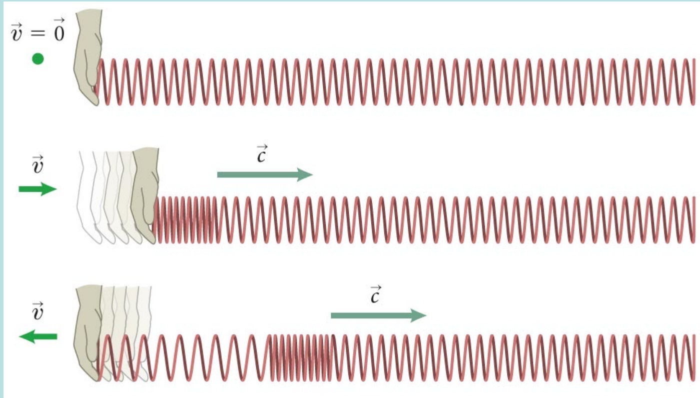
連續介質的波方程式Wave Equation

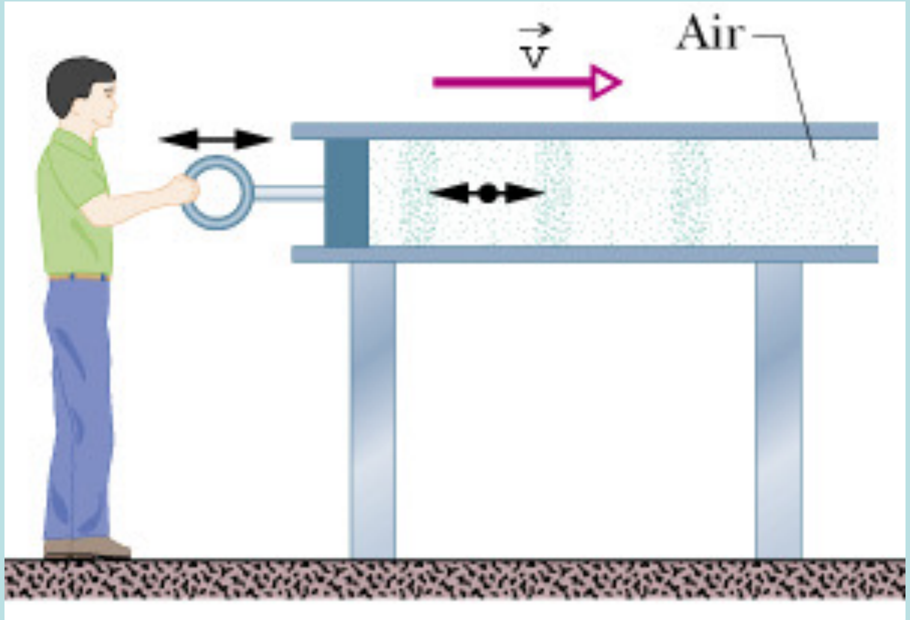
$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

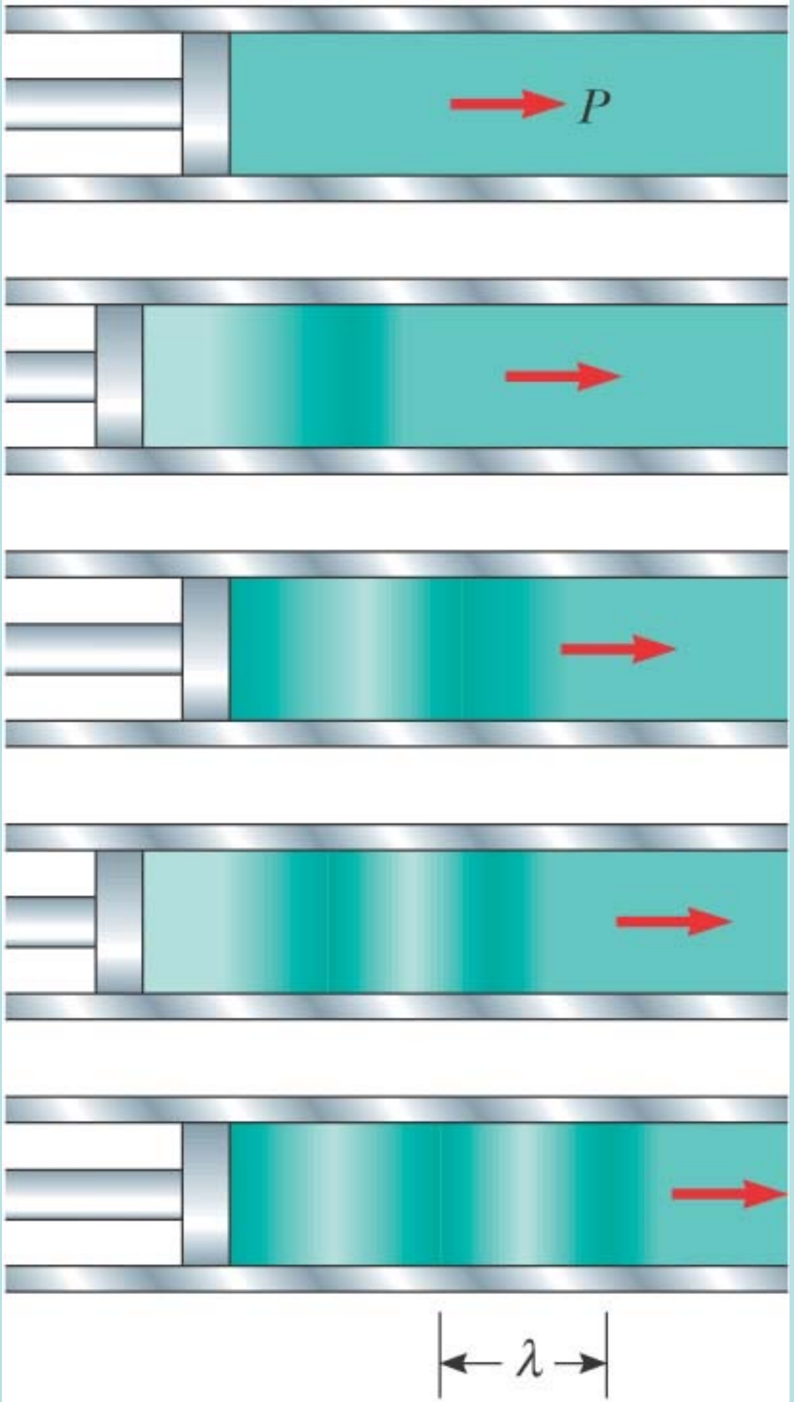


Initial Condition: 起始的弦位移$\phi(x,0)$，起始的弦垂直方向速度$\frac{\partial \phi}{\partial t}(x,0)$。

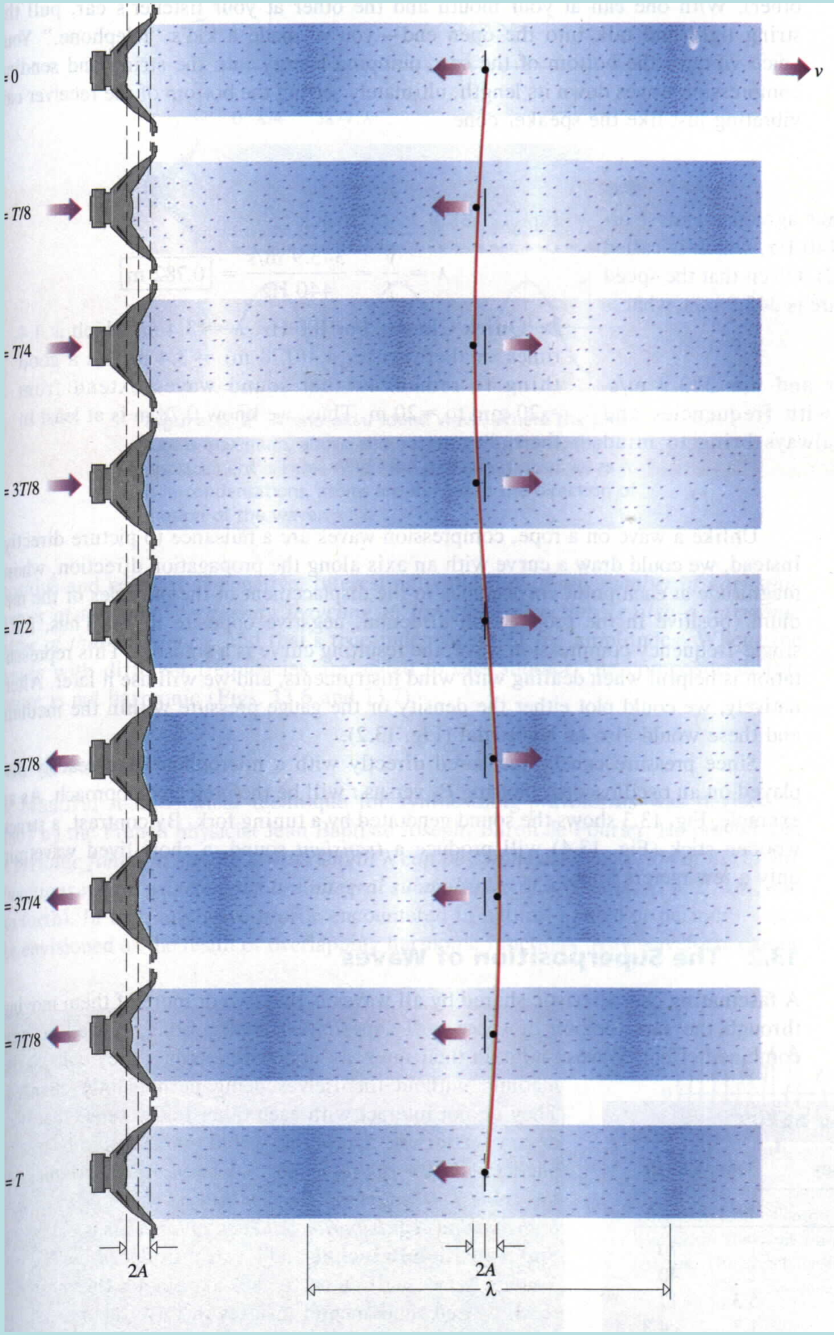
比較抽象的例子：聲波。

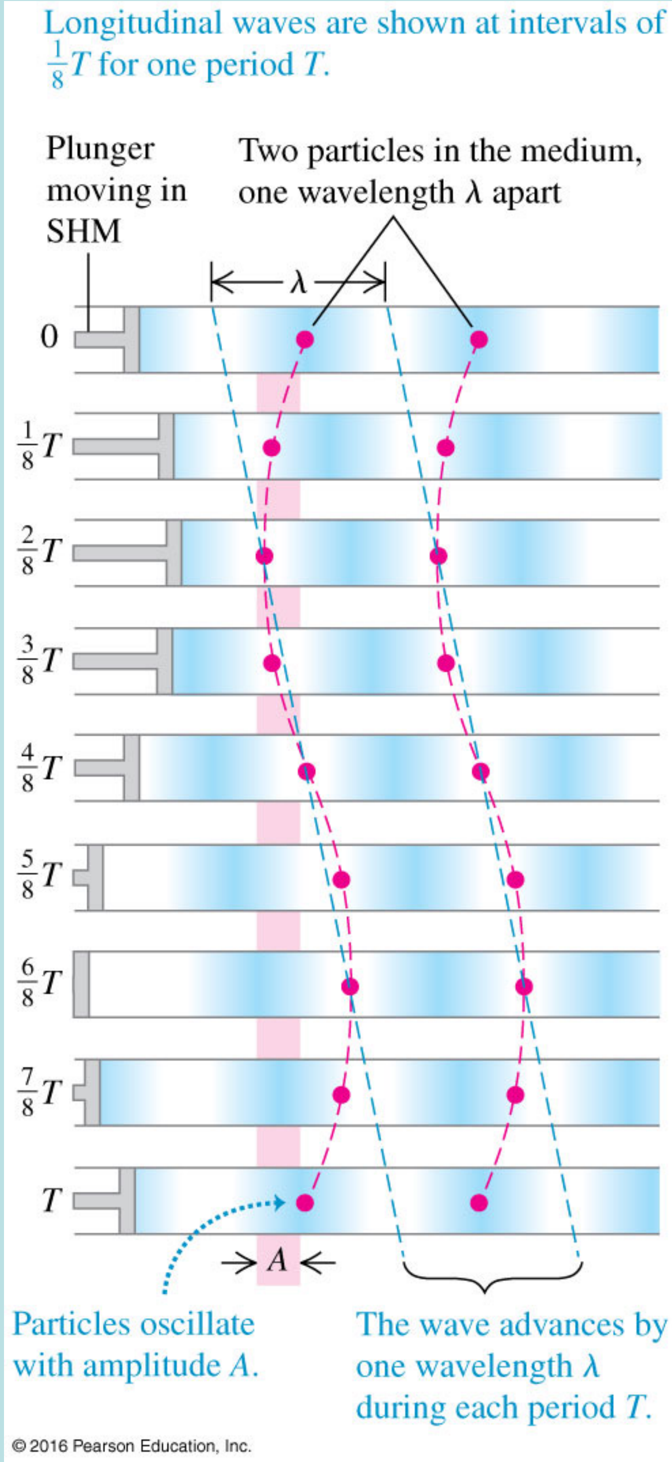
空氣柱中的聲波





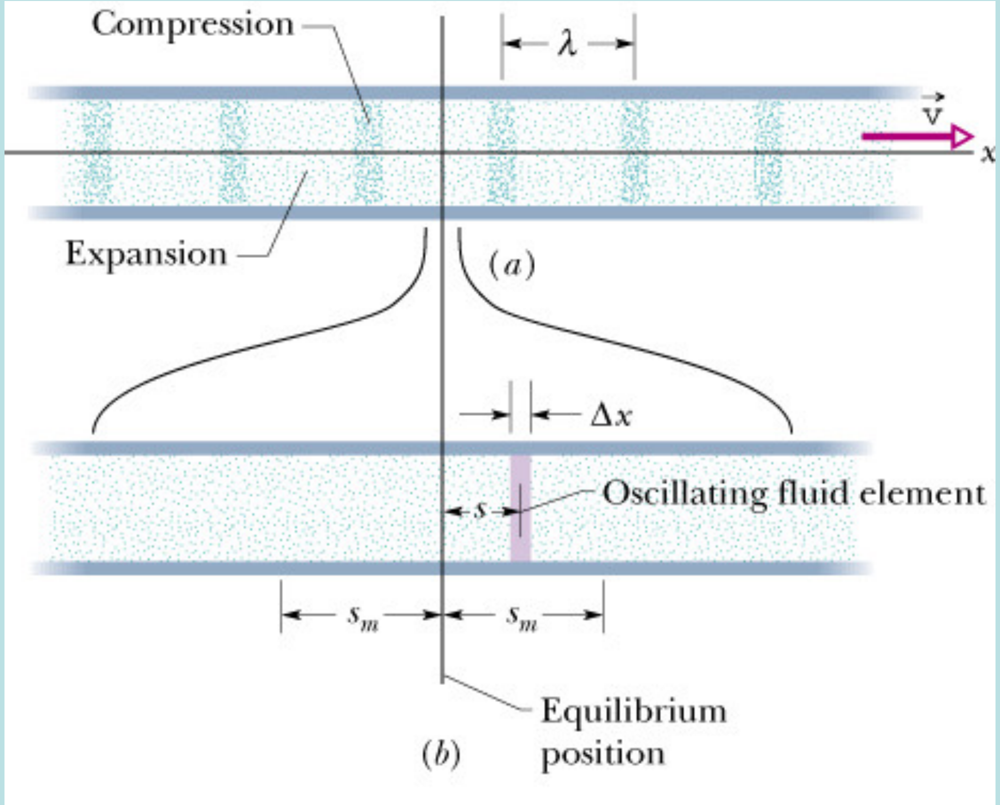
P
λ





縱波：介質粒子的震動方向與波的方向一致。

聲波的波函數



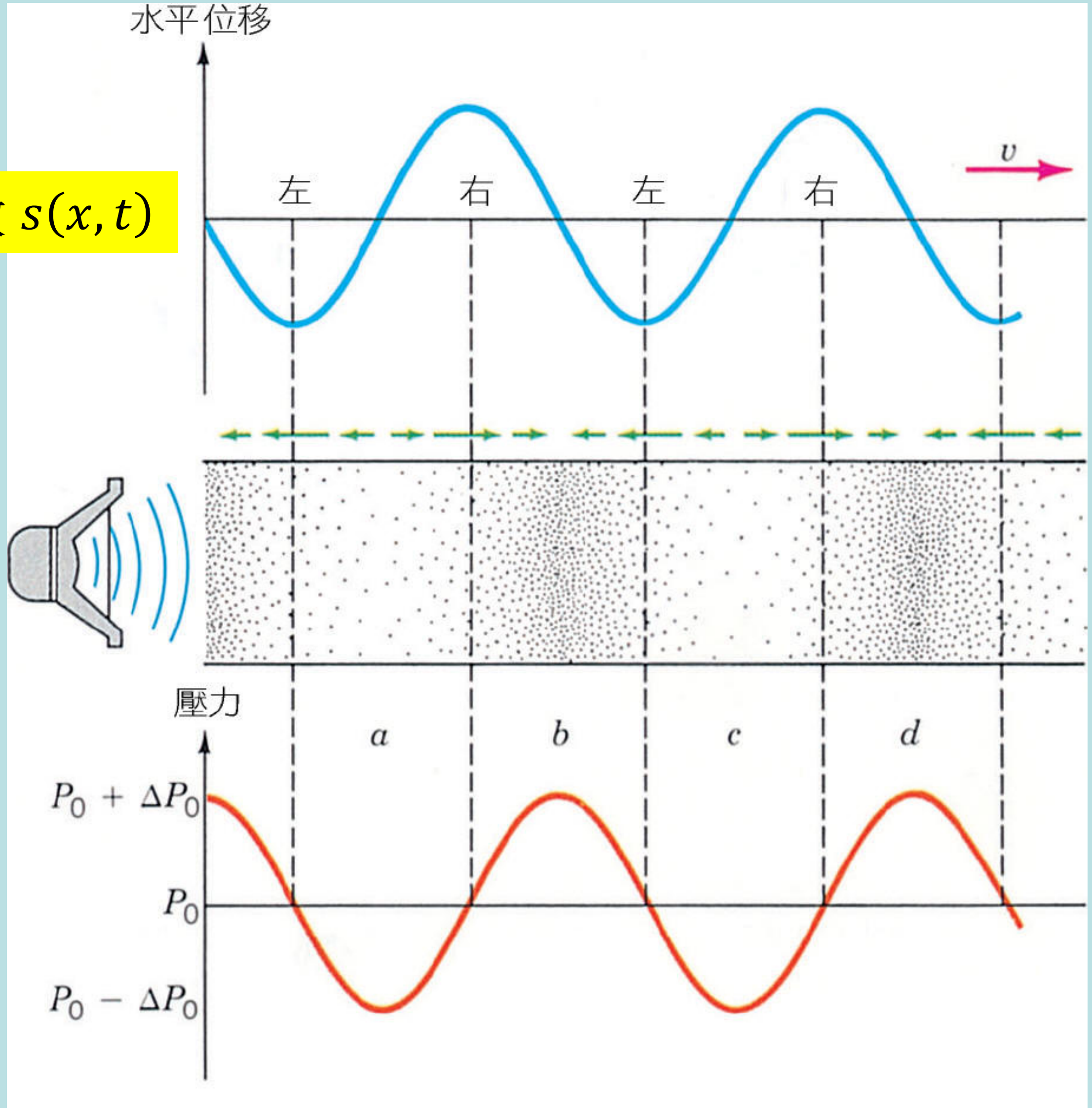
波函數 $s(x,t)$ 為以x為平衡點的氣體分子，在時間 t 時的位移

因此，時間為t時，此分子的位置為$x+s(x,t)$。

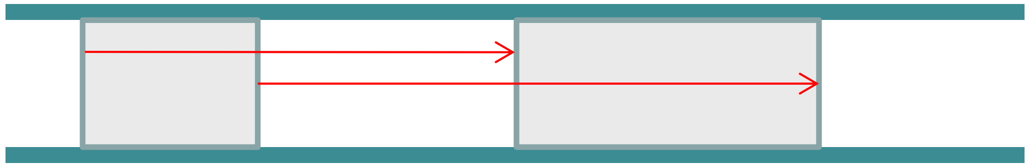
波函數可以是水平位移，直接但難測，以壓力表示時測量較為容易！

波函數 $s(x,t)$

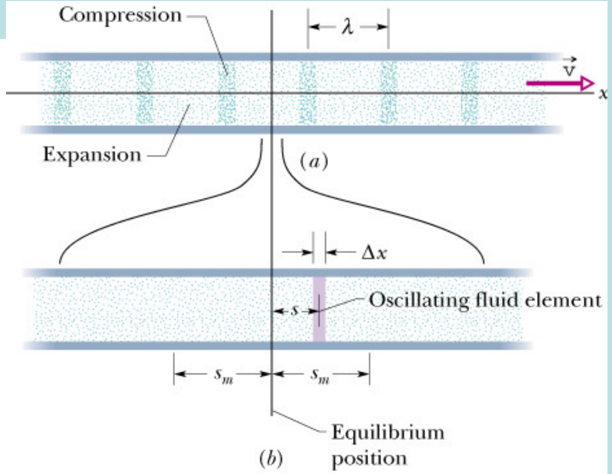
波函數 $\Delta P(x,t)$



s與ΔP兩個波函數是相關的！



x $\quad$ $x+\Delta x$ $\quad$ $x+s(x,t)$ $\quad$ $x+\Delta x+s(x+\Delta x\,,t)$



此塊氣體的體積變化等於：

$$\Delta V = A\cdot[s(x+\Delta x\,,t)-s(x\,,t)] = A\cdot\Delta x\cdot\frac{\partial s}{\partial x} = V\frac{\partial s}{\partial x}$$

