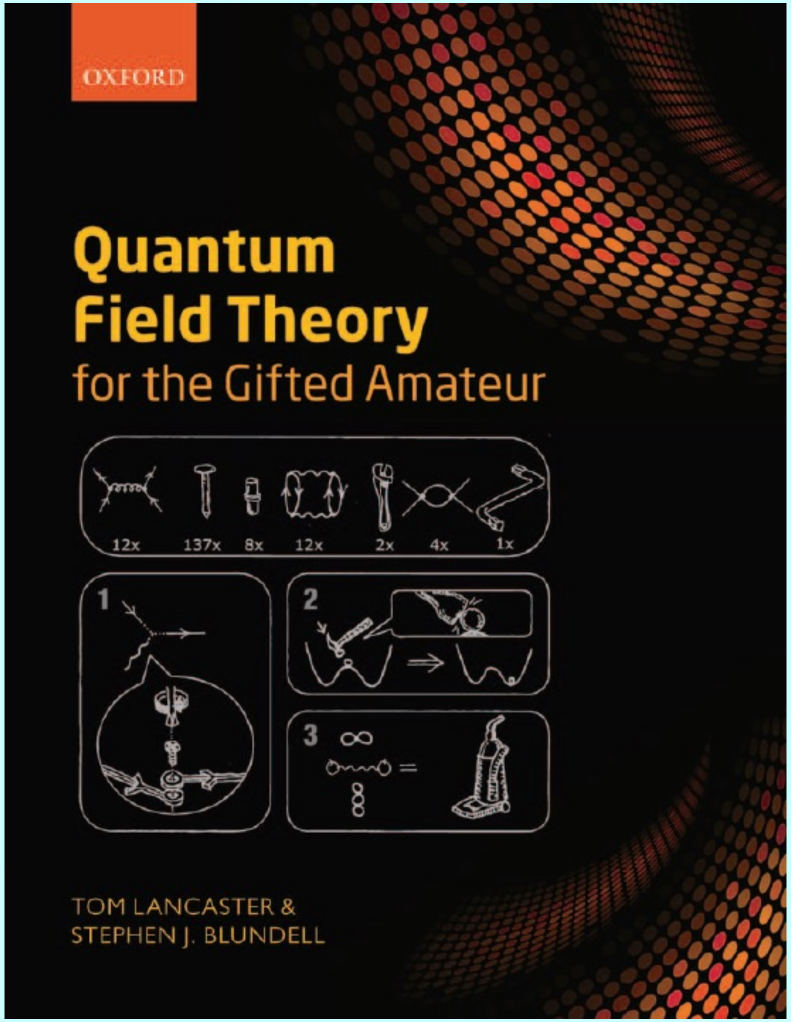
https://phy.ntnu.edu.tw/~chchang/

張嘉泓 台師大物理

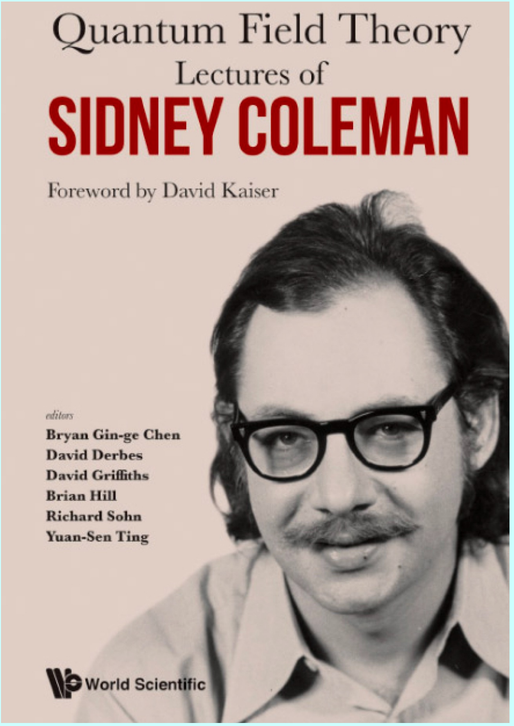


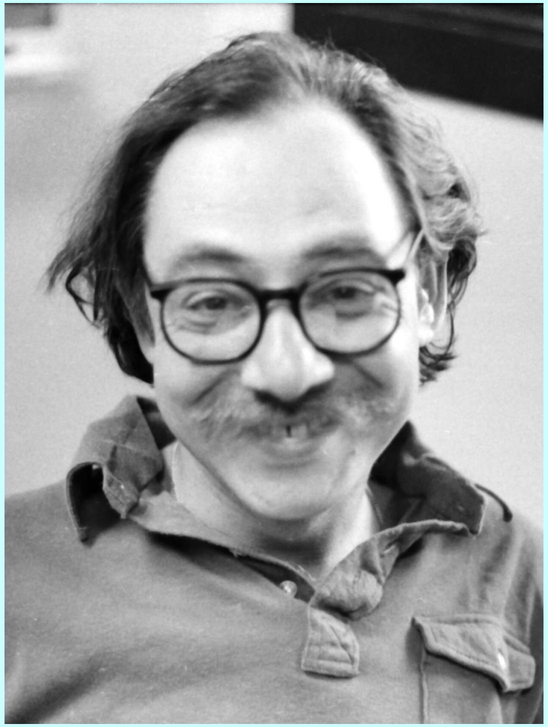
XI After P423 Non-Abelian Gauge theory (Yang-Milles)

Standard Model (Electroweak Unification), Functional Integration,

Asymptotic Freedom QCD, Fermion mass

A bit of Coleman's Style

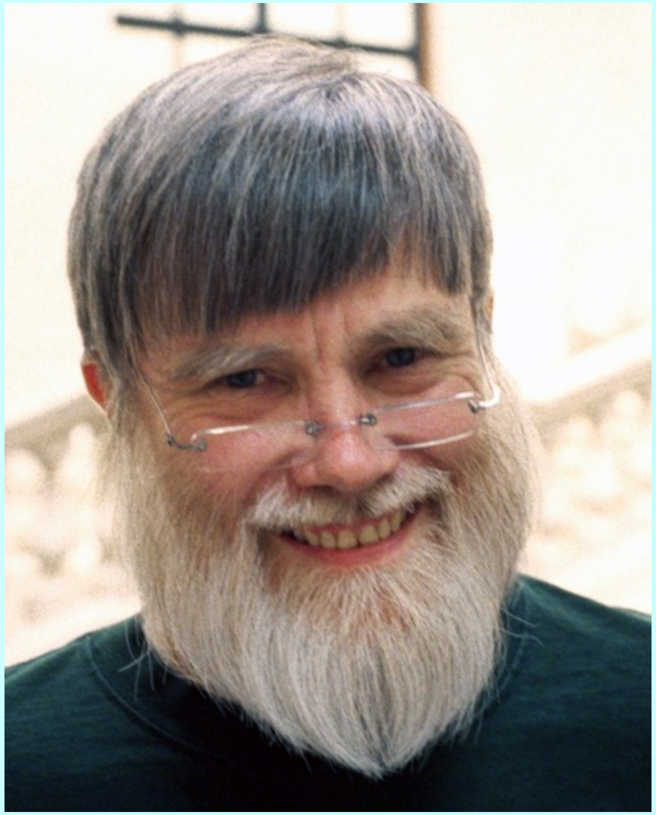


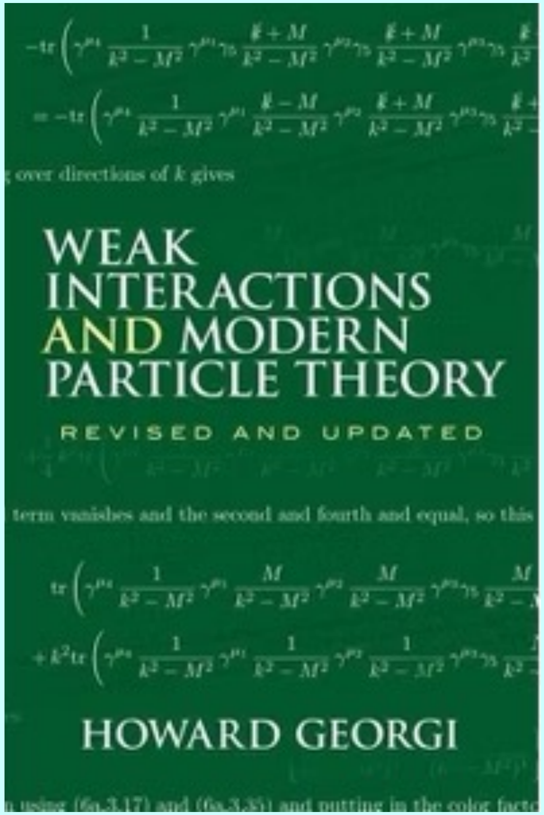


再怎麼複雜的計算，都可以站著、空著手、用說的！

His Style is: as cute as possible.

A bit of Georgi's Style





再怎麼晦澀的物理，都可以坐下來、加一點恰好的數學，用說的！

Most of all, it is a lonely desert island style

desert island discs

於一個孤島上，在腦中重建宇宙核心的秩序！

讓我跟你講道理。

Some thoughts about QFT:

古典物理與近代物理的差別是什麼？

這裡就是古典物理與近代物理的邊界！

Physics is all about particles (and fields).

In Classical Physics, no two particles are the same.

在古典物理中，沒有任何兩個粒子是一模一樣的。

You can not step into the same river twice. Heraclitus 一期一會

在量子物理中，沒有任何兩個帶電-1.6×10^{-19}C的基本粒子是不一樣的。

In Modern Physics, no two negatively charged particles are any different.

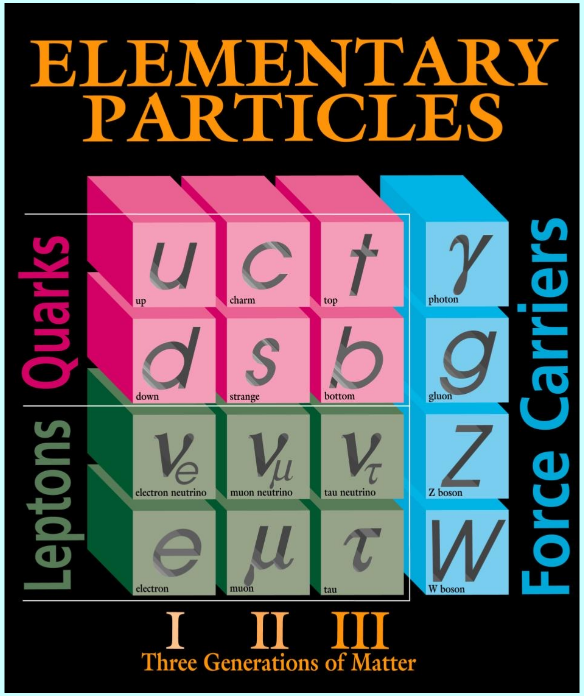
fundamental in nature

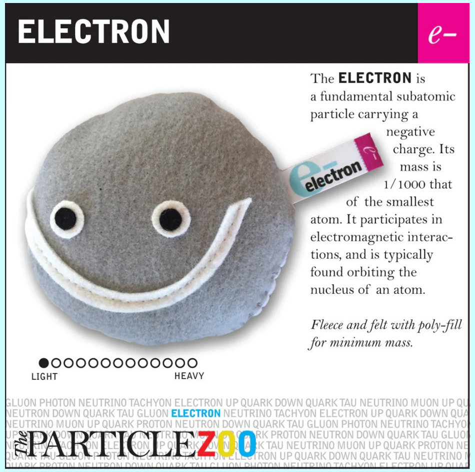
Quantum fields are particles.

Thomson發現原子可以分割為更加細微的一種物質基本組成成分，

此成分只有一種，此一種成分組成了所有的不同的原子！

電子 Electron e^{-}

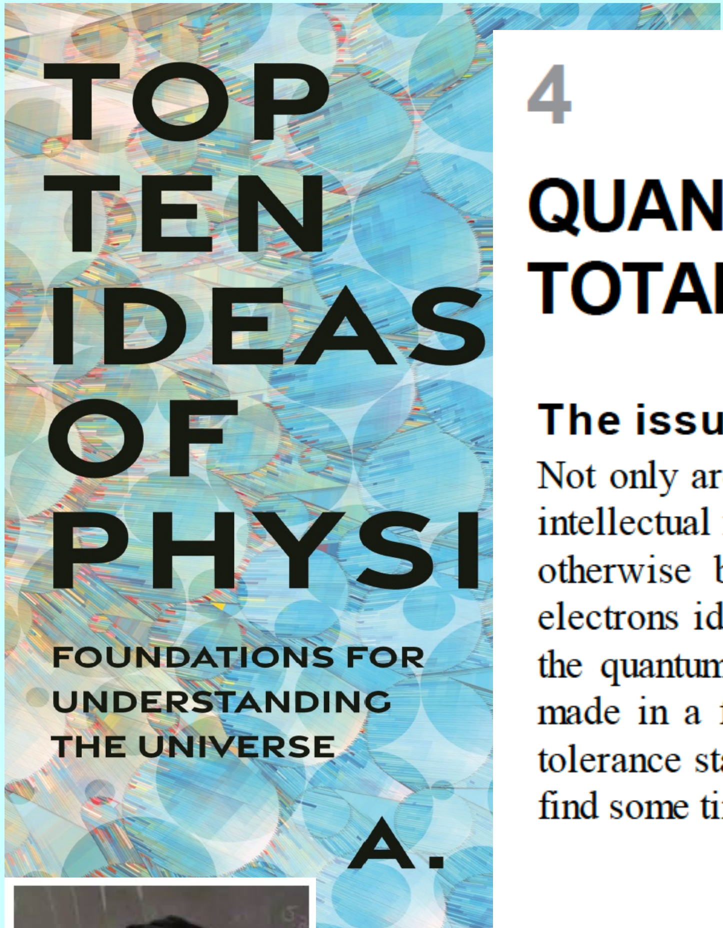




無論你在何時何地找到另一顆電子，都與你現在手中這顆電子完全沒有差異。

這深刻的奇蹟竟然有一個簡單的答案：所有的電子都屬於同一個電子場！

量子場論是近代物理核心中的核心！



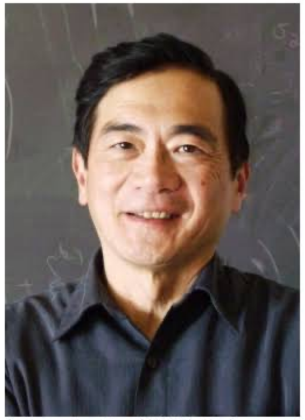
4

QUANTUM FIELDS FOREVER: EINSTEIN'S TOTAL LOVE

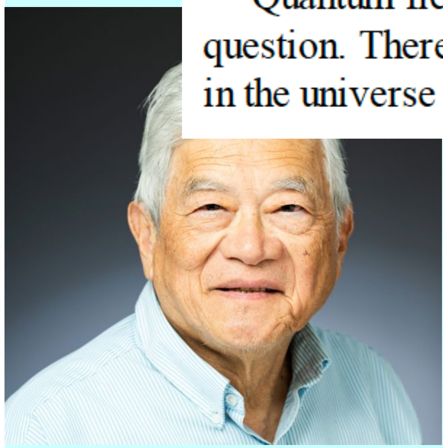
The issue of identity in the quantum world

Not only are quantum fields entirely real and undergirding our world, they provide an intellectual framework for understanding burning questions about the universe that would otherwise be unanswerable and deeply bother theorists like me. Why are all the electrons identical? And identical means truly identical; indeed, the issue of identity in the quantum world is central to quantum physics.[24] Couldn't the electrons have been made in a factory somewhere, with unavoidable manufacturing defects? Although the tolerance standards are presumably set high beyond any human capability, could we not find some tiny differences between two given electrons if we look hard enough?

Quantum field theory provides an elegantly simple answer to this purely intellectual question. There exists only one electron field, and the zillions of electrons we encounter in the universe are but excitations in this one single field.



