theta = 0$$

We can then choose to normalize θ so that:

$$\int d\theta\, \theta = 1$$

Integration of a function:

$$\int d\theta\, f(\theta) = \int d\theta (a + b\theta) = a \int d\theta + b \int d\theta\, \theta = b$$

Grassman number integration is identical to taking derivative.

θ has a complex conjugate $\bar{\theta}$.

$\theta, \bar{\theta}$ anticommute $\{\theta, \bar{\theta}\} = 0$ $\bar{\theta}\bar{\theta} = 0, \bar{\theta}^{n>1} = 0$

$$\int d\theta d\bar{\theta} = 0$$

$$\int d\theta d\bar{\theta}\, \theta = 0 \qquad \int d\theta d\bar{\theta}\, \bar{\theta} = 0$$

$$\int d\theta d\bar{\theta} \cdot \bar{\theta}\theta = \int d\theta\, \theta \cdot \int d\bar{\theta}\, \bar{\theta} = 1$$

$$\int d\theta d\bar{\theta} \cdot e^{-\frac{1}{2}\bar{\theta} a \theta} = \int d\theta d\bar{\theta} \cdot \left(1 - \frac{a}{2}\bar{\theta}\theta\right) = \frac{a}{2}$$

Now we can calculate the Gaussian Integration:

$$\theta = (\theta_1, \cdots \theta_i \cdots \theta_n)$$

$$(d\theta) = d\theta_1 d\theta_2 \cdots d\theta_n$$

$$\int (d\theta)(d\bar{\theta})\, e^{-\frac{1}{2}\langle\bar{\theta}, A\theta\rangle} = \int (d\theta)(d\bar{\theta})\, e^{-\frac{1}{2}\sum_{i,j=1}^{n} \bar{\theta}_i A_{ij} \theta_j}$$

$$= \int (d\theta)(d\bar{\theta}) \left[1 - \bar{\theta}_i A_{ij} \theta_j + \frac{1}{2}(\bar{\theta}_i A_{ij} \theta_j)(\bar{\theta}_i A_{ij} \theta_j) + \cdots\right]$$

The only term that will survive is the one with all n θ_j and all n $\bar{\theta}_i$.

$$= \int (d\theta)(d\bar{\theta}) \left[\frac{1}{n!} \sum_{\text{permutations }\{i_n\}} \pm A_{i_1 i_2} \cdots A_{i_{n-1} i_n}\right] (\theta)(\bar{\theta})$$

This is a sum over all elements $\{i, j\}$ where we choose each row and column once.

Exactly how you compute A's determinant.

$$\int (d\theta)(d\bar{\theta})\, e^{-\frac{1}{2}\langle\bar{\theta}, A\theta\rangle} = \det A$$

In contrast:

$$\int (dx)\, e^{-\frac{1}{2}\langle x, Ax\rangle} = (\det A)^{-\frac{1}{2}}$$

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Now I got a fancy and efficient way to write determinant of an operator.

$$\int (D\psi)(D\bar{\psi})\, e^{i\int d^4x[\bar{\psi}A\psi]} \sim \det A$$

You may still remember we need it in non-abelian gauge theory quantization.

$$\int (DW_\mu^k)\det\left(\frac{\delta G^k}{\delta\alpha^i}\right)\delta(\partial^\mu W_\mu^k)e^{iS}$$

$$\det\left(\frac{\delta G^k}{\delta\alpha^i}\right)\sim\det\left[\partial_\mu\partial^\mu\delta^{(4)}(x-y)\delta^{ki}-g\delta^{(4)}(x-y)\varepsilon^{jik}W_\mu^j\right]$$

We simply identify A with

$$A=\partial_\mu\partial^\mu\delta^{(4)}(x-y)\delta^{ki}-g\delta^{(4)}(x-y)\varepsilon^{jik}W_\mu^j$$

Introduce 3 ghost fermion η^j, one for every W_μ^j.

$$\det\left(\frac{\delta G^k}{\delta\alpha^i}\right)\sim\int (D\eta)(D\bar{\eta})\, e^{i\int d^4x[\bar{\eta}A\eta]}$$

$$\boldsymbol{\eta}\equiv(\eta^1,\eta^2,\eta^3)$$

$$\bar{\eta}A\eta=\overline{\eta^j}\Box\eta^j-g\overline{\eta^k}\varepsilon^{jik}\partial_\mu\left(W_\mu^j\eta^i\right)=\bar{\boldsymbol{\eta}}\partial_\mu D^\mu\boldsymbol{\eta}=\bar{\boldsymbol{\eta}}\Box\boldsymbol{\eta}+g\partial_\mu\bar{\boldsymbol{\eta}}\boldsymbol{W}_\mu\times\boldsymbol{\eta}$$

