

數物期中考

Apr 2026

1. The energy levels of the hydrogen atom can be written as $E_n = -\frac{13.6}{n^2}$ eV.

If some external electrons, with fixed kinetic energy 13.1 eV, hit hydrogen atoms at ground state $n = 1$ (assume that one electron hit one atom at a time), with all or part of external electron energy absorbed by the atoms, the atoms will emit a photon or photons afterwards. What are the largest and the smallest frequencies of these emitted photons? $h = 6.582 \times 10^{-16}$ eV·s.

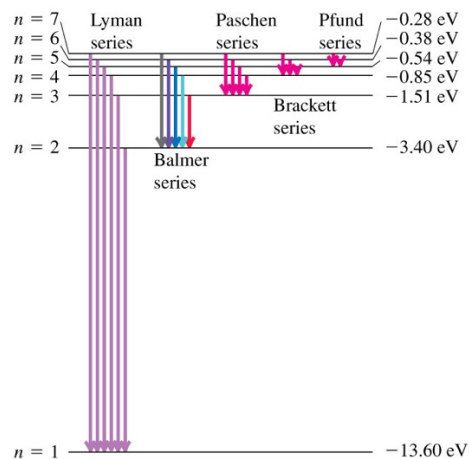
Solution:

The energy of the first three levels are: $E_1 = -13.6$ eV, $E_2 = -3.4$ eV, $E_3 = -1.5$ eV, $E_4 = -0.85$ eV, $E_5 = -0.54$ eV. The external electron can only have enough energy for the internal electron to reach $n = 2, 3, 4, 5$. Hence the largest and the smallest frequencies of these emitted photons transit: $5 \rightarrow 1$, and $5 \rightarrow 4$, respectively. The photon energies are

$$E_5 - E_1 = 13.06 \text{ eV}, E_5 - E_4 = 0.31 \text{ eV}$$

The corresponding frequencies:

$$f_{51} = \frac{E_5 - E_1}{h} = \frac{13.06 \text{ eV}}{6.582 \times 10^{-16} \text{ eV} \cdot \text{s}} = 1.98 \times 10^{16} \text{ Hz}, f_{54} = 4.71 \times 10^{14} \text{ Hz}$$



2. The Heat Equations is the equation for the temperature distribution in a heat conductivity thin ring. The ring is thin and its temperature depends only on x ($0 < x < 2\pi R$), the location along the circumference of the ring. Denote the temperature difference from a constant temperature T_0 at x and time t as $u(x, t)$. R is the radius of

the ring. Using the law of conductivity and the first law of thermodynamics, it can be shown that $u(x, t)$ satisfies the following partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t}$$



σ is a fixed real positive constant. It is almost identical to the Schrodinger Wave Equation except that the coefficient σ in front of $\frac{\partial u}{\partial t}$ is real now instead of imaginary. It is reasonable to think that there exist solutions that are separable just like in the wave equations we discussed in class:

$$u(x, t) = X(x) \cdot T(t)$$

Since it is a ring, the temperature satisfies a Periodic Boundary condition at all times:

$$u(t, 0) = u(t, 2\pi R)$$

This condition is widely used in solid state physics.

A. Find the Ordinary Differential Equation satisfied by $T(t)$. Assume that $u(x, t)$ will not grow indefinitely as $t \rightarrow \infty$. Show that the exponentially decreasing function:

$$T(t) = e^{-\frac{\lambda}{\sigma}t}$$

is a solution for $T(t)$. λ is a positive constant to be determined.

B. Find the Ordinary Differential Equation satisfied by $X(x)$. This equation is identical to the time independent Schrodinger Equation for free space and hence the solution can be written as:

$$X(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} = C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x$$

Let's concentrate on $X(x) = D \sin \sqrt{\lambda}x$ and ignore the Cosine part for now. The periodic boundary condition would be satisfied if

$$X(x) = X(x + 2\pi R)$$

The condition for $\sqrt{\lambda}$ to satisfy the periodic condition is

$$\lambda_n = cn^2$$

Calculate the constant c . Write down the $u(x, t)$ for λ_n . Observe that it will vanish $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$, meaning the temperature distribution on the ring will eventually become uniform ($= T_0$).

