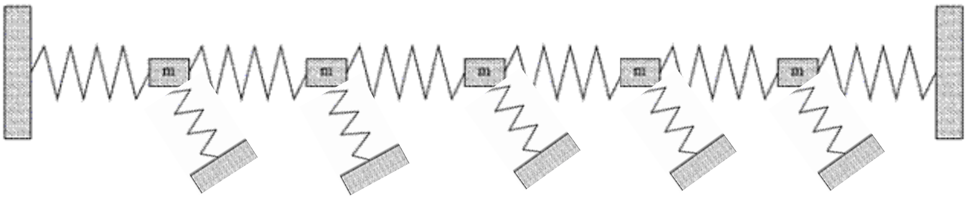
Homework VII

1. Consider the following wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} - V(x)\phi$$

Here we add a term $V(x)\phi$ to the right-hand side of the wave function we studied in class. $V(x) = v^2\alpha^2x^2$ is a known function. This could be done by adding a position dependent restoring force to the particles on the string.



It is reasonable to think that there exists solutions that are separable:

$$\phi(x,t) = X(x) \cdot T(t).$$

A. Find the Ordinary Differential Equation satisfied by $X(x)$ and $T(t)$. Show that $T(t)$ satisfies the ODE of a simple harmonic oscillator and the solution can be written as:

$$T(t) = a_m \cos(\omega t + \phi) \sim f\cos(\omega t) + g\sin(\omega t)$$

B. Check that $e^{-\alpha\frac{x^2}{2}}$ is a solution for $X(x)$. Write down the corresponding $T(t)$. What is the value of ω for this solution?

Sol:

A. Plug $\phi(x,t) = X(x) \cdot T(t)$ into $\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} - V(x)\phi$:

$$X\frac{\partial^2 T}{\partial t^2} = Tv^2\frac{\partial^2 X}{\partial x^2} - VXT$$

Divide the whole equation by v^2XT:

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} - \frac{V}{v^2} = \lambda$$

It can be equal to a constant λ. Therefore the equation for T:

$$\frac{d^2T}{dt^2} = v^2\lambda \cdot T \equiv -\omega^2 T$$

This is the same equation as a SHM for $\omega = v\sqrt{-\lambda}$. The solution is the same as a SHM:

$$T(t) = a_m \cos(\omega t + \phi) \sim f\cos(\omega t) + g\sin(\omega t)$$

The equation for X:

$$\frac{d^2X}{dx^2} - \frac{V}{v^2}X = \frac{d^2X}{dx^2} - \alpha^2 x^2 X = \lambda \cdot X$$

B. Plug $e^{-\alpha\frac{x^2}{2}}$ into

$$\frac{d^2X}{dx^2} - \alpha^2 x^2 X = \lambda \cdot X$$

$$-\alpha e^{-\alpha\frac{x^2}{2}} + (\alpha x)^2 e^{-\alpha\frac{x^2}{2}} - \alpha^2 x^2 e^{-\alpha\frac{x^2}{2}} = \lambda e^{-\alpha\frac{x^2}{2}}$$

If $\lambda = -\alpha$, $e^{-\alpha\frac{x^2}{2}}$ is a solution of X. The whole solution:

$$\phi(x,t) = X(x) \cdot T(t) = e^{-\alpha\frac{x^2}{2}}[f\cos(\omega t) + g\sin(\omega t)]$$

$$\omega = v\sqrt{\alpha}$$