數物期末考

Dec 2025

1. We mentioned in class that through the diagonalization of a matrix, we can represent a matrix by its eigenvectors and eigenvalues:

$$\boldsymbol{A} = \boldsymbol{U\Lambda U}^{-1} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^{-1}$$

$$\boldsymbol{U} \equiv \begin{pmatrix} \boldsymbol{a}^{(1)} & \boldsymbol{a}^{(2)} \end{pmatrix}$$

Given any one column vector $\boldsymbol{a}$ and any two numbers λ_1, λ_2 we can construct a matrix which has $\boldsymbol{a}$ as one of the eigenvectors and λ_1, λ_2 as eigenvalues.

Let's start with $\boldsymbol{a}^{(1)} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. (25)

A. For a symmetric matrix, eigenvectors are orthogonal to each other. Use this property to find a normalized $\boldsymbol{a}^{(2)}$ such that $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(1)} = 0, \boldsymbol{a}^{T(2)}\boldsymbol{a}^{(2)} = 1$.

B. Construct $\boldsymbol{U}$ and calculate $\boldsymbol{U}^{-1}$ using $\boldsymbol{U}^{-1} = \frac{1}{\det \boldsymbol{U}}\begin{pmatrix} \boldsymbol{U}_{22} & -\boldsymbol{U}_{12} \\ -\boldsymbol{U}_{21} & \boldsymbol{U}_{11} \end{pmatrix}$.

C. Choose the two eigenvalues as $\lambda_1 = 10, \lambda_2 = 20$. Find the matrix $\boldsymbol{A}$.

D. Check indeed $\boldsymbol{A} \cdot \boldsymbol{a}^{(1)} = \lambda_1 \boldsymbol{a}^{(1)}$.

Sol:

A. Assume that $\boldsymbol{a}^{(2)} = \begin{pmatrix} b \\ c \end{pmatrix}$. Since $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(1)} = 0,\ b - 2c = 0.$ Hence $\boldsymbol{a}^{(2)} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Using $\boldsymbol{a}^{T(2)}\boldsymbol{a}^{(2)} = 1, \boldsymbol{a}^{(2)} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

B. $\boldsymbol{U} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \boldsymbol{U}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

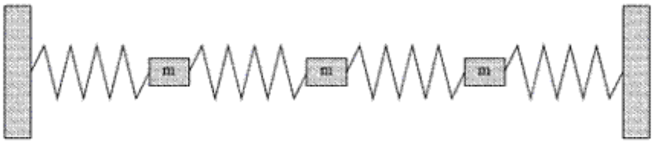
C. $\boldsymbol{A} = \boldsymbol{U\Lambda U}^{-1} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{U}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$$\frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & -20 \\ 40 & 20 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 90 & 20 \\ 20 & 60 \end{pmatrix} = \begin{pmatrix} 18 & 4 \\ 4 & 12 \end{pmatrix}$$

D. $\begin{pmatrix} 18 & 4 \\ 4 & 12 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 10 \\ -20 \end{pmatrix} = 10 \times \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2. Consider the spring system of $N = 3$ coupled masses. Assume that the spring constants are all equal to k and the masses are all equal to m. $\omega_0 = \sqrt{\frac{k}{m}}$.

If we move the first particle to the right by a and let go as all three particles are at rest, that is, the initial condition is $x_1(0) = a, x_2(0) = x_3(0) = 0, x_1'(0) = x_2'(0) = x_3'(0) = 0$, solve the $x_1(t)$. Write the solution in terms of a, ω_0.



有了以上三個模式的解，一般解將是三者的線性組合：

$\omega_1 \sim \sqrt{2-\sqrt{2}\omega_0}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t) + g_i \sin(\omega_i t)]$$

$\omega_2 \sim \sqrt{2}\omega_0$

$\omega_3 \sim \sqrt{2+\sqrt{2}\omega_0}$

三個未定常數$f_{1,2,3}$,將由三個起使條件$\boldsymbol{x}_{1,2,3}(0)$ 決定：

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} f_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

根據對稱矩陣本徵向量的展開定理，三個本徵向量彼此正交： $\boldsymbol{a}^{(m)T}\boldsymbol{a}^{(n)} = \delta_{mn}$

可以如上選$\boldsymbol{a}^{(n)}$使其向量長度為1。

任一起始條件的行向量$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$，都可以展開成三個本徵向量的線性組合。

$f_{1,2,3}$可以解出： $\boldsymbol{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$

另外三個未定常數 $g_{1,2,3}$則由另外三個起使條件$\boldsymbol{x}'_{1,2,3}(0)$ 決定。

Sol:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} f_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} f_3$$

$$\boldsymbol{a}^{(i)T} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = f_i$$

$$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_1 = \frac{a}{2}$$

$$\frac{1}{\sqrt{2}}(1,0,-1) \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_2 = \frac{1}{\sqrt{2}}a$$

$$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right)\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = f_3 = \frac{a}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i=1}^{3} \boldsymbol{a}^{(i)} \cdot [f_i \cos(\omega_i t)]$$

$$= \frac{a}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\cos(\omega_1 t) + a\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\cos(\omega_2 t) + \frac{a}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\cos(\omega_3 t)$$

$$x_1 = \frac{a}{2}\left(\frac{1}{2}\cos\sqrt{2-\sqrt{2}}\omega_0 t + \cos\sqrt{2}\omega_0 t + \frac{1}{2}\cos\sqrt{2+\sqrt{2}}\omega_0 t\right)$$