Afterword 後話: To put $\cos \sqrt{\lambda}x$ back into consideration, we need one more periodic condition: $X'(x) = X'(x + 2\pi R)$. Apparently, the first derivative is continuous along the whole ring. Then the condition for $\sqrt{\lambda}$ will be the same.

Sol:

A. 將 $u(x, t) = X(x) \cdot T(t)$ 代入波方程式：

$$\frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t}$$

左右都除以 $X(x) \cdot T(t) \cdot \sigma$ ：

$$\frac{1}{X(x)} \left[\frac{d^2 X(x)}{dx^2} \right] = \sigma \frac{1}{T(t)} \frac{dT(t)}{dt}$$

在左邊只與 x 有關，右邊只與 t 有關，兩者是獨立變數，唯一可能是左右兩式與兩個變數都無關，是一常數，設為 $-\lambda$ ：

$$\frac{1}{X(x)} \left[\frac{d^2 X(x)}{dx^2} \right] = \sigma \frac{1}{T(t)} \frac{dT(t)}{dt} \equiv -\lambda$$

$$\frac{dT(t)}{dt} = -\frac{\lambda}{\sigma} T(t)$$

$$T(t) = e^{-\frac{\lambda}{\sigma} t}$$

B. 空間部分 $\psi(x)$ 滿足：

$$\frac{1}{X(x)} \left[\frac{d^2 X(x)}{dx^2} \right] = -\lambda$$

也就是：

$$\frac{d^2 X(x)}{dx^2} = -\lambda X(x)$$

$$X(x) = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} = C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x$$

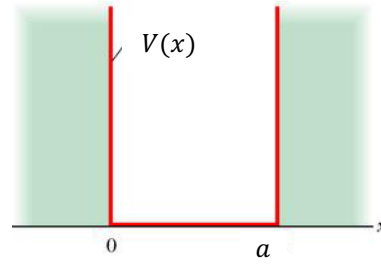
$$X(x) = X(x + 2\pi R)$$

$$\sqrt{\lambda} \cdot 2\pi R = 2n\pi, \quad n = 1, 2, 3 \dots$$

The condition is $\sqrt{\lambda} = \frac{n}{R}$ or $\lambda = \left(\frac{n}{R}\right)^2$.

3. Consider an infinite potential box, with boundaries at $x = 0$ and $x = a$:

$$V(x) = \infty, x > a, x < 0 \text{ and } V(x) = 0, 0 < x < a.$$



As we have shown in class, in this potential the stationary states 定態 state function can

be written as $u_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ with eigenvalues $E_n = \left(\frac{\hbar^2}{2m}\right) \frac{\pi^2}{a^2} n^2$.

- A. Assume the wavefunction of an electron at $t = 0$ is (, its total probability already normalized to one):

$$\begin{aligned} \Psi(x, 0) &= \left(\sqrt{\frac{2}{a}} \sin \frac{5\pi x}{a} \right) \quad 0 < x < a, \\ &= 0 \quad x < 0, x > a \end{aligned}$$

For a later time $t > 0$, write down the wave function $\Psi(x, t)$, in terms of E_5 and t .

Calculate the Probability of finding the electron between $0 < x < \frac{a}{8}$ at time t . (15)

Hint: You might need this formula: $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$.

- B. Assume the wavefunction of another electron at $t = 0$ is: (, its probability already normalized to one)

$$\begin{aligned} \Psi(x, 0) &= \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \right) + \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a} \right) \quad 0 < x < a, \\ &= 0 \quad x < 0, x > a \end{aligned}$$

For a later time $t > 0$, at $x = \frac{a}{4}$, write down the wave function $\Psi\left(\frac{a}{4}, t\right)$, in terms of

$E_{1,3}$ and t . It can be shown that its probability density at $x = \frac{a}{4}$ is an oscillating

function of time t :

$$P\left(\frac{a}{4}, t\right) \propto 1 + \cos \omega t$$

Calculate the angular frequency ω of this oscillation in terms of \hbar, m, a .

