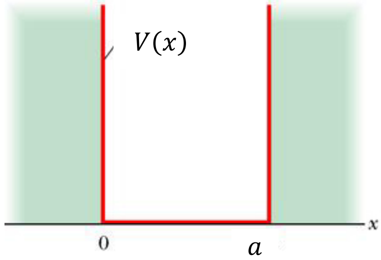
習題十二

1. Consider an infinite potential box, with boundaries at $x = 0$ and $x = a$:
$V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$.



As we have shown in class, in this potential the energy eigenstate can be written as
$u_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ with eigenvalues $E_n = \left(\frac{\hbar^2}{2m}\right)\frac{\pi^2}{a^2}n^2$.

Calculate the expectation value $\langle x^2\rangle$ and uncertainty Δx for the ground state $n = 1$.

Hint:

$$\int_0^\pi du\ u^2\cos\alpha u = -\frac{d^2}{d\alpha^2}\int_0^\pi du\ \cos\alpha u = -\frac{d^2}{d\alpha^2}\frac{\sin\alpha\pi}{\alpha} =$$

$$\frac{(\alpha^2\pi^2 - 2)\sin\alpha\pi}{\alpha^3} + \frac{2\pi\cos\alpha\pi}{\alpha^2}$$

解答：$\langle x^2\rangle = \frac{2}{a}\int_0^a dx\ x^2\sin^2\frac{n\pi x}{a} = \frac{2}{a}\int_0^a dx\ x^2\left(\frac{1-\cos\frac{2n\pi x}{a}}{2}\right)$

Define $\frac{\pi x}{a} = u, x = \frac{a}{\pi}u$

$$\langle x^2\rangle = \frac{2a^2}{\pi^3}\int_0^\pi du\ u^2\left(\frac{1-\cos 2nu}{2}\right) = \frac{a^2}{3} - \frac{a^2}{\pi^3}\int_0^\pi du\ u^2\cos 2nu$$

$$\int_0^\pi du\ u^2\cos\alpha u = -\frac{d^2}{d\alpha^2}\frac{\sin\alpha\pi}{\alpha} = \frac{d}{d\alpha}\left(\frac{\sin\alpha\pi}{\alpha^2} - \frac{\pi\cos\alpha\pi}{\alpha}\right)$$

$$= \frac{\alpha^2\pi\cos\alpha\pi - 2\alpha\sin\alpha\pi}{\alpha^4} - \frac{-\alpha\pi^2\sin\alpha\pi - \pi\cos\alpha\pi}{\alpha^2} \xrightarrow[\alpha=2n]{} \frac{2\pi}{\alpha^2} = \frac{\pi}{2n^2}$$

$$\langle x^2\rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}} = \sqrt{\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}}$$

Here we use $\langle x\rangle = \frac{a}{2}$ since the potential is symmetric about $x = \frac{a}{2}$.

2. Legendre Equation is written as

$$(1 - z^2)\frac{d^2P(z)}{dz^2} - 2z\frac{dP(z)}{dz} + l(l+1)P(z) = 0$$

The solution of this equation for positive integer l are called Legendre Polynomial and is of degree l. Consider the case $l = 4$. $P(z) = a_0 + a_2z^2 + a_4z^4$. There are no odd order terms. Plug the expression into the equation, assume that $a_0 = 1$, find the Polynomial $P(z) = a_0 + a_2z^2 + a_4z^4$.

Sol:

$$\frac{d^2P}{dz^2} = 2a_2 + 12a_4z^2$$

$$(1 - z^2)\frac{d^2P(z)}{dz^2} = 2a_2 + [12a_4 - 2a_2]z^2 - 12a_4z^4$$

$$-2z\frac{dF}{dz} = -4a_2z^2 - 8a_4z^4$$

$$l(l+1)P(z) = 20a_0 + 20a_2z^2 + 20a_4z^4$$

Equate the coefficients: $a_2 = 10$, $12a_4 - 2a_2 - 4a_2 + 20a_2 = 0$

$$a_4 = \frac{35}{3}$$

$$P(z) = 1 + 10z^2 + \frac{35}{3}z^4$$