Homework III

1. First consider the homogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 3x = 0$$

   A. Find the general solutions x_{h} of this ODE.

   B. Consider the inhomogeneous ODE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 3x = 12e^{-2t}$$

   Find one solution. Hint: $x \propto e^{-2t}$ would work.

   C. Find the general solutions x_{inh} of the inhomogeneous ODE.

   D. Assume the initial condition $x(0) = 0, \frac{dx}{dt}(0) = 0$, find the solution $x(t)$.

Sol:

A. 取$x = ce^{\alpha t}$，代入微分方程式，得到α的代數方程式：$\alpha^2 + 3\alpha + 3 = 0$：

求解：$\alpha = \frac{-3\pm i\sqrt{3}}{2}$，取$e^{\alpha t}$的實數部與虛數部，就能得到兩個實數解，取線性組合，就得到一般解：

$$x_{\mathrm{h}} = c_1 e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{3}}{2}t$$

B. 取$x = de^{-2t}$，代入微分方程式：$(4-6+3)de^{-2t} = 12e^{-2t}$，$d = 12$

得到一個特解：$x = 12e^{-2t}$

C. 一般解：

$$x_{inh} = c_1 e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3t/2}\sin\frac{\sqrt{3}}{2}t + 12e^{-2t}$$

D. 代入起始條件$x(0) = c_1 + 12 = 0$，$c_1 = -12$。

$\frac{dx}{dt}(0) = -\frac{3}{2}c_1 + \frac{\sqrt{3}}{2}c_2 - 24 = 0$，$c_2 = \frac{12}{\sqrt{3}} = 4\sqrt{3}$。

$$x_{inh} = -12e^{-3t/2}\cos\frac{\sqrt{3}}{2}t + 4\sqrt{3}e^{-3t/2}\sin\frac{\sqrt{3}}{2}t + 12e^{-2t}$$

2.

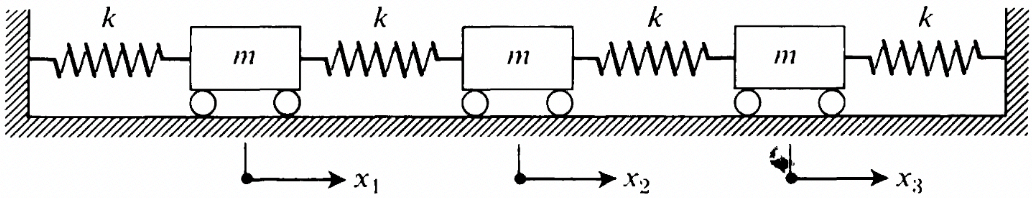

Figure 10.13

Consider the problem of three statically coupled masses (Fig. 10.13) analogous to those in Section 10.1, Example 1. Let x_1, x_2, and x_3 be their displacements from equilibrium positions.

a) Set up the equations of motion on the basis of Newton's second law.

b) Evaluate the potential energy of the system and show that it reduces to

$$V = kx_1^2 + kx_2^2 + kx_3^2 - kx_1x_2 - kx_2x_3.$$

d) Show that the characteristic frequencies can be obtained by solving

$$\det \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

e) Find the characteristic frequencies and the normal modes.

Sol:

A. 運動方程式：

$$m\frac{d^2x_1}{dt^2} = k(x_2 - x_1) - kx_1 = -2kx_1 + kx_2$$

$$m\frac{d^2x_2}{dt^2} = k(x_3 - x_2) - k(x_2 - x_1) = -2kx_2 + kx_3 + kx_1$$

$$m\frac{d^2x_3}{dt^2} = -k(x_3 - x_2) - kx_3 = -2kx_3 + kx_2$$

B. 計算彈簧的彈力位能：

$$V = \frac{k}{2}[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2] = k(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

C. 若以矩陣語言來書寫：

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \boldsymbol{A} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

取$\boldsymbol{x} \equiv \boldsymbol{a}e^{i\omega t}$，代入得本徵值方程式：$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} = \lambda\omega_0^2\boldsymbol{a}$。特徵方程式為以下行列式為零：

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

得代數方程式：

$$(2-\lambda)^3 - 2(2-\lambda) = (2-\lambda)(2-4\lambda+\lambda^2) = 0$$

解得本徵值：

$$\lambda = 2, 2\pm\sqrt{2}, \omega^2 = 2\omega_0^2, \left(2\pm\sqrt{2}\right)\omega_0^2$$

D. 計算本徵向量：

For $\lambda = 2$: 解聯立方程組：$(\boldsymbol{A} - \lambda\omega_0^2)\cdot\boldsymbol{a} = 0$

可以寫成：

$$-a_2 = 0$$
$$-a_1 - a_3 = 0$$
$$-a_2 = 0$$

可以得：$\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

For $\lambda = 2+\sqrt{2}$:

$$\sqrt{2}a_1 + a_2 = 0$$
$$a_1 + \sqrt{2}a_2 + a_3 = 0$$
$$a_2 + \sqrt{2}a_3 = 0$$

$\boldsymbol{a} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

For $\lambda = 2-\sqrt{2}$:

$$\sqrt{2}a_1 - a_2 = 0$$
$$-a_1 + \sqrt{2}a_2 - a_3 = 0$$
$$-a_2 + \sqrt{2}a_3 = 0$$

$\boldsymbol{a} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$