Hint: $|A + B|^2 = (A^* + B^*)(A + B) = |A|^2 + |B|^2 + A^*B + B^*A$

解答：

A.

$$\begin{aligned}
 P &= \int_0^{\frac{a}{8}} \left| \sqrt{\frac{2}{a}} \sin \frac{5\pi x}{a} \right|^2 \cdot dx = \frac{2}{a} \int_0^{\frac{a}{8}} \left(\frac{1 - \cos \frac{10\pi x}{a}}{2} \right) \cdot dx \\
 &= \frac{2}{a} \left(\frac{x}{2} - \frac{a}{10\pi} \frac{\sin \frac{10\pi x}{a}}{2} \right) \Bigg|_0^{\frac{a}{8}} = \frac{1}{12} - \frac{a}{10\pi} \frac{1}{2\sqrt{2}}
 \end{aligned}$$

B.

$t = 0$ 時此狀態可以視為定態 $u_{1,2}$ 的如上疊加，接著定態隨時間個自演化，位能下薛丁格方程式要求 u_n 乘上 $e^{-i\frac{E_n}{\hbar}t}$ 。乘完之後依同樣方式疊加，整個波函數也就滿足薛丁格波方程式。因此

$$\begin{aligned}
 \Psi(x, t) &= \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \right) e^{-i\frac{E_1}{\hbar}t} + \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a} \right) e^{-i\frac{E_3}{\hbar}t} \\
 \Psi\left(\frac{a}{4}, t\right) &= \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{\pi}{4} \right) e^{-i\frac{E_1}{\hbar}t} + \sqrt{\frac{1}{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{3\pi}{4} \right) e^{-i\frac{E_3}{\hbar}t} \\
 &= \sqrt{\frac{1}{a}} \left(\frac{1}{\sqrt{2}} e^{-i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{2}} e^{-i\frac{E_3}{\hbar}t} \right) \\
 P\left(\frac{a}{4}, t\right) &= \left| \Psi\left(\frac{a}{4}, t\right) \right|^2 = \frac{1}{2a} \left| e^{-i\frac{E_1}{\hbar}t} + e^{-i\frac{E_3}{\hbar}t} \right|^2 \\
 &= \frac{1}{2a} \left\{ \left| e^{-i\frac{E_1}{\hbar}t} \right|^2 + \left| e^{-i\frac{E_3}{\hbar}t} \right|^2 + e^{i\frac{E_1}{\hbar}t} e^{-i\frac{E_3}{\hbar}t} + e^{i\frac{E_3}{\hbar}t} e^{-i\frac{E_1}{\hbar}t} \right\} \\
 &= \frac{1}{2a} \left\{ 1 + 1 + e^{-i\frac{(E_3-E_1)}{\hbar}t} + e^{i\frac{(E_3-E_1)}{\hbar}t} \right\} \\
 &= \frac{1}{2a} \left\{ 2 + \cos \frac{(E_3-E_1)}{\hbar}t - i \sin \frac{(E_3-E_1)}{\hbar}t \right. \\
 &\quad \left. + \cos \frac{(E_3-E_1)}{\hbar}t + i \sin \frac{(E_3-E_1)}{\hbar}t \right\} \\
 &= \frac{1}{2a} \left\{ 2 + 2 \cos \frac{(E_3-E_1)}{\hbar}t \right\} \propto 1 + \cos \omega t
 \end{aligned}$$

$$\omega = \frac{(E_3 - E_1)}{\hbar} = \frac{\left(\frac{\hbar^2}{2m}\right) \frac{\pi^2}{a^2} 8}{\hbar} = \frac{4\pi^2 \hbar}{ma^2}$$