Anthony Zee joined UCSB's Kavli Institute for Theoretical Physics, and the Department of Physics, in 1985.
Photo Credit: COURTESY PHOTO



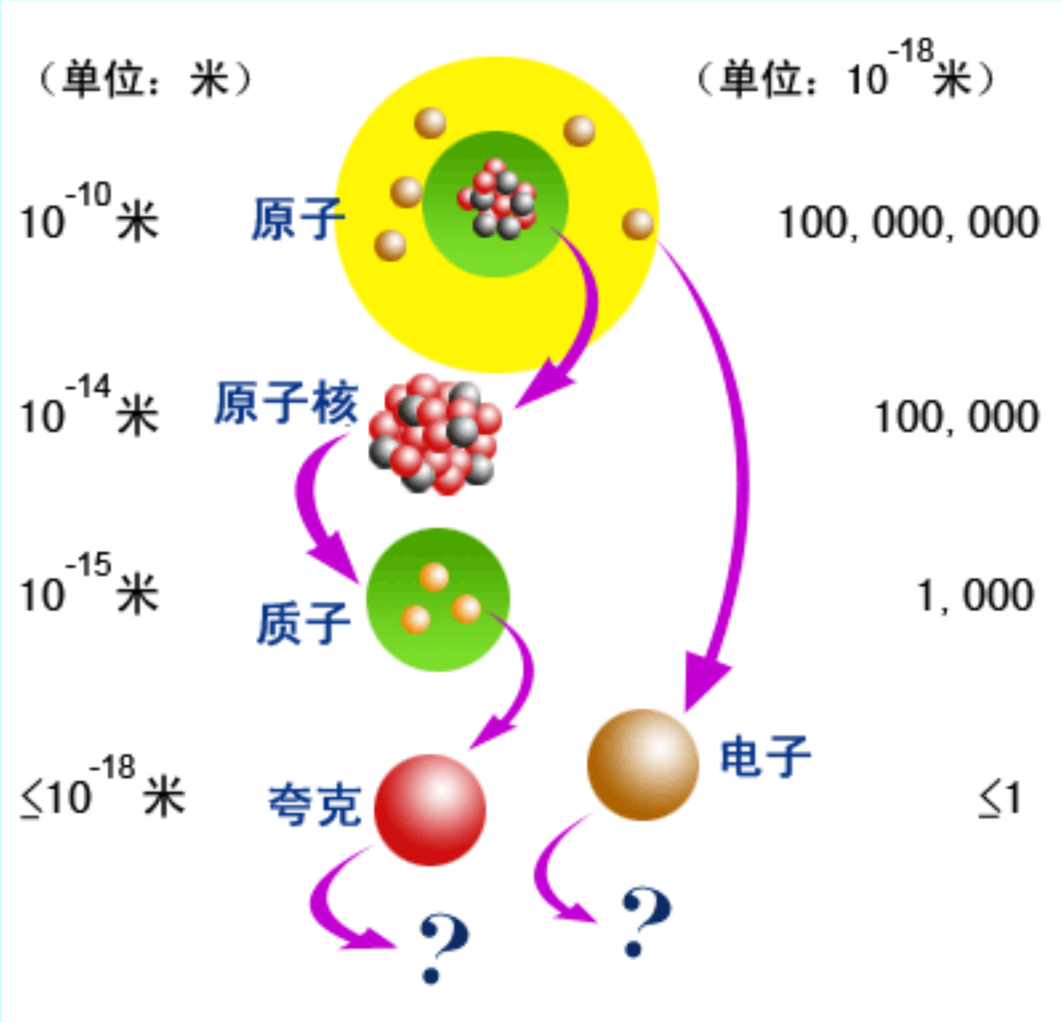
Quantum field theory is the most versatile theory of the world!

It is the basic tool of particle physics,

It the basic tool of many-body physics in condensed matter physics.

It could be used in thermal field theory of statistical mechanics.

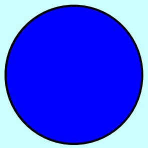
Fundamental Particles



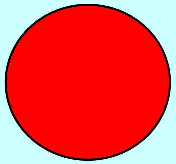
假如質子的直徑是1公分，夸克就比一根頭髮的直徑還小，

而原子則比30個足球場還大！

五種基本粒子組成了自然

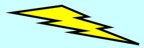


Neutron



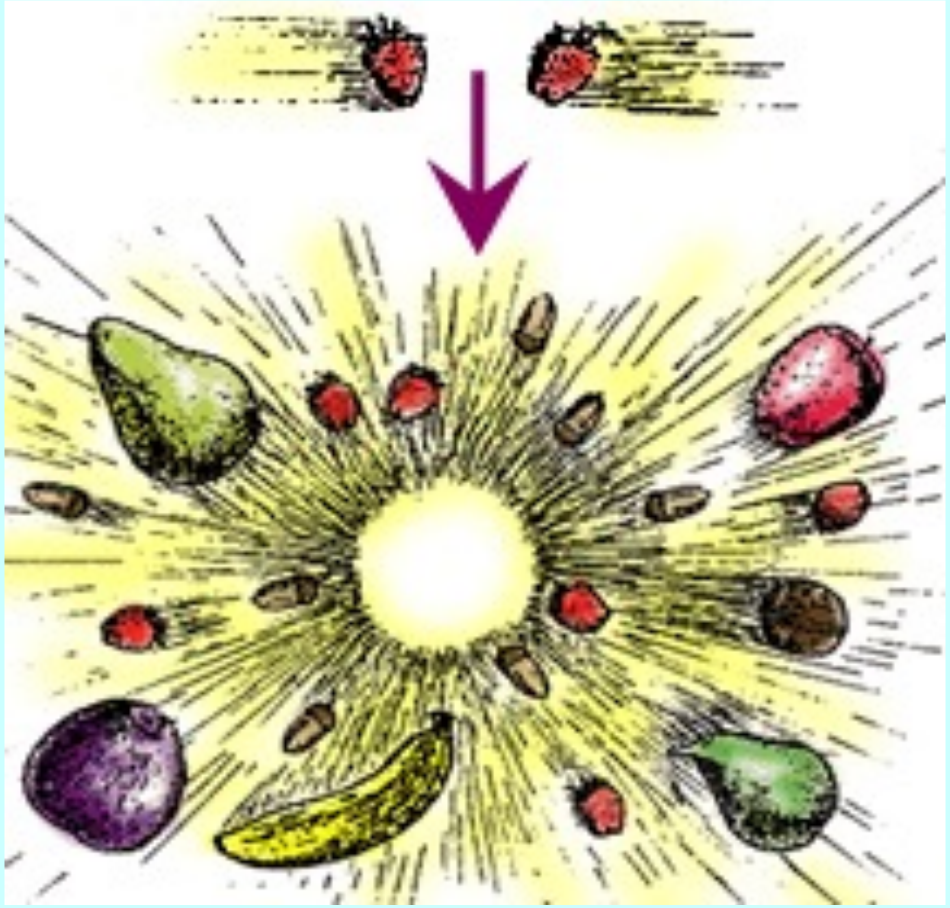
Proton

Electron



Photon

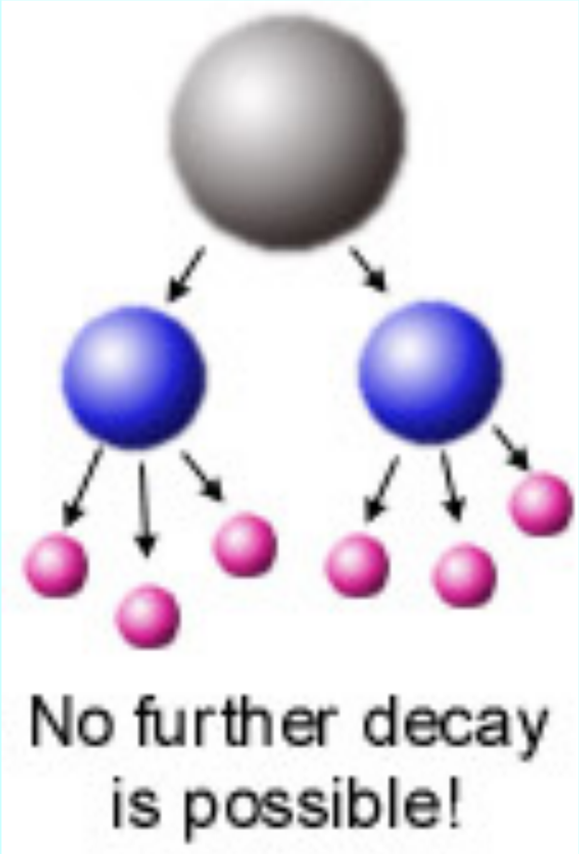
Neutrino

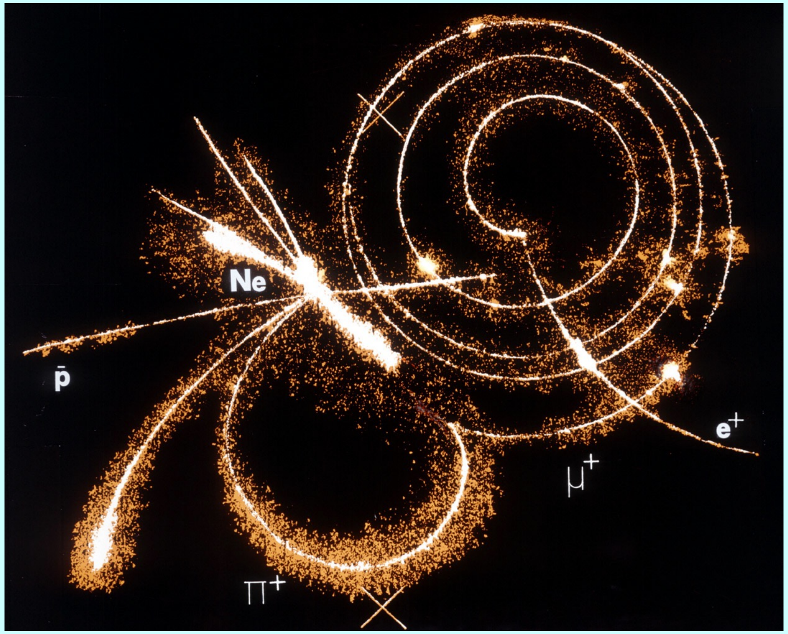


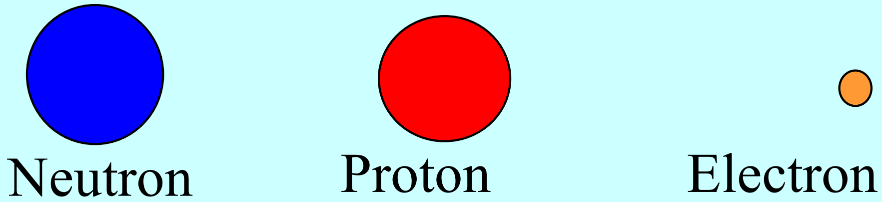
$$E = mc^2$$

能量可以轉化為質量，在撞擊中可以產生新粒子。

越重的粒子需要越多的能量才能產生。







但倒過來，新粒子也可以衰變消失，將質量轉化為產物的運動能量。

所以這些新的基本粒子，在自然界是不存在的。

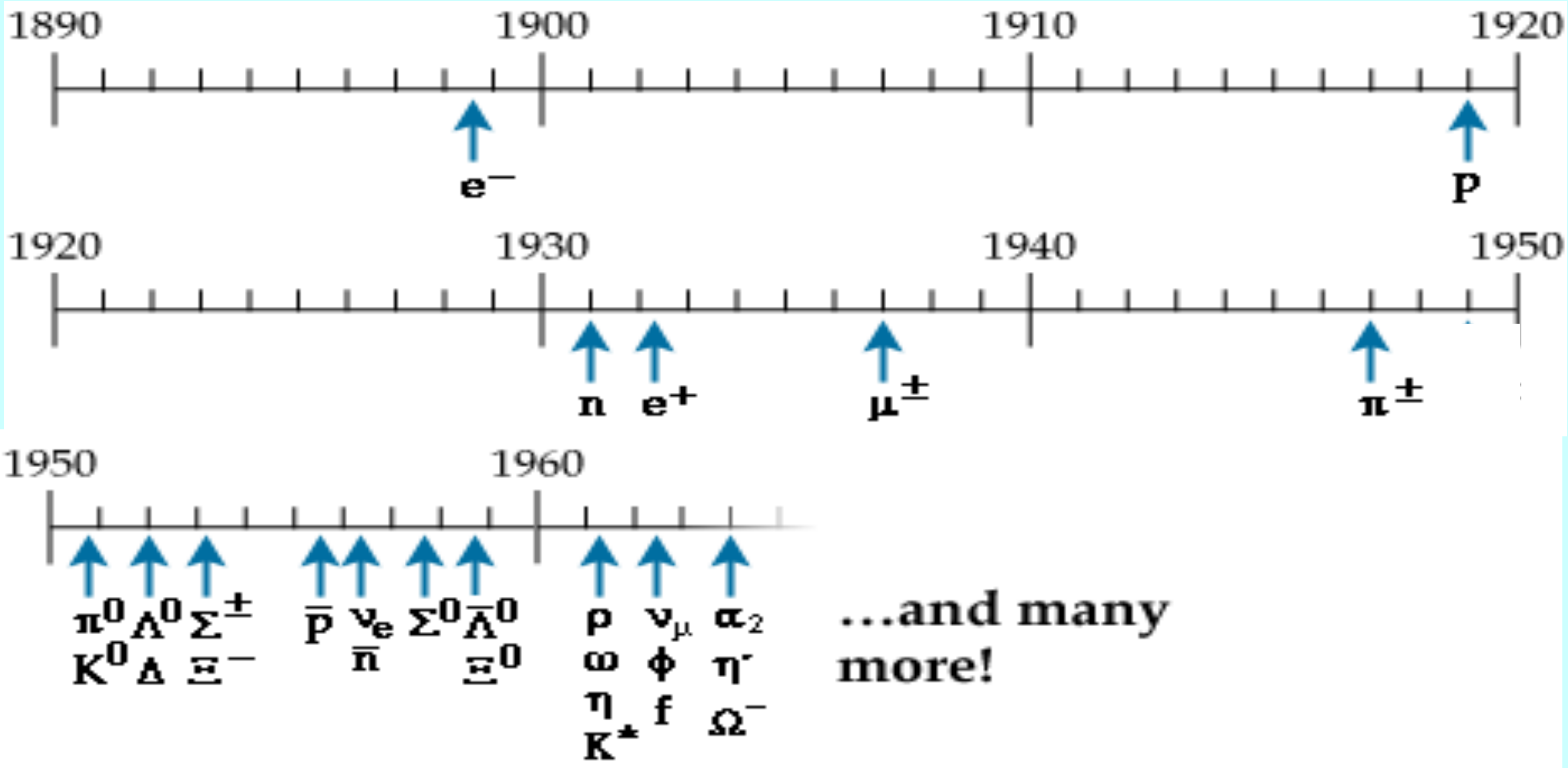
一生成很快就衰變消失。留下電子、質子、中子這些無法衰變的基本粒子。

它們構成了我們日常自然世界。

什麼！基本粒子也會衰變？

對！幾乎所有基本粒子都會衰變！

如果基本粒子也會衰變，人世間還有什麼會是永恆的？



And more and more and more …越來越多，越來越多

until they don't feel fundamental at all………..

