Homework IV

1. Find the eigenvalues and eigenvectors (normalized to length one) of the following matrices:
   A. $\boldsymbol{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$
   B. $\boldsymbol{C} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$
   C. You can find the eigenvectors are the same as

$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

   which we discussed in detail in class. And checking the corresponding eigenvalues, we will find that $\boldsymbol{B} = \boldsymbol{A}^m$ and $\boldsymbol{C} = \boldsymbol{A}^n$**.** Find m, n.
   D. Find the matrix $\boldsymbol{U}$ that will diagonalize $\boldsymbol{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$:

$$\boldsymbol{U}^{-1} \cdot \boldsymbol{B} \cdot \boldsymbol{U} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   with $\lambda_{1,2}$ the two eigenvalues of $\boldsymbol{B}$.

Sol:

A. $\boldsymbol{B} = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}$, 特徵方程式：$\begin{vmatrix} 0.7-\lambda & 0.45 \\ 0.3 & 0.55-\lambda \end{vmatrix} = 0$，

$\lambda^2 - 1.25\lambda + 0.25 = 0$, 本徵值 $\lambda_1 = 1$，$\lambda_2 = 0.25 = \frac{1}{4}$，

$\lambda_1 = 1$，本徵向量滿足$-0.3a_1 + 0.45a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$

$\lambda_2 = 0.25$，本徵向量滿足$0.45a_1 + 0.45a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

B. $\boldsymbol{C} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$，特徵方程式：$\begin{vmatrix} 0.6-\lambda & 0.6 \\ 0.4 & 0.4-\lambda \end{vmatrix} = 0$

$\lambda^2 - 1.0\lambda = 0$, 本徵值 $\lambda_1 = 1$，$\lambda_2 = 0$，

$\lambda_1 = 1$，本徵向量滿足$-0.3a_1 + 0.45a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$

$\lambda_2 = 0$，本徵向量滿足$0.6a_1 + 0.6a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

C. $\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$的本徵值$\lambda_1 = 1$，$\lambda_2 = \frac{1}{2}$，

本徵向量：$\boldsymbol{a}_1 = c_1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \propto \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$，$\boldsymbol{a}_2 = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

可見$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$本徵向量相同，根據本徵值：$\boldsymbol{B} = \boldsymbol{A}^2, \boldsymbol{C} = \boldsymbol{A}^\infty$。

D. $\boldsymbol{U} \equiv (\boldsymbol{a}_1 \quad \boldsymbol{a}_2) = \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix}$

$$\boldsymbol{U}^{-1} \cdot \boldsymbol{B} \cdot \boldsymbol{U} = \begin{pmatrix} 1 & 1 \\ 0.4 & -0.6 \end{pmatrix} \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix} \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix} = \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 0.25 \end{pmatrix}$$

2. Find the eigenvalues of the following matrices: $\boldsymbol{A}$ and $\boldsymbol{B}$ and $\boldsymbol{AB}$ and $\boldsymbol{BA}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$$

Are the eigenvalues of $\boldsymbol{AB}$ equal the eigenvalues of $\boldsymbol{A}$ times the eigenvalue of $\boldsymbol{B}$?

Sol: $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$，特徵方程式：$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda + 2 = 0,$ 本徵值 $\lambda_1 = 1$，$\lambda_2 = 2$，

$\lambda_1 = 1$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$，本徵向量滿足$-a_1 + a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$，特徵方程式：$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\lambda^2 - 7\lambda + 12 = 0,$ 本徵值 $\lambda_1 = 3$，$\lambda_2 = 4$，

$\lambda_1 = 3$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 4$，本徵向量滿足$-a_1 + a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$$

$\boldsymbol{AB} = \boldsymbol{BA} = \begin{pmatrix} 3 & 5 \\ 0 & 8 \end{pmatrix}$特徵方程式：$\begin{vmatrix} 3-\lambda & 5 \\ 0 & 8-\lambda \end{vmatrix} = 0$

$\lambda^2 - 11\lambda + 24 = 0,$ 本徵值 $\lambda_1 = 3$，$\lambda_2 = 8$，

$\lambda_1 = 3$，本徵向量滿足$a_2 = 0$，$\boldsymbol{a}_1 \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$，注意：$3 = 1 \times 3$

$\lambda_2 = 8$，本徵向量滿足$-5a_1 + 5a_2 = 0$，$\boldsymbol{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$，$8 = 2 \times 4$

$\boldsymbol{AB} = \boldsymbol{BA}$與矩陣$\boldsymbol{B}$，與矩陣$\boldsymbol{A}$，具有相同的本徵向量，$\boldsymbol{AB} = \boldsymbol{BA}$的本徵值等於矩陣$\boldsymbol{A}$的本徵值乘矩陣$\boldsymbol{B}$的本徵值。

3. The electron spin corresponds to a 2×2 matrix. When the spin is along the direction $\hat{n} = (\sin\theta, 0, \cos\theta)$, the matrix is: $S_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Find the eigenvalues and eigenvectors (normalized to length one) of this matrix.

   Hint: Ignore the $\frac{\hbar}{2}$ first and do it for $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Multiplying the eigenvalues found by $\frac{\hbar}{2}$ would give you the eigenvalues of S_n. Eigenvectors are the same.

Sol: 特徵方程式：$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$，$\lambda^2 - 1 = 0$，$\lambda = \pm 1$

For $\lambda = 1$: $(\cos\theta - 1)a_1 + \sin\theta\, a_2 = 0$

$$\boldsymbol{a} = \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix} \frac{1}{\sqrt{2 - 2\cos\theta}} = \begin{pmatrix} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\sin\frac{\theta}{2} \end{pmatrix} \frac{1}{2\sin\frac{\theta}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

For $\lambda = -1$: $(\cos\theta + 1)a_1 + \sin\theta\, a_2 = 0$

$$\boldsymbol{a} = \begin{pmatrix} \cos\theta + 1 \\ -\sin\theta \end{pmatrix} \frac{1}{\sqrt{2 + 2\cos\theta}} = \begin{pmatrix} 2\cos\frac{\theta}{2}\cos\frac{\theta}{2} \\ -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{pmatrix} \frac{1}{2\cos\frac{\theta}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}$$