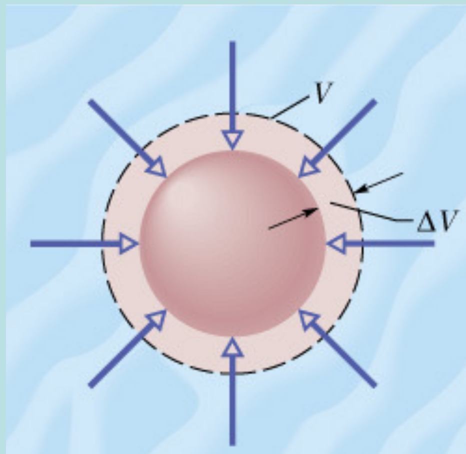
$$\Delta P = -B\cdot\frac{\Delta V}{V}$$

體積變化比例與壓力變化成正比！

B是Bulk Modulus

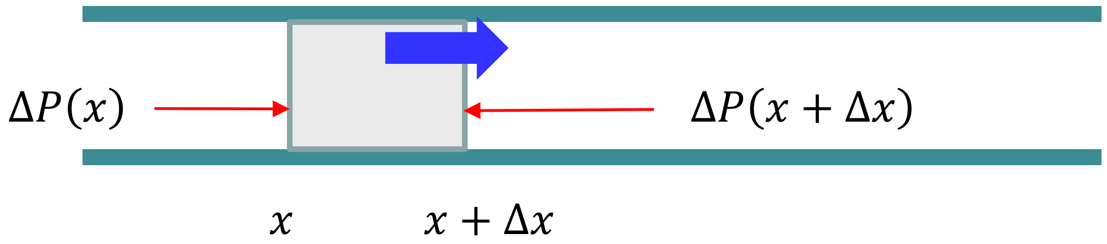
體積彈性係數

$$B = 1.4P$$



$$\Delta P = -B\frac{\partial s}{\partial x}$$

質量: ρV 加速度: $\dfrac{\partial^2 s}{\partial t^2}$



空氣柱中一小塊氣體的加速度等於$\dfrac{\partial^2 s}{\partial t^2}$，加速度正比於受力：

此塊氣體的受力來自兩邊的壓力差：

$$F=-A\cdot[\Delta P(x+\Delta x)-\Delta P(x)]=-A\cdot\Delta x\cdot\frac{\partial \Delta P}{\partial x}=-V\frac{\partial \Delta P}{\partial x}$$

力 = 質量 × 加速度 →

牛頓第二定律

$$\rho V\cdot\frac{\partial^2 s}{\partial t^2}=-V\frac{\partial \Delta P}{\partial x}$$

$$\Delta P=-B\frac{\partial s}{\partial x}$$

$$\rho\frac{\partial^2 s}{\partial t^2}=B\frac{\partial^2 s}{\partial x^2}$$

聲波的波方程式

$$\frac{\partial^2 s}{\partial x^2} = \frac{\rho}{B}\frac{\partial^2 s}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 s}{\partial t^2}$$

$$v = \sqrt{\frac{B}{\rho}}$$

聲速由彈性係數B決定。

$$\Delta P = -B \cdot \frac{\Delta V}{V}$$

將此式運用於很小的壓縮過程： $\Delta \rightarrow d$

$$dP = -B \cdot \frac{dV}{V}$$

$$B = -\frac{1}{V} \cdot \frac{dP}{dV}$$

若聲波傳送是等溫的壓縮過程：

$$P = nRT \cdot V^{-1}$$

$$B = -\frac{dP}{dV} \cdot V = nRTV^{-1} = P$$

$$v = \sqrt{\frac{P}{\rho}} \sim 288.1 \text{ m/s}$$

$$P = 1.01 \times 10^5 \text{Pa}$$
$$\rho = 1.21 \text{ kg/m}^3$$

比測量結果小！

The Speed of Sound[a]

Medium	Speed (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater[b]	1522
Solids	
Aluminum	6420
Steel	5941
Granite	6000

[a]At 0°C and 1 atm pressure, except where noted.
[b]At 20°C and 3.5% salinity.

$$v = \sqrt{\frac{B}{\rho}}$$

但聲波的傳播不是定溫而是絕熱過程，壓縮膨脹過程太快來不及傳熱。

$$P = c \cdot V^{-\gamma}$$

$$B = -\frac{dP}{dV} \cdot V = \gamma c V^{-\gamma} = \gamma P = 1.4P$$

$$B = 1.4P$$

$$P = 1.01 \times 10^5 \mathrm{Pa}$$
$$\rho = 1.21\ \mathrm{kg/m^3}$$

$$v = \sqrt{\frac{1.4P}{\rho}} \sim 341.8\ \mathrm{m/s}$$

The Speed of Sound[a]

Medium	Speed (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater[b]	1522
Solids	
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Steel	5941
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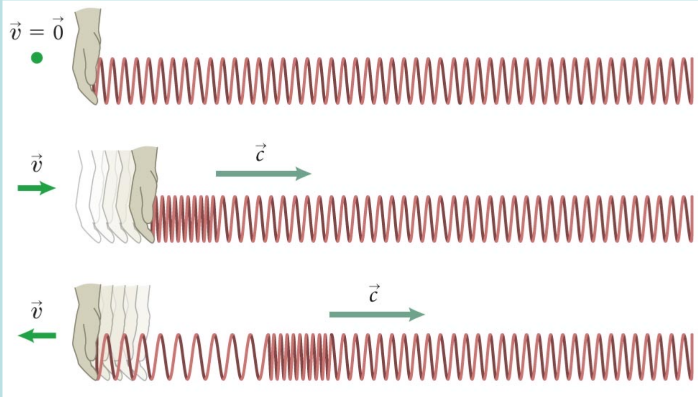
[a]At 0°C and 1 atm pressure, except where noted.
[b]At 20°C and 3.5% salinity.

連續介質的波方程式Wave Equation

$$\frac{\partial^2\phi}{\partial t^2} = v^2\frac{\partial^2\phi}{\partial x^2}$$



Initial Condition: 起始的弦位移$\phi(x,0)$，起始的弦垂直方向速度$\frac{\partial\phi}{\partial t}(x,0)$。

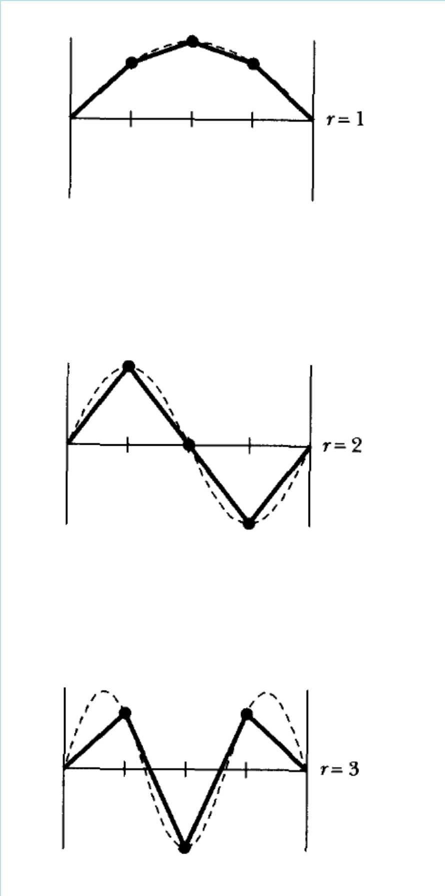
是不是要考慮邊界差別很大！

若不需考慮邊界、離開邊界很遠，介質中會有行進波的傳播 d'Alembert solution：

$$\phi(x,t) = f(x - vt) + g(x + vt)$$

若介質大小有限，連續介質就如大數目的彈簧組，有一系列模式振動。

彈簧組運動：一個本徵向量就對應一個可獨立振盪的模式。

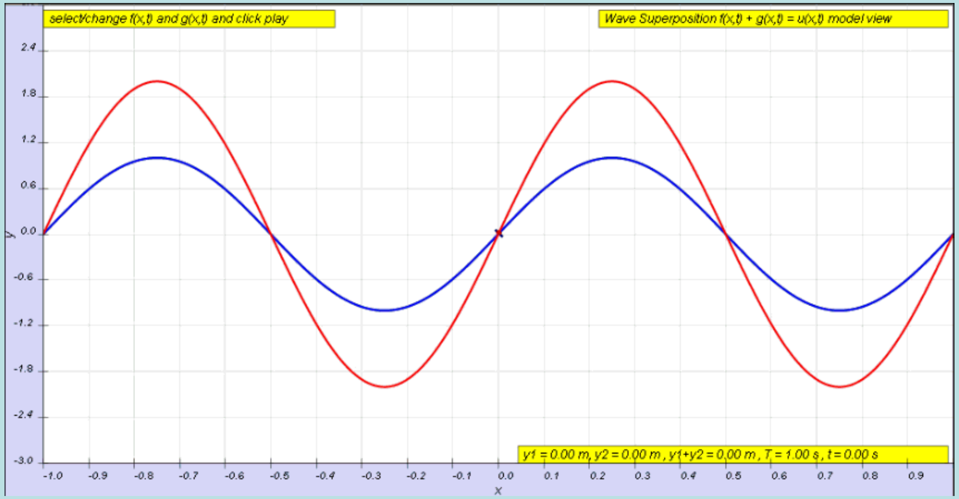


$$\boldsymbol{a}^{(1)} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{a}^{(2)} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{a}^{(3)} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

有分離的模式是因邊界條件：Boundary Condition 邊界條件，例如



$$\phi(0,t) = \phi(L,t) = 0$$

導體的邊界對電場就會強制電場為零的邊界條件。

Suppose that the conductor occupies the half of space $x > 0$. We start by shining the light head-on onto the surface. This means an incident plane wave, travelling in the x-direction,

$$\mathbf{E}_{\mathrm{inc}} = E_0\,\hat{\mathbf{y}}\,e^{i(kx-\omega t)}$$

where, as before, $\omega = ck$. Inside the conductor, we know that we must have $\mathbf{E} = 0$. But the component $\mathbf{E}\cdot\hat{\mathbf{y}}$ lies tangential to the surface and so, by continuity, must also vanish just outside at $x = 0^-$. We achieve this by adding a reflected wave, travelling in the opposite direction

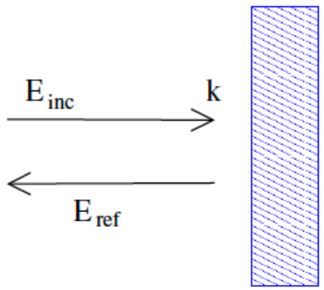


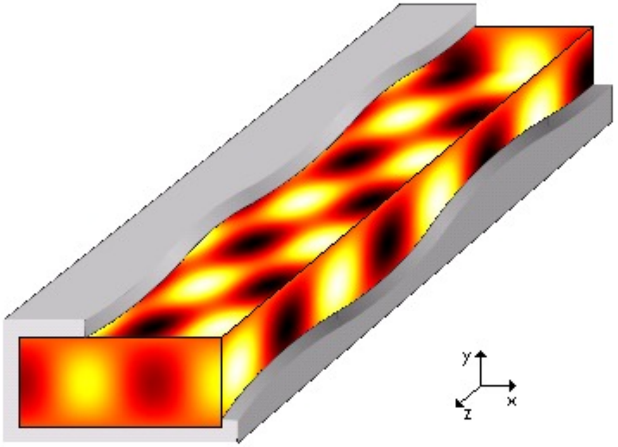
Figure 44:

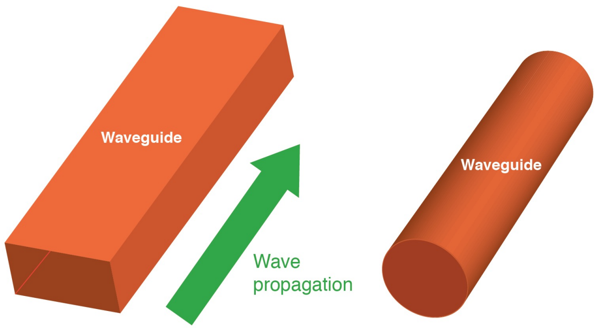
$$\mathbf{E}_{\mathrm{ref}} = -E_0\,\hat{\mathbf{y}}\,e^{i(-kx-\omega t)}$$

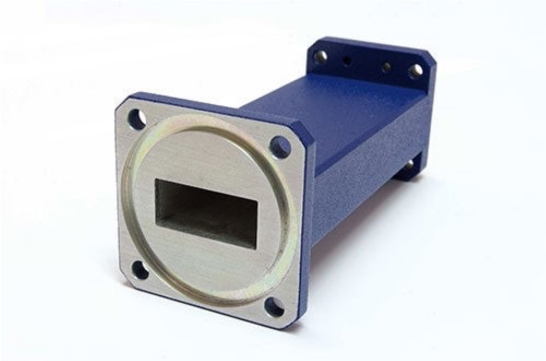
So that the combination $\mathbf{E} = \mathbf{E}_{\mathrm{inc}} + \mathbf{E}_{\mathrm{ref}}$ satisfies $E(x = 0) = 0$ as it must. This is illustrated in the figure. (Note, however, that the figure is a little bit misleading: the two waves are shown displaced but, in reality, both fill all of space and should be superposed on top of each other).

