Homework I

1. Put a potato into an oven. The temperature of the potato at $t = 0$ is T_0 and it will heat up. Denote the temperature at t as $T(t)$. The temperature of the oven is a constant T_e. Newton has a law stating that the heat transfer from oven to potato (proportional to the change of $T(t)$) per unit time is proportional to the temperature difference $T_e - T(t)$. Therefore, we can write down an equation (for positive h):

$$\frac{dT}{dt} = -h[T(t) - T_e]$$

Solve $T(t)$.

Sol: Collect factors of T on the left hand side and factors of t on the right hand side:

$$\frac{dT}{T_e - T} = hdt$$

Keep the denominator positive. Integrate and add a constant C:

$$\int \frac{dT}{T_e - T} = \int h \cdot dt + C$$

$$\ln[T_e - T] = -ht - C$$

$$T = T_e - e^{-ht}e^{-C}$$

The constant C can be obtained by the initial condition: $T(0) = T_0$: $e^{-C} = T_e - T_0$.

$$T = T_e - (T_e - T_0)e^{-ht}$$

2. Arfken Exercise 7.2.3

Sol:

$$\frac{dN}{dt} = -kN^2$$

Collect factors of N on the left hand side and factors of t on the right hand side:

$$\frac{dN}{N^2} = -kdt$$

Integrate and add a constant C:

$$\int \frac{dN}{N^2} = -\int k \cdot dt + C$$

$$-\frac{1}{N} = -kt + C$$

$$N = \frac{1}{kt - C}$$

The constant C can be obtained by the initial condition: $N(0) = N_0$

$$C = -\frac{1}{N_0}$$

$$N = \frac{1}{kt + \frac{1}{N_0}} = N_0 \frac{1}{1 + \frac{t}{kN_0}} = N_0 \frac{1}{1 + \frac{t}{\tau_0}}$$

3. The function $x(t)$ satisfies

$$\frac{dx}{dt} = 2tx^2, \quad x(0) = 1$$

Solve $x(t = 2)$.

Sol: Collect factors of x on the left hand side and factors of t on the right hand side:

$$\frac{dx}{x^2} = 2t \cdot dt$$

Integrate and add a constant C:

$$\int \frac{dx}{x^2} = \int 2t \cdot dt + C$$

$$-\frac{1}{x} = t^2 + C$$

$$x = -\frac{1}{t^2 + C}$$

The constant C can be obtained by the initial condition $x(0) = 1$: $C = -1$.

$$x = -\frac{1}{t^2 - 1}$$

4. Solve the equation of $x(t)$

$$x' + \frac{1}{t}x = 1, \quad x(1) = 3$$

Sol: The integrating factor would be $\alpha(t) = \exp\left[\int \frac{1}{t} dt\right] = \exp \ln t = t$.

Multiply the whole equation by the integrating factor $\alpha(x)$:

$$tx' + x = \frac{d}{dt}(tx) = t$$

Integrate and add a constant C:

$$tx = \int dt \cdot t + C = \frac{t^2}{2} + C$$

$$x = \frac{t}{2} + \frac{C}{t}$$

The constant C can be obtained by the initial condition $x(1) = 3$: $C = \frac{5}{2}$.

$$x = \frac{t}{2} + \frac{5}{2t}$$