$a_2(1320)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
3π	(70.1 ±2.7) %	S=1.2	624
$\eta\pi$	(14.5 ±1.2) %		535
$\omega\pi\pi$	(10.6 ±3.2) %	S=1.3	366
$K\overline{K}$	(4.9 ±0.8) %		437
$\eta'(958)\pi$	(5.3 ±0.9) $\times 10^{-3}$		288
$\pi^{\pm}\gamma$	(2.68±0.31) $\times 10^{-3}$		652
$\gamma\gamma$	(9.4 ±0.7) $\times 10^{-6}$		659
e^+e^-	< 6 $\times 10^{-9}$	CL=90%	659

$f_0(1370)$ [l] $I^G(J^{PC}) = 0^+(0^{++})$

Mass m = 1200 to 1500 MeV
Full width Γ = 200 to 500 MeV

$f_0(1370)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	672
4π	seen	617
$4\pi^0$	seen	617
$2\pi^+2\pi^-$	seen	612
$\pi^+\pi^-2\pi^0$	seen	615
$\rho\rho$	dominant	†
$2(\pi\pi)_{S\text{-wave}}$	seen	–
$\pi(1300)\pi$	seen	†
$a_1(1260)\pi$	seen	35
$\eta\eta$	seen	411
$K\overline{K}$	seen	475
$K\overline{K}n\pi$	not seen	†
6π	not seen	508
$\omega\omega$	not seen	†
$\gamma\gamma$	seen	685
e^+e^-	not seen	685

$\pi_1(1400)$ [o] $I^G(J^{PC}) = 1^-(1^{-+})$

Mass m = 1351 ± 30 MeV (S = 2.0)
Full width Γ = 313 ± 40 MeV

$\pi_1(1400)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\eta\pi^0$	seen	555
$\eta\pi^-$	seen	554

$\eta(1405)$ [p] $I^G(J^{PC}) = 0^+(0^{-+})$

Mass m = 1409.8 ± 2.5 MeV [n] (S = 2.2)
Full width Γ = 51.1 ± 3.4 MeV [n] (S = 2.0)

$\eta(1405)$ DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)

$\omega(1420)$ DECAY MODES
$\rho\pi$
$\omega\pi\pi$
$b_1(1235)\pi$
e^+e^-

$a_0(1450)$ [l]

Mass m = 1474
Full width Γ =

$a_0(1450)$ DECAY MODES
$\pi\eta$
$\pi\eta'(958)$
$K\overline{K}$
$\omega\pi\pi$

$\rho(1450)$ [s]

Mass m = 146
Full width Γ =

$\rho(1450)$ DECAY MODES
$\pi\pi$
4π
e^+e^-
$\eta\rho$
$a_2(1320)\pi$
$K\overline{K}$
$K\overline{K}^*(892)$ + c.c.
$\eta\gamma$

$\eta(1475)$ [p]

Mass m = 147
Full width Γ =

$\eta(1475)$ DECAY MODES
$K\overline{K}\pi$
$K\overline{K}^*(892)$ + c.c.
$a_0(980)\pi$
$\gamma\gamma$

$f_0(1500)$ [o]

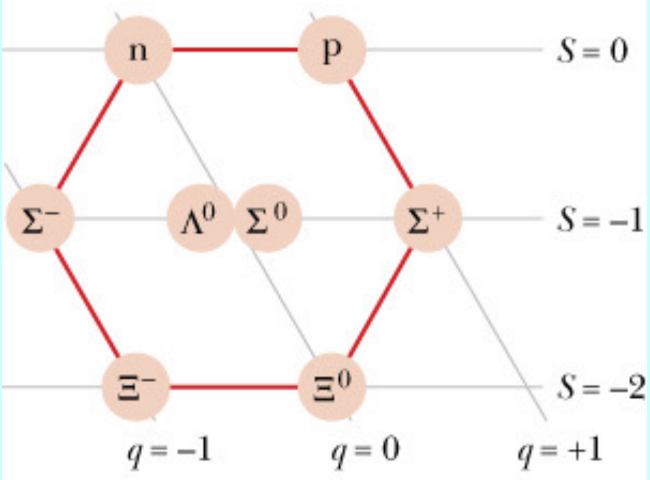
....早知道是這樣，當年還不如做個動物學家.........

費米

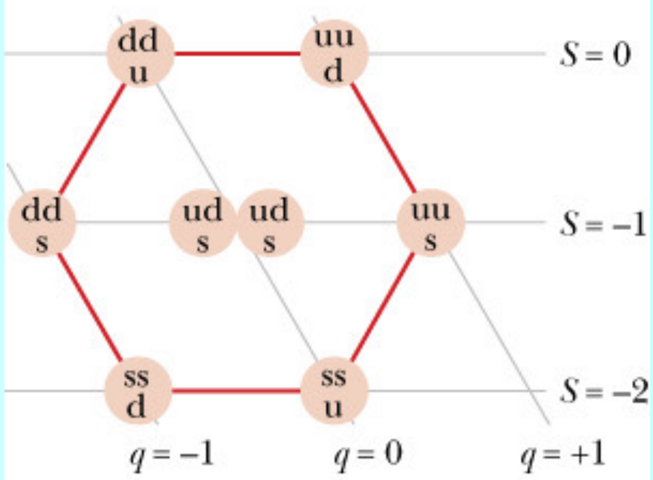
兩年後，Gellmann自己提出一個極簡單的解釋！

數學上，Octet可以看成這三個性質相似物件 u, d, s ，所作以下兩種組合的所有可能：

三個物體的組合

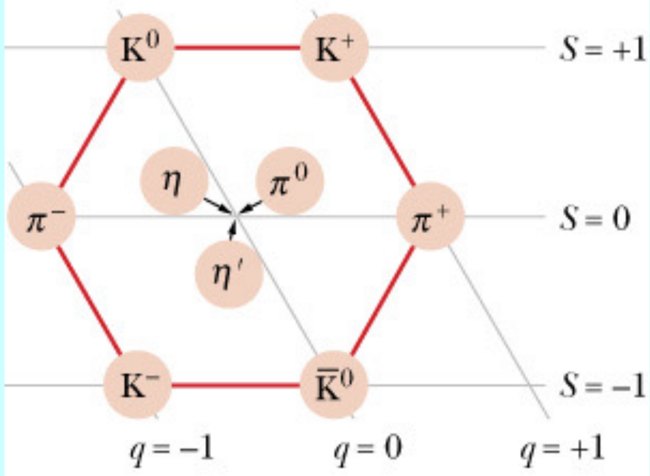


(a)

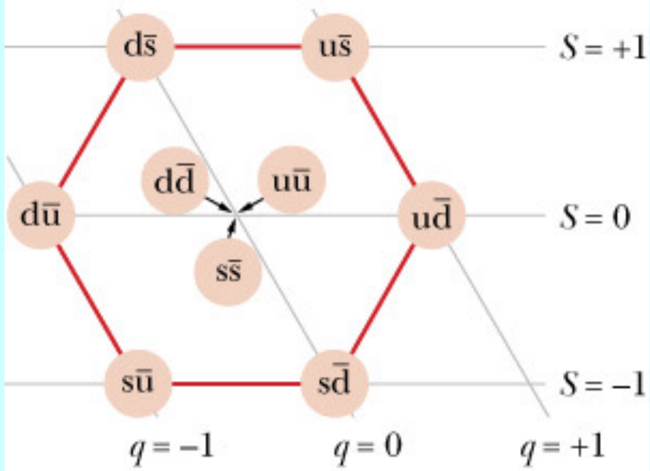


(a)

物體與反物體的組合



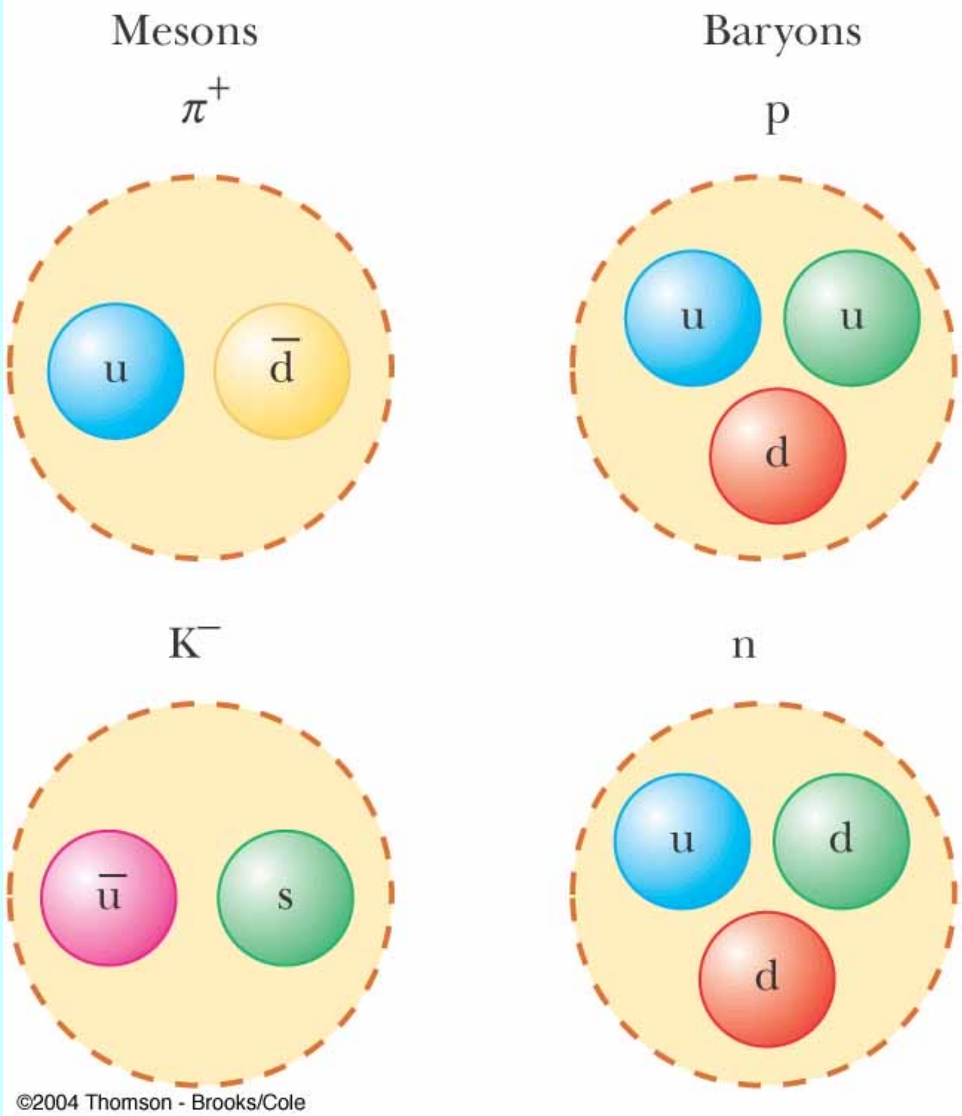
(b)



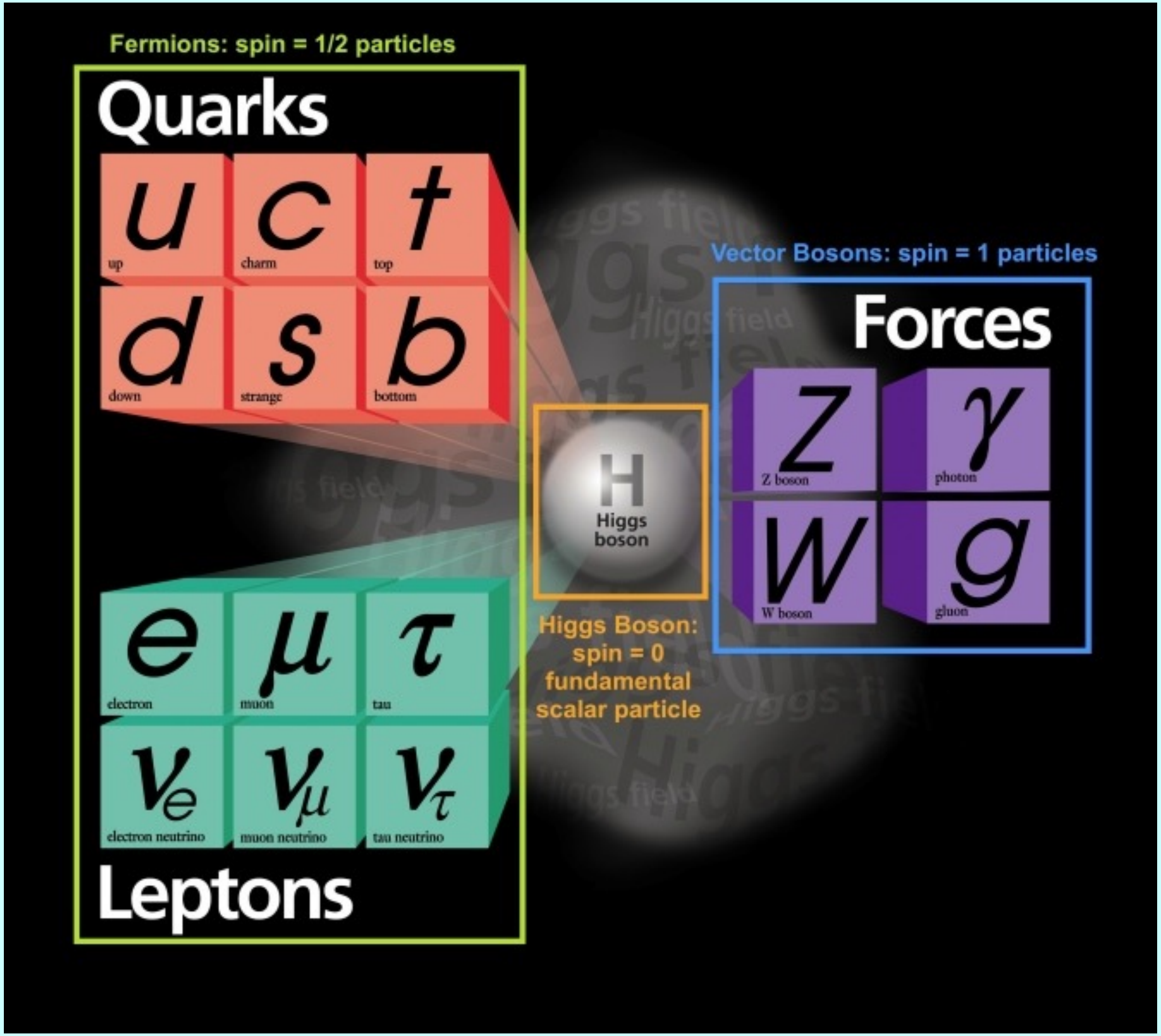
(b)

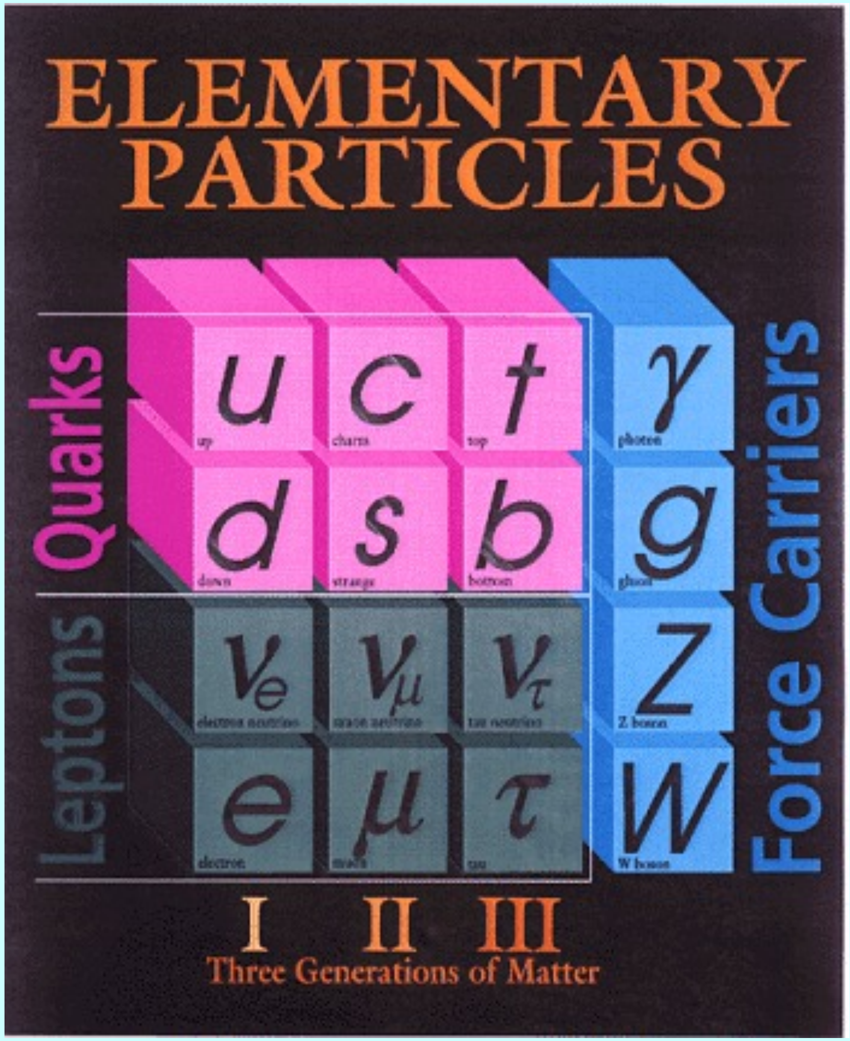
質子與中子是由三顆夸克組成，
口味是uud與udd.