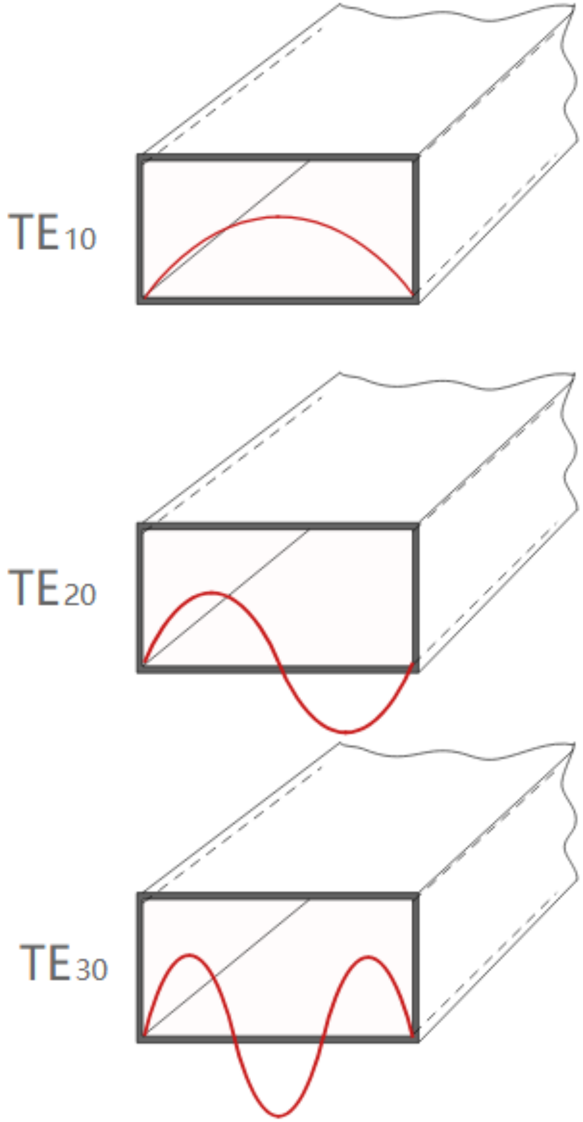
在三度空間的波，邊界就可能有形狀，如球、圓柱。解會非常不同。

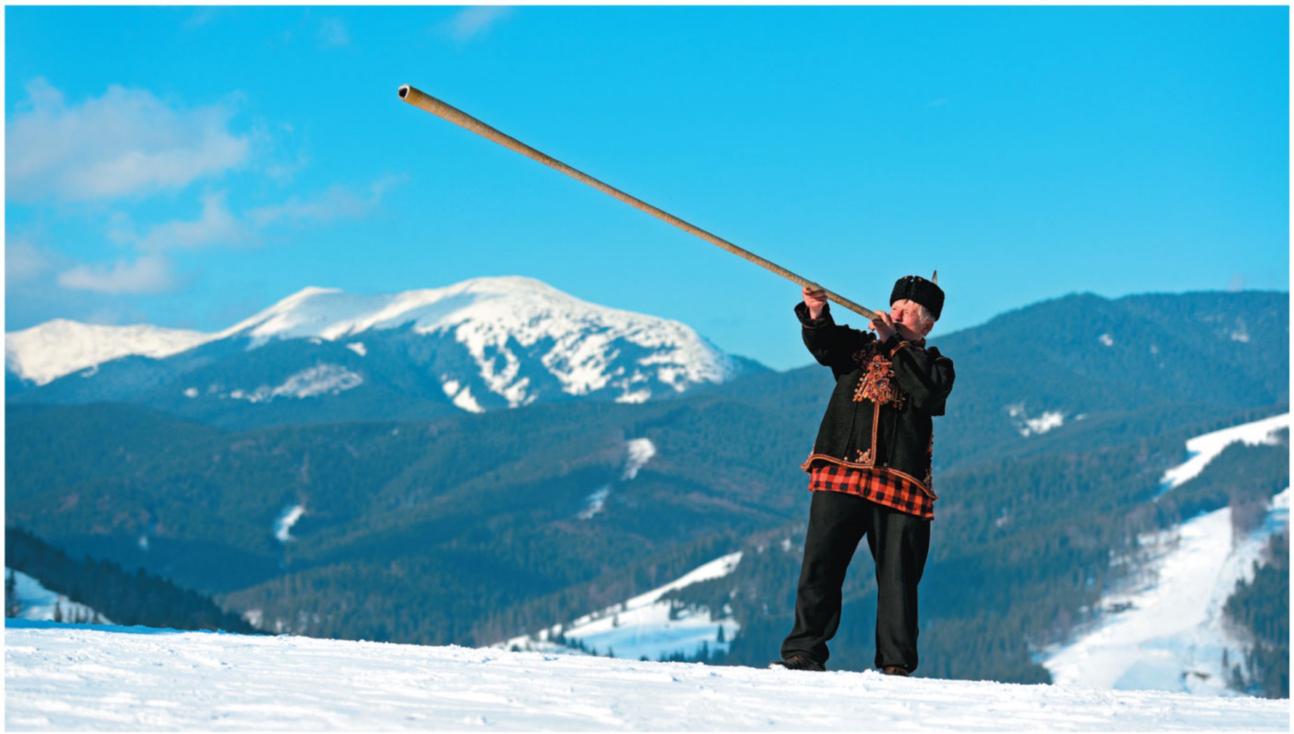
波導則是在三度空間，有二維的邊界。



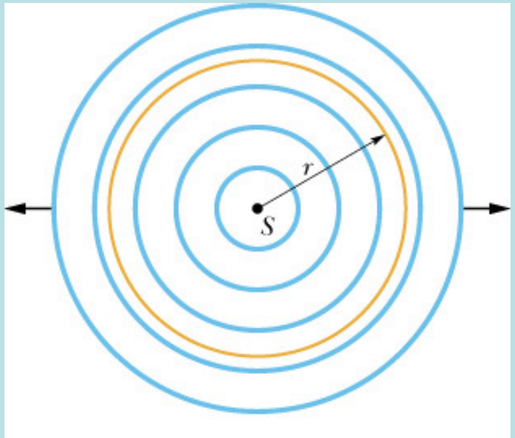


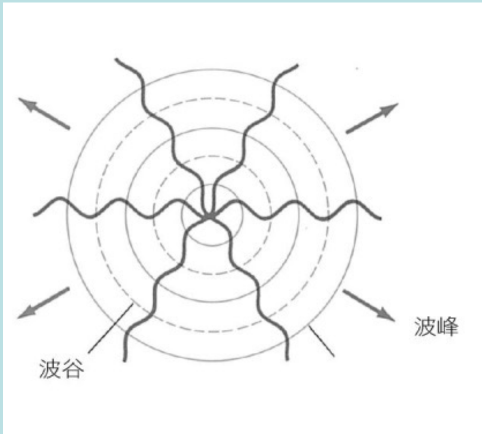






廣大空間的聲音，就是無邊界的聲波傳播。





樂器空腔內的空氣的振盪就是邊界條件下的模式。

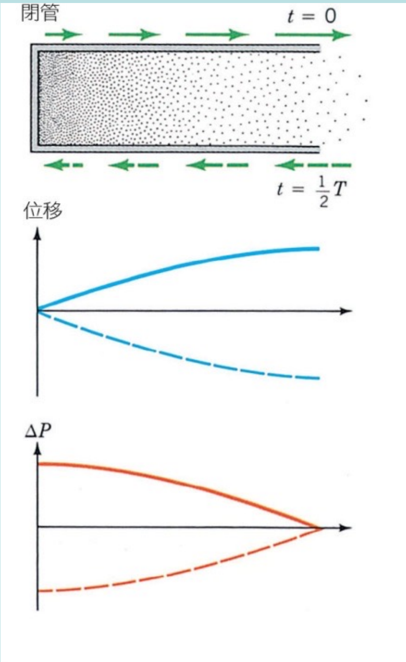


圖 17.4
閉管的基本模式。封閉端為一位移的節點，而且是壓力的腹點。實線說明在 $t=0$ 的情形，而虛線則說明在 $t=T/2$ 之情形。

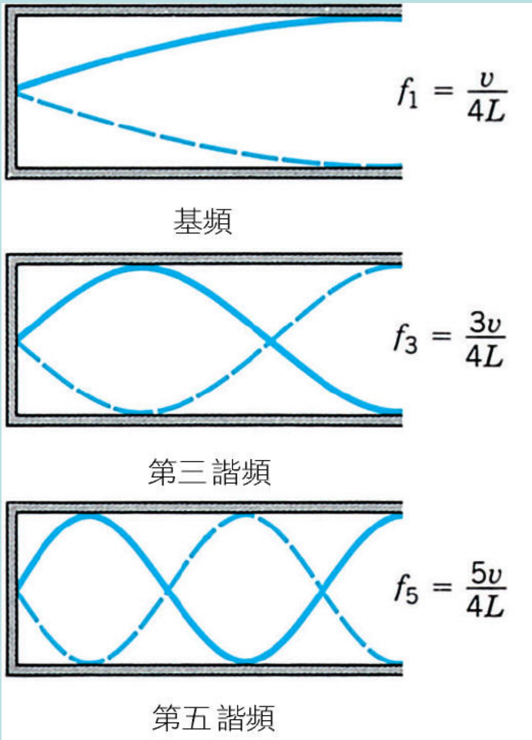
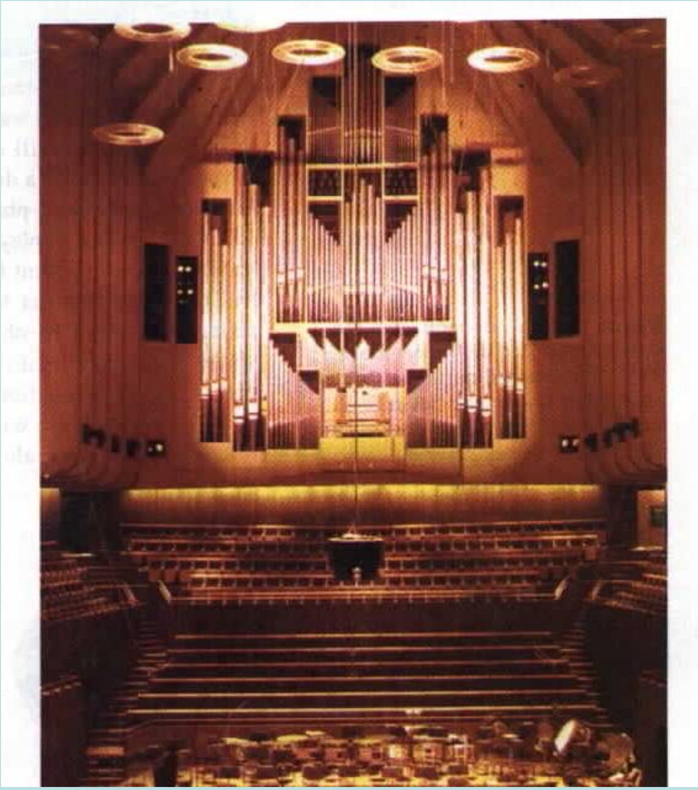
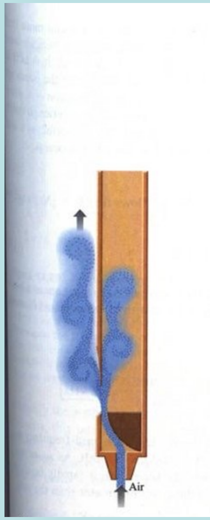


圖 17.5
閉管中前三種共振模。實線及虛線表示時差半週期的兩組縱波。

樂器空腔內的空氣的振盪就是邊界條件下的模式。





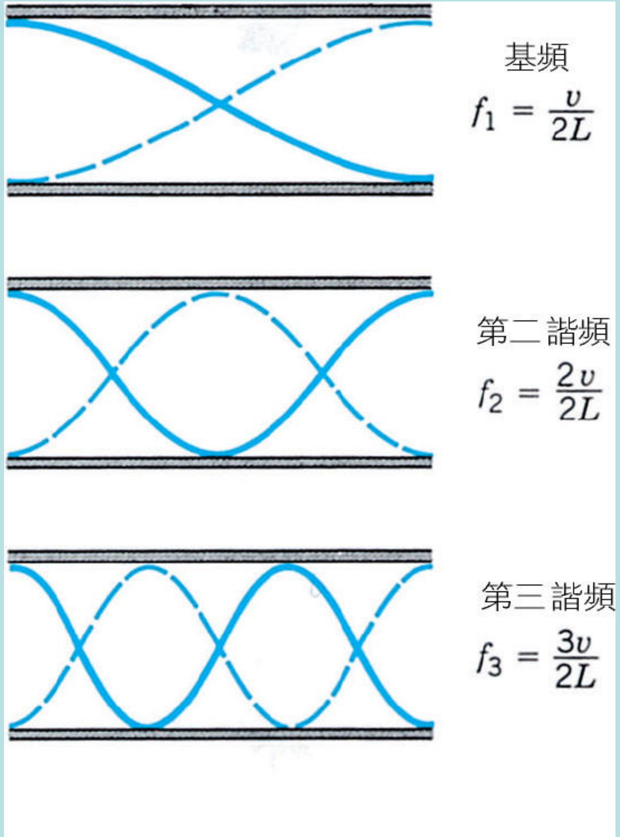
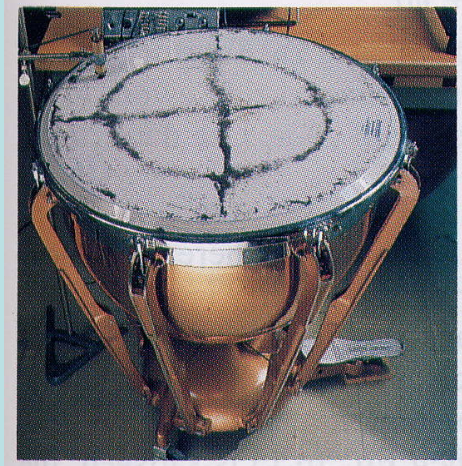
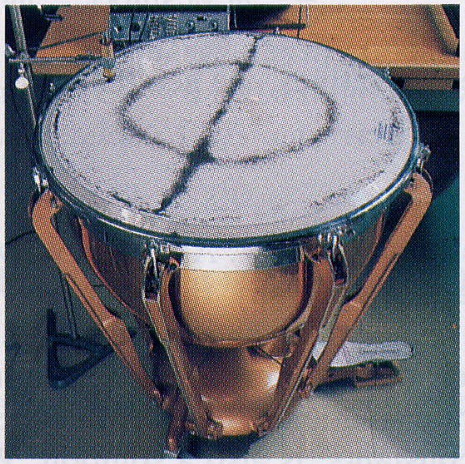
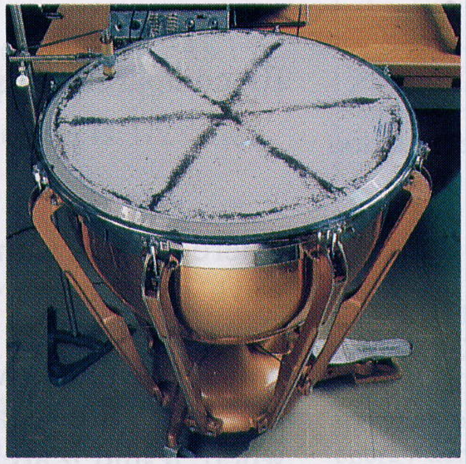


圖 17.6
開放管中前三種共振模。

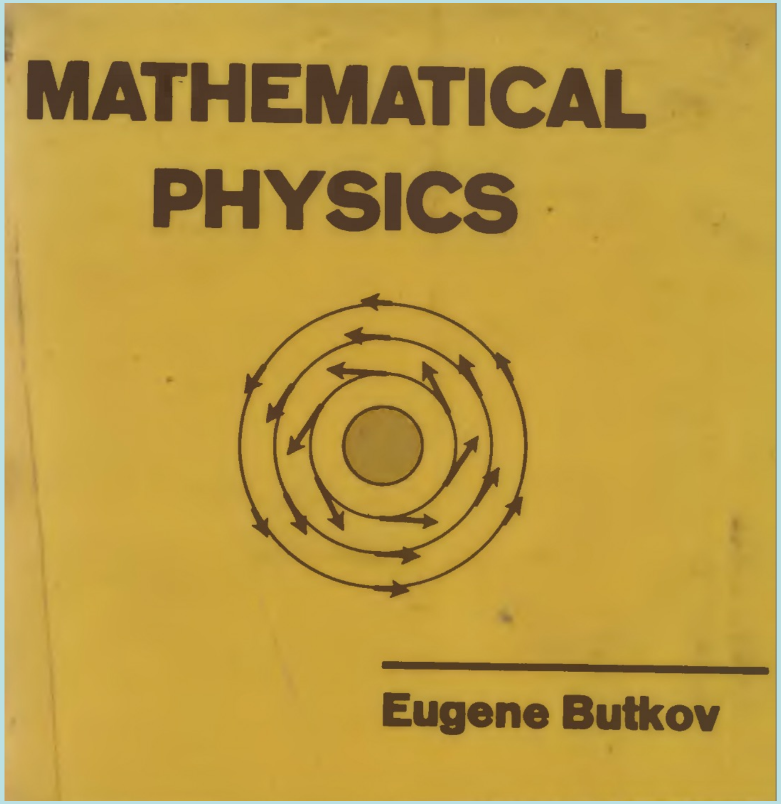






樂器空腔內的空氣的振盪就可以是三度空間邊界條件下的模式。

有邊界條件的波方程式的求解



8.2 THE METHOD OF SEPARATION OF VARIABLES

Since the additional conditions imposed on $u(x, t)$ in our string problem fall into two groups, (a) those involving x (boundary conditions) and (b) those involving t (initial conditions), it may be reasonable to seek solutions of the PDE in the form

$$u(x, t) = X(x)T(t),$$

where X is a function of x only and T is a function of t only. If $X(x)$ is chosen to satisfy the conditions

$$X(0) = 0, \qquad X(L) = 0,$$

then the function $u(x, t)$ will satisfy the same conditions. Then $T(t)$ may, perhaps, be chosen to satisfy the initial conditions.

We now require that $u(x, t)$ satisfy the PDE. We have

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{d^2 X(x)}{dx^2} T(t), \qquad \frac{\partial^2 u(x, t)}{\partial t^2} = X(x) \frac{d^2 T(t)}{dt^2}.$$

Therefore

$$\frac{d^2 X}{dx^2} T = \frac{1}{c^2} X \frac{d^2 T}{dt^2}.$$

Dividing both sides by $X(x)T(t)$, we obtain

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2}.$$

The left-hand side of this equation depends on x alone; the right-hand side depends on t alone. If this equality is to hold for all x and t, it is evident that either side must be a constant (same for both sides):

$$1 \ d^2 X \qquad 1 \ 1 \ d^2 T$$

The constant λ is known as a *separation constant*. The equation for $X(x)$ can be written as

$$\frac{d^2 X}{dx^2} = \lambda X$$

and will lead to exponential functions if $\lambda > 0$, to trigonometric functions if $\lambda < 0$, and to a linear function if $\lambda = 0$:

$$X(x) = \begin{cases} Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}} & (\lambda > 0), \\ A' \cos(x\sqrt{-\lambda}) + B' \sin(x\sqrt{-\lambda}) & (\lambda < 0), \\ A''x + B'' & (\lambda = 0). \end{cases}$$

It is not difficult to verify that the boundary conditions $X(0) = 0$, $X(L) = 0$ can be satisfied *only* if $\lambda < 0$ and, moreover, *only* if A' is set equal to zero and the values of λ satisfy the condition

$$\sqrt{-\lambda} = n\pi/L \qquad (n = 1, 2, 3, \ldots).$$

Exercise. Show, in detail, that it is possible to satisfy either $X(0) = 0$ or $X(L) = 0$, but not both, if $\lambda \geq 0$. Also, prove the statement made for the case $\lambda < 0$.

These "allowed" values of the separation constant λ,

$$\lambda_n = -n^2\pi^2/L^2 \qquad (n = 1, 2, 3, \ldots),$$

are usually called the *eigenvalues*, or *characteristic values*, of the problem under consideration.* By this we mean the problem of finding functions satisfying the given DE *and* the given boundary conditions. In our case there is an infinity of such functions, called *eigenfunctions*, and they read

$$X_n(x) = B'_n \sin(n\pi x/L) \qquad (n = 1, 2, 3, \ldots),$$

where B'_n is an arbitrary (nonzero) constant which may, in general, be different for different eigenfunctions.

In our problem of the stretched string the function $T(t)$ which is multiplied by $X(x)$ must satisfy the DE with the same separation constant as $X(x)$. Therefore, to each eigenfunction $X_n(x)$ there corresponds a function $T_n(t)$ satisfying

$$\frac{1}{c^2}\frac{1}{T_n}\frac{d^2T_n}{dt^2} = \lambda_n = -\frac{n^2\pi^2}{L^2}.$$

This yields

$$T_n(t) = C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L},$$

where C_n and D_n are arbitrary constants.

---

* The word *eigenvalue* is an adaptation of the German term *eigenwert* derived from *eigen* = proper and *wert* = value.

Summarizing our results we may say that the attempt to find the solution of our PDE with given boundary conditions and initial conditions in the form

$$u(x, t) = X(x)T(t)$$

leads us, so far, to an *infinite number* of such functions which may be written as

$$u_n(x, t) = \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right) \sin \frac{n\pi x}{L},$$

where $A_n = B_n' C_n$ and $B_n = B_n' D_n$ are arbitrary constants. Each of these functions $u_n(x, t)$ satisfies the PDE and the boundary conditions. It remains for us to select from among these functions, adjusting the constants A_n and B_n, those functions that will also satisfy the desired initial conditions. Before we do this, however, note that each function $u_n(x, t)$ represents, on its own, *some* kind of possible motion of the stretched string (corresponding to some special initial conditions). These types of motion are known as the *characteristic modes* (or *normal modes*) of vibration of the string. Each one represents a harmonic motion (vibration) with the characteristic frequency (or "eigenfrequency")

$$\omega_n = n\pi c/L \qquad (n = 1, 2, 3, \ldots).$$

A bit of reflection will show that it is not possible to satisfy *arbitrary* initial conditions

$$u(x, 0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) = v_0(x)$$

by any *single* function $u_n(x, t)$. Indeed, any particular function $u_n(x, t)$ will satisfy initial conditions of the form

$$u_0(x) = A_n \sin \frac{n\pi x}{L}, \qquad v_0(x) = B_n \frac{cn\pi}{L} \sin \frac{n\pi x}{L}$$

and *no others*. To overcome this difficulty we can lean back on the powerful *principle of superposition*. Our PDE is *linear homogeneous*, and so are our boundary conditions. It is trivial to verify that if two functions $u_n(x, t)$ and $u_m(x, t)$ satisfy the PDE and the boundary conditions, then their linear combination

$$f(x, t) = C_1 u_n(x, t) + C_2 u_m(x, t)$$

will also satisfy the PDE and the boundary conditions. By induction this will also be true for a linear combination of a *finite* number of functions $u_n(x, t)$. It is not unreasonable to conjecture that the same properties will hold for an *infinite series* formed by the functions $u_n(x, t)$:

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

provided the series converges (or if not, provided it can be treated as a distribution, in conformity with the principles stated in Chapter 6).

