習題九

1. We consider String Wave Equation last term: $\tau \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}$. It can be generalized to the case with x-dependent $\tau(x), \mu(x)$ and an extra restoring force $v(x)$. The equation is now written as:

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] - v(x)y = \mu(x)\frac{\partial^2 y}{\partial t^2}$$

also known as generalized string wave equation.

It can be solved in ways quite similar to wave equation by separation of variables.

Assume the separable mode solutions can be written as

$$y(x,t) = X(x) \cdot T(t).$$

   A. Find the Ordinary Differential Equation satisfied by $X(x)$ and $T(t)$. Show that $T(t)$ satisfies the same equation of motion as SHM and solution can be written as

$$T(t) = a_m \cos(\omega t + \phi)$$

   The constant ω (unspecified yet) can be interpreted as the oscillating frequency of the mode solution while a_m, ϕ can be used to fit the initial conditions. The ODE satisfied by $X(x)$ is called Sturm-Liouville Eigenfunction Problem. Almost the spatial part of classical wave equations and QM time independent Schrodinger Equations are examples of SL problems.

   B. Consider the case:

$$v(x) = ax^2, \tau(x) = \mu(x) = 1$$

   Write down the Ordinary Differential Equation, or Sturm-Liouville Eigenfunction Equation, satisfied by $X(x)$ in terms of ω, a. Observe that it is the same as time independent Schrodinger Equations for SHM:

$$\frac{d^2 u}{dx^2} = \frac{m^2\omega_0^2}{\hbar^2}x^2 u - \frac{2mE}{\hbar^2}u$$

   if we replace $\frac{m^2\omega_0^2}{\hbar^2}$ in QM SHM with a in generalized string equation, and $\frac{2mE}{\hbar^2}$ with ω^2. What are the possible values of oscillating frequencies ω in terms of the parameter a (just a, since it is the only parameter in our generalized string equation, and no $\hbar, m, \omega_0$, which exist only in analogy QM equation)? What is the wavefunction $y(x,t)$ for the smallest possible ω.

時間部分T很快可以解出：

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = \lambda$$

$$\frac{d^2T}{dt^2} = v^2\lambda T \equiv -\omega^2 T$$

若$\lambda < 0$，那麼$T(t)$就是以$\omega = v\sqrt{-\lambda}$為角頻率的簡諧運動！

$$T(t) = a_m \cos(\omega t + \phi) \sim f\cos(\omega t) + g\sin(\omega t)$$

因此這樣的可分解的解，從時間演化看，也可以稱為模式振盪解。

To motivate our discussion of the general Sturm-Liouville eigenfunction theory, we first derive a more *general* string equation by including an additional restoring force per unit length of the form

$-v(x)u(x, t)$ along the string. This can be realized physically, for example, by attaching springs continuously along the string, as illustrated in Fig. 40.2. Here $v(x)$ is the force constant per unit length, and the Hooke's-law restoring force is proportional to the displacement from equilibrium and opposes it. The lagrangian density for this system is obtained by generalizing Eq. (38.2) to the form

$$\mathscr{L} = \mathscr{T} - \mathscr{V} \qquad (40.6a)$$

$$\mathscr{T} = \tfrac{1}{2}\sigma(x)\left(\frac{\partial u}{\partial t}\right)^2 \qquad (40.6b)$$

$$\mathscr{V} = \tfrac{1}{2}\tau(x)\left(\frac{\partial u}{\partial x}\right)^2 + \tfrac{1}{2}v(x)u^2 \qquad (40.6c)$$

where, as previously, the mass density $\sigma(x)$ and the tension $\tau(x)$ may vary along the string. In Eq. (40.6), $\mathscr{L}$ has been separated into a kinetic-energy density $\mathscr{V}$ and a potential-energy density $\mathscr{V}$ that includes the extra restoring force. The Euler-Lagrange equation (38.8) then leads to the equation of motion

$$\sigma(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right] - v(x)u \qquad (40.7)$$

which will now be called the *general string equation*. We again seek normal-mode solutions

