Homework VI

1. Prove that $\vec{\nabla} r^p = p r^{p-1}\hat{r}$, as p is an integer. You can use this formula in the following problems.

   Sol: $\frac{\partial}{\partial x} r^p = \frac{\partial r}{\partial x}\frac{d}{dr} r^p = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} p r^{p-1} = p r^{p-1}\frac{x}{\sqrt{x^2+y^2+z^2}} = p r^{p-1}\hat{r}_x$

   The same applied to $\frac{\partial}{\partial y,z} r^p = p r^{p-1}\hat{r}_{y,z}$. Hence $\vec{\nabla} r^p = p r^{p-1}\hat{r}$.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

2. The electric potential of a point charge Q fixed at the origin can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

   Calculate $-\vec{\nabla}V$ and check it is the same as the electric field $\vec{E}$ of a point charge.

   Sol: $-\vec{\nabla}V = -\frac{Q}{4\pi\varepsilon_0}\vec{\nabla} r^{-1} = \frac{Q}{4\pi\varepsilon_0} r^{-2}\hat{r}$. We have used $\vec{\nabla} r^p = p r^{p-1}\hat{r}$ for $p = -1$.

3. The electric field $\vec{E}$ of a point charge Q fixed at the origin can be written as:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

   A. Calculate $\vec{\nabla}\times\vec{E}$.

   B. Calculate $\vec{\nabla}\cdot\vec{E}$ and show it is equal to zero except at the origin $r = 0$.

   Sol:

   A. $\vec{\nabla}\times\vec{E} = \frac{Q}{4\pi\varepsilon_0}\vec{\nabla}\times\frac{1}{r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right)\times\vec{r} + \frac{1}{r^3}\vec{\nabla}\times\vec{r}\right]$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-3r^{-4}\hat{r}\times\vec{r} + \frac{1}{r^3}\vec{\nabla}\times\vec{r}\right]$$

   The first term equals zero.

   The second term: $\vec{\nabla}\times\vec{r} = \left(\frac{\partial y}{\partial z} - \frac{\partial z}{\partial y}, \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}, \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right) = 0.$

$$\vec{\nabla}\times\vec{E} = 0$$

   B. $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0}\left[\left(\vec{\nabla}\frac{1}{r^3}\right)\cdot\vec{r} + \frac{1}{r^3}\vec{\nabla}\cdot\vec{r}\right]$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-3r^{-4}\hat{r}\cdot\vec{r} + \frac{1}{r^3}\vec{\nabla}\cdot\vec{r}\right] == \frac{Q}{4\pi\varepsilon_0}\left[\frac{-3}{r^3} + \frac{3}{r^3}\right] = 0$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

   $\vec{\nabla}\cdot\vec{E} = 0$ except for $r = 0$, where the two terms diverge $\to\infty$.

4. You would have found in problem 3 that $\vec{\nabla} \cdot \vec{E}$ is infinite at $r = 0$. This divergence can be avoided if we replace an infinitely small point charge with a small sphere of radius R and constant charge density ρ. We learned from general physics outside the sphere $\vec{E}$ is just like outside a point charge but inside the sphere the electric field equals:

$$\vec{E} = \frac{\rho}{3\varepsilon_0} r\hat{r}$$

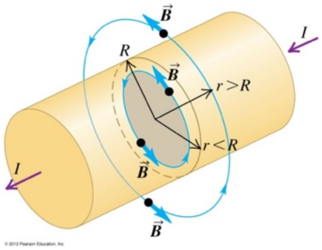
Calculate $\vec{\nabla} \cdot \vec{E}$ again and check it is consistent with Gauss's law.

Sol:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{\rho}{3\varepsilon_0} r\hat{r} = \frac{\rho}{3\varepsilon_0} \vec{\nabla} \cdot \vec{r} = \frac{\rho}{3\varepsilon_0}$$

$\vec{\nabla} \cdot \vec{E}$ is a costant for constant charge distribution.

5. The magnetic field inside a cylindrical current along the z axis with radius R and constant current density j can be written as $(r < R)$:



$$\vec{B} = \frac{\mu_0 j}{2}(y, -x, 0)$$

Calculate $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$ and check it is consistent with Maxwell Equations.

Sol:

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0 j}{2}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0 j}{2}\left(-\frac{\partial x}{\partial z}, -\frac{\partial y}{\partial z}, \frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}\right) = \frac{\mu_0 j}{2}(0,0,2) = (0,0,\mu_0 j)$$