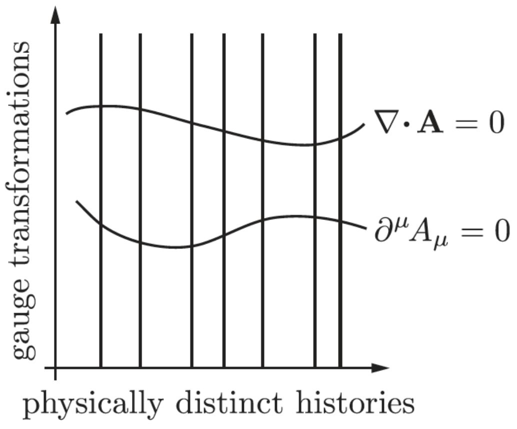


Figure 31.3: Fixing the gauge

You can no longer do that in non-abelian gauge theory.

$$\delta \boldsymbol{W}_\mu = \frac{1}{g}\partial_\mu \boldsymbol{\alpha} + \boldsymbol{\alpha}\times\boldsymbol{W}_\mu$$

or in components:

$$\delta W_\mu^k = \frac{1}{g}\partial_\mu \alpha^k + \varepsilon^{ijk}\alpha^i W_\mu^j$$

Take Lorentz gauge: $G^k = \partial^\mu W_\mu^k = 0$

$$\int (DW_\mu^k)\det\left(\frac{\delta G^k}{\delta\alpha^i}\right)\delta(\partial^\mu W_\mu^k)e^{iS}$$

The condition is transformed into:

$$G^k\left[W_\mu^k + \frac{1}{g}\partial_\mu\alpha^k + \varepsilon^{ijk}\alpha^i W_\mu^j\right] = \partial^\mu W_\mu^k + \frac{1}{g}\partial_\mu\partial^\mu\alpha^k + \partial_\mu\left(\varepsilon^{ijk}\alpha^i W_\mu^j\right)$$

$$= \partial^\mu W_\mu^k + \frac{1}{g}\partial_\mu\left[\partial^\mu\alpha^k - g\partial_\mu\left(\varepsilon^{jik}W_\mu^j\alpha^i\right)\right]$$

Note that the difference depends on gauge field and can’t be ignored.

$$\det\left(\frac{\delta G^k}{\delta\alpha^i}\right) = \det\left(\frac{\delta\Box\alpha}{\delta\alpha}\right) \sim \det\left[\partial_\mu\partial^\mu\delta^{(4)}(x-y)\delta^{ki} - g\delta^{(4)}(x-y)\varepsilon^{jik}W_\mu^j\right]$$

Det term involves gauge bosons.

$$\bar{\eta}A\eta = \overline{\eta^j}\Box\eta^j - g\overline{\eta^k}\varepsilon^{jik}\partial_\mu\left(W_\mu^j\eta^i\right) = \bar{\boldsymbol{\eta}}\partial_\mu D^\mu\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}\Box\boldsymbol{\eta} + g\partial_\mu\bar{\boldsymbol{\eta}}\boldsymbol{W}_\mu\times\boldsymbol{\eta}$$

Adding ghost fields is the final Fadeev-Popov prescription:

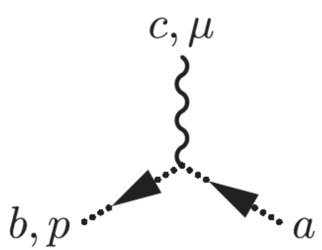
$$\int (D\boldsymbol{W}_\mu)(D\boldsymbol{\eta})(D\bar{\boldsymbol{\eta}})\,\delta\left(\partial^\mu\boldsymbol{W}_\mu\right)e^{i\int d^4x[\bar{\boldsymbol{\eta}}\partial_\mu D^\mu\boldsymbol{\eta}]+iS[\boldsymbol{W}_\mu]}$$

Note that we one ghost η^j for every W_μ^j and η^j couple only to W_μ^j.

Ghost propagate like a scalar without spin :

$$\frac{i\delta^{ab}}{p^2 + i\epsilon}$$

But they are fermions (it certainly breaks CPT theorem, it's ghost anyway.)



$$ip^\mu g\varepsilon^{abc}$$