To be more precise, the function defined above certainly satisfies the boundary conditions

$$y(0, t) = y(L, t) = 0$$

(since each term satisfies them). It will also satisfy the PDE, *provided* it can be differentiated twice term by term with respect to both x and t. From the point of view of the classical theory of functions, this may impose considerable restrictions on A_n and B_n; e.g., the twice differentiated series $y(x, t)$ must be uniformly convergent. On the other hand, from the point of view of the theory of distributions it is sufficient* that A_n and B_n are of the order of some power of n:

$$|A_n| < An^N, \qquad |B_n| < Bn^N \qquad \text{(some } N\text{)},$$

and this point of view may be adequate for physical applications.

In any case, however, the justification of the method depends on the properties of A_n and B_n, and it is usually convenient to *assume* its validity and *justify it later*, after the solution has been developed.

It is needless to emphasize that the function $y(x, t)$ is a Fourier sine series in x (it is also a Fourier series in t, but this fact is of much less importance). Setting $t = 0$ and using the first initial condition, we obtain

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

If the function $u_0(x)$ can be expanded into a Fourier sine series, the coefficients A_n can be determined. In physical problems $u_0(x)$ is invariably continuous, piecewise very smooth, and vanishes at $x = 0$ and $x = L$. Therefore, it can be represented as above.

Similarly, calculating $(\partial u/\partial t)(x, t)$ and using the second initial condition, we can obtain

$$v_0(x) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{L} \sin \frac{n\pi x}{L}.$$

In physical problems $v_0(x)$ is sometimes assumed to be discontinuous. However, it is invariably piecewise continuous and piecewise very smooth, and the coefficients B_n can be determined as well.

Consequently, we have constructed a solution to our problem in the form of a series

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

which satisfies the boundary conditions and initial conditions for all physically reasonable functions $u_0(x)$ and $v_0(x)$.

有邊界的連續介質與大數目的彈簧組，有非常密切的對應。

對照表

$$\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A} \cdot \boldsymbol{X}$$

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

存在一系列振盪模式解， $\boldsymbol{X} = \boldsymbol{a}^{(i)} e^{i\omega_i t}$

存在一系列振盪模式解，

特徵是x, t兩變數的函數可以分離。

$$\phi(x,t) = \tilde{\phi}^{(i)}(x) e^{i\omega_i t}$$

普遍解即是模式解的線性組合！

普遍解即是模式解的線性組合！

解波方程式的第一步：尋找x, t兩變數的函數可以分離的模式解。

$$\phi(x,t) = X(x) \cdot T(t)$$

Method of separation of variables

Solving Wave Equation: $\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$ and Schrodiner Wave Equation

8.2 THE METHOD OF SEPARATION OF VARIABLES

Since the additional conditions imposed on $u(x, t)$ in our string problem fall into two groups, (a) those involving x (boundary conditions) and (b) those involving t (initial conditions), it may be reasonable to seek solutions of the PDE in the form

因為起始條件與邊界條件是分開的，

首先尋找時間部分與空間部分，可以分離為x, t兩變數個別的函數的模式解：

$$\phi(x,t) = X(x) \cdot T(t)$$

一系列這樣的解的線性組合就構成所有可能的一般解。

作法：將起始條件分解為$X(x)$的線性組合(永遠可以)。

$X(x)$各自作時間演化$T(t)$(簡諧振盪)後，最後再組合！

首先尋找時間部分與空間部分，可以分離為x,t兩變數個別的函數的模式解：

$$\phi(x,t)=X(x)\cdot T(t)$$

代入波方程式：

$$\frac{\partial^2\phi}{\partial t^2}=v^2\frac{\partial^2\phi}{\partial x^2}$$

得到：

$$X\frac{\partial^2 T}{\partial t^2}=Tv^2\frac{\partial^2 X}{\partial x^2}$$

一般會把 v^2 移到左邊，同時左右邊都除以 $X(x)\cdot T(t)$ 。偏微分可以寫成常微分：

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{X}\frac{d^2X}{dx^2}$$

現在左邊只與x有關，右邊只與t有關，但兩者是獨立變數！相等並不可能。

唯一的例外：左右兩式都與各自的變數無關，是一常數。設此常數為λ。

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{X}\frac{d^2X}{dx^2}\equiv\lambda \quad\Rightarrow\quad \frac{d^2X}{dx^2}=\lambda\cdot X \qquad \frac{d^2T}{dt^2}=v^2\lambda\cdot T$$

我們就得到空間部分X與時間部分T各自需要滿足的常微分方程式。

將偏微分方程式分解Reduce為常微分方程式，這是常見的作法。

$\frac{d^2X}{dx^2} = \lambda \cdot X$　$X(x)$滿足的二次常微分方程式特別重要：

對照表

$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A} \cdot \boldsymbol{X}$	$\frac{\partial^2\phi}{\partial t^2} = v^2 \frac{\partial^2\phi}{\partial x^2}$
$-\boldsymbol{A} \cdot \boldsymbol{X}$	$\frac{\partial^2}{\partial x^2}\phi$
$\boldsymbol{A}$	$\frac{d^2}{dx^2}$ 微分運算稱為算子operator
$\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$	$\phi(x,t) = X(x) \cdot T(t)$

$\Downarrow$

矩陣的本徵值方程式！ $-\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$

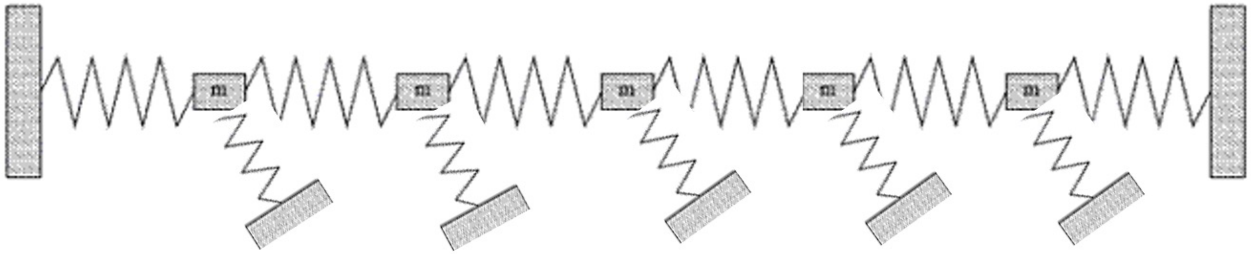
$\frac{d^2X}{dx^2} = \lambda X$ 算子 $\frac{d^2}{dx^2}$ 的本徵值方程式！

$X(x)$就稱為微分算子Differential Operator $\frac{d^2}{dx^2}$ 的本徵函數！

1. Consider the following wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} - V(x)\phi$$

Here we add a term $V(x)\phi$ to the right-hand side of the wave function we studied in class. $V(x) = \alpha^2 x^2$ is a known function. This could be done by adding a position dependent restoring force to the particles on the string.



It is reasonable to think that there exists solutions that are separable:

$$\phi(x,t) = X(x) \cdot T(t).$$

A. Find the Ordinary Differential Equation satisfied by $X(x)$ and $T(t)$. Show that $T(t)$ satisfies the ODE of a simple harmonic oscillator and the solution can be written as:

same as in wave equation

$$T(t) = a_m \cos(\omega t + \phi) \sim f\cos(\omega t) + g\sin(\omega t)$$

B. Check that $e^{-\alpha\frac{x^2}{2}}$ is a solution for $X(x)$. Write down the corresponding $T(t)$. What is the value of ω for this solution?

CHAPTER 8

STURM-LIOUVILLE THEORY

Characterization of the general features of eigenproblems arising from second-order differential equations is known as **Sturm-Liouville theory**. It therefore deals with eigenvalue problems of the form

$$\mathcal{L}\psi(x) = \lambda\psi(x), \tag{8.7}$$

where $\mathcal{L}$ is a linear second-order differential operator, of the general form

$$\mathcal{L}(x) = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x). \tag{8.8}$$

The key matter at issue here is to identify the conditions under which $\mathcal{L}$ is a Hermitian operator.

Self-Adjoint ODEs

$\mathcal{L}$ is known in differential equation theory as **self-adjoint** if

$$p_0'(x) = p_1(x). \tag{8.9}$$

This feature enables $\mathcal{L}(x)$ to be written

$$\mathcal{L}(x) = \frac{d}{dx}\left[p_0(x)\frac{d}{dx}\right] + p_2(x), \tag{8.10}$$

若是在三度空間，波函數可以分離為$\vec{r},t$個別的函數的模式解：

$$\phi(\vec{r},t)=\psi(\vec{r})\cdot T(t)$$

Method of Separation of Variables

Hermann von Helmholtz 1821-1894

代入波方程式：

$$\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi$$

得到：

$$\frac{1}{v^2}\psi\frac{\partial^2 T}{\partial t^2}=T\nabla^2\psi$$

一般會把v^2移到左邊，同時左右邊都除以$\psi(\vec{r})\cdot T(t)$：

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{\psi}\nabla^2\psi$$

現在左邊只與$\vec{r}$有關，右邊只與t有關，但兩者是獨立變數！相等並不可能。

唯一的例外：左右兩式都與各自的變數無關，是一常數。設此常數為λ。

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{\psi}\nabla^2\psi\equiv\lambda \quad\Rightarrow\quad \frac{d^2T}{dt^2}=v^2\lambda\cdot T$$

$$\nabla^2\psi=\lambda\cdot\psi$$

Helmholtz Equation

$X(x)$滿足標準的二次常微分方程式：

$$\frac{d^2X}{dx^2} = \lambda X$$

我們稱這是算子 $\frac{\partial^2}{\partial x^2}$ 的本徵值方程式！

它的解與λ是正或負有關：

$$X(x) = \begin{cases} Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} & \lambda > 0 \\ A'\cos\sqrt{-\lambda}x + B'\sin\sqrt{-\lambda}x & \lambda < 0 \\ A''x + B'' & \lambda = 0 \end{cases}$$

λ若是正值，$X(x)$是指數函數，λ若是負值，$X(x)$則是三角函數。

如果考慮邊界條件，只能選$\lambda < 0, A' = 0$。一般會定義角波數： $k \equiv \sqrt{-\lambda}$

$X(x) = \sin kx$ 自動滿足原點邊界條件 $\phi(0,t) = 0$

另一端邊界條件就對k、也就是λ有限制：

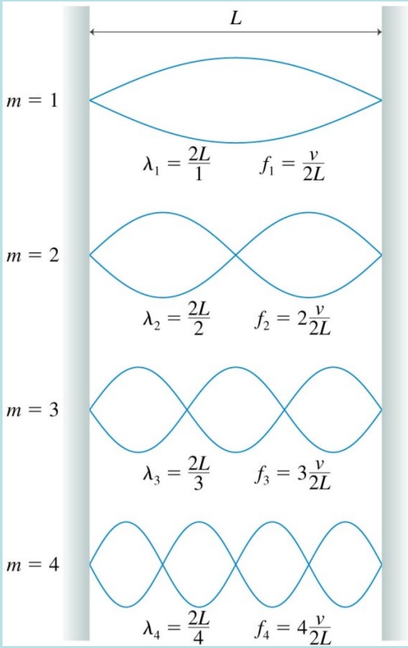
$X(L,0) \sim \sin(kL) = 0$ $\quad$ $k_n = \frac{n\pi}{L} = \sqrt{-\lambda_n}$

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,\cdots\infty$$

$$\frac{d^2X_n}{dx^2} = \lambda_n X_n$$

我們稱這些解是算子 $\frac{\partial^2}{\partial x^2}$ 的本徵函數 Eigenfunction！

物理上對應一系列模式(駐波)，離散以n標定，但有無限多個。



$$\frac{\partial^2 X}{\partial x^2} = \lambda X$$ 微分算子 $\frac{\partial^2}{\partial x^2}$ 的本徵值方程式！ $\nabla^2 \psi = \lambda \cdot \psi$

加了邊界條件後，方程式的解就是離散分布，以$n = 1 \cdots \infty$標定 。

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$ 算子 $\frac{\partial^2}{\partial x^2}$ 解出來的本徵函數。

$$\lambda_n = -\frac{n^2\pi^2}{L^2}$$ 算子 $\frac{\partial^2}{\partial x^2}$ 解出來的本徵值。

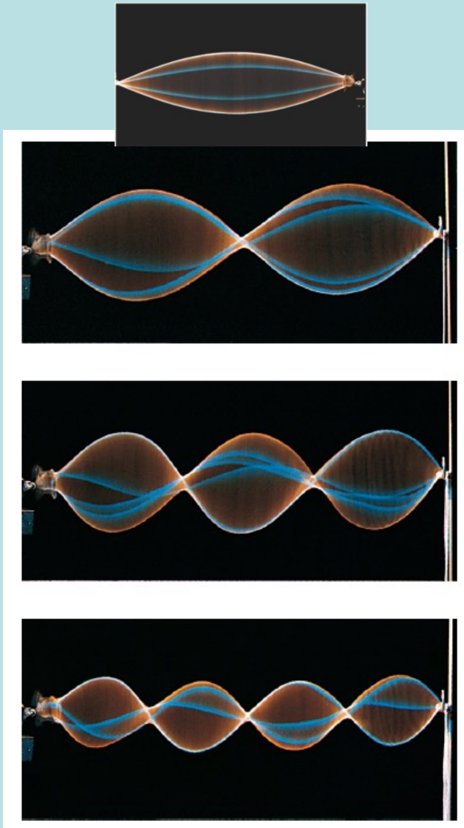
$$\frac{d^2 X_n}{dx^2} = \lambda_n X_n$$

我們從矩陣的例子就已經可以猜測$X_n(x)$的重要性質：

不同本徵值的本徵函數彼此正交(何謂正交?)。

所有滿足邊界條件的函數都可分解為$X_n(x)$的線性組合！Completeness.完備性。

有了λ_n及k_n，時間部分T也能解出：



$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{X}\frac{d^2X}{dx^2}\equiv\lambda_n=-\frac{n^2\pi^2}{L^2}$$

$$\frac{d^2T}{dt^2}=v^2\lambda_n T\equiv-\omega_n^2 T$$

$T(t)$就是以為$\omega_n=v\sqrt{\lambda}=vk$為角頻率的簡諧運動！

$$T(t)=a_m\cos(\omega_n t+\phi)\sim f\cos(\omega_n t)+g\sin(\omega_n t)$$

這個模式的解，命名為$u^{(n)}$，現在可以完整寫出：

$$u^{(n)}(x,t)=X_n(x)\cdot T(t)=\ \sin(k_n x)(A_n\cos\omega_n t+B_n\sin\omega_n t)$$

時間部分是角頻率為ω_n的振盪函數，空間部分是角波數為k_n的週期函數。

$$k_n=\frac{n\pi}{L}$$

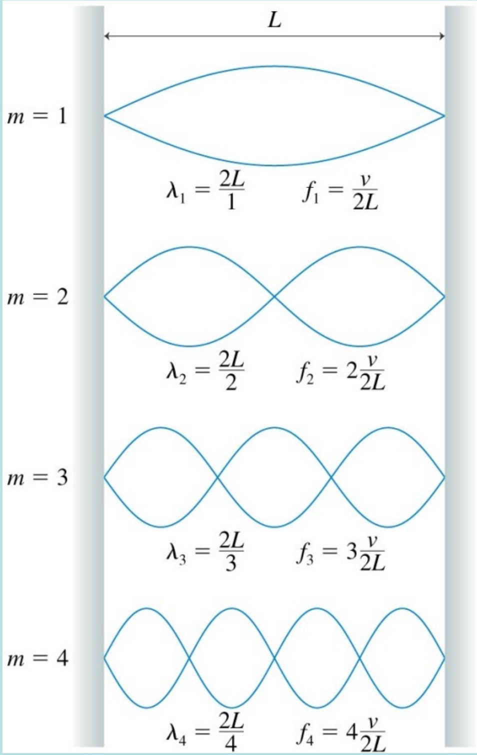
$$\omega_n=vk_n=\frac{n\pi}{L}v$$

這就是一個典型的駐波！

這個解稱為Bernoulli Solution.

Bernoulli Solution與d'Alembert solution等價。

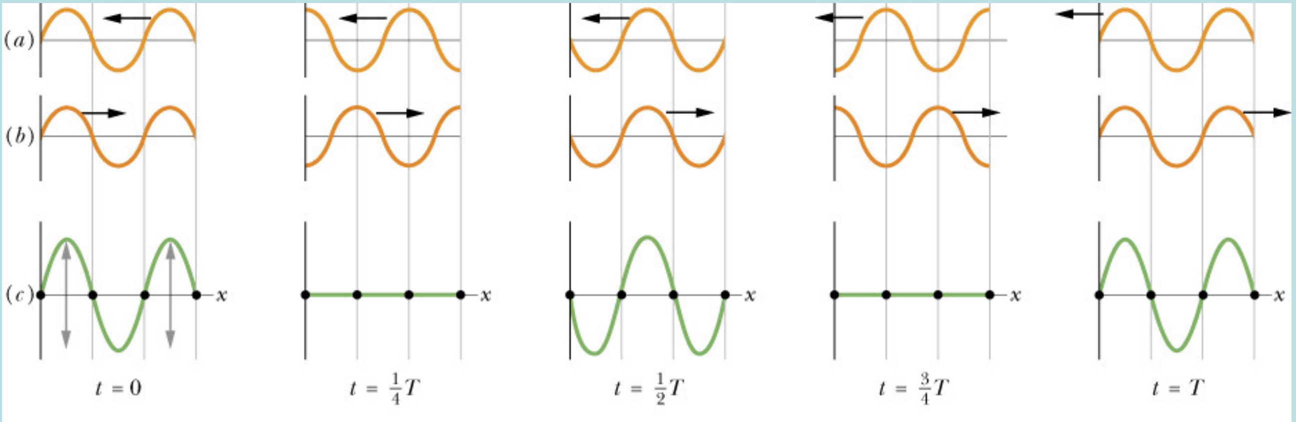
Daniel Bernoulli 1700-1782



之前駐波被視為向左正弦波與向右正弦波 d'Alembert solution 的疊加。

現在駐波同時是振盪弦的模式之波函數。

Normal Modes



在一模式下，所有粒子是一起以同一個時間函數$(A_n \cos\omega_n t + B_n \sin\omega_n t)$振盪的。

任兩個位置的波函數比例固定與時間無關。

$$\frac{\phi(x_1,t)}{\phi(x_2,t)} = \frac{\sin\left(\frac{n\pi}{L}x_1\right)}{\sin\left(\frac{n\pi}{L}x_2\right)}$$

