Midterm Oct 2025

1. Solve the equation of $x(t)$ with initial condition:

$$x' - 3x = e^{-t}, \quad x(1) = 3$$

Hint: Use integrating factor.

Sol: The integrating factor would be $\alpha(t) = \exp[-\int 3dt] = e^{-3t}$.

Multiply the whole equation by the integrating factor $\alpha(x)$:

$$e^{-3t}x' - 3e^{-3t}x = \frac{d}{dt}(e^{-3t}x) = e^{-4t}$$

Integrate and add a constant C:

$$e^{-3t}x = \int dt \cdot e^{-4t} + C = -\frac{1}{4}e^{-4t} + C$$

$$x = -\frac{1}{4}e^{-t} + Ce^{3t}$$

The constant C can be obtained by the initial condition

$x(1) = -\frac{1}{4}e^{-1} + Ce^{3} = 3.\ C = 3e^{-3} + \frac{1}{4}e^{-4}$

$$x = -\frac{1}{4}e^{-t} + 3e^{3t-3} + \frac{1}{4}e^{3t-4}$$

2. Consider the 2nd order ODE:

$$2x^2y'' + 3xy' - 15y = 0$$

We know one of the solutions is x^{-3}.

Use the method of Wronskian to solve the equation.

Sol: Write the ODE into the standard form:$y'' + \frac{3}{2x}y' - \frac{15}{2x^2}y = 0.\ y_1 = x^{-3}$

$$y_2 \sim y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$

$$\exp\left(-\int_0^x P(x') \cdot dx'\right) = \exp\left(-\int_0^x \frac{3}{2x'} \cdot dx'\right) = \exp\left(-\frac{3}{2}\ln x\right) = x^{-\frac{3}{2}}.$$

$$y_2 \sim x^{-3} \int dx \frac{x^{-\frac{3}{2}}}{x^{-6}} \propto x^{-3}x^{\frac{11}{2}} \sim x^{5/2}$$

$$y = C_1y_1(x) + C_2y_2(x) = C_1x^{-3} + C_2x^{5/2}$$

3. First consider the homogeneous ODE:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$$

A. Find the general solutions x_{h} of this ODE.

B. Consider the inhomogeneous ODE:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 2\cos t$$

Find first one solution and then the general solutions x_{inh} of the inhomogeneous ODE. (10)

Hint: $2\cos t$ is the real part of $2e^{-it}$.

Sol:

A. For the complex homogeneous ODE:

$$\frac{d^2z}{dt^2} + 4\frac{dz}{dt} + 3z = 0$$

Guess the solution $z_0 e^{\alpha t}$. Plug in: The unknown α satisfies the algebraic equation:

$$\alpha^2 + 4\alpha + 3 = 0$$

The solutions are

$$\alpha = -1, -3$$

The general solutions of the ODE are linear combinations of the corresponding solutions:

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

B. Since $2\cos t$ is the real part of $2e^{-it}$, the corresponding complex inhomogeneous ODE can be written as:

$$\frac{d^2z}{dt^2} + 4\frac{dz}{dt} + 3z = 2e^{-it}$$

Again try $a_0 e^{\alpha t}$:

$$(\alpha^2 + 4\alpha + 3)a_0 e^{\alpha t} = 2e^{-it}$$

$$\alpha = -i, a_0 = \frac{1}{1-2i}$$

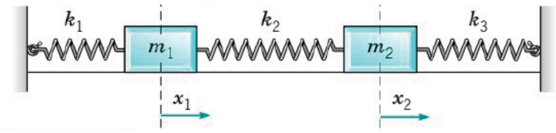
$$z = \frac{1}{1-2i}e^{-it}$$

$$x = \operatorname{Re} z = \operatorname{Re}\frac{1}{1-2i}e^{-it} = \operatorname{Re}\frac{(1+2i)}{5}(\cos t - i\sin t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

C. General solutions x_{inh} of the inhomogeneous ODE

$$x_{\text{inh}} = \frac{1}{5}\cos t + \frac{2}{5}\sin t + c_1 e^{-t} + c_2 e^{-3t}$$

4. Consider a coupled oscillation of two particles as shown below:



with equations of motion:

$$\frac{d^2x_1}{dt^2} = -\frac{k_1+k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2, \qquad \frac{d^2x_2}{dt^2} = \frac{k_2}{m_2}x_1 - \frac{k_2+k_3}{m_2}x_2$$

which can be written in the notations of matrices:

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

Assume that the matrix $\boldsymbol{A}$ equals:

$$\boldsymbol{A} \equiv \omega_0^2\begin{pmatrix}5 & 4\\ 4 & 5\end{pmatrix}$$

The general solutions can be written as:

$$\boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = c_1\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}\cos(\omega_1 t + \phi_1) + c_2\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}\cos(\omega_2 t + \phi_2)$$

A. Find the numbers $\omega_1, \begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}$ and $\omega_2, \begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}$.

B. If the initial condition is $x_1(0) = a_m, x_2(0) = 0, x_1'(0) = x_2'(0) = 0$, find the solution. Hint: $\phi_1 = \phi_2 = 0$.

Sol:

A. Guess the solutions are $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$, $\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$**.** This is eigenvalue problem of $\boldsymbol{A}$**.** The characteristic equation:

$$\mathbf{det}\,(\boldsymbol{A} - \lambda\boldsymbol{I}) = \mathbf{det}\begin{bmatrix}5-\lambda & 4\\ 4 & 5-\lambda\end{bmatrix} = \lambda^2 - 10\lambda + 9 = 0$$

$$\lambda = \omega^2 = \lambda_1 = \omega_1^2 = 1 \text{ or } \lambda_2 = \omega_2^2 = 9$$

$$\omega_1 = 1,\ \omega_2 = 3$$

For $\omega_1 = 1$, $(\boldsymbol{A} - \lambda\boldsymbol{I})\cdot\boldsymbol{a_1} = \begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}\cdot\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix} = 0$, $\boldsymbol{a_1} = c_1\begin{pmatrix}1\\ -1\end{pmatrix}$

For $\omega_2 = 3$, $(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a_2} = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0$, $\boldsymbol{a_2} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(t + \phi_1) + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(3t + \phi_2)$$

B. $x_1(0) = c_1 + c_2 = a_m, x_2(0) = c_1 - c_2 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{a_m}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t + \frac{a_m}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 3t$$

5. Diagonalization of a matrix: We have calculated in class that the eigenvectors of the matrix $\boldsymbol{A} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ are $c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 10$. The diagonalizing matrix can be written as: $\boldsymbol{U} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

A. Calculate $\boldsymbol{U}^{-1}$ using $\boldsymbol{U^{-1}} = \frac{1}{\det \boldsymbol{U}} \begin{pmatrix} \boldsymbol{U_{22}} & -\boldsymbol{U_{12}} \\ -\boldsymbol{U_{21}} & \boldsymbol{U_{11}} \end{pmatrix}$.

B. Calculate $\boldsymbol{A} \cdot \boldsymbol{U}$ and $\boldsymbol{U} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and show they are equal.

C. Check $\boldsymbol{U}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{U} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Sol:

A. $\boldsymbol{U^{-1}} = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

B. $\boldsymbol{A} \cdot \boldsymbol{U} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix}$,

$\boldsymbol{U} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix}$

C. $\boldsymbol{U}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{U} = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 30 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$