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Lecture 1:
Simple Harmonic Oscillators

1 Introduction

The simplest thing that can happen in the physical universe is nothing. The next simplest thing, which doesn't get too far away from nothing, is an oscillation about nothing. This course studies those oscillations. When many oscillators are put together, you get waves.

Almost all physical processes can be explained by breaking them down into simple building blocks and putting those blocks together. As we will see in this course, oscillators are the building blocks of a tremendous diversity of physical phenomena and technologies, including musical instruments, antennas, patriot missiles, x-ray crystallography, holography, quantum mechanics, 3D movies, cell phones, atomic clocks, ocean waves, gravitational waves, sonar, rainbows, color perception, prisms, soap films, sunglasses, information theory, solar sails, cell phone communication, molecular spectroscopy, acoustics and lots more. Many of these topics will be covered first in lab where you will explore and uncover principles of physics on your own.

The key mathematical technique to be mastered through this course is the **Fourier transform**. Fourier transforms, and **Fourier series**, play an absolutely crucial role in almost all areas of modern physics. I cannot emphasize enough how important Fourier transforms are in physics.

The first couple of weeks of the course build on what you've covered in 15a (or 16 or 11a or AP50) – balls and springs and simple oscillators. These are described by the differential equation for the **damped, driven oscillator**:

$$\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}+\omega_0^2\,x(t)=\frac{F(t)}{m} \tag{1}$$

Here $x(t)$ is the displacement of the oscillator from equilibrium, ω_0 is the natural angular frequency of the oscillator, γ is a damping coefficient, and $F(t)$ is a driving force. We'll start with $\gamma=0$ and $F=0$, in which case it's a simple harmonic oscillator (Section 2). Then we'll add γ, to get a damped harmonic oscillator (Section 4). Then add $F(t)$ (Lecture 2).

The damped, driven oscillator is governed by a **linear** differential equation (Section 5). Linear equations have the nice property that you can add two solutions to get a new solution. We will see how to solve them using complex exponentials, $e^{i\alpha}$ and $e^{-i\alpha}$, which are linear combinations of sines and cosines (Section 6). A review of complex numbers is given in Section 7. Studying multiple coupled oscillators will lead to the concept of **normal modes**, which lead naturally to the **wave equation**, the Fourier series, and the Fourier transform (future lectures).

2 Why waves? Why oscillators?

Recall **Hooke's law**: if your displace a spring a distance x from its equilibrium position, the restoring force will be $F=-kx$ for some constant k. You probably had this law told to you in high school or 15a or wherever. Maybe it's an emperical fact, deduced from measuring springs, maybe it was just stated as true. Why is it true? Why does Hooke's law hold?

To derive Hooke's law, you might image you need a microscopic description of a spring (what is it made out of, how does it bend, how are the atoms arranged, etc.). Indeed, if you hope to compute k, yes, absolutely, you need all of this. In fact you need so much detail that generally it's impossible to compute k in any real spring. But also generally, we don't care to compute k, we just measure it. That's not the point. We don't want to compute k. What we want to know is why is the force is proportional to displacement. Why is Hooke's law true?

First of all, it is true. Hooke's law applies not just to springs, but to just about everything:

Figure 1. The restoring force for pretty much anything (bending trees, swings, balls, tires, etc) is **linear** close to equilibrium.

You can move any of these systems, or pretty much anything else around you, a little bit away from its equilibrium and it will want to come back. The more you move it, the stronger the restoring force will be. And often to an excellent approximation, the distance and force are directly proportional.

To derive Hooke's law, we just need a little bit of calculus. Let's say we dispace some system, a spring or a tire or whatever a distance x from its equilibrium and measure the function $F(x)$. We define $x=0$ as the equibrium point, so by definition, $F(0)=0$. Then, we can use Taylor's theorem

$$F(x)=F(0)+xF'(0)+\frac{1}{2}x^2F''(0)+\cdots \tag{2}$$

Now $F(0)=0$ and $F'(0)$ and $F''(0)$ etc are just fixed numbers. So no matter what these numbers are, we can always find an x small enough so that $F'(0)\gg\frac{1}{2}xF''(0)$. Then we can neglect the $\frac{1}{2}x^2F''(0)$ term compared to the $xF'(0)$ term. Similarly, we can always take x small enough that all of the higher derivative terms are as small as we want. And therefore,

$$F(x)=-kx \tag{3}$$

with $k=-F'(0)$. We have just derived Hooke's law! Close enough to equilibrium, the restoring force for *anything* will be proportional to the displacement. Since $y=-kx$ is the equation for a line, we say systems obeying Hooke's law are linear. Thus, everything is linear close to equilibrium. More about linearity in Section 5.