Pion粒子則是一正一反兩顆夸克組成。



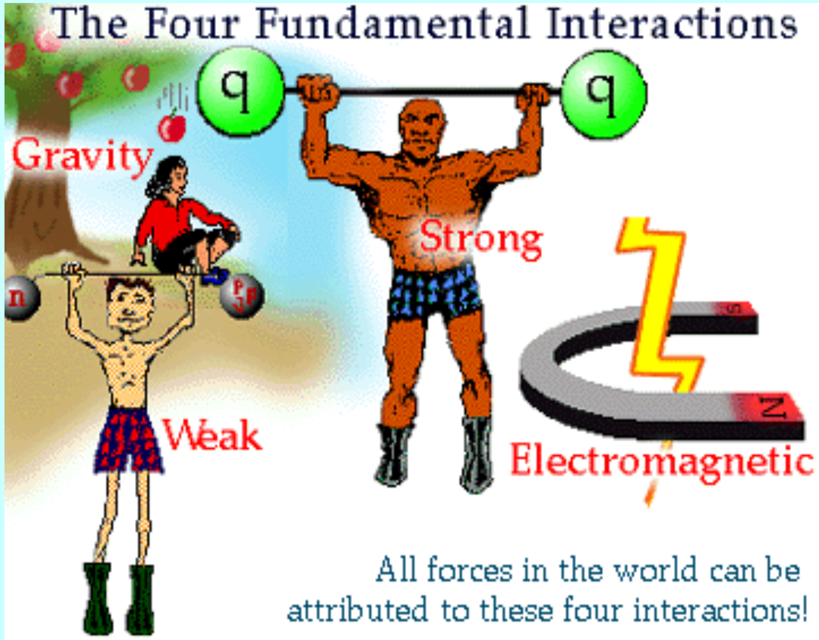
We now know there is a fundamental structure of the quantum world.

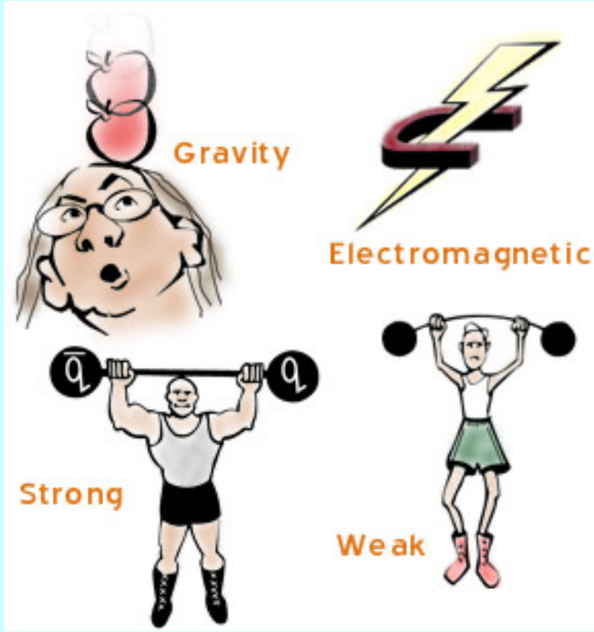


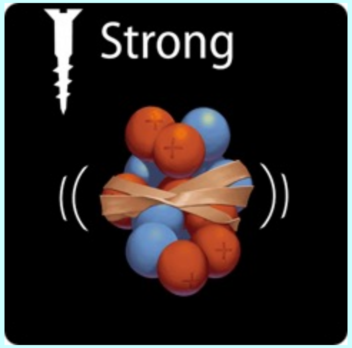

ELEMENTARY
PARTICLES
Quarks
Leptons
Force Carriers
u c t γ
d s b g
ν_e ν_μ ν_τ Z
e μ τ W
I II III
Three Generations of Matter

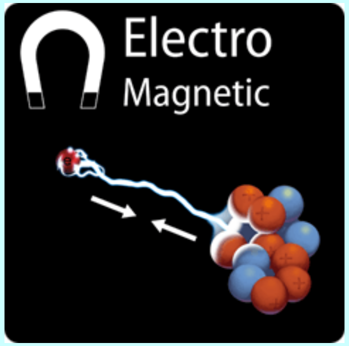
牛頓力學裡的力，現代我們喜歡稱為交互作用

宇宙間的所有交互作用都可以分解為四個基本交互作用









Non-Abelian Gauge Thoery

電、弱、強交互作用竟然可以用同一種規範場論來描述！

Gaue field theory

In EM you have the freedom to choose gauge.

$A^{\mu} = (\phi, \boldsymbol{A})$ A^{μ}是4-vector向量位：

You can choose a different gauge by the following gauge transformation:

$\phi \rightarrow \phi + \partial_t \theta$ $\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\theta$ or $A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\theta$

Note that $\theta(t, \boldsymbol{x})$ is function of both time and space. This is a localized transformation.

The observable electromagnetic fields do not change!

$$\boldsymbol{E} = -\nabla\phi + \frac{\partial \boldsymbol{A}}{\partial t} \rightarrow -\nabla\phi - \nabla\frac{\partial\theta}{\partial t} + \frac{\partial \boldsymbol{A}}{\partial t} + \frac{\partial}{\partial t}\nabla\theta = \boldsymbol{E}$$

$$\boldsymbol{B} \rightarrow \nabla\times\boldsymbol{A} + \nabla\times\nabla\theta = \boldsymbol{B}$$

Field strength tensor $F^{\mu\nu}$ is invariant under gauge transformation.

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \rightarrow \partial^{\mu}A^{\nu} + \partial^{\mu}\partial^{\nu}\theta - \partial^{\nu}A^{\mu} - \partial^{\nu}\partial^{\mu}\theta = F^{\mu\nu}$$

A^{μ} is not observable. We can pretend gauge tr ool.

Choosing gauge is a freedom.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

但不幸的是，在量子力學中，A^{μ}是可觀察的。Aharonov–Bohm effect。

但同時這個gauge transformation在量力中竟然還有更大發揮：透過波函數的相角！

量子力學只與波函數的絕對值有關。 $U(1)$ symmetry

若對波函數乘上一個相角，將不影響物理結果， $\psi(x) \to e^{i\theta}\psi(x)$

薛丁格方程式將不會有變化：

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \phi(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

因為這個對稱，電荷是守恆的。

但Weyl and Fock有個怪點子 ~1918：

若要求對稱可以即興：可以任意引入與時間有關的相角： $\psi(x) \to e^{i\theta(t)}\psi(x)$

Vladimir Fock 1898-1974

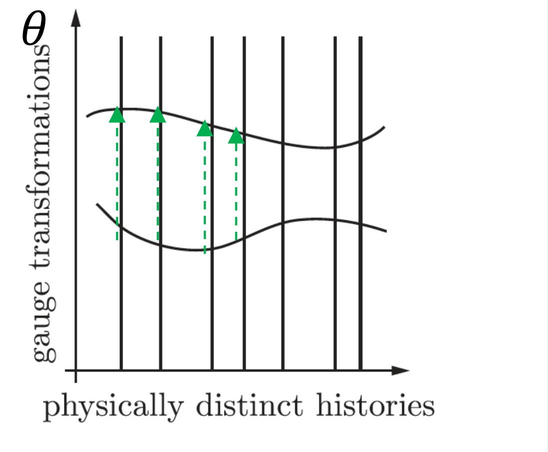
Hermann Weyl 1885-1955

若要求對稱可以即興：可以任意引入與時間有關的相角： $\psi(x) \to e^{i\theta(t)}\psi(x)$

如此我可以在任何時間，利用此自由度，改變物理量隨時間的變化。

下圖下方的歷史軌跡，現在變成上方的曲線。

因為變化是任意的，所以歷史軌跡的曲線也可以任意的。



若這樣的即興變換不影響物理結果，那就表示不同曲線代表同樣的物理。

不難看出垂直的這一個量是多餘的Redundant!

$$\psi(x) \to e^{i\theta(t)}\psi(x)$$

但如果我們堅持在地化對稱、橫柴入灶！

$$\partial_t\psi \to e^{i\theta}\partial_t\psi + i\partial_t\theta \cdot e^{i\theta}\psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + e\phi\psi = i\hbar\partial_t\psi \xrightarrow[\text{消掉}e^{i\theta}]{} -\frac{\hbar^2}{2m}\nabla^2\psi + e\phi\psi = i\hbar\partial_t\psi + i\partial_t\theta \cdot \psi$$

多了

只要電位也跟著變！要求電位變化正好消掉多出來的項！

$$\phi \to \phi + \frac{1}{e}\partial_t i\theta$$

而這就是電位的規範變換，可見規範變換就是即興的波函數相位變換。

規範變換一般可以在地：θ與地點相關，造成向量位$\boldsymbol{A}$的變換：

$$\boldsymbol{A} \to \boldsymbol{A} + \frac{1}{e}\nabla i\theta$$

如果相位在地，如此波函數空間微分會變換：

$$\nabla\psi(x) \to e^{i\theta}\nabla\psi + i\nabla\theta \cdot e^{i\theta}\psi$$

多了

現在我可以在薛丁格方程式的空間微分中引入向量位：

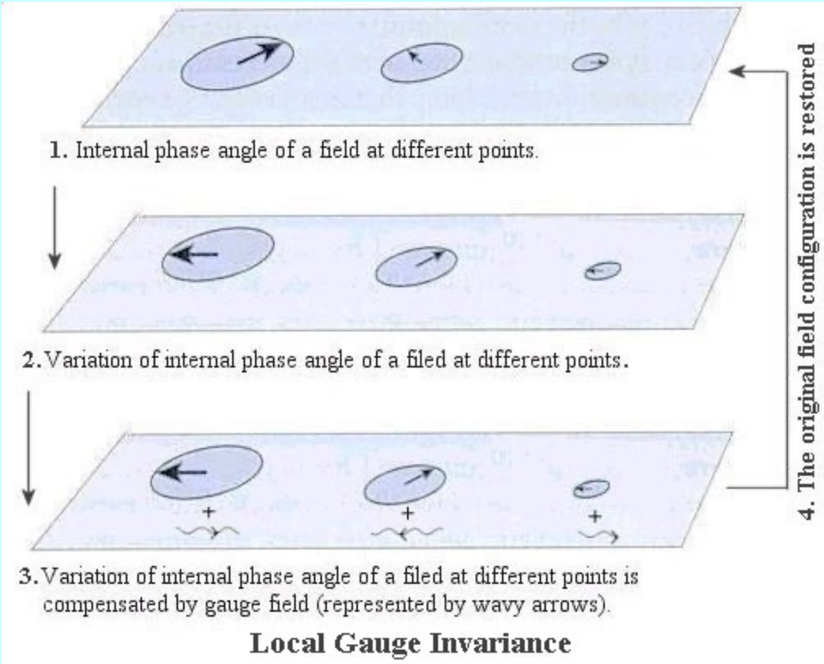
$$\nabla\psi(x) \to (\nabla - e\boldsymbol{A})\psi$$

空間微分的在地相位變換就正好抵消向量位的規範變換。

$$-\frac{\hbar^2}{2m}(\nabla - e\boldsymbol{A})^2\psi + e\phi(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

此薛丁格方程式是在地規範對稱的。對稱給定了、限制了電荷與電磁場的作用。

$$\psi(x) \to e^{i\theta(x)}\psi(x)$$



稱為$U(1)$ local 規範對稱 Gauge symmetry

Gauge Symmetry could be written in relativistic notation.

$$\phi \to \phi + \frac{1}{e}\partial_t\theta \qquad \boldsymbol{A} \to \boldsymbol{A} + \frac{1}{e}\nabla\theta \quad \Longrightarrow \quad A^\mu \to A^\mu + \frac{1}{e}\partial^\mu\theta$$

A^μ 有些自由度是多餘的Redundant!

This gauge symmetry can be also defined in QFT.

Gauge symmetry in QFT

Consider a Dirac field $\Psi(x)$.

$$\mathcal{L} = \bar{\Psi} i\gamma^{\mu}\partial_{\mu}\Psi - m\bar{\Psi}\Psi$$

The Lagrangian has a $U(1)$ symmetry when it is multiplied by a constant phase factor.

$$\Psi \to \Psi' = e^{i\theta}\Psi \qquad \partial_{\mu}\Psi' = e^{i\theta}\partial_{\mu}\Psi$$

$\partial_{\mu}\Psi$ transform exactly like Ψ.

Consider gauge transformation where the angle $\theta(x)$ is local. 在地化

$$\Psi'(x) = e^{i\theta(x)}\Psi(x)$$

Now $\partial_{\mu}\Psi$ does not transform exactly like Ψ.

$$\partial_{\mu}\Psi' = e^{i\theta}\partial_{\mu}\Psi + i\partial_{\mu}\theta \cdot e^{i\theta}\Psi(x)$$

Could we find a new "derivative" D_{μ} that act as if the transformation is global?

The recipe: add one vector field to ∂_{μ} and ask it to transform so that to cancel $\partial_{\mu}\theta$:

$$D_{\mu}\Psi = \partial_{\mu}\Psi - igA_{\mu}\Psi \qquad A^{\mu\prime} = A^{\mu} + \frac{1}{g}\partial_{\mu}\theta$$

$$D'_{\mu}\Psi' = \partial_{\mu}\Psi' - igA'_{\mu}\Psi' = e^{i\theta}\partial_{\mu}\Psi + i\partial_{\mu}\theta \cdot e^{i\theta}\Psi - igA^{\mu}e^{i\theta}\Psi - i\partial_{\mu}\theta \cdot e^{i\theta}\Psi$$

$$D'_{\mu}\Psi' = e^{i\theta(x)}D_{\mu}\Psi$$

Covariant derivative $D_{\mu}\Psi$ transform exactly like Ψ.

This is called Abelian Gauge Theory since symmetry is a commuting Abelian group $U(1)$.

The recipe is enough for us to write down gauge invariant QFT right away.

Simply replace ∂_μ by covariant derivative D_μ:

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \quad \Rightarrow \quad \mathcal{L} = \bar{\Psi} i\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi$$

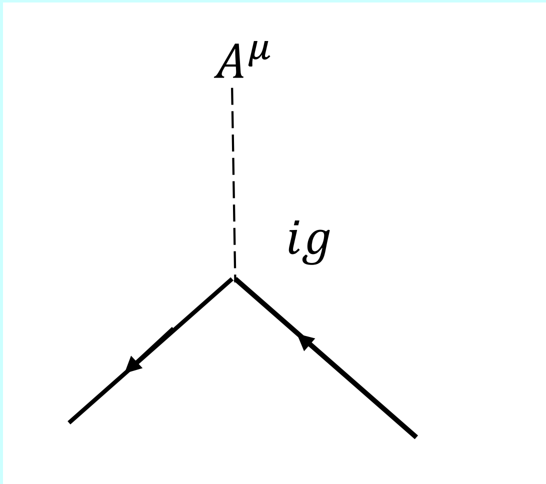
We are sure it's invariant since $D_\mu\Psi$ transform exactly like Ψ.

$$\rightarrow \bar{\Psi} e^{-i\theta(x)} e^{i\theta(x)} i\gamma^\mu D_\mu \Psi = \bar{\Psi} i\gamma^\mu D_\mu \Psi$$

Plug in the recipe: $\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi + ig\bar{\Psi}\gamma^\mu A^\mu \Psi$

The vertex is the interaction of charged particle and photon.

$$\mathcal{L}_I = ig\bar{\Psi}\gamma^\mu A^\mu \Psi \quad \Rightarrow$$



如果我們橫柴入灶堅持在地化對稱呢？加一個規範場吧！規範場因對稱而生。

光的存在是有道理的！

Let there be light!

In the name of gauge symmetry! 為了在地的規範對稱！

Gauge field has his own gauge invariant dynamics.