$$u(x, t) = \rho(x)\cos(\omega t + \phi) \qquad (40.8)$$

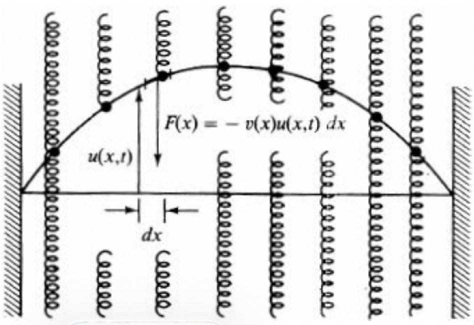
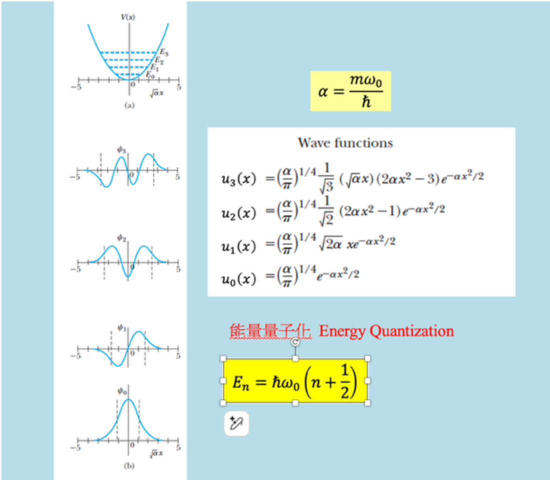


Figure Screenshot ation of additional restoring force $-v(x)u(x, t)$ along the string. Here $v(x)$ is the force constant per unit length.



Solution: Sol:

A. 將$y(x,t) = X(x) \cdot T(t)$代入波方程式：

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] - v(x)y = \mu(x)\frac{\partial^2 y}{\partial t^2}$$

左右都除以 $X(x) \cdot T(t) \cdot \mu(x)$：

$$\frac{1}{\mu(x)X(x)}\left[\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial X}{\partial x}\right] - v(x)X\right] = \frac{1}{T(t)}\frac{d^2T}{dt^2}$$

在左邊只與x有關，右邊只與t有關，兩者是獨立變數！唯一可能是左右兩式與兩個變數都無關，是一常數，設為λ。

$$\frac{1}{\mu(x)X(x)}\left[\frac{d}{dx}\left[\tau(x)\frac{dX}{dx}\right] - v(x)X\right] = \frac{1}{T(t)}\frac{d^2T}{dt^2} \equiv \lambda$$

$$\frac{d^2T}{dt^2} = \lambda T \equiv -\omega^2 T$$

$\lambda < 0$，

$$T(t) = a_m \cos(\omega t + \phi)$$

空間部分$X(x)$滿足

$$\frac{1}{\mu(x)X(x)}\left[\frac{d}{dx}\left[\tau(x)\frac{dX}{dx}\right] - v(x)X\right] = -\omega^2$$

That is:

$$\left[\frac{d}{dx}\left[\tau(x)\frac{dX}{dx}\right] - v(x)X\right] = -\omega^2\mu(x)X(x)$$

This is the so-called Sturm-Liouville Eigenfunction Problem. ω^2 is the eigenvalue.

B. In the case: $v(x) = ax^2, \tau(x) = \mu(x) = 1$, the SL equation becomes:

$$\left[\frac{d^2X}{dx^2} - ax^2X\right] = -\omega^2 X(x)$$

Compare it to QM SHM:

$$\frac{d^2u}{dx^2} - \frac{m^2\omega_0^2}{\hbar^2}x^2u = -\frac{2mE}{\hbar^2}u$$

They are identical if we replace $\frac{m^2\omega_0^2}{\hbar^2}$ with a and $\frac{2mE}{\hbar^2}$ with ω^2.

$$\frac{m^2\omega_0^2}{\hbar^2} = a$$

$$\frac{2mE}{\hbar^2} = \omega^2$$

We know the possible values of E are $E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) = \hbar\left(n + \frac{1}{2}\right)\frac{\hbar}{m}\sqrt{a}$. The possible values of ω are: $\omega_n = \frac{2mE_n}{\hbar^2} = (2n+1)\sqrt{a}$.