You might also ask, why does F depend only on x? Well, what else could it depend on? It could, for example, depend on velocity. Wind resistance is an example of a velocity-dependent force. However, sinec we are assuming that the object is close to equilibrium, its speed must be small (or else our assumption would quickly be violated). So $\dot{x}$ is small. Thus we can Taylor exapnd in $\dot{x}$ as well

$$F(x,\dot{x})=x\frac{\partial F(x,\dot{x})}{\partial x}\bigg|_{x=\dot{x}=0}+\dot{x}\frac{\partial F(x,\dot{x})}{\partial\dot{x}}\bigg|_{x=\dot{x}=0}+\cdots \tag{4}$$

where the terms $\cdots$ are higher order in x or $\dot{x}$, so they are subleading close to equilbrium. Writing $\frac{\partial F(x,\dot{x})}{\partial\dot{x}}\Big|_{x=\dot{x}=0}=-m\gamma$ we then have

$$F(x)=-kx-m\gamma\dot{x} \tag{5}$$

Then $F=ma$ with $a=\ddot{x}$ gives

$$\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}+\omega_0^2\,x(t)=0 \tag{6}$$

as in Eq. (1) with $\omega_0=\sqrt{\frac{k}{m}}$. γ is called a damping coefficient, since the velocity dependence tends to slow the system down (as we will see).

The other piece of Eq. (1), labeled $F(t)$, is the driving force. It represents the action of something external to the system, like a woman pushing the swing with the girl on it, or the car tire being compressed by the car.

3 Simple harmonic motion

We have seen that Eq. (1) describes universally any system close to equilibrium. Now let's solve it. First, take $\gamma = 0$. Then Eq. (1) becomes

$$\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = 0 \tag{7}$$

For a spring, $\omega_0 = \sqrt{\frac{k}{m}}$, for a pendulum $\omega_0 = \sqrt{\frac{g}{L}}$. Other systems have different expressions for ω_0 in terms of the relevant physical parameters.

We can solve this equation by hand, by plugging into Mathematica, or just by guessing. Guessing is often the easiest. So, we want to guess a function whose second derivative is proportional to itself. You know at least two functions with this property: sine and cosine. So let us write as an *ansatz* (ansatz is a sciency word for "educated guess"):

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{8}$$

This solution has 3 free parameters A, B and ω. Plugging in to Eq. (7) gives

$$-\omega^2[A\sin(\omega t) + B\cos(\omega t)] + \omega_0^2[A\sin(\omega t) + B\cos(\omega t)] = 0 \tag{9}$$

Thus,

$$\omega = \omega_0 \tag{10}$$

That is, the angular frequency ω of the solution must be the paramter $\omega_0 = \sqrt{\frac{k}{m}}$ in the differential equation. We get no constraint on A and B.

ω is called the **angular frequency**. It has units of radians per second. The **frequency** is

$$\nu = \frac{\omega}{2\pi} \tag{11}$$

units of 1/sec. The solution $x(t)$ we found goes back to itself after $t \to t + T$ where

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega} \tag{12}$$

is the **period**. T has units of seconds. The function $x(t) = A\sin(\omega t) + B\cos(\omega t)$ satisfies $x(t) = x(t + nT)$ for any integer n. In other words, the solutions *oscillate*!

A and B are the **amplitudes** of the oscillation. They can be fixed by boundary conditions. For example, you specify the position and velocity at any given time, you can determine A and B. To be concrete, suppose we start with $x(0) = 1m$ and $x'(0) = 2\frac{m}{s}$. Then,

$$1m = x(0) = A\sin(\omega 0) + B\cos(\omega 0) = B \tag{13}$$

$$2\frac{m}{s} = x'(0) = \omega \mathrm{A}\cos(\omega 0) - \omega B\sin(\omega 0) = \omega A \tag{14}$$

So we find $A = \frac{2}{\omega}\frac{m}{s}$ and $B = 1m$.