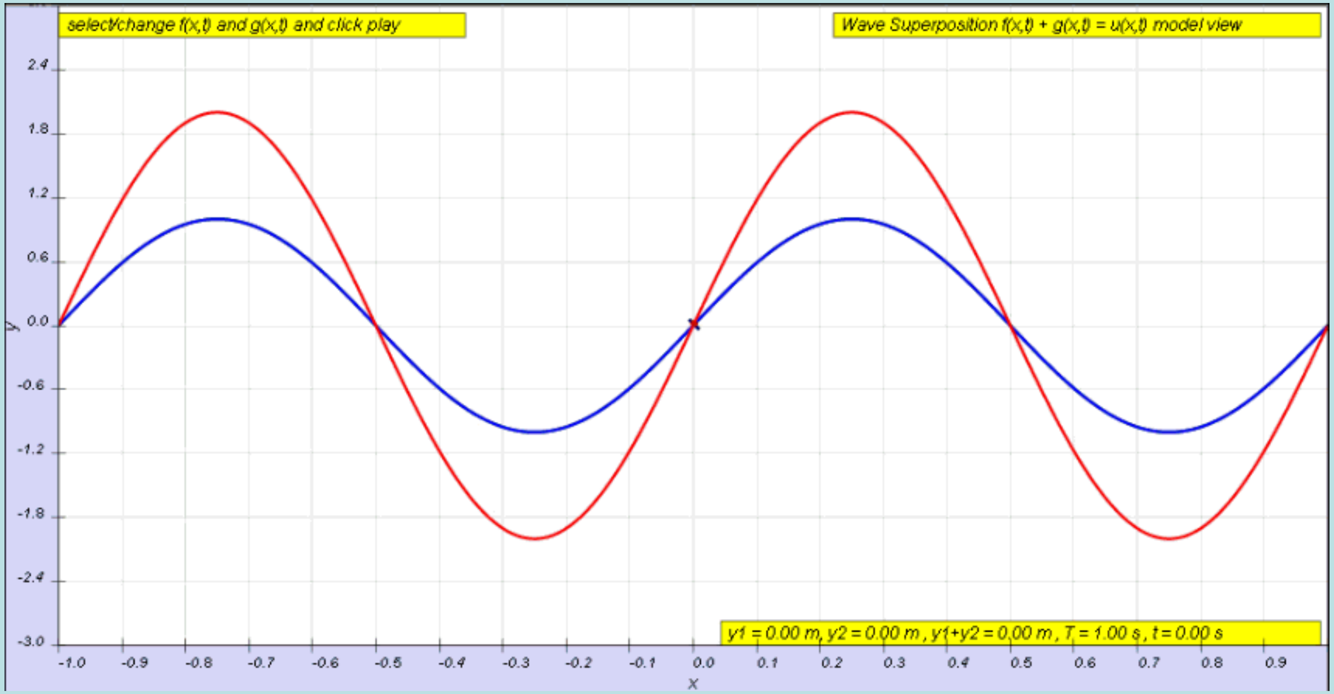
$$u^{(n)}(x,t) = \sin\left(\frac{n\pi}{L}x\right)(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

波函數若一開始就與某一$\sin(k_n x)$成正比，接下來波函數就會維持此比例關係。

之前對coupled mass system模式的描述

位移行向量若一開始就與一本徵向量成正比，接下來位移就會維持此比例關係。

可分解波函數就對應模式的位移行向量， $\sin(k_n x)$就對應本徵向量！



對照表

$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$	$\frac{\partial^2\phi}{\partial t^2} = v^2\frac{\partial^2\phi}{\partial x^2}$ 右手邊稱為算子operator
$-\boldsymbol{A}\cdot\boldsymbol{X}$	$v^2\frac{\partial^2\phi}{\partial x^2}$
$\boldsymbol{A}$	$\frac{\partial^2}{\partial x^2}$
$\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$	$\phi(x,t) = X(x)\,e^{i\omega t}$
$a_j \sim e^{-ipj}$ or 虛數部 $\sin pj$	$X(x) \sim e^{-ikx}$ or 虛數部 $\sin kx$
$-\boldsymbol{A}\cdot\boldsymbol{a} = \lambda\boldsymbol{a}$	$\frac{\partial^2 X}{\partial x^2} = \lambda X$
矩陣的本徵值方程式！	算子 $\frac{\partial^2}{\partial x^2}$ 的本徵值方程式！

在波的現象中的算子operator，根源事實上是矩陣matrix。

算子operator的本徵值，與矩陣matrix的本徵值一樣將扮演重要角色。

Solving Wave Equation: $\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$ and Schrodiner Wave Equation

首先尋找時間部分與空間部分，可以分離為x, t兩變數個別的函數的模式解：

$\phi(x,t) = X(x) \cdot T(t)$

要求滿足邊界條件，如此$X_i(x), i = 1, \cdots \infty$ 會是一系列可數的函數。

$u_i(x,t) = X_i(x) \cdot T_i(t)$

關鍵：波方程式是線性方程式，$X_i(x)T_i(t)$作線性組合，還是波方程式的解。

將$t = 0$的起始條件分解為$X(x)T_i(0)$的線性組合，可以證明這永遠可以作到。

可以分解任意起始條件這個性質非常關鍵，$X(x)$稱為具有完備性！

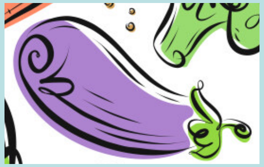
讓組合中的$X_i(x)$各自依照$T_i(t)$作時間演化 (簡諧振盪)後，到要求的時間t再組合！

起始各個配料$X_i(x)$依配方c_i收集

各個配料按分離烹煮$T_i(t)\sim\cos(\omega_i t)$

各自演化後，最後合體！

$=$



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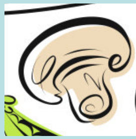

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$\vdots$

$$\phi(x,t)=\sum_{i=1}^{\infty}c_iX_i(x)\cdot T_i(t)=\sum_{i=1}^{\infty}c_iX_i(x)(A_i\cos\omega_i t+B_i\sin\omega_i t)$$

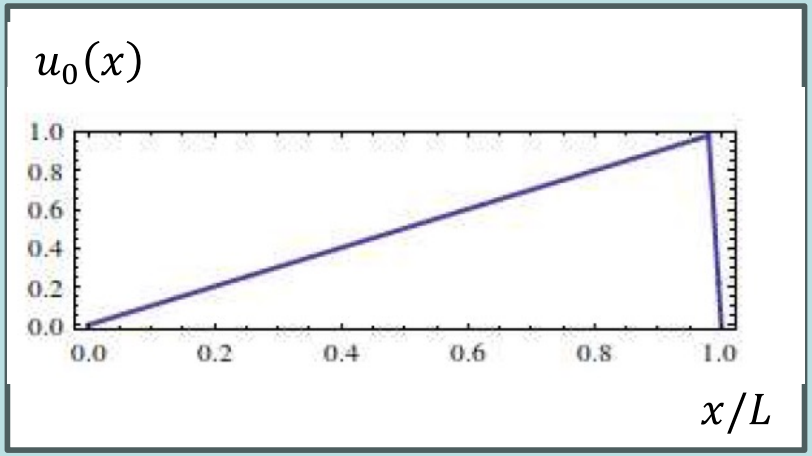
這樣的波函數顯然滿足波方程式，又滿足起始條件，就是唯一解。

$u_n(x,t)$的線性組合是一般解。

$$\phi(x,t)=\sum_{n=1}^{\infty}c_nX_n(x)(A_n\cos\omega_n t+B_n\sin\omega_n t)$$
$$=\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)(A_n\cos\omega_n t+B_n\sin\omega_n t)$$

係數由起始條件決定：

$$\phi(x,0)=u_0(x)\qquad \frac{\partial}{\partial t}\phi(x,0)=v_0(x)$$



起始條件代入前式(將c_nA_n改稱A_n)：

$$u_0(x)=\sum_{n=1}^{\infty}A_nX_n(x)=\sum_{n=1}^{\infty}A_n\cdot\sin\left(\frac{n\pi}{L}x\right)$$

這是將起始條件$u_0(x)$分解為本徵函數$X_n(x)=\sin\left(\frac{n\pi}{L}x\right)$的線性組合。

$$v_0(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}v\cdot\sin\left(\frac{n\pi}{L}x\right)$$

這是將起始條件$v_0(x)$分解為本徵函數$X_n(x)=\sin\left(\frac{n\pi}{L}x\right)$的線性組合。

為討論簡化起見，設起始速度為零：$v_0(x) = 0$，因此$B_n = 0.$

只需將起始條件$u_0(x)$分解為本徵函數$X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$的線性組合。

在之前的三個粒子彈簧組的coupled masses例子

任一起始條件的行向量$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$，都可以展開成三個本徵向量的線性組合。

關鍵是：根據對稱矩陣本徵向量的定理，三個本徵向量彼此正交 $\boldsymbol{a}^{(m)T}\boldsymbol{a}^{(n)} = \delta_{mn}$

猜測如同矩陣的本徵向量，算子的本徵函數彼此也正交！

函數的推廣的內積

$$\sum_{j=1}^{3} \boldsymbol{a}_j^{(m)T}\boldsymbol{a}_j^{(n)} = \delta_{mn} \quad \Rightarrow \quad \sum_{j=1}^{\infty} X_m(dj)X_n(dj) \to \int_0^L dx\ X_m(x)X_n(x) =?\ \delta_{mn}$$

Method 0

直接計算推廣的內積：$\int_0^L dx\ X_m(x)X_n(x)$

若$m \neq n$

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \frac{1}{2}\int_0^L dx\left[\cos\frac{\pi}{L}(m-n)x - \cos\frac{\pi}{L}(m+n)x\right]$$

$$= \frac{L}{2\pi}\left[\frac{1}{m-n}\sin\frac{\pi}{L}(m-n)x + \frac{1}{m+n}\sin\frac{\pi}{L}(m+n)x\right]\Bigg|_0^L = 0$$

正弦函數是週期函數，上式為零。 $\int_0^L dx\ X_m(x)X_n(x) = 0$

算子不同本徵值的本徵函數彼此正交！

若$m = n$，

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = \frac{1}{2}\int_0^L dx\left[1-\cos\frac{2n\pi}{L}x\right]$$

$$= \frac{L}{2} + \frac{2n\pi}{L}\left[\sin\frac{2n\pi}{L}x\right]\Bigg|_0^L = \frac{L}{2}$$

我們可以重新定義本徵函數，使它的推廣內積為1。

$$X_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L dx\ X_n^2 = 1$$

本徵函數為正交歸一Orthonormal的一系列函數。

$$\int_0^L dx\ X_m(x)X_n(x) = \delta_{mn}$$

Kronecker δ Function

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

若有Orthonormal的一系列函數，任一函數很容易分解為此系列的線性組合。

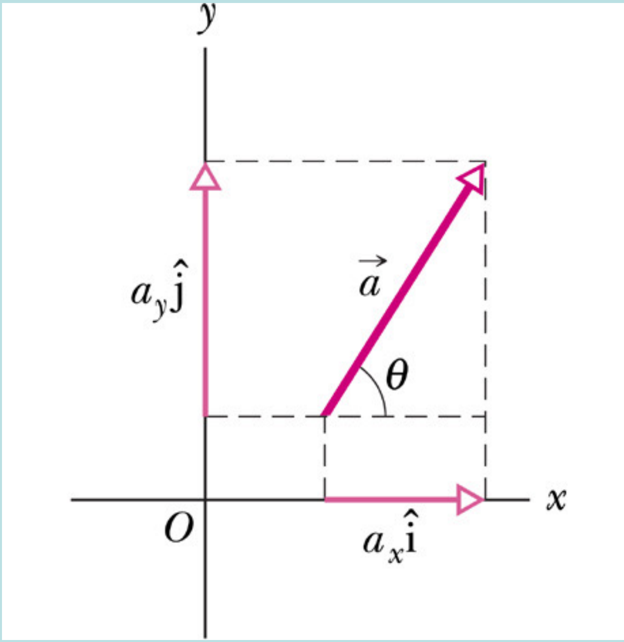
展開定理

給定任一向量：$\boldsymbol{v}$，可以展開如下：

$$\boldsymbol{v} = c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{v} = \boldsymbol{u}^{(1)} \cdot \boldsymbol{v} = c_1$$

$$\boldsymbol{u}^{(2)T}\boldsymbol{v} = \boldsymbol{u}^{(2)} \cdot \boldsymbol{v} = c_2$$



向量可以用它的分量表示！

把一向量投影在選取的座標軸上即是它的分量！

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (a_x, a_y)$　　分量法

$a_x = \hat{\imath}\cdot\vec{a}, \qquad a_y = \hat{\jmath}\cdot\vec{a}$

這兩個orthonormal的本徵向量$\boldsymbol{u}_1, \boldsymbol{u}_2$，類似組成一組座標軸的單位向量$\hat{\imath}, \hat{\jmath}$！

這是將起始條件$u_0(x)$分解為本徵函數$X_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$的線性組合。

$$u_0(x)=\sum_{n=1}^{\infty}A_n\cdot\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$\boldsymbol{u}^{(1)T}\boldsymbol{v}=\boldsymbol{u}^{(1)}\cdot\boldsymbol{v}=c_1$$

$$\Updownarrow$$

若是這樣的分解是正確的，取函數u_0與本徵函數X_n的推廣內積就能得到分解的係數：

$$\sqrt{\frac{2}{L}}\int_0^L dx\,u_0(x)\sin\left(\frac{n\pi}{L}x\right)=\frac{2}{L}\int_0^L dx\left[\sum_{m=1}^{\infty}A_m\cdot\sin\left(\frac{m\pi}{L}x\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

$$=\frac{2}{L}\sum_{m=1}^{\infty}\left[A_m\int_0^L dx\,\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\right]=\sum_{m=1}^{\infty}[A_m\delta_{mn}]=A_n$$

$$A_n=\sqrt{\frac{2}{L}}\int_0^L dx\;u_0(x)\sin\left(\frac{n\pi}{L}x\right)\quad\Longleftrightarrow\quad\boldsymbol{u}^{(1)T}\boldsymbol{v}=\boldsymbol{u}^{(1)}\cdot\boldsymbol{v}=c_1$$

這就是Fourier Series！

$$\phi(x,t)=\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

$$\frac{d^2X_n}{dx^2}=\lambda_n X_n$$

乘上X_m後，兩邊作積分

Method 1

$$\int_0^L dx\, X_m \frac{d^2X_n}{dx^2}=\int_0^L dx\;\lambda_n X_m X_n=\int_0^L dx \frac{dX_m}{dx}\frac{dX_n}{dx}+\int_0^L dx\,\frac{d}{dx}\left(X_m\frac{dX_n}{dx}\right)$$

$$=\int_0^L dx \frac{dX_m}{dx}\frac{dX_n}{dx}+\left(X_m\frac{dX_n}{dx}\right)\Bigg|_0^L=\int_0^L dx \frac{dX_m}{dx}\frac{dX_n}{dx}$$

$$\lambda_n\int_0^L dx\;X_m X_n=\int_0^L dx \frac{dX_m}{dx}\frac{dX_n}{dx}$$

將n,m互換，等式依舊成立。

$$\lambda_m\int_0^L dx\;X_m X_n=\int_0^L dx \frac{dX_n}{dx}\frac{dX_m}{dx}$$

兩式相減，右邊抵消。

$$(\lambda_n-\lambda_m)\int_0^L dx\;X_m X_n=0$$

$$\int_0^L dx\;X_m X_n=0$$

算子的本徵函數為正交歸一Orthonormal的一系列函數。

$$u_0(x) = \sum_{n=1}^{\infty} A_n \cdot X_n(x)$$

以上的結果也可以直接由本徵函數正交歸一的特性直接導出：

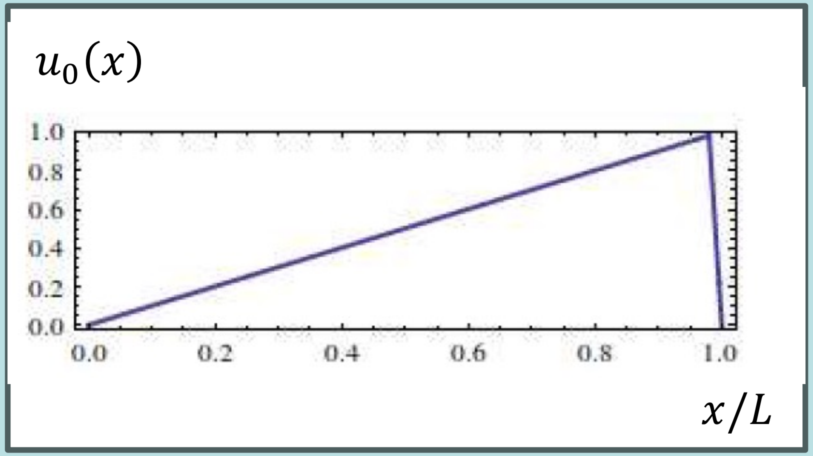
若是這樣的分解是正確的

$$\int_0^L dx\, u_0(x)\, X_n(x) = \frac{2}{L}\int_0^L dx \left[\sum_{m=1}^{\infty} A_m \cdot X_m(x)\right] \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{m=1}^{\infty}\left[A_m \int_0^L dx\, X_m(x) X_n(x)\right] = \sum_{m=1}^{\infty}\left[A_m \delta_{mn}\right] = A_n$$

$$A_n = \int_0^L dx\, u_0(x)\, X_n(x) \quad \Leftrightarrow \quad \boldsymbol{u}^{(1)T}\boldsymbol{v} = \boldsymbol{u}^{(1)} \cdot \boldsymbol{v} = c_1$$

考慮起始條件為一個Sawtooth鋸齒波形



$$u_0(x) = \frac{1}{L}x$$

$$0 < x < L$$

$$A_n = \sqrt{\frac{2}{L}}\int_0^L dx\ \ u_0(x)\sin\left(\frac{n\pi}{L}x\right) = \sqrt{\frac{2}{L}}\int_0^L dx\frac{1}{L}x\cdot\sin\left(\frac{n\pi}{L}x\right)$$

$$= \sqrt{\frac{2}{L}}\frac{1}{n\pi}\left[x\cos\left(\frac{n\pi}{L}x\right)\right]\Bigg|_0^L + \sqrt{\frac{2}{L}}\frac{1}{n\pi}\int_0^L dx\ \ \cos\left(\frac{n\pi}{L}x\right)$$

$$= \sqrt{2L}\frac{1}{n\pi}\cos(n\pi) + \sqrt{2L}\left(\frac{1}{n\pi}\right)^2\left[x\sin\left(\frac{n\pi}{L}x\right)\right]\Bigg|_0^L = \sqrt{2L}\frac{(-1)^n}{n\pi}$$

$$\phi(x,t) =?\sum_{n=1}^{\infty} 2\frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\cdot\cos\omega_n t$$

這個解既滿足波方程式，又滿足起始條件，因此是唯一解。

若是本徵函數的分解是正確的，可計算任一合邊界條件的起始條件函數的分解係數。

但這樣的分解真的是正確的嗎？分解所得的級數真的與被分解的函數相等嗎？

物理直覺：合理的起始條件下，波方程式應該都有解，因此這樣的分解一定可行 。

數學上相等與否的判準：這些係數組成的$X_n(x)$的線性組合，會趨近被分解的函數。

$$\lim_{N\to\infty}\int_0^L dx\left[u_0(x)-\sum_{n=1}^{N}A_n\cdot X_n(x)\right]^2=0$$

兩者差距的平方和(積分)$N\to\infty$時，趨近於零。

這稱為Completeness.完備性，可以嚴格證明的。

但在波方程式的例子中，起始條件的展開就是傅立葉級數的一部分。

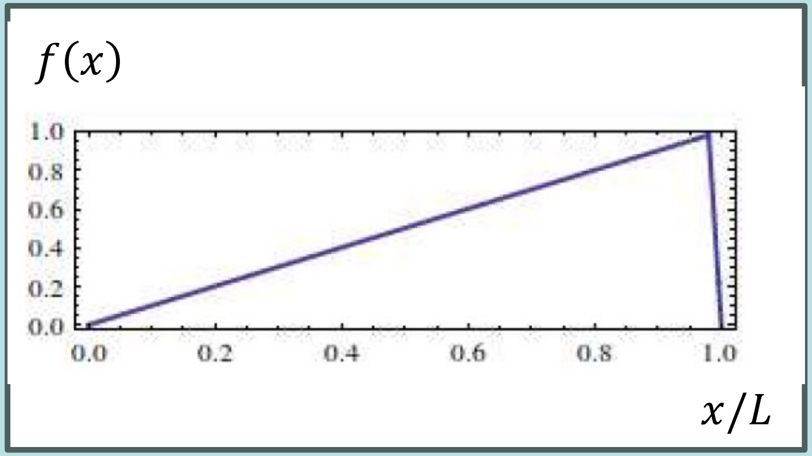
猜測且相信 ：三角函數是Complete完備的。傅立葉級數可以趨近任一週期函數！