The smallest possible ω is $n = 0, \omega = \sqrt{a}$. The corresponding $X(x) = e^{-\alpha\frac{x^2}{2}} = e^{-\frac{m\omega_0}{\hbar}\frac{x^2}{2}} = e^{-\sqrt{a}\frac{x^2}{2}}$. The wavefunction is

$$y(x,t) = X(x)\cdot T(t) = e^{-\sqrt{a}\frac{x^2}{2}}a_m\cos\left(\sqrt{a}t + \phi\right)$$

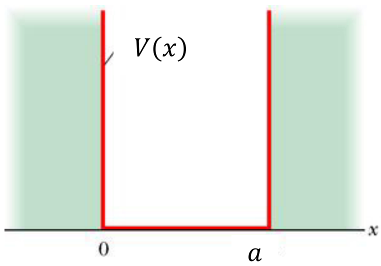
2. Consider the wavefunction of an electron at a certain moment:

$$\psi(x) = \sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}, 0 < x < a,$$

$$\psi(x) = 0, x > a, x < 0$$

This is the stationary state wavefunction of an electron located within an infinite potential box, with boundaries at $x = 0$ and $x = a$:

(The potential is $V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$. We will discuss this wavefunction in details in class later.)



A. Prove that it is normalized: $\int_0^a |\psi(x)|^2 \cdot dx = 1$.(15)

B. Calculate the Probability of finding the electron between $0 < x < \frac{a}{12}$.(15)

Hint: You might need this formula: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

解答：

A.

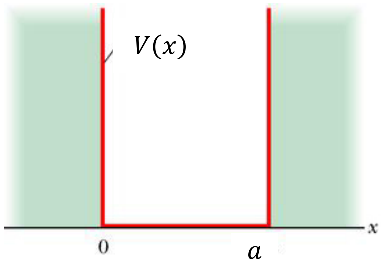
$$\int_0^a \left|\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right|^2 \cdot dx = \frac{2}{a}\int_0^a \left(\sin\frac{3\pi x}{a}\right)^2 \cdot dx = \frac{2}{a}\int_0^a \left(\frac{1-\cos\frac{6\pi x}{a}}{2}\right)dx =$$

$$\frac{2}{a}\left(\frac{x}{2}-\frac{a}{6\pi}\frac{\sin\frac{6\pi x}{a}}{2}\right)\Bigg|_0^a = \frac{2}{a}\frac{a}{2} - 0 = 1$$

B.

$$P = \int_0^{\frac{a}{12}} \left|\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right|^2 \cdot dx = \frac{2}{a}\left(\frac{x}{2}-\frac{a}{6\pi}\frac{\sin\frac{6\pi x}{a}}{2}\right)\Bigg|_0^{\frac{a}{12}} = \frac{1}{12}-\frac{1}{6\pi}$$

3. Consider an infinite potential box, with boundaries at $x = 0$ and $x = a$:
$V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$.



As we have shown in class, in this potential the energy eigenstate can be written as $\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ with eigenvalues $E_n = \left(\frac{\hbar^2}{2m}\right)\frac{\pi^2}{a^2}n^2$ (you can use the notation E_n to simplify your answers) . Assume the wavefunction of a particle at $t = 0$ (probability already normalized to one) is:

$$\Psi(x,0) = \sqrt{\frac{4}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right) + \sqrt{\frac{1}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right) \quad 0 < x < a,$$