Keep in mind that the angular frequency ω is *not* fixed by boundary conditions. It is determined by the physical problem: $\omega = \sqrt{\frac{k}{m}}$ where $k = -F'(0)$ and m is the mass of the thing oscillating. That is why if you start a pendulum from any height and give it any sort of initial kick, it will oscillate with the same frequency.

Another representation of the general solution $x(t) = A\sin(\omega t) + B\cos(\omega t)$ is often convenient. Instead of using A and B we can write

$$x(t) = C\sin(\omega t + \phi) \tag{15}$$

using trig identities, we find

$$C\sin(\omega t + \phi) = C\cos(\phi)\sin(\omega t) + C\sin(\phi)\cos(\omega t) \tag{16}$$

and so

$$A = C\cos(\phi) \quad B = C\sin(\phi) \tag{17}$$

Thus we can swap the amplitudes A and B for the sine and cosine components for a single amplitude C and a phase ϕ.

4 Damped oscillators

A damped oscillator dissipates its energy, returning eventually to the equilibrium $x(t)=\text{const}$ solution. When the object is at rest, the damping force must vanish. For small velocities, the damping force should be proportional to velocity: $F=-\gamma\frac{dx}{dt}$ with γ some constant. Contributions to the force proportional to higher powers of velocity, like $F=-\kappa\left(\frac{dx}{dt}\right)^2$ will be suppressed when the object is moving slowly. Thus the generic form for damped motion close to equilibrium is

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2 x=0 \tag{18}$$

Indeed, this equation describes a great many physical systems: vibrating strings, sound waves, etc. Basically everything we study in this course will have damping.

Neither $\sin(\omega t)$ nor $\cos(\omega t)$ solve the damped oscillator equation. Sines and cosines are proportional to their second derivatives, but here we have also a first derivative. Since $\frac{d}{dt}\sin(\omega t)\propto\cos(\omega t)$ and vice versa, neither sines nor cosines alone will solve this equation. However, the exponential function is proportional to its *first* derivative. Thus exponentials are a natural guess, and indeed they will work.

So, let's try plugging

$$x(t)=Ce^{\alpha t} \tag{19}$$

into Eq. (18). We find

$$\alpha^2Ce^{\alpha t}+\gamma\alpha Ce^{\alpha t}+\omega_0^2Ce^{\alpha t}=0 \tag{20}$$

Dividing out by $Ce^{\alpha t}$ we have reduced this to an algebraic equation:

$$\alpha^2+\gamma\alpha+\omega_0^2=0 \tag{21}$$

The solutions are

$$\alpha=-\frac{\gamma}{2}\pm\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2} \tag{22}$$

And therefore the general solution to the damped oscillator equation is

$$x(t)=e^{-\frac{\gamma}{2}t}\left(C_1e^{t\sqrt{(\frac{\gamma}{2})^2-\omega_0^2}}+C_2e^{-t\sqrt{(\frac{\gamma}{2})^2-\omega_0^2}}\right) \tag{23}$$

The cases when $\gamma>2\omega_0$, $\gamma=2\omega_0$ and $\gamma<2\omega_0$ give very different physical behavior.

4.1 Underdamping: $\gamma<2\omega_0$

The case $\gamma<2\omega_0$ includes the case when $\gamma=0$. For $\gamma=0$ the damping vanishes and we should regain the oscillator solution. Increasing γ from zero should slowly damp the oscillator. Let's see how this works mathematically.