我們將先說明X_n的展開就是傅立葉級數，因此這一系列的X_n也是完備的。

下學期再說明滿足某些條件的算子的本徵函數也都是完備的。

任一滿足邊界條件的起始條件 $f(x)$ 都可分解：

$$f(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$



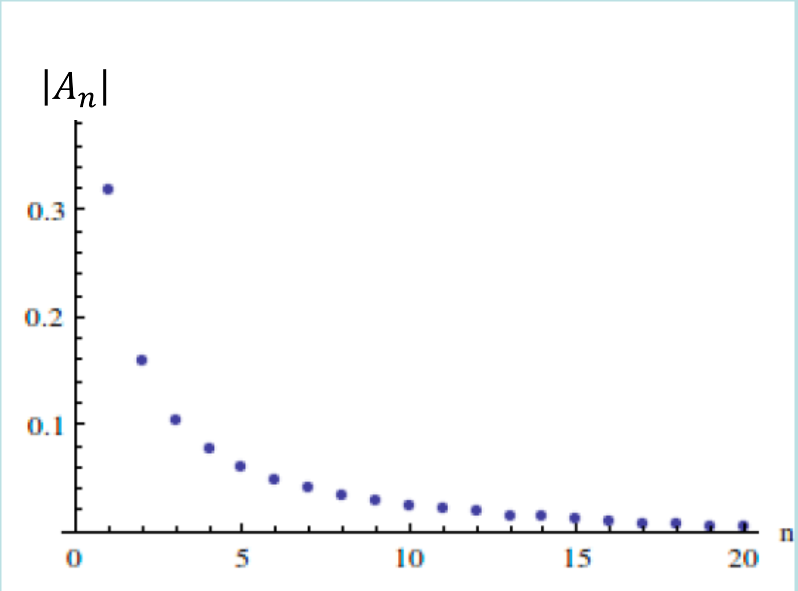
線段內的連續函數等價於$n \to \infty$維向量$R^{n=\infty}$！

$$f(x) \sim (A_1, A_2 \cdots A_j \cdots) \in R^{n=\infty}$$

函數空間等價於無限但可數維度的向量空間。

Jean-Baptiste Joseph Fourier

1768-1830



Chapter 3
Fourier Series

首先是傅立葉級數

Just before 1800, the French mathematician/physicist/engineer Jean Baptiste Joseph Fourier made an astonishing discovery, [**42**]. Through his deep analytical investigations into the partial differential equations modeling heat propagation in bodies, Fourier was led to claim that "every" function could be represented as an infinite series of elementary trigonometric functions: sines and cosines. For example, consider the sound produced by a musical instrument, e.g., piano, violin, trumpet, or drum. Decomposing the signal into its trigonometric constituents reveals the fundamental frequencies (tones, overtones, etc.) that combine to produce the instrument's distinctive timbre. This Fourier decomposition lies at the heart of modern electronic music; a synthesizer combines pure sine and cosine tones to reproduce the diverse sounds of instruments, both natural and artificial, according to Fourier's general prescription.

Fourier's claim was so remarkable and counterintuitive that most of the leading mathematicians of the time did not believe him. Nevertheless, it was not long before scientists came to appreciate the power and far-ranging applicability of Fourier's method, thereby opening up vast new realms of mathematics, physics, engineering, and beyond. Indeed, Fourier's discovery easily ranks in the "top ten" mathematical advances of all time, a list that would also include Newton's invention of the calculus, and Gauss and Riemann's differential geometry, which, 70 years later, became the foundation of Einstein's general relativity. Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytic tool for solving a broad range of linear partial differential equations. Applications in physics, engineering, biology, finance, etc., are almost too numerous to catalogue: typing the word "Fourier" in the subject index of a modern science library will dramatically demonstrate just how ubiquitous these methods are. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CDs and DVDs, cell phones, movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded on the many ramifications of Fourier theory. In your career as a mathematician, scientist, or engineer, you will find that Fourier theory, like calculus and linear algebra, is one of the most basic weapons in your mathematical arsenal. Mastery of the subject is essential.

傳立葉級數Fourier Series

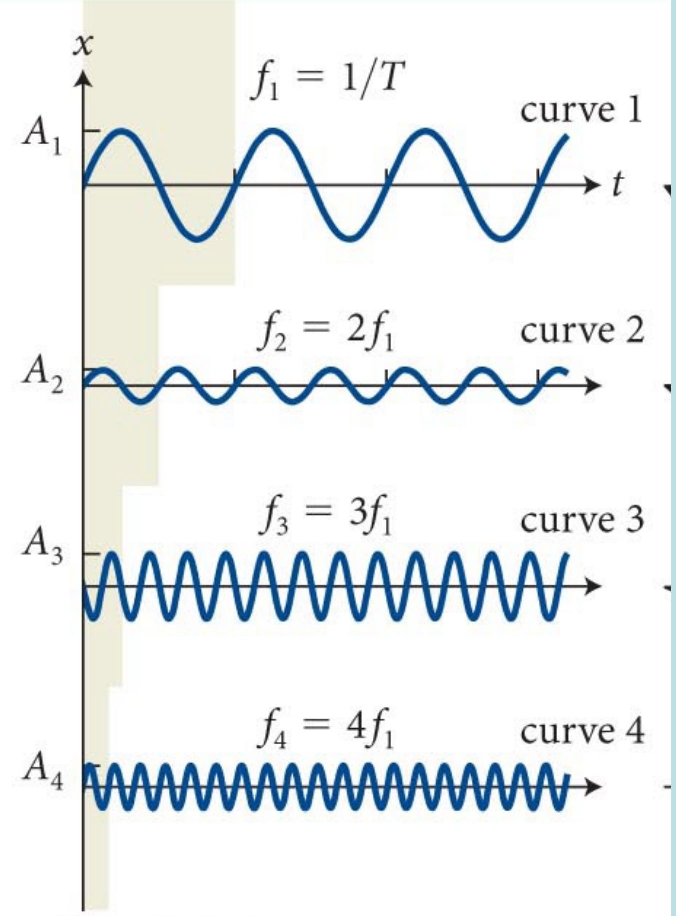
任一週期函數可以分解成一系列正弦函數與餘弦函數的級數和：

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \sin nx + b_n \cdot \cos nx]$$

以上的函數$f(x)$週期是2π，定義於$-\pi < x < \pi$。

週期若如絃段為$2L$的週期函數，上式可以改為：

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cdot \sin \frac{n\pi}{L} x + b_n \cdot \cos \frac{n\pi}{L} x\right]$$



Jean-Baptiste Joseph Fourier

1768-1830

如果運用於時間的函數，例如訊號或聲音：

任一週期為T的週期函數(訊號)可以分解為：

$$x(t) = \sum_{n=1}^{\infty} [a_n \cdot \cos \omega_n t + b_n \cdot \sin \omega_n t]$$

各成分的頻率f_n是訊號函數頻率f的整數倍！

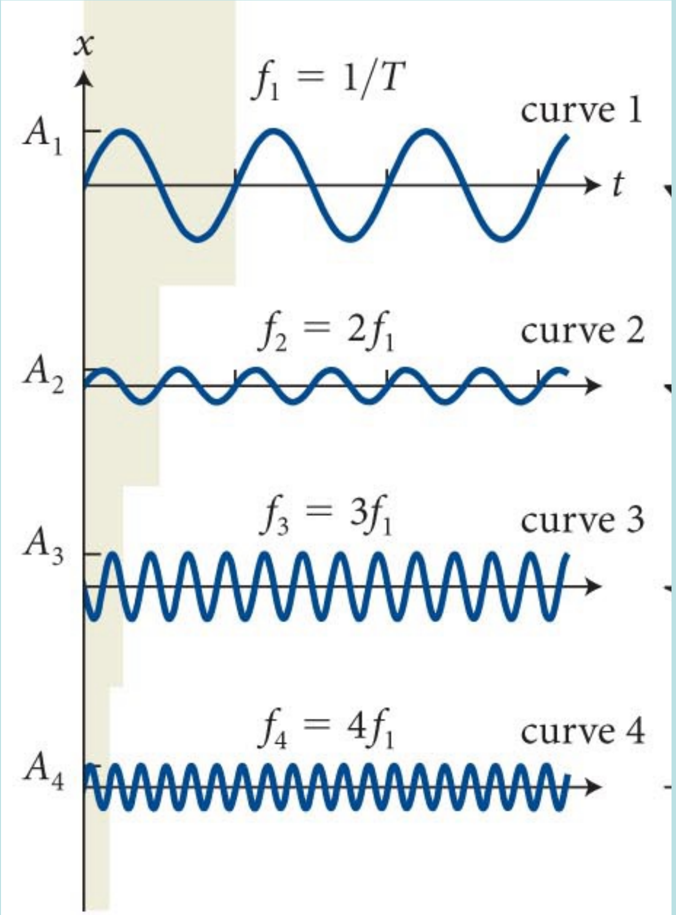
$$\omega_n = n\omega = n\frac{2\pi}{T}$$

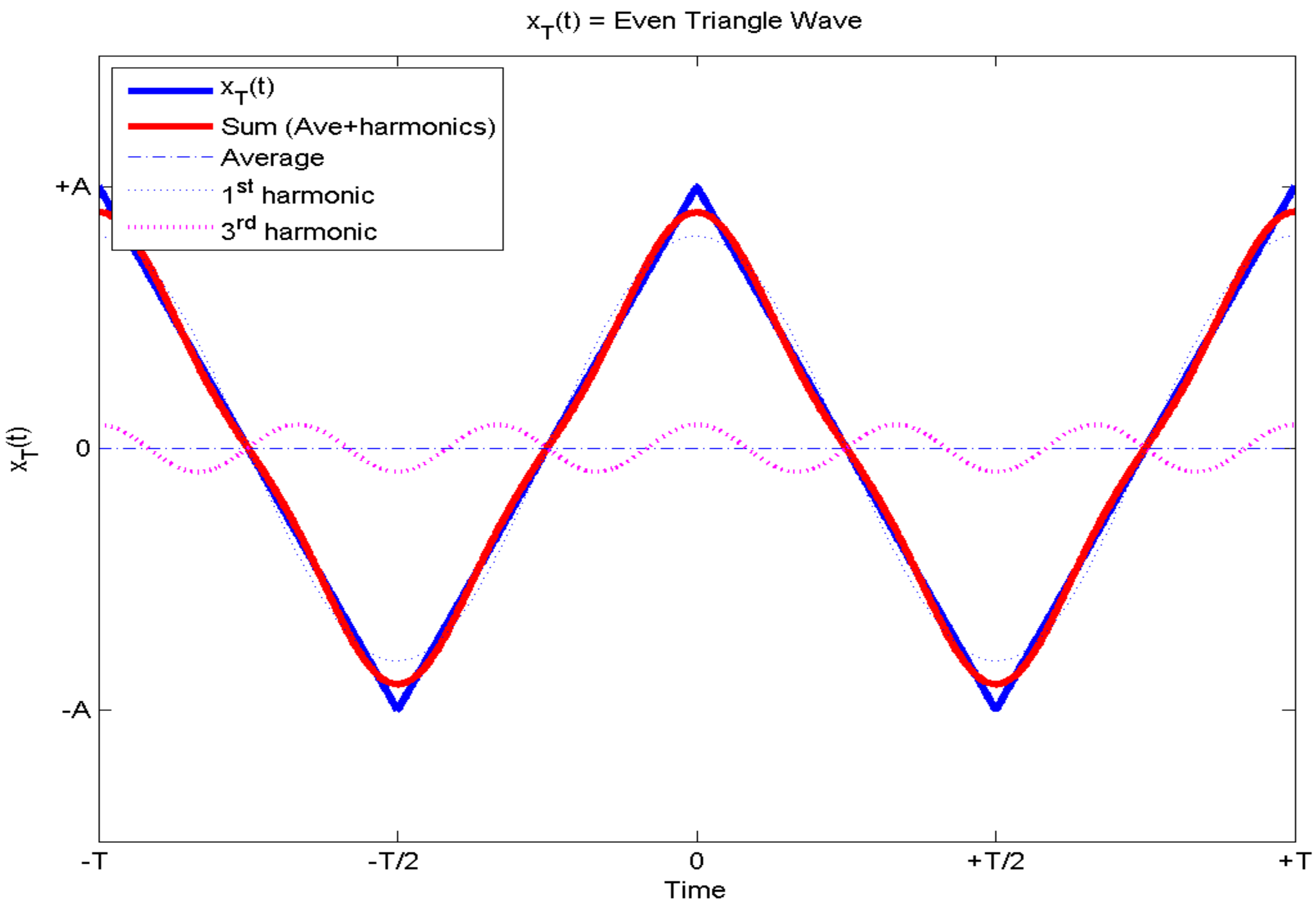
$$T_n = \frac{T}{n}$$

各成分的週期$T_n = \frac{T}{n}$是函數週期的整數分之一！

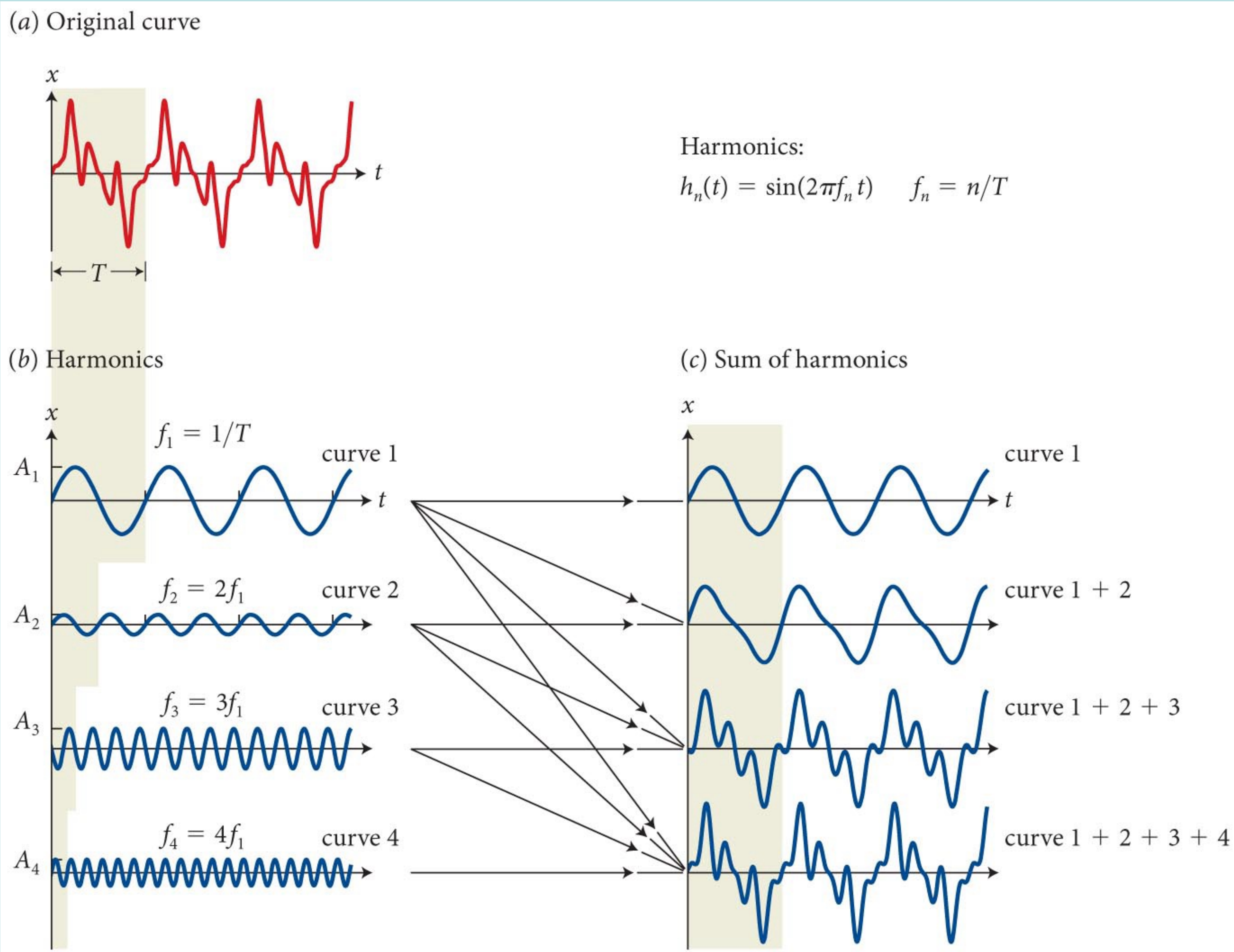
a_n, b_n即是各個成分的強度。

研究三角函數就等於研究所有週期函數。





xT(t) = Even Triangle Wave
xT(t)
Sum (Ave+harmonics)
Average
1st harmonic
3rd harmonic
+A
0
-A
xT(t)
-T
-T/2
0
+T/2
+T
Time


(a) Original curve
x
t
T
Harmonics:
$h_n(t) = \sin(2\pi f_n t) \quad f_n = n/T$
(b) Harmonics
x
$f_1 = 1/T$
curve 1
A_1
t
$f_2 = 2f_1$
curve 2
A_2
$f_3 = 3f_1$
curve 3
A_3
$f_4 = 4f_1$
curve 4
A_4
(c) Sum of harmonics
x
curve 1
t
curve 1 + 2
curve 1 + 2 + 3
curve 1 + 2 + 3 + 4

首先三角函數形成正交歸一的一系列函數：

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \frac{1}{2}\int_{-\pi}^{\pi} dx[\cos(m+n)x + \cos(m-n)x]$$

若$m \neq n$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m+n}\sin(m-n)x\right]\Bigg|_{-\pi}^{\pi}$$

正弦函數是週期函數，上式為零。

若$m = n$，

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \frac{1}{2}\int_{-\pi}^{\pi} dx[\cos(m+n)x + 1]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi$$

綜合兩式可以表示為：

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \pi\delta_{mn}$$

Kronecker δ Function

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

同理：

$$\int_{-\pi}^{\pi} dx \sin mx \sin nx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = \frac{1}{2}\int_{-\pi}^{\pi} dx[\sin(m+n)x + \sin(m-n)x]$$

若$m \neq n$

$$= \frac{1}{2}\left[\frac{1}{m+n}\cos(m+n)x + \frac{1}{m+n}\cos(m-n)x\right]\Bigg|_{-\pi}^{\pi} = 0$$

若$m = n$，

$$= \frac{1}{2}\left[\frac{1}{m+n}\cos(2m)x\right]\Bigg|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = 0$$

三角函數形成正交歸一的一系列函數：

$$\int_{-\pi}^{\pi} dx \cos mx \cos nx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \sin mx \sin nx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} dx \cos mx \sin nx = 0$$

The coefficients can be easily computed:

我們只要對$f(x)$成三角函數後積分即可。

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left[a_0 + \sum_{m=1}^{\infty} [a_m \cdot \cos mx + b_m \cdot \sin mx]\right] \cos nx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left[a_0 + \sum_{m=1}^{\infty} [a_m \cdot \cos mx + b_n \cdot \sin mx]\right] \cos nx = \sum_{m=1}^{\infty} [a_m \delta_{mn}] = a_n$$

完整的傅立葉級數公式：

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos nx + b_n \cdot \sin nx]$$