$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under gauge transformation.

$\mathcal{L} \supset \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is invariant under gauge transformation.

$$= \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\partial_\nu A_\mu \partial^\mu A^\nu$$

The first term looks like a Klein-Gordon $\mathcal{L}$ for each A^ν.

If we use it to write the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2$$

It is to be integrated over $\int d^4x$. After integration by part, it can be written as:

$$\mathcal{L} \supset -\frac{1}{2} A_\nu \partial_\mu \partial^\mu A^\nu - \frac{1}{2} A_\mu \partial^\mu \partial_\nu A^\nu$$

$$= -\frac{1}{2} A_\mu [g^{\mu\nu} \Box + \partial^\mu \partial^\nu] A_\nu$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$U(1)$ global symmetry

$$\Psi(x) \to e^{-i\theta}\Psi(x)$$

$$\theta$$

$$J^{\mu}$$

Current conservation

$U(1)$ local symmetry 規範對稱 Gauge symmetry

$$\Psi(x) \to e^{-i\theta(x)}\Psi(x)$$

$$\theta(x)$$

$$A^{\mu}$$

After quantization, we expect a gauge boson.

規範場、電磁場

Globalization

在地化 Localization

Yang and Mills

Yang-Mills Field

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Conservation of Isotopic Spin and Isotopic Gauge Invariance*

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(Received June 28, 1954)

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistant with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance and the existence of a **b** field which has the same relation to the isotopic spin that the electromagnetic field has to the electric charge. The **b** field satisfies nonlinear differential equations. The quanta of the **b** field are particles with spin unity, isotopic spin unity, and electric charge $\pm e$ or zero.

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Yang was thinking to extend Abelin $U(1)$ gauge theory into non-abelian $SU(2)$ Gauge theory.

If photon, $U(1)$ gauge boson, mediate EM, we may find mediation particles of strong int.

想到$SU(2)$你可能先想到自旋，那是空間的旋轉變換。

楊想的則是數學結構與自旋完全相同的isospin $SU(2)$ Internal Symmetry

中子與質子幾乎等重！　質量差只有 10^{-3}

p $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Mass $m = 938.272046 \pm 0.000021$ MeV [a]

n $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Mass $m = 939.565379 \pm 0.000021$ MeV [a]

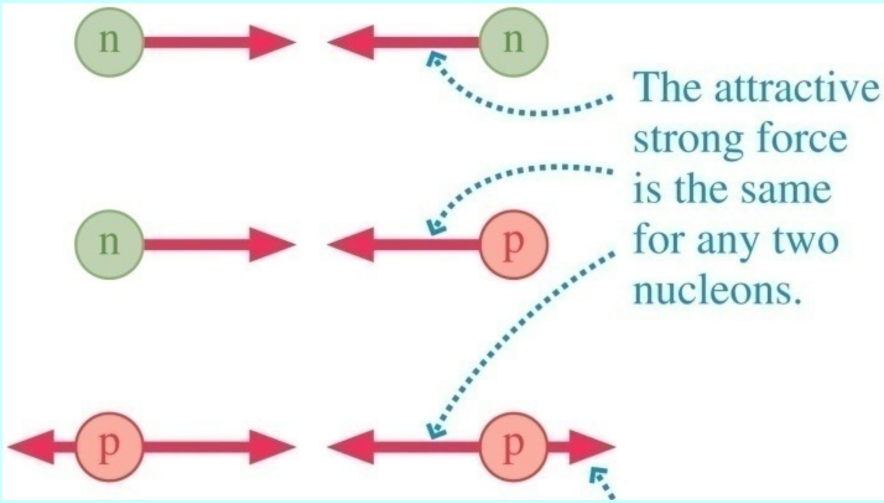
$$m_n - m_p = 1.2933322 \pm 0.0000004 \text{ MeV}$$

物理學家相信自然是沒有巧合的，巧合必定有深意。

質子與中子不只質量相似，兩者的核力、就是強交互作用也相同。

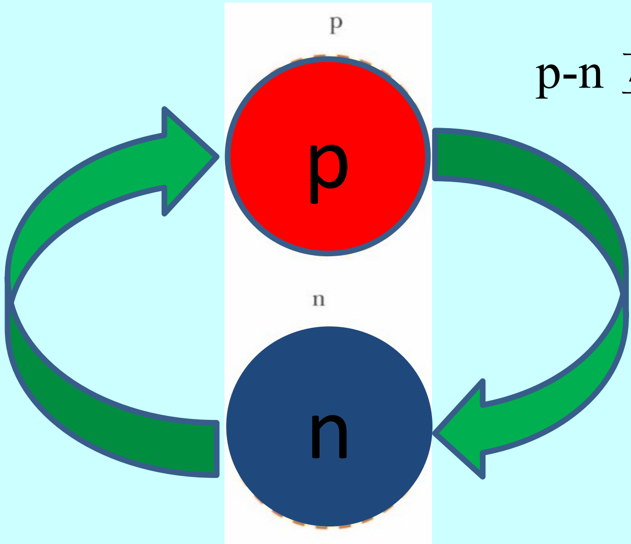
許多原子核的性質非常接近 $^{7}_{3}\mathrm{Li} \approx ^{7}_{4}\mathrm{Be}$ $^{11}_{5}\mathrm{B} \approx ^{11}_{6}\mathrm{C}$ 它們差別只是交換了質子與中子。