Since $\gamma<2\omega_0$ then

$$\omega_u=\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2} \tag{24}$$

is a real number. In terms of ω_u, the general solution is then

$$x(t)=e^{-\frac{\gamma}{2}t}(C_1e^{i\omega_u t}+C_2e^{-i\omega_u t}) \tag{25}$$

Since $x(t)$ must be real, we must also have $C_1=C_2^\star$. Thus we can write

$$C_1=\frac{1}{2}Ae^{i\phi},\quad C_2=\frac{1}{2}Ae^{-i\phi} \tag{26}$$

for two real constants A and ϕ. This leads to

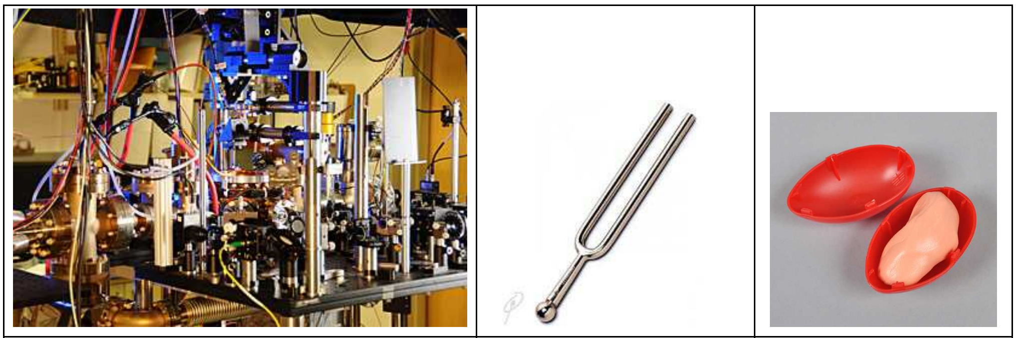
$$x(t) = A e^{-\frac{\gamma}{2}t} \cos(\omega_u t + \phi) \tag{27}$$

Thus we see that in the underdamped case, the object still oscillates, but at an angular frequency $\omega_u = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$ and the amplitude slowly goes down over time.

Note that both ω_0 and γ have dimensions of $\frac{1}{\text{seconds}}$. Their relative size determines how much the amplitude gets damped in a single oscillation. To quantify this, we define the **Q-factor** (or ***Q*--value**) as

$$Q \equiv \frac{\omega_0}{\gamma} \tag{28}$$

The smaller the Q the more the damping. Q stands for quality. The higher Q is, the higher quality, and the less resistance/friction/damping is involved.For example, a tuning fork vibrates for a long time. It is a very high quality resonator with $Q \sim 1000$. Here are some examples:



Atomic clock: $Q \approx 10^{11}$	Tuning fork: $Q \approx 1000$	Silly putty: $Q \sim 0.01$

Figure 2. Some Q-factors

Q is roughly the number of complete oscillations a system has gone through before it's amplitude goes down by a factor of around 20. To see this, note that due to the $\cos(\omega_u t)$ factor, it takes a time $t_Q = \frac{2\pi}{\omega_u} Q$ to go through Q cycles. Then due to the $e^{-\frac{\gamma}{2}t}$ factor, the amplitude has decayed by a factor of

$$\exp\left(-\frac{\gamma}{2}t\right) = \exp\left(-\frac{\gamma}{2} Q \frac{2\pi}{\omega_u}\right) = \exp\left(-\frac{\omega_0}{\omega_u}\pi\right) \approx \exp(-\pi) = 0.043 \tag{29}$$

In the next-to-last step, we have used that $\omega_u \approx \omega_0$ when $Q \gg 1$. (If Q is not large, then the system is highly damped and couting oscilations is not so useful). Since $0.043 \approx \frac{1}{23}$ we get the $\frac{1}{20}$ rule.

4.2 Overdamping: $\gamma > 2\omega_0$

In the over damped case, $\gamma > 2\omega_0$, then $\left(\frac{\gamma}{2}\right)^2 - \omega_0^2$ is positive so the roots in Eq. (22) are real. Thus the general solution is simply

$$x(t) = C_1 e^{-u_1 t} + C_2 e^{-u_2 t} \tag{30}$$

with

$$u_1 = \frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}, \tag{31}$$

and

$$u_2 = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} \tag{32}$$

Both solutions have exponential decay. Since $u_1 > u_2$, the u_1 solution will die away first, leaving the u_2 solution. Overdamped systems have $Q < \frac{1}{2}$.