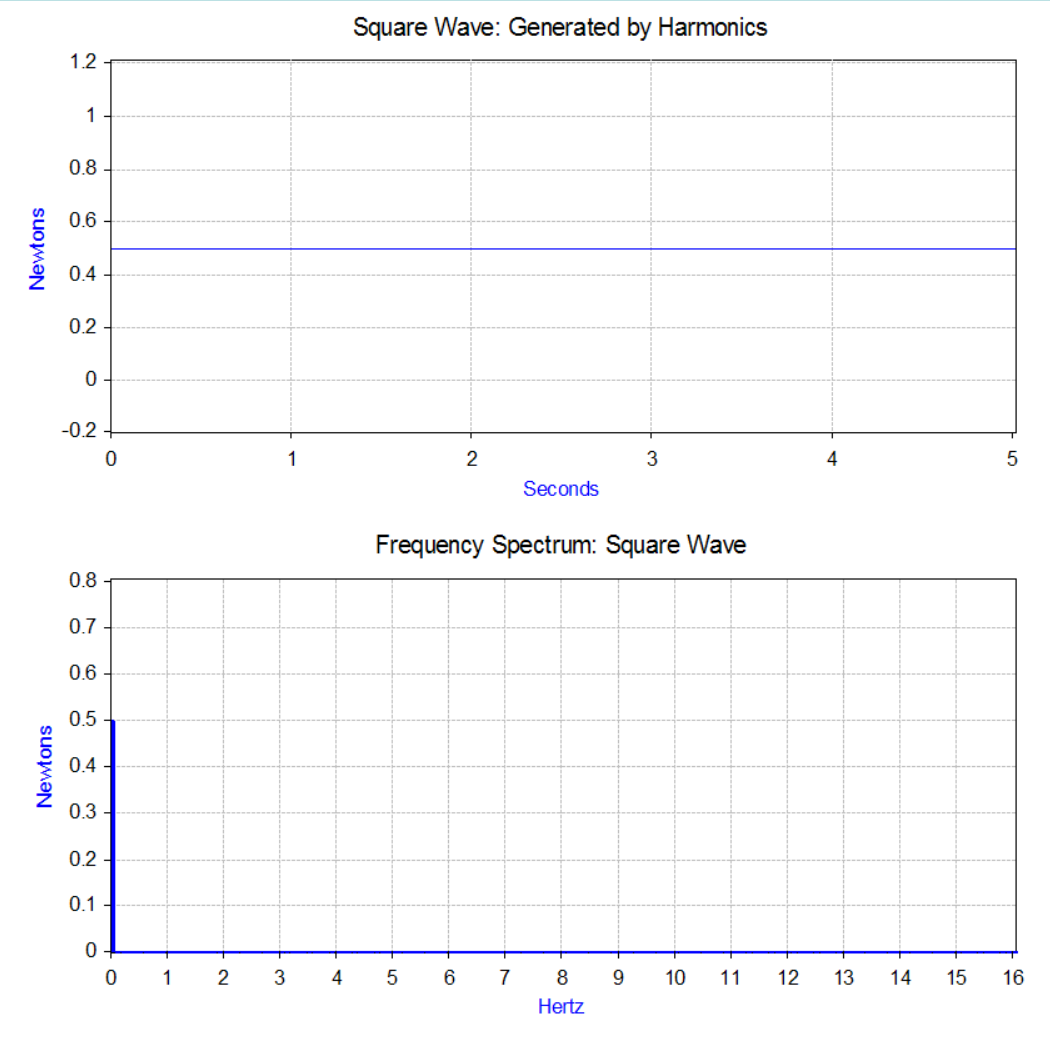
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, f(x)$$

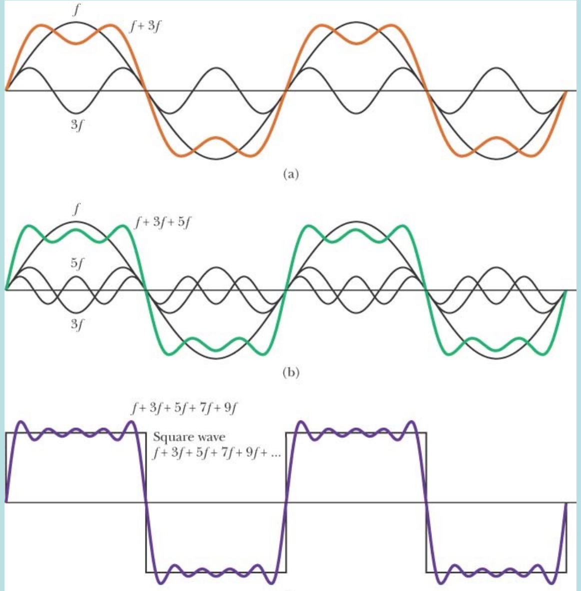
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\, f(x) \cos nx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\, f(x) \sin nx$$

三角函數是完備的。

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} dx \left[f(x) - a_0 + \sum_{n=1}^{N} [a_n \cdot \cos nx + b_n \cdot \sin nx] \right]^2 = 0$$



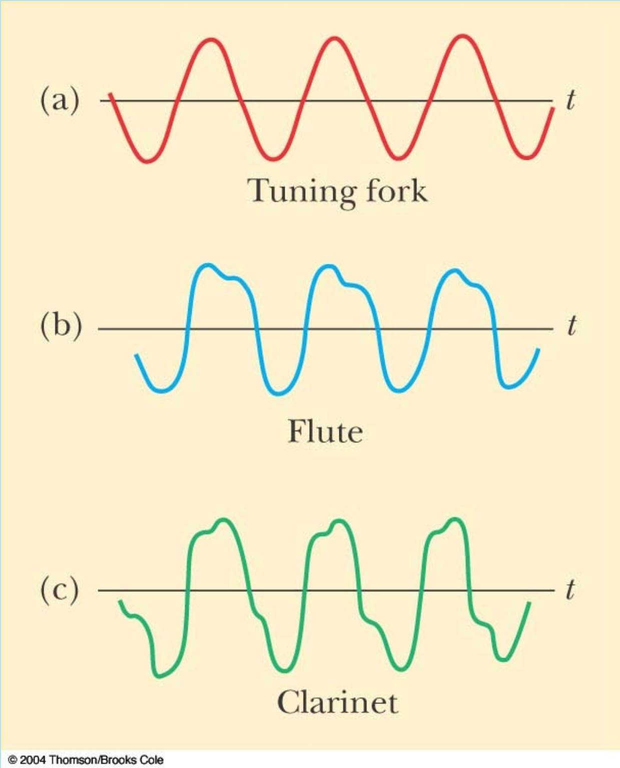


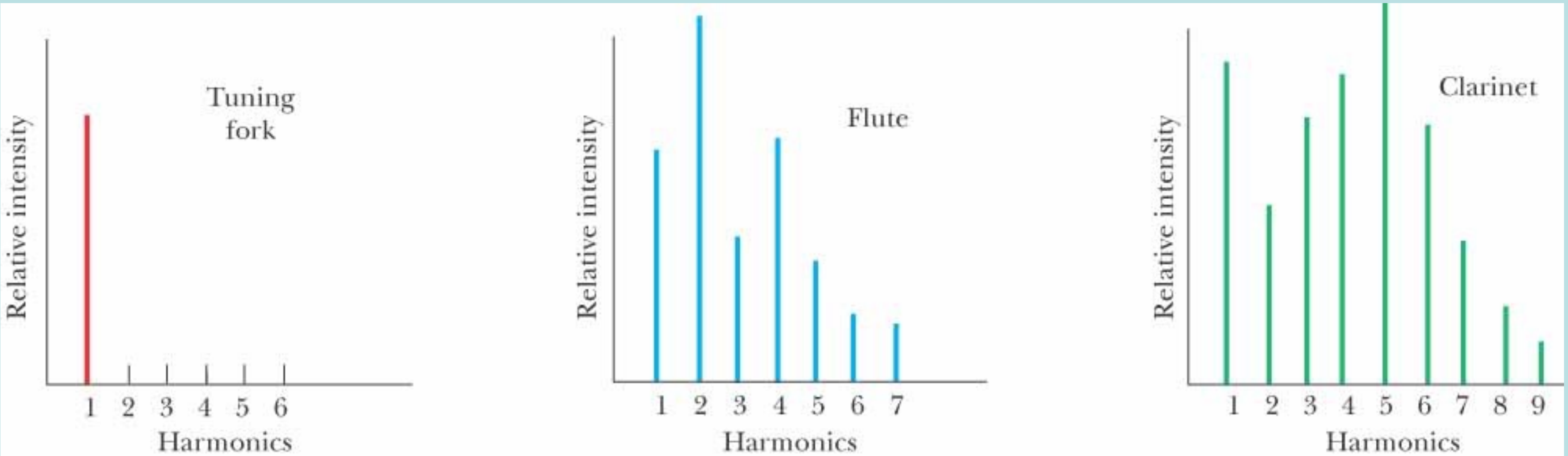
一個連續的函數被分解成一系列分離的數

線段內的連續函數等價於$n \to \infty$維向量$R^{n=\infty}$！

$$f(x) \sim (A_1, A_2 \cdots A_j \cdots) \in R^{n=\infty}$$

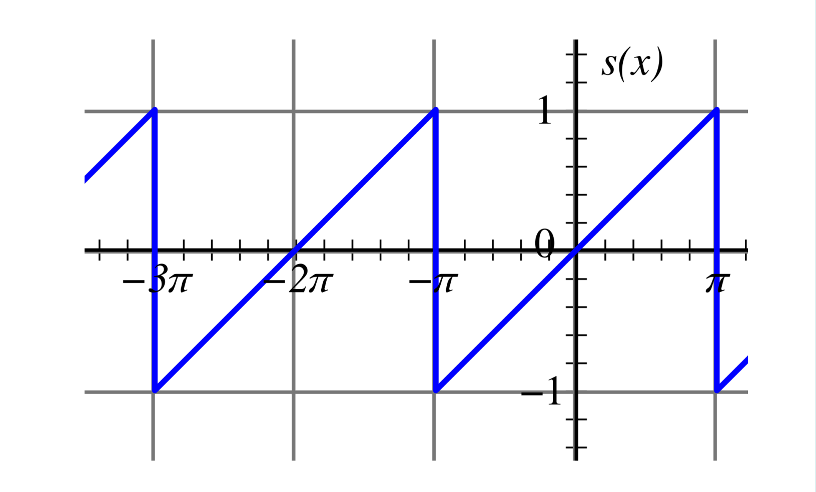
函數空間等價於無限但可數維度的向量空間。





一般來說，n 越大，強度越小，所以通常只要有限個成分就能近似一個周期運動！

例如：Sawtooth 鋸齒函數



$$u_0(x) = \frac{1}{\pi}x, \qquad -\pi < x < \pi$$

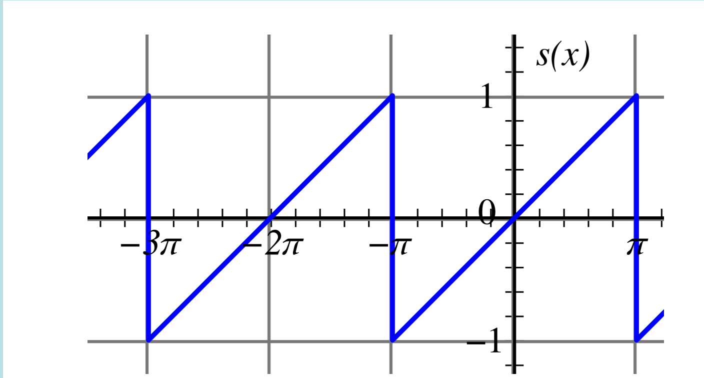
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \frac{1}{\pi} x\sin(nx) =$$

$$= \frac{1}{\pi^2}\frac{1}{n}[x\cos(nx)]|_{-\pi}^{\pi} + \frac{1}{\pi^2}\frac{1}{n}\int_{-\pi}^{\pi} dx\ \cos(nx)$$

$$= 2\frac{1}{n\pi}\cos(n\pi) + \left(\frac{1}{n\pi}\right)^2 \left[x\sin\left(\frac{n\pi}{L}x\right)\right]\Big|_{-\pi}^{\pi}$$

$$= 2\frac{(-1)^n}{n\pi}$$

$$u_0(x) = \frac{1}{\pi} x, \qquad -\pi < x < \pi$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, \frac{1}{\pi} x \cos(nx) = \frac{1}{\pi} \int_{-\pi}^{0} dx \, \frac{1}{\pi} x \cos(nx) + \frac{1}{\pi} \int_{0}^{\pi} dx \, \frac{1}{\pi} x \cos(nx)$$

$$= \frac{1}{\pi} \int_{0}^{\pi} dx \, \frac{1}{\pi} (-x) \cos(-nx) + \frac{1}{\pi} \int_{0}^{\pi} dx \, \frac{1}{\pi} x \cos(nx) = 0$$

餘弦函數是偶函數Even function $\cos(-nx) = \cos(nx)$

起始條件是奇函數Odd function $u_0(-x) = -u_0(x)$

相乘後為奇函數，在原點兩邊積分將互相抵消！

鋸齒函數的傅立葉級數展開如下：

$$\frac{1}{\pi} x = \sum_{n=1}^{\infty} \left(2 \frac{(-1)^n}{n\pi} \cdot \sin nx \right)$$

起始條件若是奇函數Odd function，展開只有奇函數的正弦函數有貢獻！

$$f(-x) = -f(x)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \sin nx$$

起始條件若是偶函數Even function，展開只有偶函數的餘弦函數有貢獻！

$$f(-x) = f(x)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \cdot \cos nx$$

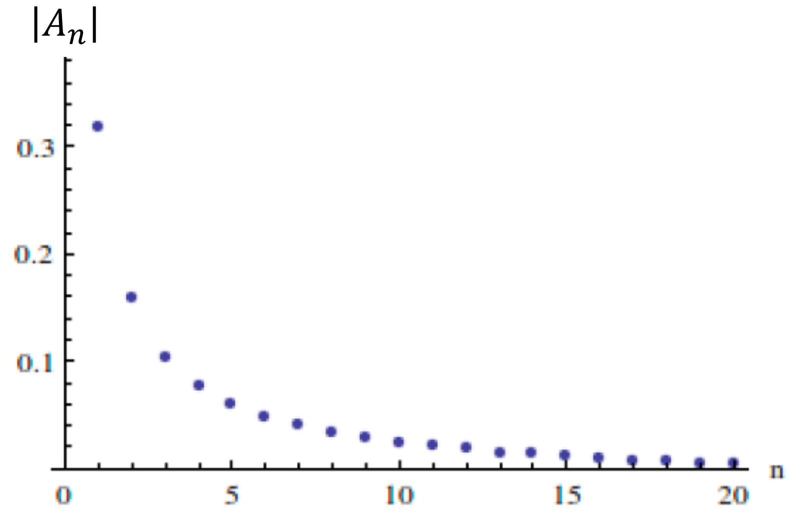
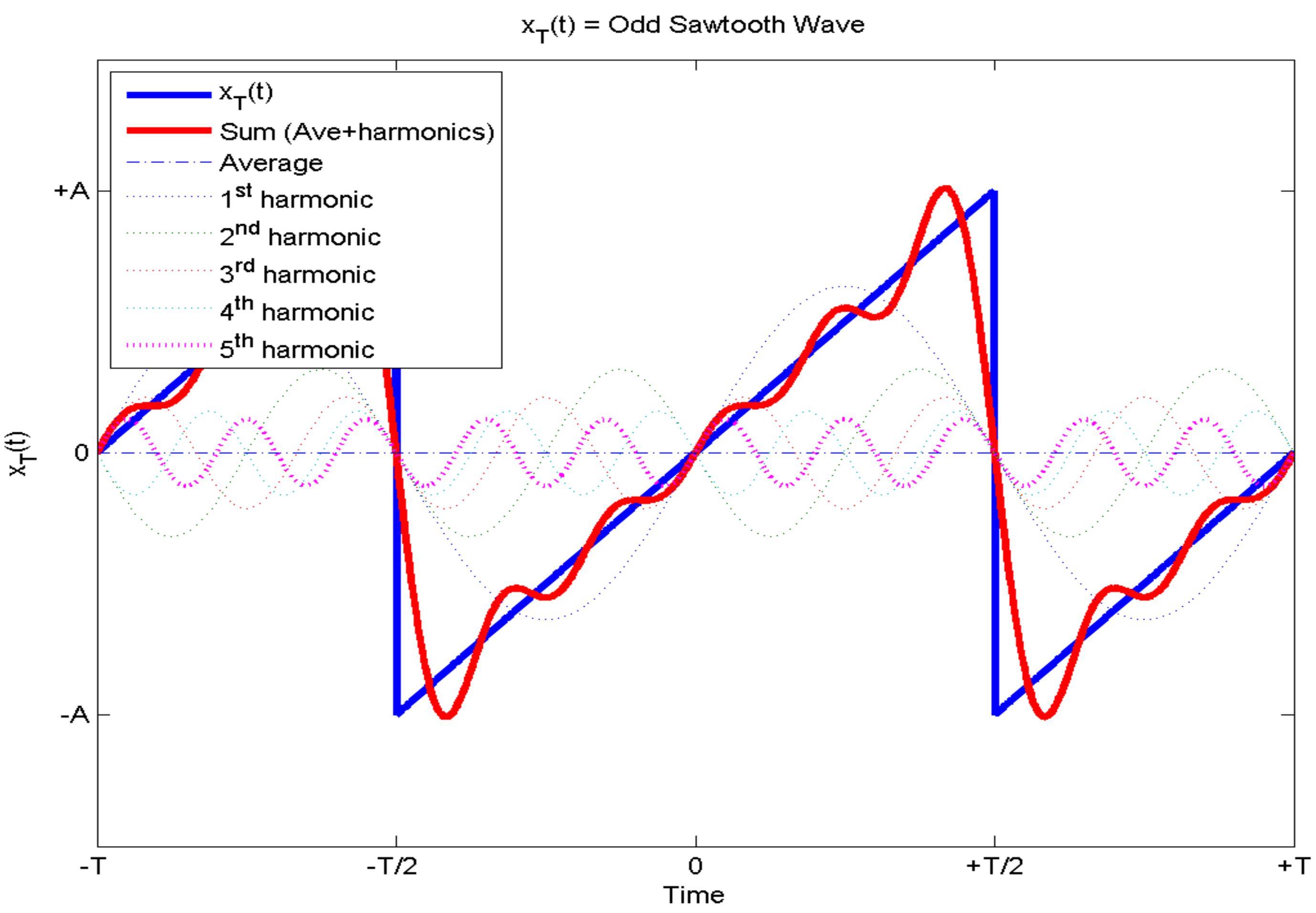
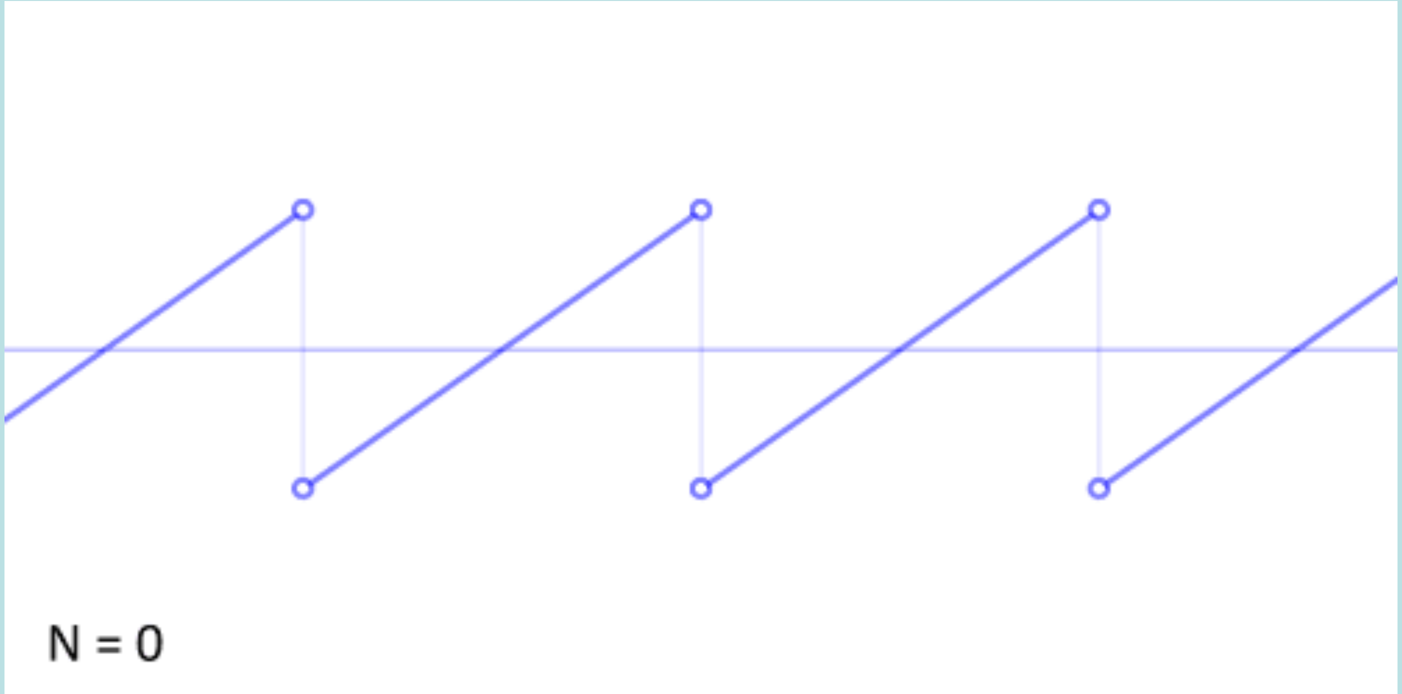