3. The Maxwell Equation in vacuum can be written as:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Consider a plane wave propagating in the x direction. The electric field of the wave is along the y direction and both the electric field and magnetic field are independent of coordinates y, z. The electric field can be written as:

$$\vec{E} = \left(0, E_y(x,t), 0\right) = (0, E_0 \cos(kx - \omega t), 0)$$

A. Calculate $\vec{\nabla} \times \vec{E}$ and $\frac{\partial \vec{E}}{\partial t}$.

B. Use Maxwell Equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ to find the magnetic field $\vec{B}(x,t)$ (write it in three component notation).

Hint: Only one component is nonzero.

Comment: If you further plug the result into the other Maxwell Equation $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, you would find that $\frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ as expected.

Sol:

A.

$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = \left(\frac{\partial E_y}{\partial z}, 0, -\frac{\partial E_y}{\partial x}\right)$$

$$= \left(0, 0, -\frac{\partial E_y}{\partial x}\right) = (0, 0, kE_0 \sin(kx - \omega t))$$

$$\frac{\partial \vec{E}}{\partial t} = (0, -\omega E_0 \sin(kx - \omega t), 0)$$

B. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = (0,0, kE_0 \sin(kx - \omega t))$

$$\frac{\partial B_z}{\partial t} = -kE_0 \sin(kx - \omega t)$$

$$B_z(x,t) = \frac{k}{\omega} E_0 \cos(kx - \omega t)$$

4. One of the first Schrodinger Wave Equations to be written down and solved is that of an electron in the simple harmonic potential $V(x) = \frac{\hbar^2}{2m}\alpha^2 x^2$:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$$

α is actually $\frac{m\omega}{\hbar} = \frac{\sqrt{mk}}{\hbar}$. But calculation would be easier to keep α as above. $\hbar$ is the Planck Constant h divided by 2π. Here $\Psi(x,t)$ is the famous wave function. As you can see from the equation, the left-hand side contains an i and hence the wavefunction must be complex valued. It is reasonable to think that there exist solutions that are separable just like in the classical wave equation we discussed in class:

$$\Psi(x,t) = \psi(x) \cdot \phi(t).$$

A. Find the Ordinary Differential Equation satisfied by $\psi(x)$ and $\phi(t)$. Show that:

$$\phi(t) = e^{-i\frac{E}{\hbar}t}$$

is the solution for $\phi(t)$. E is a constant to be determined.

B. $\psi(x)$ would have solutions for a discreet set of infinite number of positive values of E. Prove that $e^{-\alpha\frac{x^2}{2}}$ is a solution for $\psi(x)$. Write down the corresponding E (this is the smallest E and Ecorresponds to energy. All the other solutions have larger E and $e^{-\alpha\frac{x^2}{2}}$ is the ground state) and $\phi(t)$ in terms of $\alpha, m, \hbar$.

Sol:

A. 將$\Psi(x,t) = \psi(x) \cdot \phi(t)$代入波方程式：

$$-\frac{\hbar^2}{2m}\phi(t)\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\phi(t) = i\hbar\psi(x)\frac{d\phi(t)}{dt}$$

左右都除以 $\psi(x) \cdot \phi(t)$ ：

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)\right]=i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}$$

唯一可能是左右兩式與兩個變數都無關，是一常數。設為E。

在左邊只與x有關，右邊只與t有關，兩者是獨立變數！

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)\right]=i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}\equiv E$$

$$i\hbar\frac{d\phi(t)}{dt}=E\phi(t)$$

$$\phi(t)=e^{-i\frac{E}{\hbar}t}$$

B. 空間部分$\psi(x)$滿足：

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)\right]=E$$

也就是：

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{\hbar^2}{2m}\alpha^2x^2\psi(x)=E\psi(x)$$

代入$\psi(x)=e^{-\alpha\frac{x^2}{2}}$

$$-\frac{d^2\psi}{dx^2}+\alpha^2x^2\psi=\frac{2m}{\hbar^2}E\cdot\psi$$

$$\alpha e^{-\alpha\frac{x^2}{2}}-(\alpha x)^2e^{-\alpha\frac{x^2}{2}}+\alpha^2x^2e^{-\alpha\frac{x^2}{2}}=\alpha e^{-\alpha\frac{x^2}{2}}=\frac{2m}{\hbar^2}E\cdot\psi$$

If $E=\frac{\hbar^2}{2m}\alpha$, $e^{-\alpha\frac{x^2}{2}}$ is a solution. The whole solution:

$$\Psi(x,t)=\psi(x)\cdot\phi(t)=e^{-\alpha\frac{x^2}{2}}e^{-i\frac{E}{\hbar}t}$$

$$E=\frac{\hbar^2}{2m}\alpha$$