4.3 Critical damping: $\gamma = 2\omega_0$

In the critically damped case, the two solutions in Eq. (23) reduce to one:

$$x(t) = Ce^{-\omega_0 t} \tag{33}$$

What happened to the other solution? That is, a second-order differential equation is supposed to have two independent solutions, but we have only found one. To find the other solution, let's look at the damped oscillator equation again, but set $\gamma = 2\omega_0$ to begin with. Then the equation is

$$\frac{d^2x}{dt^2} + 2\omega_0 \frac{dx}{dt} + \omega_0^2 x = 0 \tag{34}$$

Solving this with Mathematica, we find the general solution is

$$x(t) = (C + Bt)\, e^{-\omega_0 t} \tag{35}$$

You should check yourself that this ansatz satisfies Eq. (34).

A comparison of over-damping, underdamping and critical damping is shown in Figure 3. One thing to note is that the critically damped curve goes to zero faster than the overdamped curve! Can you think of an application for which you'd want a critically damped oscillator?

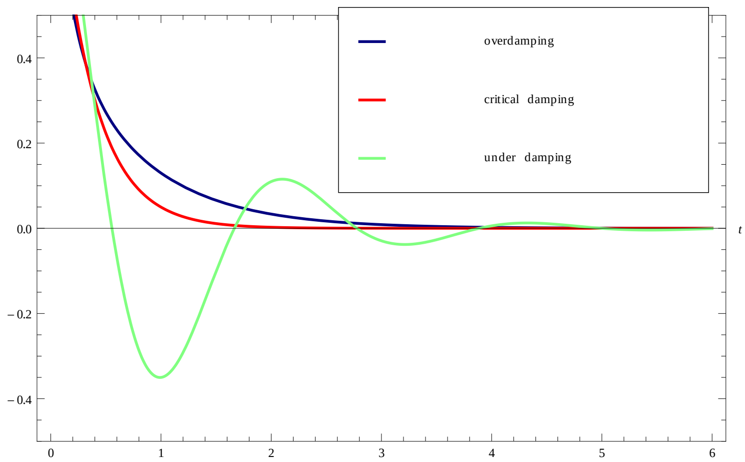


Figure 3. Comparison of underdamping, overdamping, and critical damping. We have taken $\omega_0 = 3$ and $\gamma = 8, 2$ and 6.

5 Linearity

The oscillator equation we have been solving has a very important property: **linearity**. Differential equations with at most single powers of x are **linear differential equations**. For example,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{36}$$

is linear. If there is no constant (x^0 term), the differential equations are **homogeneous**.

Linearity is important because it implies that if $x_1(t)$ and $x_2(t)$ are solutions to the equations of motion for a homogenous linear system then

$$x(t) = x_1(t) + x_2(t) \tag{37}$$

is also a solution. Let's check this for Eq. (36). By assumption $x_1(t)$ and $x_2(t)$ satisfy:

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = 0 \tag{38}$$

$$\frac{d^2x_2}{dt^2} + \omega^2 x_2 = 0 \tag{39}$$

Adding these equations, we find

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{40}$$

So that x satisfies Eq. (36) as well. So, *solutions to homogeneous linear differential equations add.*

5.1 Examples of linear systems

The damped oscillator in Eq. (18) is a linear system: $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$.

If you have a string of tension T and mass density μ going along the x direction, then its displacement in a transverse direction $y(x,t)$ satisfies

$$\mu\frac{\partial^2}{\partial t^2}y(x,t) - T\frac{\partial^2}{\partial x^2}y(x,t) = 0 \tag{41}$$

This is the **wave equation**. It is linear. (We'll derive this in a couple of weeks).

Electromagnetic waves are described by Maxwell's equations. With a little work, you can combine two of Maxwell's equations into an equation for the electric field $\vec{E}(x,\, t)$ of the form (we'll derive this soon too):

$$c^2\frac{\partial^2}{\partial t^2}\vec{E}(x,t) - \frac{\partial^2}{\partial x^2}\vec{E}(x,t) = 0 \tag{42}$$

with c the speed of light. Thus each component of $\vec{E}$ satisfies the **wave equation**.

Sound waves, water waves, etc., all satisfy linear differential equations.

5.2 Forced oscillation

What happens if the equation is not homogeneous? For example, what if we had

$$\frac{d^2x}{dt^2} = F_1(t) \tag{43}$$

Here, $F(t)$ represents some force from a motor or the wind or someone pushing you on a swing or the electromagnetic waves from the cell-phone tower transmitting that critical text message to you during class.

