Homework II

1. This is a similar problem to Problem 1 of Homework I. Again put a potato into an oven. The temperature of the potato at $t = 0$ is T_0 and it will heat up. Denote the temperature at t as $T(t)$. But this time the temperature of the oven **is not a constant** but $Q(t) = Q_0 e^{1.1 \cdot ht}$ (for positive h). $Q_0 > T_0$. Likewise, we can write down an equation for the change of potato temperature:

$$\frac{dT}{dt} = -h[T(t) - Q(t)]$$

Solve $T(t)$.

Hint: Use the method of integrating factor.

Sol: The integrating factor would be $\alpha(t) = \exp[-\int h dt] = \exp(-ht)$.

Multiply the whole equation by the integrating factor $\alpha(x)$:

$$\frac{d}{dt}(e^{-ht}T) = hQ_0 e^{1.1 \cdot ht} e^{-ht} = hQ_0 e^{0.1 \cdot ht}$$

Integrate and add a constant C:

$$e^{-ht}T = Q_0 \frac{e^{1.1 \cdot ht}}{0.1} + C$$

$$T = Q_0 \frac{e^{2.1 \cdot ht}}{0.1} + Ce^{ht}$$

$$T = Q_0 \frac{e^{2.1 \cdot ht}}{0.1} + \left(T_0 - \frac{Q_0}{0.1}\right) e^{ht}$$

2. Find the general solution $x(t)$ of the equation of

$$x' + \frac{2}{t}x = t$$

Hint: Use the method of integrating factor.

Sol: The integrating factor would be $\alpha(t) = \exp\left[\int \frac{2}{t} dt\right] = \text{ex } 2\ln t = t^2$.

Multiply the whole equation by the integrating factor $\alpha(x)$:

$$t^2 x' + 2tx = \frac{d}{dt}(t^2 x) = t^3$$

Integrate and add a constant C:

$$t^2 x = \int dt \cdot t^3 + C = \frac{t^4}{4} + C$$

$$x = \frac{t^2}{4} + \frac{C}{t^2}$$

3. Arfken Exercise 7.6.16 (p372)

   Hint: Use the method of Wronskian.

   Sol:

$$y_2 \sim r^m \int dr \frac{\exp\left(-\int \frac{1}{r'} \cdot dr'\right)}{r^{2m}} = r^m \int dr \frac{\frac{1}{r}}{r^{2m}} = -r^m \frac{r^{-2m}}{2m} \sim r^{-m}$$

   Note that constants do not matter.

4. Arfken Exercise 7.6.19 (p373)

   a. $y_2 \sim 1 \int dx \frac{\exp(-\int 2x' \cdot dx')}{1} = \int dx\, e^{-x^2} = \int dx \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \right)$

$$= x - \frac{2}{3!}x^3 + \frac{4}{5!}x^5 + \cdots$$

   b. $y_2 \sim x \int dx \frac{\exp(-\int 2x' \cdot dx')}{x^2} = x \int dx \frac{e^{-x^2}}{x^2} =$

$$x \int dx \left(\frac{1}{x^2} - 1 + \frac{1}{2!}x^2 - \frac{1}{3!}x^4 + \frac{1}{4!}x^8 + \cdots\right) = 1 - x^2 + \frac{4}{4!}x^4 + \cdots$$

5. Find the general solution $x(t)$ of the equation of

$$x'' + 2x' + 5x = 0$$

   using the damp oscillation formula we gave in class.

   Sol: Guess the solution is :$z = z_0 e^{\alpha t}$ and plug it into the equation:

$$\alpha^2 e^{\alpha t} + 2\alpha e^{\alpha t} + 5 e^{\alpha t} = 0$$

   The unknown α satisfies the algebraic equation:

$$\alpha^2 + 2\alpha + 5 = 0$$

   There are two real solutions:

$$\alpha = \frac{-2 \pm 4i}{2}$$

   For $\alpha = \frac{-2+4i}{2}$, give us one complex solution:

$$z = z_0 e^{-t} e^{4it} = z_0 e^{-t} \cos 4t + i z_0 e^{-t} \sin 4t$$

The real part and the imaginary part are two real solutions. Note that $\alpha = \frac{-2-4i}{2}$ gives us the same real solutions.

Hence The general solutions are their linear combination:

$$x = c_1 e^{-t} \cos 4t + c_2 e^{-t} \sin 4t$$

6. Solve the equation of

$$x'' + 4x' + 3x = 0$$

And initial conditions: $x(0) = 1, x'(0) = 0$. (After class on Sep. 23)

Sol: Guess the solution is :$z = z_0 e^{\alpha t}$ and plug it into the equation:

$$\alpha^2 e^{\alpha t} + 4\alpha e^{\alpha t} + 3e^{\alpha t} = 0$$

The unknown α satisfies the algebraic equation:

$$\alpha^2 + 4\alpha + 3 = 0$$

There are two real solutions:

$$\alpha = -1, -3$$

We get two solutions and the general solutions are their linear combination:

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

Put in the initial conditions: $x(0) = 1, x'(0) = 0$:

$$c_1 + c_2 = 1, c_1 + 3c_2 = 0$$

$$x = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t}$$