同時散射實驗顯示 p-p, p-n, n-n之間的核力完全相等！



當然質子與中子電荷不同，弱力也不同。

假想我們將電磁力及弱力關掉，然後將所有的質子與中子互換，



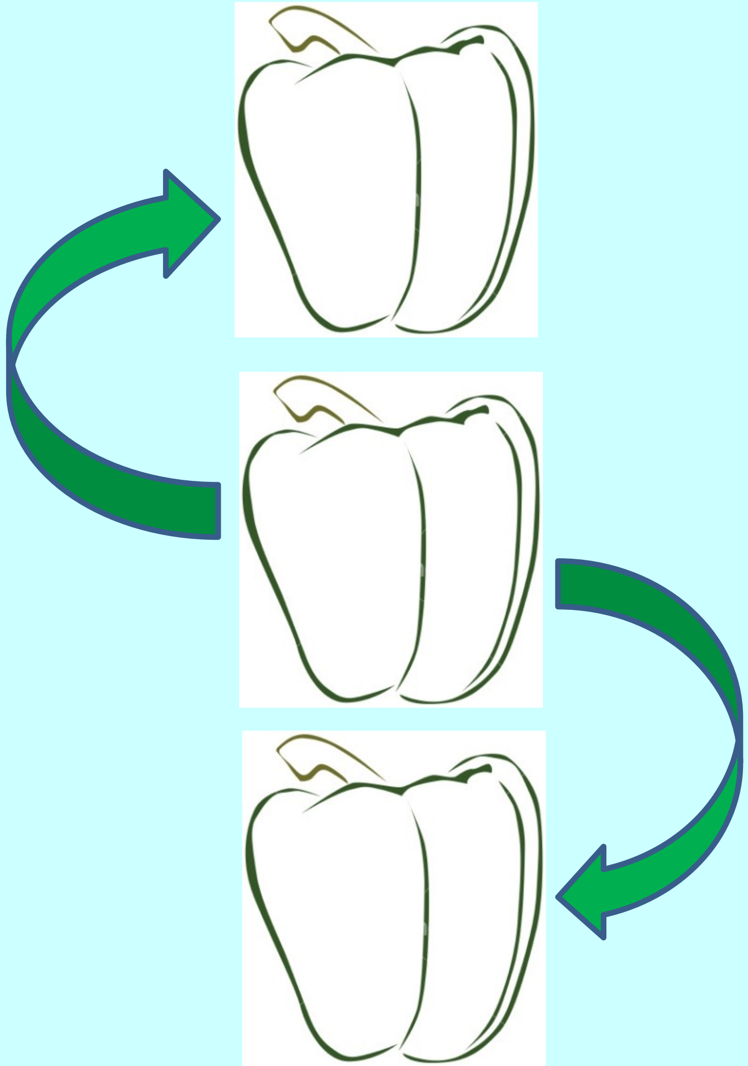
p-n 互換

質子與中子質量幾乎相等！

此時宇宙中僅存的作用：強交互作用也相同。

在 p-n 互換之後，整個自然世界將運作如常，並沒有變化！

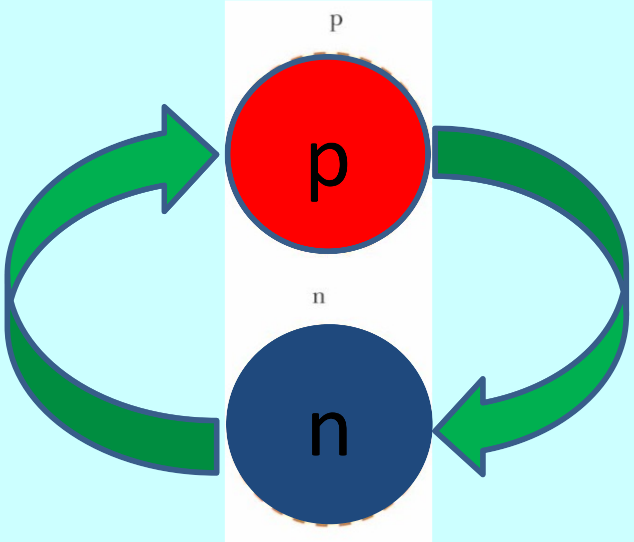
透過物理實驗，你無法辨別互換是否已經發生！

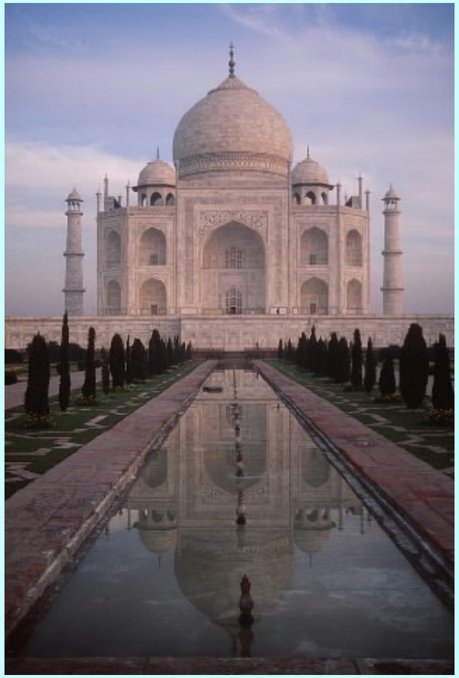


色盲將無法分辨青椒與紅黃椒，將其中兩者互換，他將不會發覺

關閉對顏色的感知(將電磁弱力關掉)，你才會發現它們在形狀上的相似！

這樣的p-n互換不變性是一個近似不變性，或部分不變性，只有核力遵守！

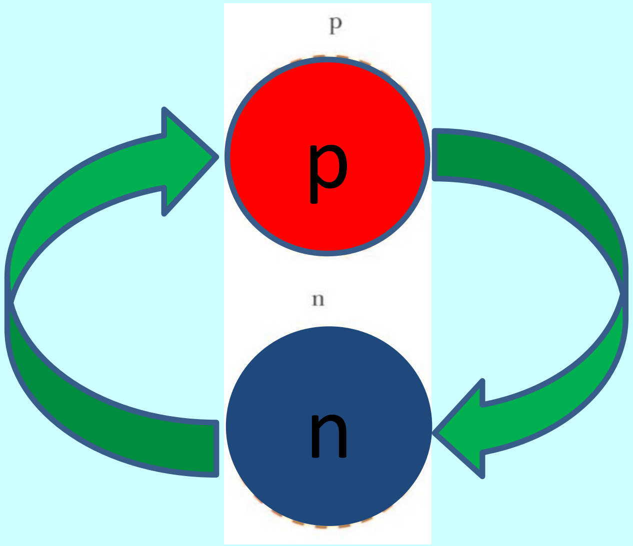


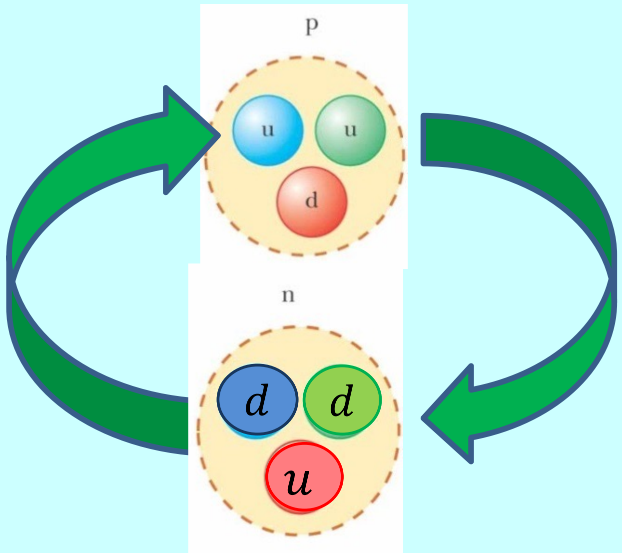


對愛因斯坦來說，不變性、對稱是物理世界美的代表！　　海森堡實用主義者！

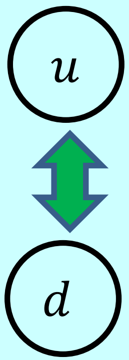
質子與中子的世界是部分美麗的！

p,n 的差別在其中一個 u夸克換成了 d夸克





p-n 互換不變性其實是 u-d 互換不變性



u-d 互換不變性

互換不變性的內涵：u 與 d 的分別是人造的命名，互換不應影響物理結果！

$u \Leftrightarrow d$

這個互換變換，需要定義得更精確一些：

想像變換後進入一新世界：其中的夸克，稱為u', d'。

但事實上，u'其實是原來的夸克d，而d'其實是u。

$u \Rightarrow u' = d$

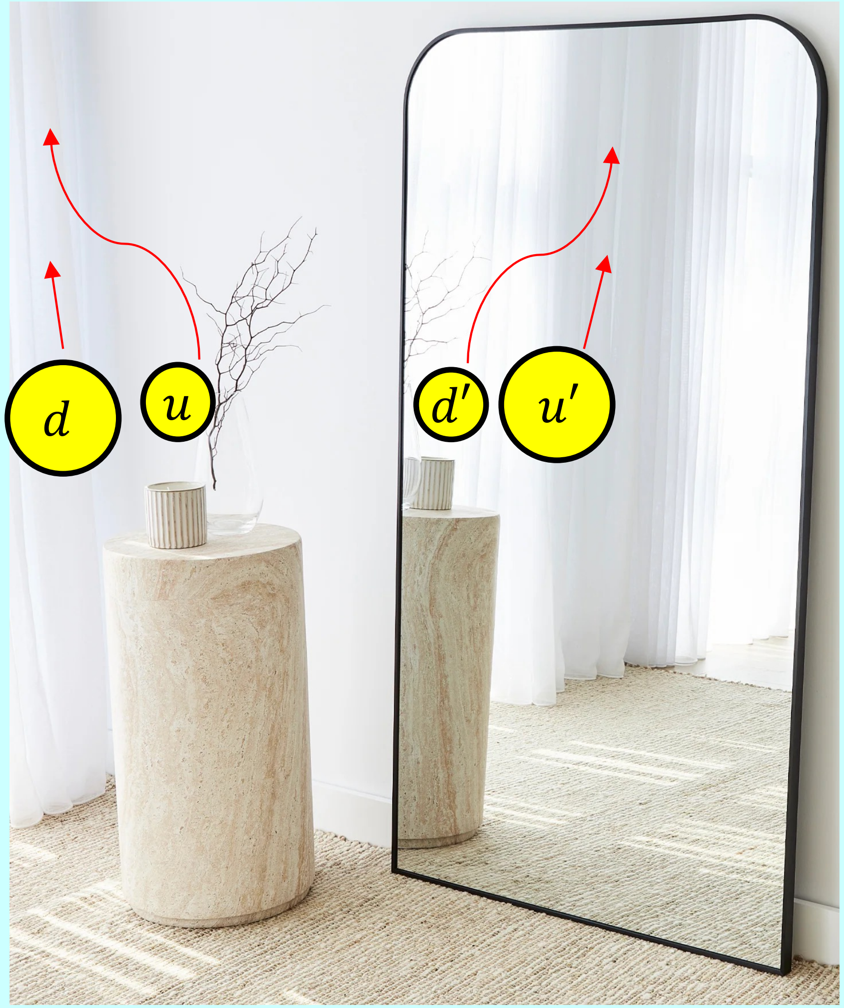
$d \Rightarrow d' = u$

如此的想像世界等價於我們世界中$u-d$作了互換。

那如果u, d的差別只是命名上的方便而不是有實質的差異，

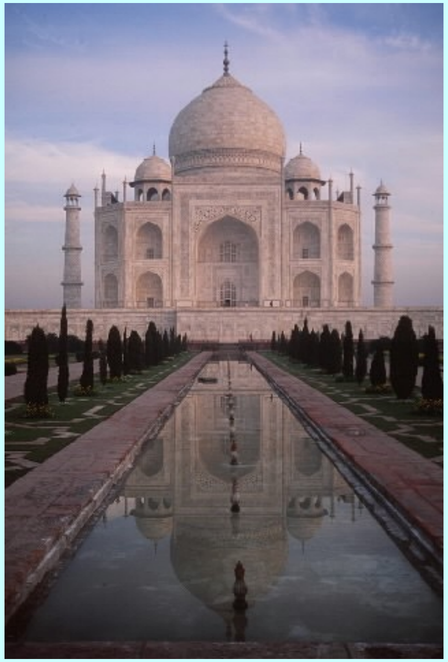
u', d'的性質與滿足的物理定律，和u, d一樣。

變換後的世界與變換前的世界將無法分辨！



若命名原則是所見右邊的夸克稱為u，鏡中右邊會稱為u'，但其實是d映像。

現在夸克依物理定律運動：



如果u, d的差別只是命名上的方便，鏡中世界的夸克路徑也會是符合定律的。

如此我們將無法分辨那一個是鏡中世界。

變換後的世界並不是一個實質存在的世界。它是一個想像的世界。

就像日常生活的對稱變換，是讓我們釐清一個物體的形狀的內在性質。

物理的想像的對稱變換，是讓我們釐清物理定律的內在數學結構與性質。

但如果物件u與d是量子狀態呢？

量子態可以做線性疊加。

現在就得推廣u', d'為兩個彼此互相垂直的u與d的疊加狀態！

就是u與d分別換成兩個正交的u, d疊加！

如果u與d性質類似：

u', d'的性質與滿足的物理定律，還是會和u, d一樣。這就稱為$SU(2)$對稱。

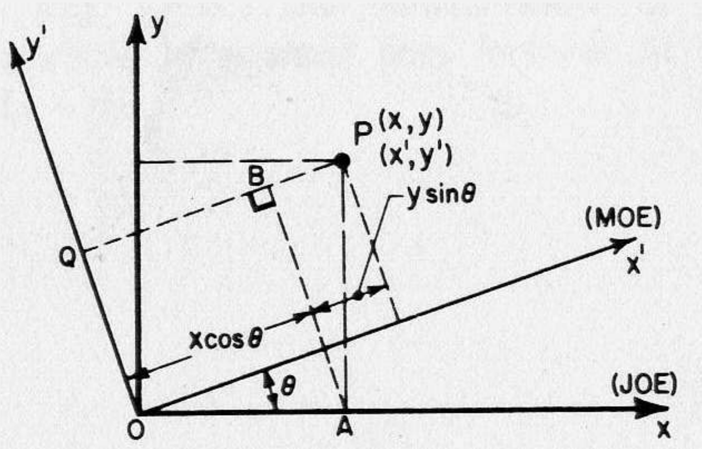
$$u \Rightarrow u' = d \qquad\qquad u \Rightarrow u' = a \cdot u + b \cdot d$$

$$d \Rightarrow d' = u \qquad\qquad d \Rightarrow d' = c \cdot u + d \cdot d$$

古典　　　　　　　　量子

古典的 u-d 互換　　　　量子力學推廣的 u-d $SU(2)$變換

這與向量的座標軸選轉變換非常類似！



因為座標軸的選取是任意的。

我可以將座標軸旋轉 θ 角，得到一組新的座標軸。

物理的結果應該與選擇哪一組座標軸無關！

$$\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

注意：此旋轉變換有連續分布的無限多個，由一個連續變數 θ 標定。

量子力學推廣的雙重態互換不變性

$$u \Rightarrow u' = a \cdot u + b \cdot d$$

$$d \Rightarrow d' = c \cdot u + d \cdot d$$

$$u \Rightarrow u' = a\cdot u + b\cdot d$$

$$d \Rightarrow d' = c\cdot u + d\cdot d$$

只要滿足適當條件的四個複數 a,b,c,d 就給定一個變換！

u及d歸一且彼此正交，因此u'及d'也必須歸一且彼此正交！

$|a|^2+|b|^2=1$ $|c|^2+|d|^2=1$ $ac^*+bd^*=0$

以矩陣來描寫這些變換更加方便，若將夸克寫成一個1×2的行向量：$\begin{pmatrix} u \\ d \end{pmatrix}$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix} \equiv U\begin{pmatrix} u \\ d \end{pmatrix}$$

$$UU^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = I$$

Any unitary 2×2 matrix U give a transformation.

矩陣U必須是unitary。任一unitary 2×2 矩陣就給定一個變換。

u-d 互換在量子力學中必須包含無限多個可能的變換.

$UU^{\dagger} = I$ ➡ $|\det U|^2 = 1$ ➡ $\det U = e^{i\theta}$

一般會把phase θ由量子態吸收，而要求 $\det U = 1$ 稱為Special Unitary矩陣。

只要滿足以上條件的2×2矩陣就給定一個變換！

不難證明：滿足以上條件的兩個矩陣的乘積依舊滿足以上條件。

因此這些連續分布 … 組成一個群，稱為$SU(2)$！

應用於u與d 夸克就 …

$SU(2)$ is just the QM … ntical (similar).

當我們要說兩個東 … ！

$UU^\dagger = I$ $\det U = 1$

2×2的矩陣有四個複數元素a, b, c, d，special unitary 2×2 matrix滿足五個條件。

因此變換由三個連續實數來標訂。那三個？

定義U的對數矩陣A：

$$U = e^{-iA} = 1 - iA + \frac{1}{2!}(-iA)^2 + \cdots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$-iA \equiv \ln U = (U-1) - \frac{1}{2}(U-1)^2 + \frac{1}{3}(U-1)^3 + \cdots$$

U矩陣是 unitary： $U^\dagger = U^{-1}$， $\left(e^{-iA}\right)^\dagger = e^{iA^\dagger} = e^{iA}$

A矩陣必須是Hermitian： $A^\dagger = A$

$\mathrm{Det}\, U = e^{-i\,\mathrm{tr}A} = 1$ A矩陣必須是Traceless： $\mathrm{tr}A = 0$

符合這兩個條件的矩陣A就一對一地對應一個U矩陣。

$A^\dagger = A$ $\mathrm{tr}A = 0$

符合這兩個條件的矩陣A就一對一地對應一個U矩陣。

$$A = \begin{pmatrix} a_{\mathrm{re}} & c_{\mathrm{re}} - ic_{\mathrm{im}} \\ c_{\mathrm{re}} + ic_{\mathrm{im}} & d_{\mathrm{re}} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}$$ $\alpha_{1,2,3}$任意連續變數。

矩陣A、因此U矩陣也是，是以三個連續變數標定的空間！

A一般可以用三個標準矩陣的線性組合描述：

$$A = -i\left(\frac{\tau_1}{2}\alpha_1 + \frac{\tau_2}{2}\alpha_2 + \frac{\tau_3}{2}\alpha_3\right)$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli Matrices

$\frac{\alpha_{1,2,3}}{2}$ 就稱為Isospin SU(2) 的Generators。A矩陣是Generators的線性組合。

任一Generators的線性組合都可以是A矩陣，取指數後就是一個U矩陣：

$$U(\alpha_1, \alpha_2, \alpha_3) = e^{-iA} = e^{-i\left(\frac{\tau_1}{2}\cdot\alpha_1 + \frac{\tau_2}{2}\cdot\alpha_2 + \frac{\tau_3}{2}\cdot\alpha_3\right)} = \exp\left(-i\sum_{i=1}^{3}\frac{\tau_i}{2}\cdot\alpha_i\right)$$

Isospin $SU(2)$的變換在數學結構上就完全等於自旋1/2 的電子態的3D 旋轉。

$SU(2){\sim}SO(3)$　　$SU(2)$ is locally isomorphic to $SO(3)$

但並不是真的旋轉，而是一個Internal Space的變換。

強交互作用在對應的 Isospin “旋轉” 變換下是不變的！

要了解 Isospin $SU(2)$ 就研究 3D 旋轉即可！

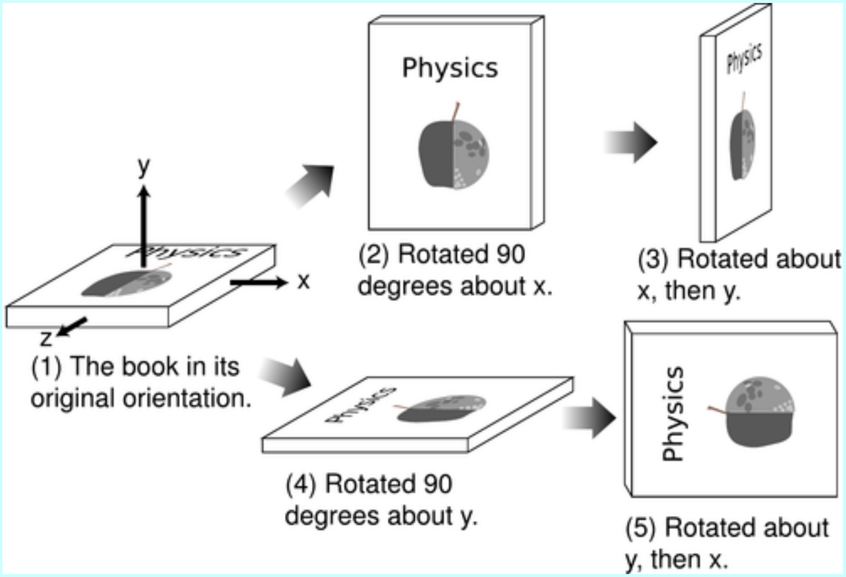
從自旋的量力知道$\frac{\tau_i}{2}$就是自旋角動量向量，

因此方便起見，我們可以以向量符號來表示，儘管Isospin $SU(2)$不是旋轉。

$\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$　　$\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$　　$U = \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)$

Isospin $SU(2)$如同3D 旋轉是不commute的。

稱為non-abelian Group or non-commutative group。



Niels Henrik Abel 1802-1829

Rotation in QM! Quantum states can be classified as:

s	Singlet
$\begin{pmatrix} u \\ d \end{pmatrix}$	Doublet 2
$\begin{pmatrix} \pi^+ \\ \pi^3 \\ \pi^- \end{pmatrix}$	Triplet 3
$\begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$	Quartet 4

Yang and Mills

Could isospin $SU(2)$ be local like $U(1)$ in EM ?

$SU(2)$ global symmetry

$$\Psi(x) \to \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)\Psi(x)$$

$$\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$$

$$\boldsymbol{J}^\mu = (J^1, J^2, J^3)$$

Current conservation

Globalization

$SU(2)$ local symmetry 規範對稱 Gauge symmetry

$$\Psi(x) \to \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)\right)\Psi(x)$$

$$\boldsymbol{\alpha}(x) = \left(\alpha^1(x), \alpha^2(x), \alpha^3(x)\right)$$

A^μ ? $\quad (W_\mu^1, W_\mu^2, W_\mu^2) \equiv \boldsymbol{W}_\mu$

規範場、電磁場

在地化 Localization

Consider an isospin $SU(2)$ doublet listed as a column: Ψ.

It could be proton and neutron as in Yang's mind: $\begin{pmatrix} p \\ n \end{pmatrix}$

Or better be u and d quark as we now know. $u(x), d(x)$ are bispinors.

$$\Psi = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \qquad \mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \equiv (\bar{u}, \bar{d}) \begin{pmatrix} (i\gamma^\mu \partial_\mu - m)u \\ (i\gamma^\mu \partial_\mu - m)d \end{pmatrix}$$

$$= \bar{u}(i\gamma^\mu \partial_\mu - m)u + \bar{d}(i\gamma^\mu \partial_\mu - m)d$$

It has a global $SU(2)$ symmetry. $\bar{\Psi} i\gamma^\mu \partial_\mu \Psi \to \bar{\Psi} U^{-1} U i\gamma^\mu \partial_\mu \Psi = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi$

But it could be anything that transforms like an iso-doublet,
ie. by a 2×2 unitary matrix U as follows:

$$\Psi' = U \cdot \Psi = \exp\left(-\frac{i}{2} \boldsymbol{\tau} \cdot \boldsymbol{\alpha}(x)\right) \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}$$

We want a space-time derivative D_μ such that $D_\mu \Psi$ transform exactly like Ψ:

$$D'_\mu \Psi' = U \cdot D_\mu \Psi$$

so that we can easily build gauge invariant Lagrangian.
Note that in addition to ∂_μ, D_μ could involve 2×2 matrices.

ISOTOPIC GAUGE TRANSFORMATION

Let ψ be a two-component wave function describing a field with isotopic spin $\frac{1}{2}$. Under an isotopic gauge transformation it transforms by

$$\psi = S\psi', \tag{1}$$

where S is a 2×2 unitary matrix with determinant unity. In accordance with the discussion in the previous section, we require, in analogy with the electromagnetic case, that all derivatives of ψ appear in the following combination:

$$(\partial_\mu - i\epsilon B_\mu)\psi.$$

B_μ are 2×2 matrices such that[7] for $\mu = 1, 2$, and 3, B_μ is Hermitian and B_4 is anti-Hermitian. Invariance requires that

$$S(\partial_\mu - i\epsilon B_\mu{}')\psi' = (\partial_\mu - i\epsilon B_\mu)\psi. \tag{2}$$

Combining (1) and (2), we obtain the isotopic gauge transformation on B_μ:

$$B_\mu{}' = S^{-1}B_\mu S + \frac{i}{\epsilon}S^{-1}\frac{\partial S}{\partial x_\mu}. \tag{3}$$

$$U(x) = \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)\right)$$

The recipe: add 3 vector fields:

$$D_\mu\Psi = \partial_\mu\Psi - \frac{i}{2}g\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu\Psi$$

Three 4-vectors $\boldsymbol{W}_\mu$ forms one isospin 3-vector.

2×2 matrices $\boldsymbol{\tau}$ acting on column Ψ.

$$D'_\mu\Psi' = U\cdot D_\mu\Psi$$

為了使 $\mathcal{L}$ 是 gauge invariant，對應於每一個連續變數α_i，即每一個generator，必須引進一個如電磁場的向量規範場$A_\mu(x)$，

也就是一個如光子一樣，自旋為1的規範粒子Gauge Boson!

$U(1)$ local 規範對稱 Gauge symmetry

θ | $\psi(x) \to e^{-i\theta}\psi(x)$

1

$\Downarrow$

A^μ 電磁場

$SU(2)$ local 規範對稱 Gauge symmetry

$\alpha_1, \alpha_2, \alpha$

$\frac{\tau_1}{2}, \frac{\tau_2}{2}, \frac{\tau_3}{2}$

$$\Psi' = \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)\right)\Psi$$

$\Downarrow$

$A_1^\mu, A_2^\mu, A_3^\mu$

$\Downarrow$

W_1, W_2, W_3

例如：媒介弱作用的W波色子就是$SU(2)$的規範粒子Gauge Boson。

$$\left(W_\mu^1, W_\mu^2, W_\mu^2\right) \equiv \boldsymbol{W}_\mu$$

The three gauge fields transform like one 3-vector under isospin rotation.