It is in general hard to solve this differential equation. But let's imagine we can do it and find a function $x_1(t)$ which satisfies

$$\frac{d^2x_1}{dt^2} = F_1(t) \tag{44}$$

Say this is your motion on a swing when you are being pushed. Now say some friend comes and pushes you too. Then

$$\frac{d^2x}{dt^2} = F_1(t) + F_2(t) \tag{45}$$

The amazing thing about linearity is that if we can find a solution $x_2(t)$ which satisfies

$$\frac{d^2x_2}{dt^2} = F_2(t) \tag{46}$$

Then $x = x_1 + x_2$ satisfies

$$\frac{d^2x}{dt^2} = F_1(t) + F_2(t) \tag{47}$$

This is *extremely important*. It is the key to this whole course. Really complicated systems are solvable by simpler systems, as long as the equations are linear.

In contrast, we cannot add solutions to *nonlinear* equations. For example, suppose we want to solve

$$a\frac{d^2}{dt^2}x + b\frac{d}{dt}x^2 = F_1(t) + F_2(t) \tag{48}$$

Although we can solve for x_1 and x_2 produced by the two forces separately, it does not then follow that $x = x_1 + x_2$ is a solution with the combined force is present. The x^2 term couples the x_1 and x_2 solutions together, so there is interference.

All the systems we study in this course will be linear systems. An important example is electromagnetism (Maxwell's equations are linear). Suppose you are making some radio waves in your radio station with $F_1 = \sin(7t)$ and some MIT student is making waves in her station with $F_2 = \sin(4t)$. Then $x_1(t) = \frac{1}{49}\sin(7t)$ satisfies Eq. (44) and $x_2(t) = \frac{1}{16}\sin(4t)$ satisfies Eq. (46). We then immediately conclude that

$$x(t) = \frac{1}{49}\sin(7t) + \frac{1}{16}\sin(4t) \tag{49}$$

must satisfy Eq. (45). We just add the oscillations! Thus if there are radio waves at frequency $\nu = 89.9$ MHz flying around and frequency $\nu = 90.3$ MHz flying around, they don't interfere with each other.

That explains why we can tune our radio – because electromagnetism is linear, we can add radio waves. There is no interference! The different frequencies don't mix with each other. All we have to do is get our radio to extract the coefficient of the $\sin(7t)$ oscillation from $x(t)$. Then we will get only the output from our radio station. As we will see, you can always find out which frequencies are present with which amplitudes using **Fourier decomposition**. We'll come back to this soon .

5.3 Summary

Linearity is a really important concept in physics. The definition of linearity is that all terms in a differential equation for $x(t)$ have at most one power of $x(t)$. So $\frac{d^3}{dt^3}x(t) = 0$ is linear, but $\frac{d}{dt}x(t)^2$ is nonlinear.

For linear systems, one can add different solutions and still get a solution. This lets us break the problem down to easier subproblems.

Linearity does not only let us solve problems simply, but it is also a universal feature of physical systems. Whenever you are close to a static solution $x(t) = x_0 = \text{constant}$, the equations for deviations around this solution will be linear. To see that, we again use Taylor theorem. We shift by $x(t) \to x(t) - x_0$ so the equilibrium point is now $x(t) = 0$. Then no matter how complicated and nonlinear the exact equations of motion for the system are, when $x - x_0 \ll 1$, the linear term, proportional to $x - x_0$ will dominate. For example, if we had

$$\frac{d^2}{dt^2}\frac{x}{x^2 - 2}e^{-x^4} + \frac{d}{dt}x^7 + (x^2 - 4)\sin^3(x) = 0 \tag{50}$$

then $x(t) = x_0 = 0$ is a solution. For $x \ll 1$ this simplifies to

$$-\frac{1}{2}\frac{d^2 x}{dt^2} - 4x = 0 \tag{51}$$

which is again linear (it's the oscillator equation again).

6 Solving general linear systems

At this point, we've defined linearity, argued that it should be universal for small deviations from equilibrium, and showed how it can help us combine solutions to a differential equation. Now we will see how to solve general linear differential equations.