$$= 0 \quad x < 0, x > a$$

A. For a later time t, write down the wave function $\psi(x,t)$. There is no need to simplify the answer. (15)

B. At $x = \frac{a}{6}$, calculate the probability density $P\left(\frac{a}{6}, t\right)$ as a function of time t.

Hint: $|A+B|^2=(A^*+B^*)(A+B)=|A|^2+|B|^2+A^*B+B^*A$

解答：

A. $t=0$時此狀態可以視為定態$u_{1,2}$的如上疊加，接著定態隨時間個自演化，位能下薛丁格方程式要求u_n乘上$e^{-i\frac{E_n}{\hbar}t}$。乘完之後依同樣方式疊加，整個波函數也就滿足薛丁格波方程式。因此

$$\Psi(x,t)=\sqrt{\frac{4}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right)e^{-i\frac{E_1}{\hbar}t}+\sqrt{\frac{1}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right)e^{-i\frac{E_2}{\hbar}t}\text{。}$$

B. $$\Psi\left(\frac{a}{6},t\right)=\sqrt{\frac{4}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi}{6}\right)e^{-i\frac{E_1}{\hbar}t}+\sqrt{\frac{1}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi}{3}\right)e^{-i\frac{E_2}{\hbar}t}$$

$$=\sqrt{\frac{2}{5a}}\left(2\frac{1}{2}e^{-i\frac{E_1}{\hbar}t}+\frac{\sqrt{3}}{2}e^{-i\frac{E_2}{\hbar}t}\right)$$

$$P\left(\frac{a}{6},t\right)=\left|\Psi\left(\frac{a}{6},t\right)\right|^2=\frac{2}{5a}\left|e^{-i\frac{E_1}{\hbar}t}+\frac{\sqrt{3}}{2}e^{-i\frac{E_2}{\hbar}t}\right|^2$$

$$=\frac{2}{5a}\left\{\left|e^{-i\frac{E_1}{\hbar}t}\right|^2+\left|e^{-i\frac{E_2}{\hbar}t}\right|^2+e^{i\frac{E_1}{\hbar}t}e^{-i\frac{E_2}{\hbar}t}+e^{i\frac{E_2}{\hbar}t}e^{-i\frac{E_1}{\hbar}t}\right\}$$

$$=\frac{2}{5a}\left\{1+1+e^{-i\frac{(E_2-E_1)}{\hbar}t}+e^{i\frac{(E_2-E_1)}{\hbar}t}\right\}$$

$$=\frac{2}{5a}\left\{2+\cos\frac{(E_2-E_1)}{\hbar}t-i\sin\frac{(E_2-E_1)}{\hbar}t\right.$$

$$\left.+\cos\frac{(E_2-E_1)}{\hbar}t+i\sin\frac{(E_2-E_1)}{\hbar}t\right\}$$

$$=\frac{2}{5a}\left\{2+2\cos\frac{(E_2-E_1)}{\hbar}t\right\}$$

$$=\frac{2}{5a}\left\{2+2\cos\frac{\left(\frac{\hbar^2}{2m}\right)\frac{\pi^2}{a^2}3}{\hbar}t\right\}=\frac{2}{5a}\left\{2+2\cos\frac{3\pi^2\hbar}{2ma^2}t\right\}$$

4.

Consider an infinite box of unknown width. In transitions between neighboring values of n, photons of various energies are emitted. It is found that the largest wavelength of the various photons seen is 450×10^{-9} m. Use this information to determine a, the width of the infinite box.

Sol:

The longest wavelength corresponds to the lowest frequency. Since ΔE is proportional to $(n + 1)^2 - n^2 = 2n + 1$, the lowest value corresponds to $n = 1$ (a state with $n = 0$ does not exist). We therefore have

$$h\frac{c}{\lambda} = 3\frac{\hbar^2\pi^2}{2ma^2}$$

If we assume that we are dealing with electrons of mass $m = 0.9 \times 10^{-30}$ kg, then

$$a^2 = \frac{3\hbar\pi\lambda}{4mc} = \frac{3\pi(1.05\times10^{-34}\,J.s)(4.5\times10^{-7}\,m)}{4(0.9\times10^{-30}\,kg)(3\times10^{8}\,m/s)} = 4.1\times10^{-19}\,m^2$$

so that $a = 6.4 \times 10^{-10}$ m.