I can define its inner product with three Pauli matrices as:

$$\sum_{i=1}^{3} \tau_i \cdot W_\mu^i = \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu$$

$\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu$ is a field dependent 2×2 matrix that can act on column Ψ:

$$\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \Psi = \begin{pmatrix} W^3 & W^1 - iW^2 \\ W^1 + iW^2 & -W^3 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The recipe can be neatly written as:

$$D_\mu \Psi = \partial_\mu \Psi - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \Psi \quad \Longleftrightarrow \quad D_\mu \Psi = \partial_\mu \Psi - igA_\mu \Psi$$

$U(1)$ gauge theory

The recipe:

$$D_\mu \Psi = \partial_\mu \Psi - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \Psi$$

Our wish:

$$D'_\mu \Psi' = U \cdot D_\mu \Psi = U \cdot \left(\partial_\mu \Psi - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \Psi \right)$$

In reality, (note that $\boldsymbol{W}_\mu$ in D_μ could change too):

$$D'_\mu \Psi' = \partial_\mu (U\Psi) - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu U \Psi$$

For wish to become reality, we just need to ask:

$$U \partial_\mu \Psi + \partial_\mu (U) \Psi - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu U \Psi = U \cdot \left(\partial_\mu \Psi - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \Psi \right)$$

$$\partial_\mu (U) - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu U = U \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu \quad \cancel{\Psi}$$

It should work for any Ψ, we can take Ψ away and multiply both sides by U^{-1} :

$$\frac{1}{2} \boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu = -\frac{i}{g} \partial_\mu (U) U^{-1} + U \frac{1}{2} \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu U^{-1}$$

This is how the gauge field should transform.

We can easily find the gauge transformation of $\boldsymbol{W}_\mu$ for infinitesimal transformation:

$$U(x) = \exp\left(-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right) \xrightarrow[\alpha\to 0]{} 1 - \frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}$$

$$\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}'_\mu = -\frac{i}{g}\partial_\mu(U)U^{-1} + U\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu U^{-1}$$

$$\sim -\frac{i}{g}\partial_\mu\left(1-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)\cdot\left(1-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)^{-1} + \left(1-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu\left(1+\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)$$

$$\sim\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu - \frac{i}{2}\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\alpha}(x) - \left[\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha},\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu\right]$$ up to order $\boldsymbol{\alpha}$.

The second equals the derivative acting on $\boldsymbol{\alpha}(x)$, as in Abelian QED.

The change of $\boldsymbol{W}_\mu$ for infinitesimal gauge transformation:

$$\boldsymbol{\tau}\cdot\left(\boldsymbol{W}'_\mu - \boldsymbol{W}_\mu\right) \equiv \boldsymbol{\tau}\cdot\delta\boldsymbol{W}_\mu = \frac{1}{g}\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\alpha} + \frac{1}{2}\left[\boldsymbol{\tau}\cdot\boldsymbol{\alpha},\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu\right]$$

α^i是純數，而同一時空點的W_μ^j彼此對易，可以看成純數。

$$[\boldsymbol{\tau}\cdot\boldsymbol{\alpha},\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu]=\sum_{ij=1}^{3}[\tau^i,\tau^j]\alpha^i W_\mu^j=\sum_{ij=1}^{3}2\epsilon_{ijk}\tau^k\alpha^i W_\mu^j=2\boldsymbol{\tau}\cdot(\boldsymbol{\alpha}\times\boldsymbol{W}_\mu)$$

The commutator turns out to be the cross product of $\boldsymbol{\alpha}$ and $\boldsymbol{W}_\mu$.

$$\boldsymbol{\tau}\cdot\delta\boldsymbol{W}_\mu=\frac{1}{g}\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\alpha}+\boldsymbol{\tau}\cdot(\boldsymbol{\alpha}\times\boldsymbol{W}_\mu)$$

$$[\tau^i,\tau^j]=\epsilon_{ijk}\tau^k$$

$$\delta\boldsymbol{W}_\mu=\frac{1}{g}\partial_\mu\boldsymbol{\alpha}+\boldsymbol{\alpha}\times\boldsymbol{W}_\mu$$

In components:

$$\delta W_\mu^k=\frac{1}{g}\partial_\mu\alpha^k+\varepsilon^{ijk}\alpha^i W_\mu^j \quad\Longleftrightarrow\quad \delta A^\mu=\frac{1}{g}\partial_\mu\theta$$

Done.

The derivation and proof is quite striking, but the real jewel is the recipe.

The recipe is enough for us to write down gauge invariant QFT right away.

Simply replace ∂_μ in the free Lagrangian by covariant derivative D_μ:

This even works for scalar.

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \quad \Rightarrow \quad \mathcal{L} = \bar{\Psi} i\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi$$

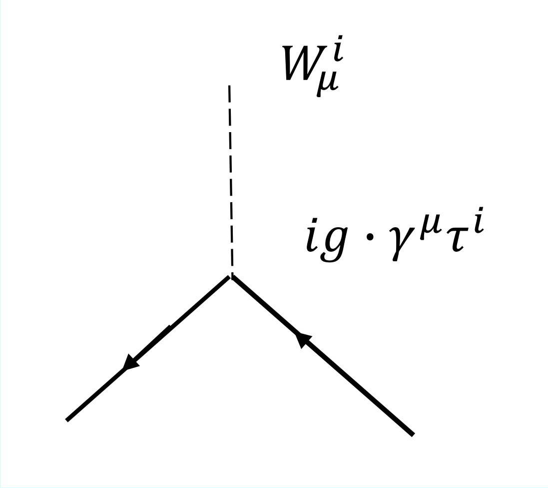
We are sure it's invariant since $D_\mu\Psi$ transform exactly like Ψ.

$$\rightarrow \bar{\Psi} U^{-1} U i\gamma^\mu D_\mu \Psi = \bar{\Psi} i\gamma^\mu D_\mu \Psi$$

Plug in the recipe: $\mathcal{L} = \bar{\Psi} i\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi + \frac{g}{2}\bar{\Psi}\gamma^\mu \boldsymbol{\tau}\cdot\boldsymbol{W}_\mu \Psi$

The vertex is determined by the matching Pauli matrix (generators) of gauge boson.

$$\mathcal{L}_I = \frac{g}{2}\bar{\Psi}\gamma^\mu \boldsymbol{\tau}\cdot\boldsymbol{W}_\mu \Psi \quad \Rightarrow$$

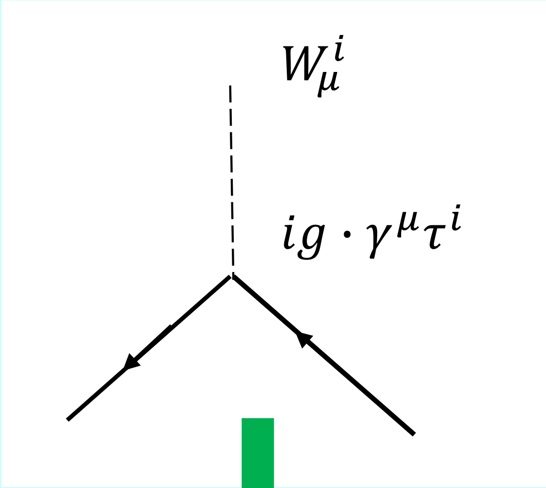


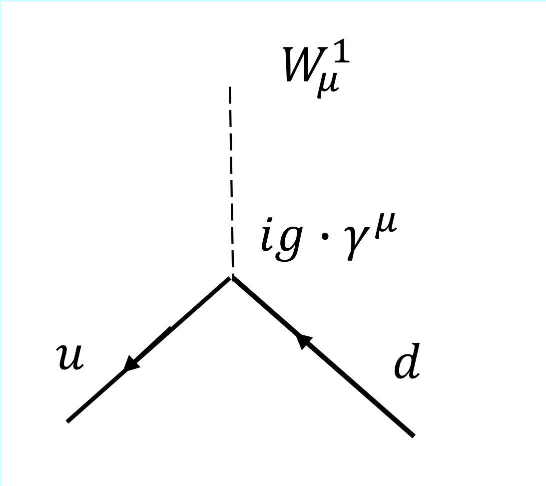
Take W_μ^1 for an example:

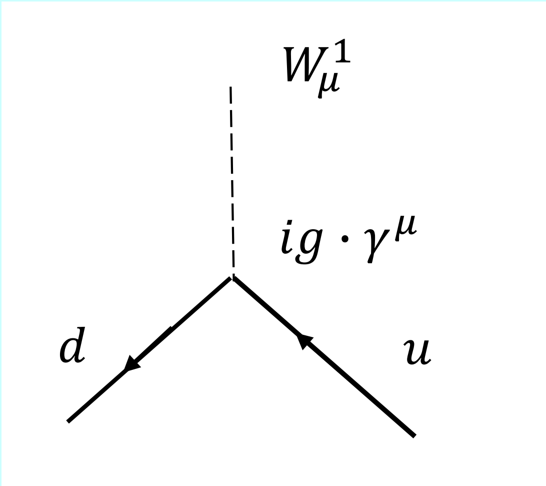
$$\mathcal{L}_I = \frac{g}{2}\bar{\Psi}\gamma^\mu\tau^1 W_\mu^1\Psi$$

$$= \frac{g}{2}(\bar{u} \quad \bar{d})\gamma^\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \cdot W_\mu^1$$

$$= \frac{g}{2}\left(\bar{u}\gamma^\mu d \cdot W_\mu^1 + \bar{d}\gamma^\mu u \cdot W_\mu^1\right)$$





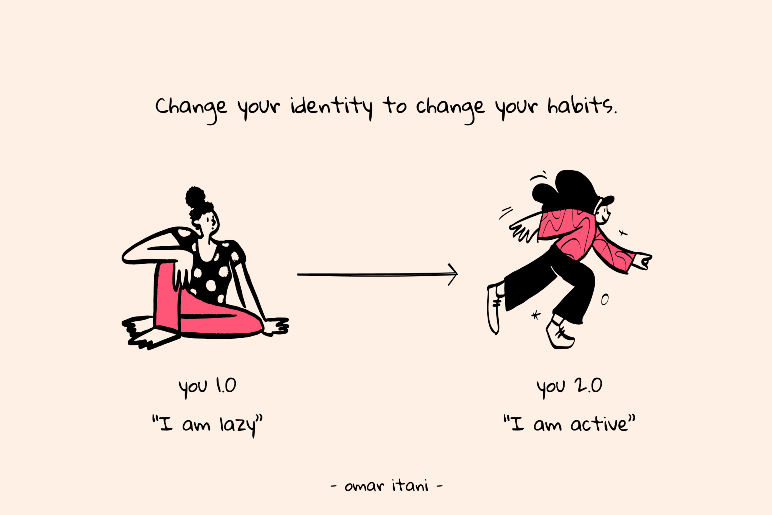


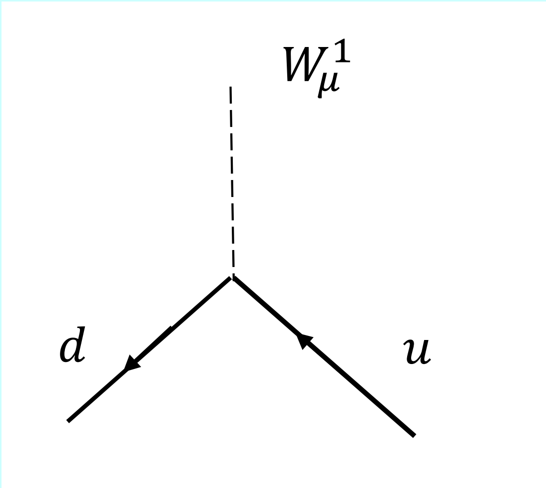
Absorbing a gauge boson W^1 change quark u into quark d, and d into u!

Gauge interaction changes a particle's identity.

The form and coupling constant of the interaction is totally fixed by gauge symmetry.

Emitting a gauge boson is an identity changing interaction.



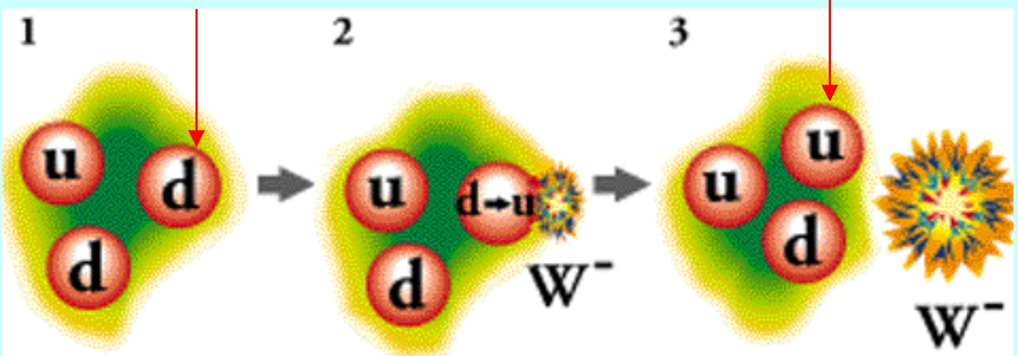


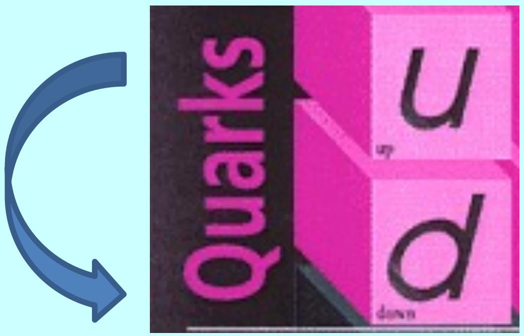
If photon, the $U(1)$ gauge boson, mediate EM,

maybe iso-$SU(2)$ non-abelian gauge boson can mediate strong interaction.

That is the original purpose of Yang's paper.

But it first works for weak interaction.



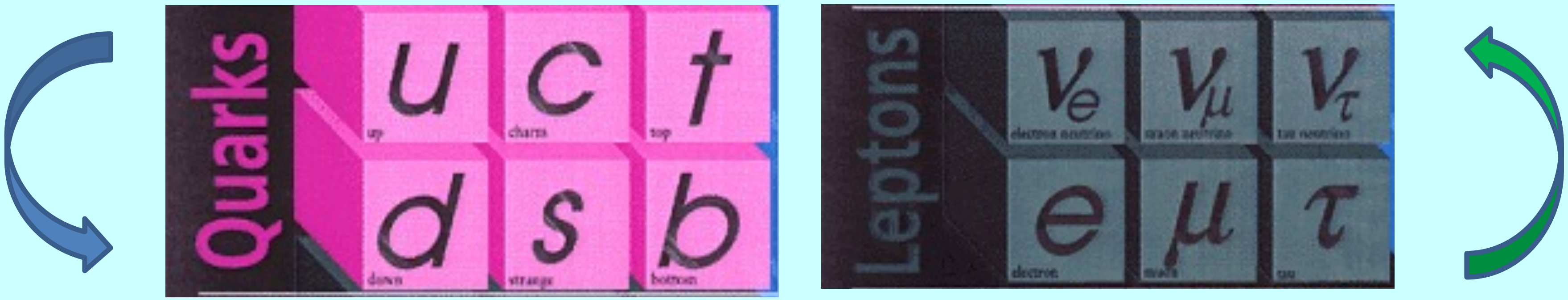


Isotopic $SU(2)$ 並不真的是一個Localized規範對稱。

Global Isotopic $SU(2)$ 依舊是一個部分的對稱。

但推廣這個不變性， 假設 u-d 互換時，所有的雙重態也會一起上下互換！

同時假設所有費米子都沒有質量，這個$SU(2)$就真的是規範對稱，



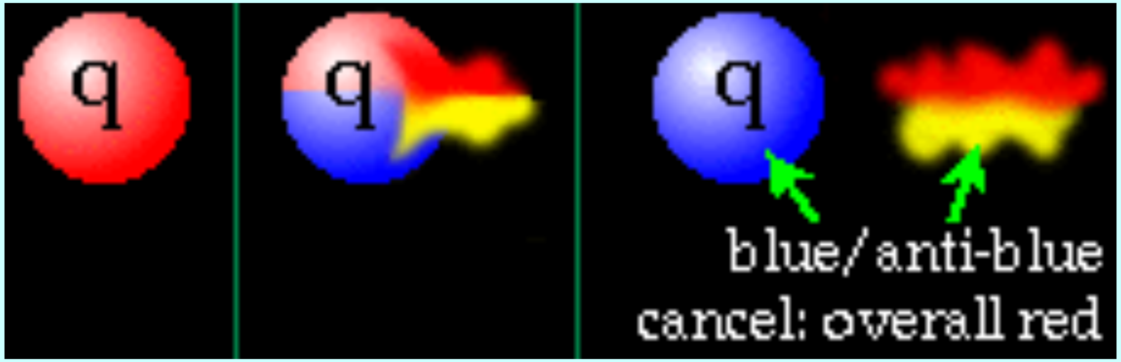
u-d 互換$SU(2)$ → 所有雙重態互換$SU(2)$規範對稱

這個$SU(2)$稱為weak $SU(2)$，規範波色子 W^1, W^2, W^2就是弱作用的媒介子。

吸收W^1後，成對的粒子可以由上方降到下方！

那W^1是帶什麼電荷？

Later it also works for strong interaction, as Yang expected, but not in the same way.