6.1 Exponentials, sines and cosines

A general linear equation has a bunch of derivatives with respect to time acting on a single function x:

$$\cdots + a_3\frac{d^3}{dt^3}x + a_2\frac{d^2}{dt^2}x + a_1\frac{d}{dt}x + a_0 x = F(t) \tag{52}$$

Let's first consider the case when $F=0$. A really easy way to solve these equations for $F=0$ is to consider solutions for which all the derivatives are proportional to each other. What is a function with this property? Sines and cosines have derivatives proportional to themselves: $\frac{d^2}{dt^2}\sin(\omega t) = -\omega^2\sin(\omega t)$, but only *second* (or even numbers of) derivatives. A function with *all* of its derivatives proportional to itself is the exponential: $x(t)=Ce^{\alpha t}$

$$\frac{d}{dt}Ce^{\alpha t}=C\alpha e^{\alpha t} \tag{53}$$

As an example, let's try this Ansatz into our oscillator equation, Eq. (8):

$$\frac{d^2x}{dt^2}=-\omega^2 x \tag{54}$$

Plugging in $x(t)=Ce^{\alpha t}$ gives

$$\alpha^2 e^{\omega t}=-\omega^2 e^{\omega t} \tag{55}$$

which implies $\alpha=\sqrt{-\omega^2}$ or $\alpha=\pm i\omega$,

Thus the solutions are

$$x(t)=C_1 e^{i\omega t}+C_2\, e^{-i\omega t} \tag{56}$$

Recalling that

$$\sin(\omega t)=\frac{e^{i\omega t}-e^{-i\omega t}}{2i}, \quad \cos(\omega t)=\frac{e^{i\omega t}+e^{-i\omega t}}{2} \tag{57}$$

We can also write

$$x(t)=i(C_1-C_2)\sin(\omega t)+(C_1+C_2)\cos(\omega t) \tag{58}$$

In summary

- Sines and cosines are useful if you have only 2nd derivatives
- Exponentials work for any number of derivatives

Now, if $F\neq 0$, then a simple exponential will not obviously be a solution. The key, however, is that we can *always* write any function $F(t)$ on the interval $0<t\leqslant T$ as a sum of exponentials

$$F(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\frac{t}{T}} \tag{59}$$

for some coefficients a_n. This is called a **Fourier decomposition**. Since we can solve the equation as if $F(t)=e^{2\pi i n\frac{t}{T}}$ for a fixed n, we can then add solutions using *linearity* to find a solution with the original $F(t)$. Don't worry about understanding this now – it's just a taste of what's to come.

6.2 Relating e^{ix} to $\sin(x)$ and $\cos(x)$

Suppose you didn't know that $\sin(\omega t)=\frac{e^{i\omega t}-e^{-i\omega t}}{2i\omega}$. How could you derive this?

One way is using the fact that we solved the oscillator equation

$$\frac{d^2x}{dt^2}+\omega^2 x=0 \tag{60}$$

two ways. On the one hand we found

$$x(t)=A\sin(\omega t)+B\cos(\omega t) \tag{61}$$

and on the other hand we found

$$x(t)=C_1 e^{i\omega t}+C_2\, e^{-i\omega t} \tag{62}$$

Since the differential equation is second order (two derivatives), the solution is given uniquely once two boundary conditions are set. Conversely, if we know the solution we can work out the boundary conditions. For example, if the solution were $x(t)=\sin(\omega t)$ then $x(0)=0$ and $x'(0)=\omega$. Plugging in $x(0)=0$ to the exponential solution implies

$$C_1+C_2=0 \tag{63}$$

plugging in $x'(0) = \omega$ implies

$$i\omega C_1 - i\omega C_2 = \omega \tag{64}$$

The solution to these two equations is $C_1 = -\frac{1}{2i}$ and $C_2 = \frac{1}{2i}$, as you can easily check. We thus conclude that the two solutions are exactly equal and so

$$\boxed{\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}} \tag{65}$$

Similarly

$$\boxed{\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}} \tag{66}$$

You should have these relationships memorized – we will use them a lot.

7 Complex numbers (mathematics)

Complex numbers are a wonderful invention. They make complicated equations look really simple. Being able to take the square root of anything is unbelievably helpful.

To see how important complex numbers are for solving equations, consider how sophisticated mathematics needs to be to solve some equations. The equation

$$3x - 4 = 0 \tag{67}$$

has a solution $x = \frac{4}{3}$ which is a simple **rational number** (rational numbers can be written as ratios of whole numbers $0, 1, -1, 2, -2, \cdots$).

