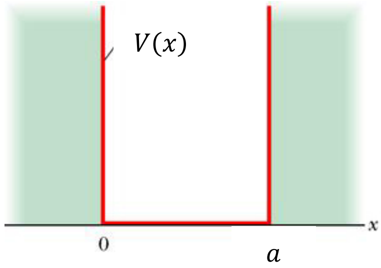
習題十

1. Consider an infinite potential as discussed in class, with boundaries at $x = 0$ and $x = a$: $V(x) = \infty, x > a, x < 0$ and $V(x) = 0, 0 < x < a$.



A particle is known to be localized in middle of the box, with the instant wavefunction as:

$$\psi(x) = \sqrt{\frac{3}{a}} \quad \frac{a}{3} < x < \frac{2a}{3},$$

$$= 0 \quad x < \frac{a}{3}, x > \frac{2a}{3}$$

We expect $\psi(x)$ could be expanded by u_n as $\psi(x) = \sum C_n u_n$. Calculate C_1, C_2, C_3.

Hint: $a_n = \int_0^a dx \cdot u_n^*(x)\psi(x) = \int_0^a dx \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)\psi(x)$

2. Consider an electron in a simple harmonic potential $V(x) = \frac{1}{2}kx^2$. The stationary states $u_n(x)$ are the eigenfunctions corresponding to the eigenvalues $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. For example:

$$u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} \equiv c e^{-\frac{m\omega x^2}{2\hbar}}, \qquad u_2(x) = c\frac{1}{\sqrt{2}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega x^2}{2\hbar}}$$

Here we use the notation $c = \sqrt[4]{\frac{m\omega}{\pi\hbar}}$ to simplify the expression.

A. Prove that $\int_{-\infty}^{\infty} dx \cdot u_0(x)^* \cdot u_0(x) = 1$.

Hint: $\int_{-\infty}^{\infty} dx \cdot e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

Assume the wavefunction of the electron at $t = 0$ is:

$$\Psi(x, 0) = \frac{1}{\sqrt{3}}u_0(x) + \frac{\sqrt{2}}{\sqrt{3}}u_2(x)$$

Its total probability has been normalized to one (You can check this after exam).

B. How many node points 節點 does $u_2(x)$ have? (5)

C. For a later time $t = t_0 > 0$, write down the wave function $\psi(x, t_0)$. There is no need to simplify the answer. Calculate the probability density at the origin $x = 0$ when $t = t_0$, in terms of $\hbar, \omega, c$. (10)

Hint: $|A + B|^2 = (A^* + B^*)(A + B) = |A|^2 + |B|^2 + A^*B + B^*A$