在色力作用中（強交互作用）中，夸克改變自己的顏色，放出一個規範子膠子。

膠子就是把夸克黏在一起的色力的媒介子！膠子同時帶一個顏色與一個反顏色。

We can write down the invariant kinetic energy of the vector field.

We need the field strength $F_{\mu\nu}$ here!

$$G^k_{\mu\nu} = \partial_\mu W^k_\nu - \partial_\nu W^k_\mu + g \sum_{ij=1}^{3} \epsilon_{ijk} W^i_\mu W^j_\nu$$

$$\boldsymbol{G}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu + g\left(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu\right)$$

define now

$$F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu} + i\epsilon(B_\mu B_\nu - B_\nu B_\mu). \qquad (4)$$

One easily shows from (3) that

$$F_{\mu\nu}' = S^{-1} F_{\mu\nu} S \qquad (5)$$

under an isotopic gauge transformation.‡ Other simple functions of B than (4) do not lead to such a simple transformation property.

$$\frac{1}{4} \boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} = \frac{1}{4} \sum_{i=1}^{3} G^i_{\mu\nu} G^{i\mu\nu}$$

is gauge invariant!

To write down the field equations for the **b** field we clearly only want to use isotopic gauge invariant quantities. In analogy with the electromagnetic case we therefore write down the following Lagrangian density:[8]

$$-\tfrac{1}{4} \mathbf{f}_{\mu\nu} \cdot \mathbf{f}_{\mu\nu}.$$

Since the inclusion of a field with isotopic spin $\frac{1}{2}$ is illustrative, and does not complicate matters very much, we shall use the following total Lagrangian density:

$$\mathfrak{L} = -\tfrac{1}{4} \mathbf{f}_{\mu\nu} \cdot \mathbf{f}_{\mu\nu} - \bar{\psi} \gamma_\mu (\partial_\mu - i\epsilon \boldsymbol{\tau} \cdot \mathbf{b}_\mu)\psi - m\bar{\psi}\psi. \qquad (11)$$

The proof is tedious. You can plug in:

$$\delta \boldsymbol{W}_\mu = \frac{1}{g} \partial_\mu \boldsymbol{\alpha} + \boldsymbol{\alpha} \times \boldsymbol{W}_\mu$$

After quantization, one field corresponds to one massless boson, called gauge boson.

Gauged $SU(2)$

$\alpha_1, \alpha_2, \alpha_3$ $\quad$ $\dfrac{\tau_1}{2}, \dfrac{\tau_2}{2}, \dfrac{\tau_3}{2}$

$$\Downarrow$$

$(W^1 \quad W^2 \quad W^3)$

There is a simpler but more formal proof:

$$D'_\mu \Psi' = U \cdot D_\mu \Psi$$

$D_\mu \Psi$ transform exactly like Ψ.

$$D'_\nu D'_\mu \Psi' = U \cdot D_\nu D_\mu \Psi$$

$D_\nu D_\mu \Psi$ also transform exactly like Ψ.

$$\left[D'_\mu, D'_\nu\right]\Psi' = U \cdot \left[D_\mu, D_\nu\right]\Psi = U\left[D_\mu, D_\nu\right]U^{-1}\Psi'$$

Considered individually the commutator transform like:

$$\left[D_\mu, D_\nu\right] \to \left[D'_\mu, D'_\nu\right] = U\left[D_\mu, D_\nu\right]U^{-1}$$

Take the trace and the unitary matrix will cancel!

$$\mathbf{tr}\left[D_\mu, D_\nu\right] \to \mathbf{tr}U\left[D_\mu, D_\nu\right]U^{-1} = \mathbf{tr}\left[D_\mu, D_\nu\right]U^{-1}U = \mathbf{tr}\left[D_\mu, D_\nu\right]$$

$\mathbf{tr}\left[D_\mu, D_\nu\right]$ is invariant.

Now write out the expression explicitly:

$$[D_\mu, D_\nu] = \left[\left(\partial_\mu - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu\right), \left(\partial_\nu - \frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu\right)\right]$$

I can write it as four commutators with the first apparently vanishing:

$$[\partial_\mu, \partial_\nu] + \left[\partial_\mu, -\frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu\right] + \left[-\frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu, \partial_\nu\right] + \left[-\frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\mu, -\frac{i}{2} g \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu\right]$$

The second and third equals the derivative acting on $\boldsymbol{W}_\mu(x)$ just as in QED.

$$[\partial_\mu, \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu] = \boldsymbol{\tau} \cdot \partial_\mu \boldsymbol{W}_\nu \qquad [\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu, \partial_\nu] = -\boldsymbol{\tau} \cdot \partial_\nu \boldsymbol{W}_\mu$$

The fourth term is a surprise, unique to Non-Abelian Gauge theory.

$$[\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu, \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu] = \sum_{ij=1}^{3} [\sigma^i, \sigma^j] W_\mu^i W_\nu^j = \sum_{ij=1}^{3} 2\epsilon_{ijk} \sigma^k W_\mu^i W_\nu^j = 2\boldsymbol{\tau} \cdot (\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu)$$

$$\frac{i}{g}[D_\mu, D_\nu] \equiv \frac{1}{2} \boldsymbol{\tau} \cdot \boldsymbol{G}_{\mu\nu} \qquad \boldsymbol{G}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu + g(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu)$$

The first two terms are the normal field strength, just as in QED.

The third term indicate gauge bosons have self interaction. They couple to themselves.

Gauge boson is self interacting!

$$\mathbf{tr}\left\{\left[D_\mu, D_\nu\right]\left[D^\mu, D^\nu\right]\right\}$$

is gauge invariant!

$$\mathbf{tr}\left\{\left(\boldsymbol{\tau}\cdot\boldsymbol{G}_{\mu\nu}\right)\left(\boldsymbol{\tau}'\cdot\boldsymbol{G}^{\mu\nu}\right)\right\} = \mathbf{tr}(\boldsymbol{\tau}\,\boldsymbol{\tau}')$$

$\mathbf{tr}(\boldsymbol{\tau}\,\boldsymbol{\tau}')$ is non-zero only for the same $\boldsymbol{\tau}$, such as $\mathbf{tr}(\tau^1\tau^1) = 2$

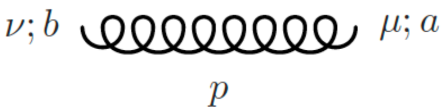
This forces the two $\boldsymbol{G}_{\mu\nu}$ into inner product: $\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu}$.

$$\frac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu} = \frac{1}{4}\sum_{i=1}^{3} G^i_{\mu\nu}G^{i\mu\nu}$$

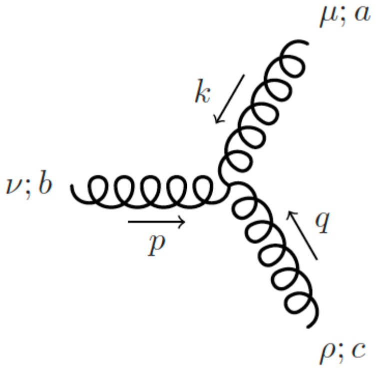
$$\boldsymbol{G}_{\mu\nu} = \partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu + g\left(\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu\right)$$

$$\frac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu} = \frac{1}{4}\sum_{i=1}^{3} G^{i}_{\mu\nu}G^{i\mu\nu}$$

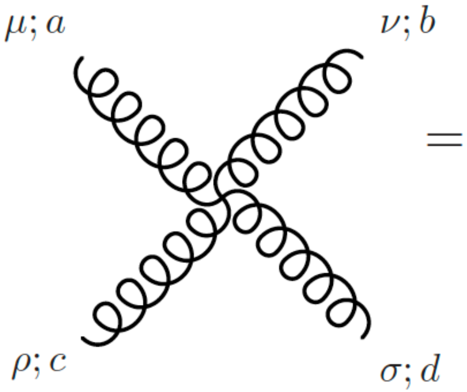
$$\boldsymbol{G}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu + g\left(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu\right)$$



$$= \quad i\frac{-g^{\mu\nu} + (1-\xi)\,\frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}\delta^{ab}.$$



$$= g f^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$$



$$\begin{aligned} = -ig^2 \times \Big[& f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ & + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ & + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right) \Big] \end{aligned}$$

Fin

(End)

Scanned at the American Institute of Physics

這樣的建議雖然優美，但還沒起飛就已墜落！

Gauge Bosons are massless because of gauge symmetry.

Gauge Bosons mass term is not gauge invariant.

$$m_W W_\mu^i W^{i\mu}$$

$$\delta W_\mu^k = \frac{1}{g}\partial_\mu \alpha^k + \varepsilon^{ijk}\alpha^i W_\mu^j$$

規範場論真的跟自然有關嗎？畢竟沒有無質量的向量波色子存在。

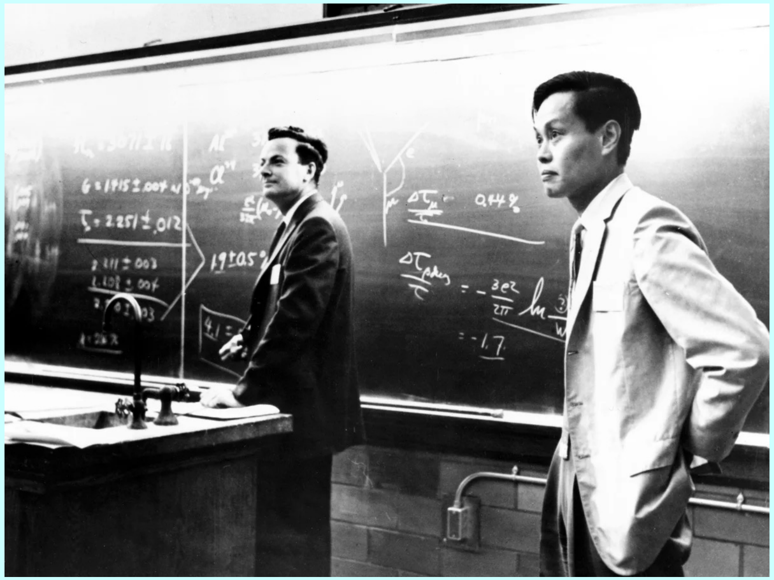
We next come to the question of the mass of the **b** quantum, to which we do not have a satisfactory answer. One may argue that without a nucleon field the

事實上，Pauli在一年前已經完成在地化Non-Abelian Symmetry的工作，

但就因為規範波色子無質量的問題，沒有發表。

好死不死，楊振寧在發表前(02/23/1954)在Princeton給了演講，Pauli就在台下。

Pauli沒有在客氣。



Despite the unresolved mass problem, Yang and Mills decided to publish. “We did not know how to make the theory fit experiment. It is our judgment that the beauty of the idea alone merited attention.” Yang 1983

初生之犢，年輕真好。

一個方程式長得美比符合實驗結果更重要。

It is more important to have beauty in one’s equation than to have them fit experiment.

Dirac

The grand unification theory is so beautiful that it is impossible to be untrue.

Georgi to Feynman

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Conservation of Isotopic Spin and Isotopic Gauge Invariance*

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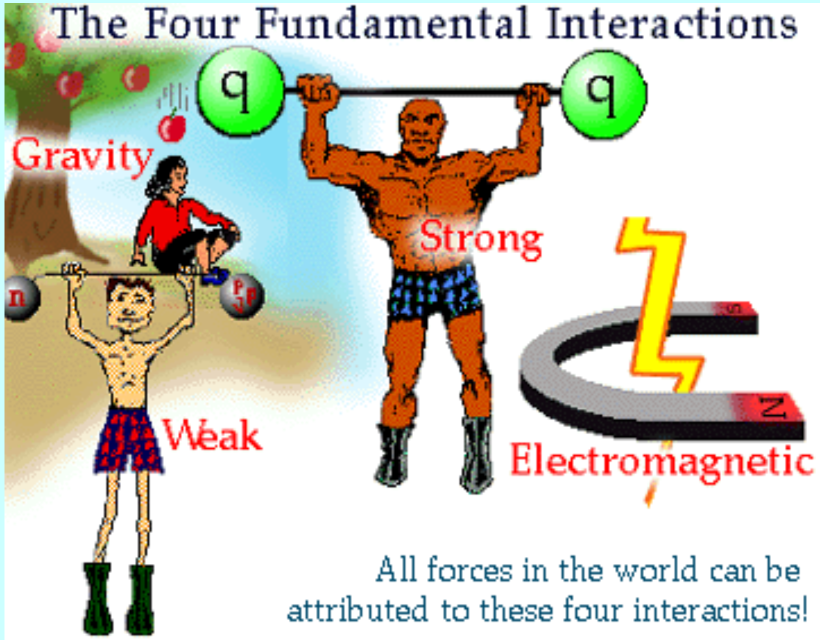
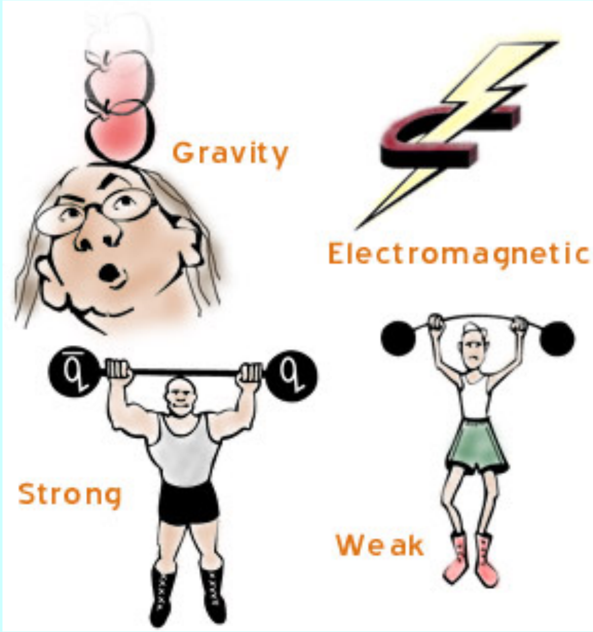
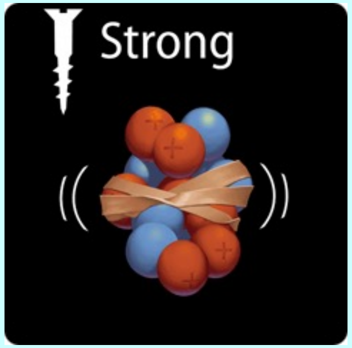
(Received June 28, 1954)

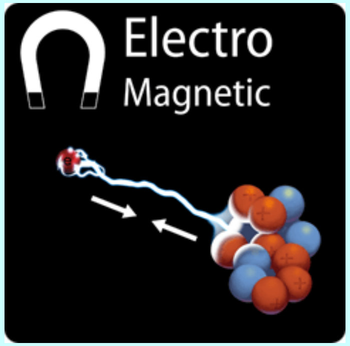
We next come to the question of the mass of the **b** quantum, to which we do not have a satisfactory answer. One may argue that without a nucleon field the Lagrangian would contain no quantity of the dimension of a mass, and that therefore the mass of the **b** quantum in such a case is zero. This argument is however subject to the criticism that, like all field theories, the **b** field is beset with divergences, and dimensional arguments are not satisfactory.

宇宙間的所有交互作用都可以分解為四個基本交互作用









基本粒子的基本交互作用幾乎全是規範理論