To solve

$$x^2 - 2 = 0 \tag{68}$$

we need **irrational numbers**: $x = \sqrt{2}$. Such numbers cannot be written as ratios of whole numbers.

To solve

$$x^2 + 4 = 0 \tag{69}$$

we need **complex numbers**. The solutions are $x = \pm 2i$, with $i = \sqrt{-1}$.

Now the punch line: to solve

$$ax^3 + bx^2 + cx + d = 0 \tag{70}$$

we still need *only* complex numbers. Complex numbers are the end of the road. Any polynomial equation can be solved with complex numbers.

$$ax^3 + bx^2 + cx + d = (x - r_1)(x - r_2)(x - r_3) = 0 \tag{71}$$

for some $r_i \in \mathbb{C}$.

Exponentials are for linear differential equations what complex numbers are for algebraic equations. Any linear differential equation can be solved by exponentials. Say we had

$$a\frac{d^3}{dt^3}x(t) + b\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + dx(t) = 0 \tag{72}$$

We can factor this into

$$(\frac{d}{dt} - r_1)(\frac{d}{dt} - r_2)(\frac{d}{dt} - r_3)x(t) = 0 \tag{73}$$

Thus if

$$(\frac{d}{dt} - r_3)x(t) = 0 \tag{74}$$

Then we have a solution. The solution is therefore a product of factors like

$$x(t) = e^{ir_3 t} \tag{75}$$

So we're always going to have exponential solutions to linear equations.

7.1 Complex number arithmetic

I hope you're already familiar with complex numbers from your math classes. If not, here's a quick review.

We can write any complex number as

$$z = a + b i \tag{76}$$

with a and b real. Then

$$z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)i \tag{77}$$

and

$$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + b_1 a_2 i + b_2 a_1\, i + b_1 b_2 i^2 \tag{78}$$

$$= (a_1 a_2 - b_1 b_2) + (b_1 a_2 + b_2 a_1)i \tag{79}$$

It's helpful to define **complex conjugation** $i \to -i$. In fact, we could have used $-i$ instead of i from the beginning. We define

$$\bar{z} = a - b i \tag{80}$$

as the **complex conjugate** of a complex number $z = a + b i$.

Then

$$z\,\bar{z} = (a + b i)(a - b i) = a^2 + b^2 \in \mathbb{R} \tag{81}$$

The trick to dividing complex numbers is to use that $z\bar{z} \in \mathbb{R}$:

$$\frac{1}{z} = \frac{\bar{z}}{\bar{z}\,z} = \frac{a - b i}{a^2 + b^2} \tag{82}$$

That is,

$$\frac{a_2 + b_2 i}{a_1 + b_1 i} = (a_2 + b_2 i)\frac{1}{a_1 + b_1 i} = (a_2 + b_2 i)\frac{a_1 - b_1\, i}{a_1^2 + b_1^2} = \frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2} + \frac{a_1 b_2 - a_2 b_1}{a_1^2 + b_1^2} i \tag{83}$$

For functions, we usually write $f^\star$ instead of $\bar{f}$ for complex conjugation. For any function $f(x) \in \mathbb{C}$, we have

$$[f(x)]\,[f(x)]^\star \in \mathbb{R} \tag{84}$$

This is easy to see using that the conjugate of a product of complex numbers is the product of the conjugates:

$$(f f^\star)^\star = f^\star f^{\star\star} = f^\star f = f f^\star \tag{85}$$

Since $f f^\star$ is invariant under complex conjugation it must be real.

Any complex number can be also written as

$$z = r e^{i\theta} = a + b i \tag{86}$$

to relate r and θ to a and b we use

$$\bar{z} = r e^{-i\theta} \tag{87}$$

$$a = \frac{z + \bar{z}}{2} = r\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = r\cos\theta \tag{88}$$

$$b = \frac{z - \bar{z}}{2i} = r\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) = r\sin\theta \tag{89}$$

$$z\,\bar{z} = r^2 = a^2 + b^2 \tag{90}$$

r is sometimes called the **modulus** of a complex number and θ the **phase** of a complex number.