Figure 4. Amplitudes of the relative harmonics of a string plucked with a sawtooth plucking.


x_T(t) = Odd Sawtooth Wave
x_T(t)
Sum (Ave+harmonics)
Average
1st harmonic
2nd harmonic
3rd harmonic
4th harmonic
5th harmonic
+A
0
-A
x_T(t)
-T
-T/2
0
+T/2
+T
Time



數學家早已證明三角函數是完備的。

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} dx\left[f(x)-a_0+\sum_{n=1}^{N}[a_n\cdot\cos nx+b_n\cdot\sin nx]\right]^2=0$$

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases}$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

Solution

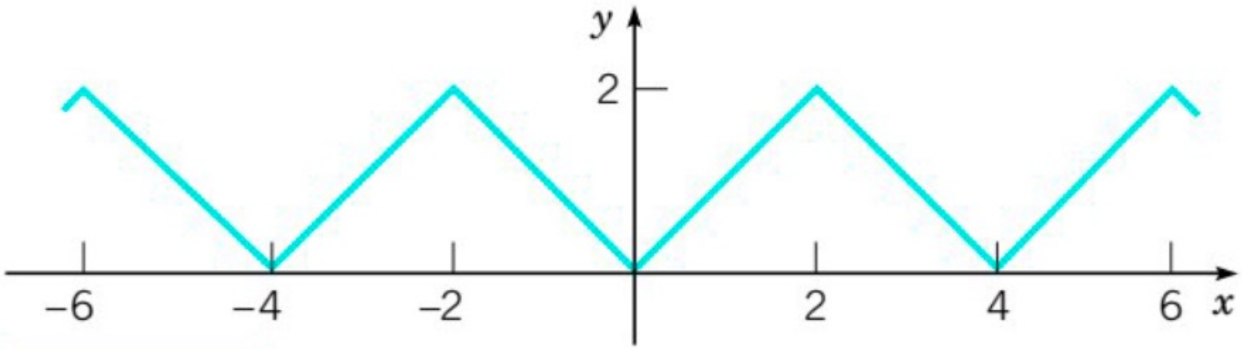


FIGURE 10.2.2 The triangular wave in Example 10.2.1.

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^2}\cos\left(\frac{m\pi x}{2}\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2}\right).$$

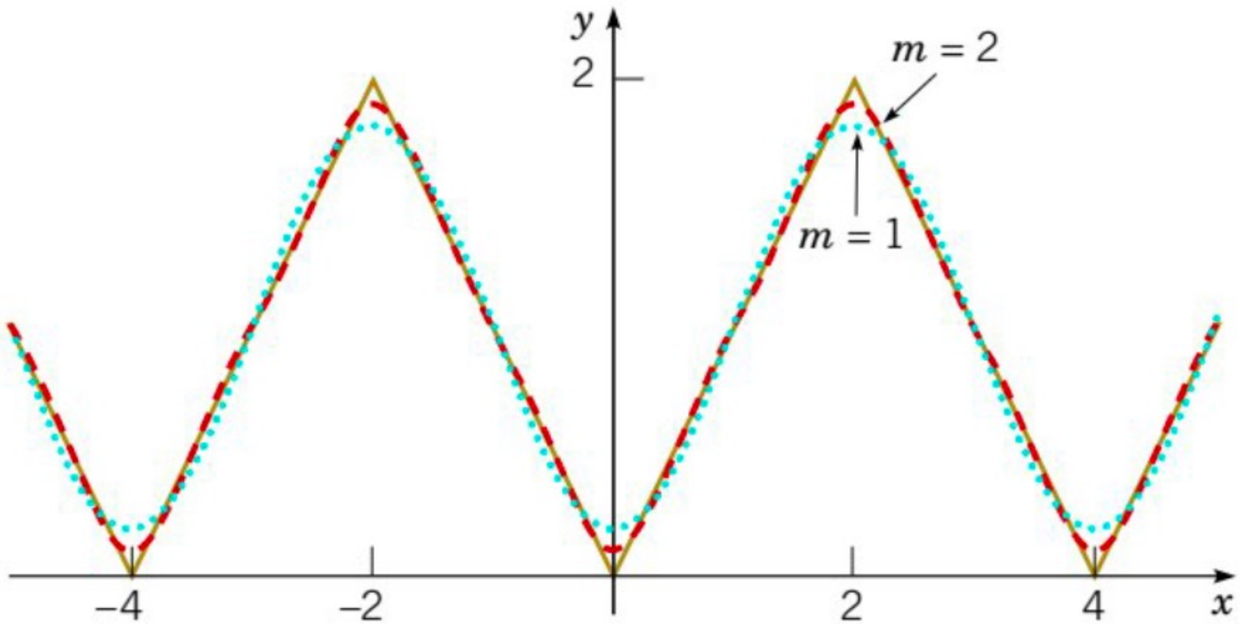


FIGURE 10.2.4 Partial sums in the Fourier series, equation (20) for $m = 1$ (dotted blue) and for $m = 2$ (dashed red), for the triangular wave (gold).

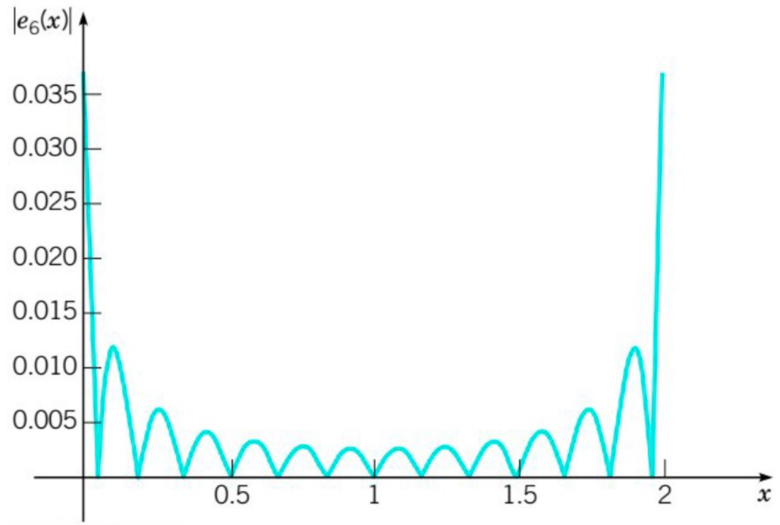


FIGURE 10.2.5 Plot of $|e_6(x)|$ versus x for the triangular wave in Example 10.2.3.

TABLE 10.2.1 Values of the Error $e_m(2)$ for the Triangular Wave

m	$e_m(2)$
2	0.09937
4	0.05040
6	0.03370
10	0.02025
15	0.01350
20	0.01013
25	0.00810

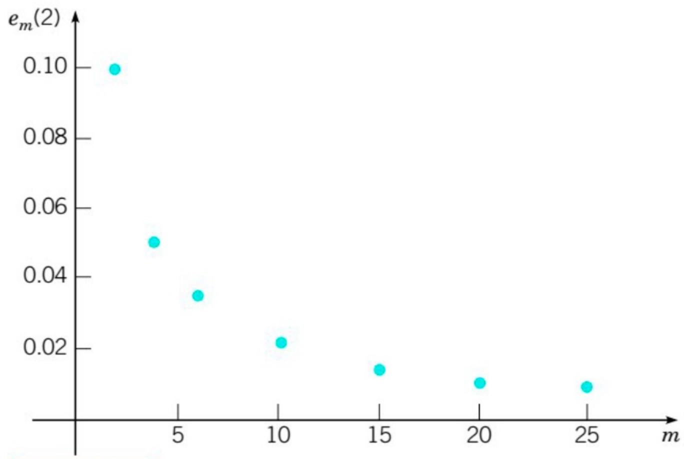
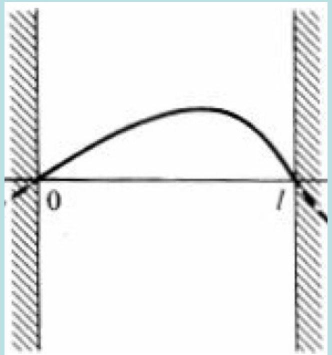


FIGURE 10.2.6 Plot of $e_m(2)$ versus m for the triangular wave.

傅立葉分析是分解週期函數為一系列正弦函數與餘弦函數的級數和。

波方程式的起始條件則是定義在一個線段中：

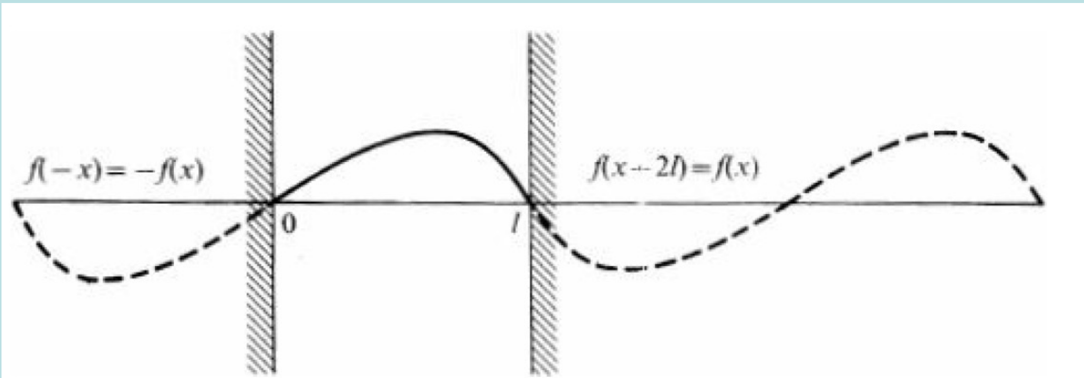
$$\phi(x,0) = u_0(x), \qquad 0 < x < L$$

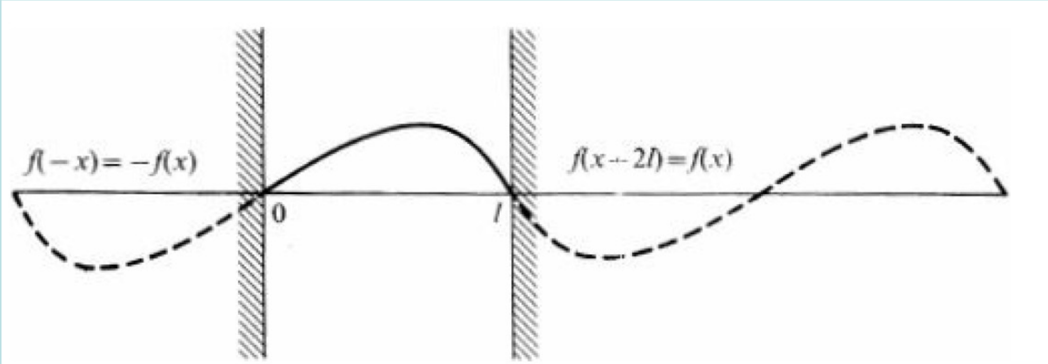


我們若將$u_0(x)$延展到整個空間，成為一週期函數，傅立葉分析就能用！

定義延展後的函數為一奇函數： $u_0(-x) = -u_0(x), \qquad -\infty < x < \infty$

同時是一週期為$2L$的週期函數： $u_0(x+2L) = u_0(x), \qquad -\infty < x < \infty$





$u_0(-x) = -u_0(x),$ ➡ $u_0(0) = -u_0(0) = 0$

$u_0(x+2L) = u_0(x),$ ➡ $u_0(L) = u_0(-L) = -u_0(L) = 0$

這個$u_0(x)$的延展，滿足邊界條件，同時在$0 < x < L$等同於原來的起始條件。

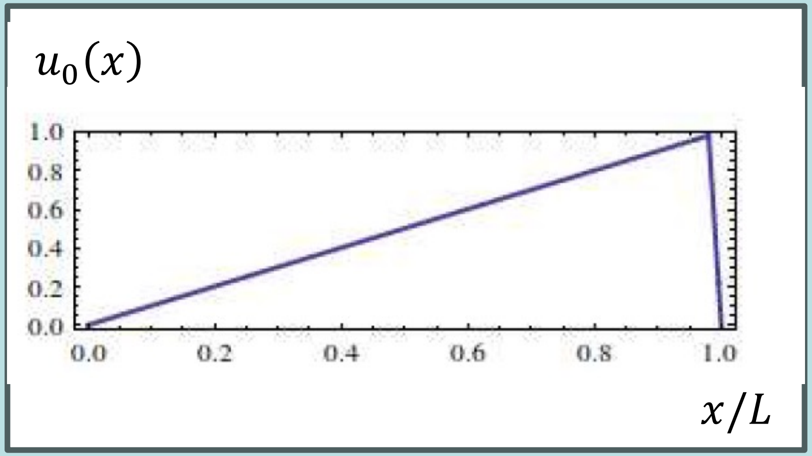
此延展為一週期函數，傅立葉分析就能用！

$$u_0(x) = \sum_{n=1}^{\infty}\left[A_n \cdot \sin\frac{n\pi}{L}x + B_n \cdot \cos\frac{n\pi}{L}x\right] \rightarrow = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi}{L}x$$

$u_0(x)$是奇函數，只有正弦函數留下。這個傅立葉數列是完備的！

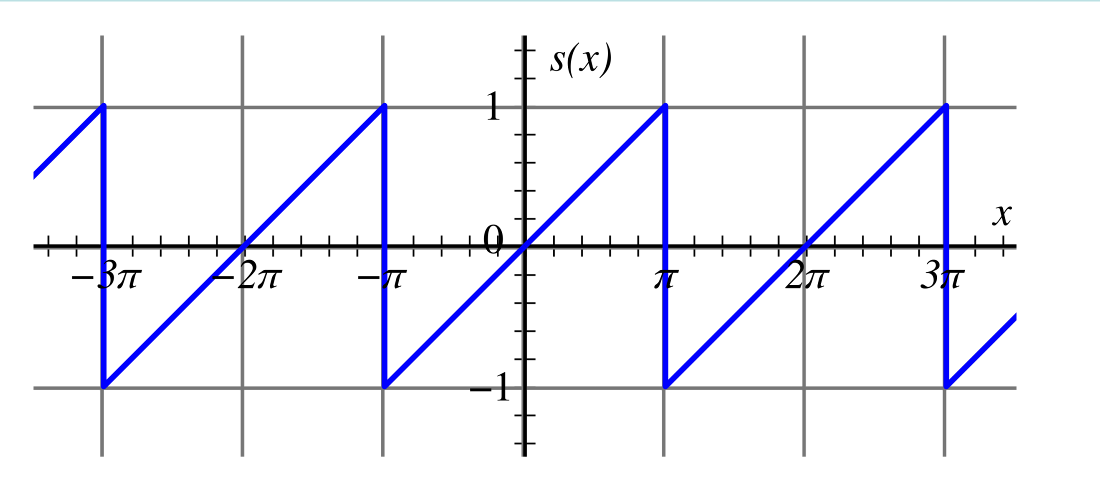
這一系列算子的本徵函數$X_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$是完備的！

考慮起始條件為一個Sawtooth鋸齒波形



$0 < x < L$

延展，滿足邊界條件，同時在$0 < x < L$等同於原來的起始條件。



$0 < x' < \pi$

傅立葉數列的結果，只要換一下變數即可直接用於固定端點的弦：

$$\frac{1}{\pi}x' = \sum_{n=1}^{\infty} 2\frac{(-1)^n}{n\pi} \cdot \sin nx'$$

$$x' = \frac{\pi}{L}x$$

$$\frac{1}{L}x = \sum_{n=1}^{\infty} 2\frac{(-1)^n}{n\pi}\sin\frac{n\pi}{L}x = \sum_{n=1}^{\infty} \sqrt{2L}\frac{(-1)^n}{n\pi} \cdot X_n(x)$$

$$A_n = \sqrt{2L}\frac{(-1)^n}{n\pi}$$

所得到的係數與直接用本徵函數的正交歸一定理所得結果相同！

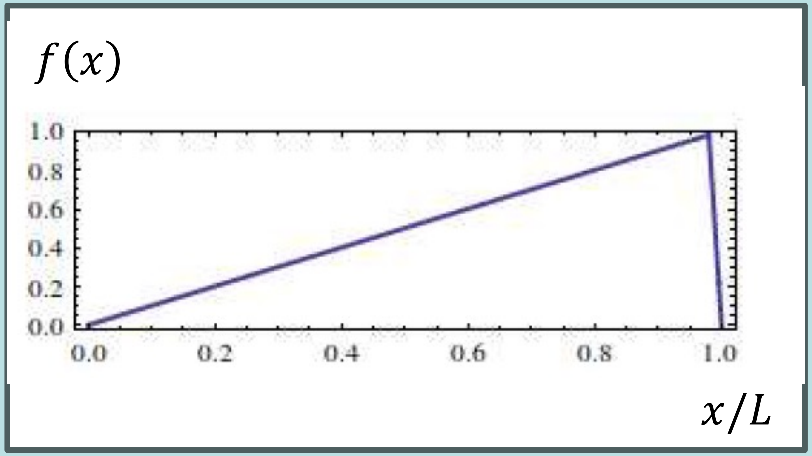
$$A_n = \int_0^L dx\, u_0(x)\, X_n(x) =$$

$$= \sqrt{2L}\frac{1}{n\pi}\cos(n\pi) + \sqrt{2L}\left(\frac{1}{n\pi}\right)^2 \left[x\sin\left(\frac{n\pi}{L}x\right)\right]\Big|_0^L = \sqrt{2L}\frac{(-1)^n}{n\pi}$$

這一系列算子的本徵函數$X_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$是完備的！

任一滿足邊界條件的起始條件 $f(x)$ 都可分解：

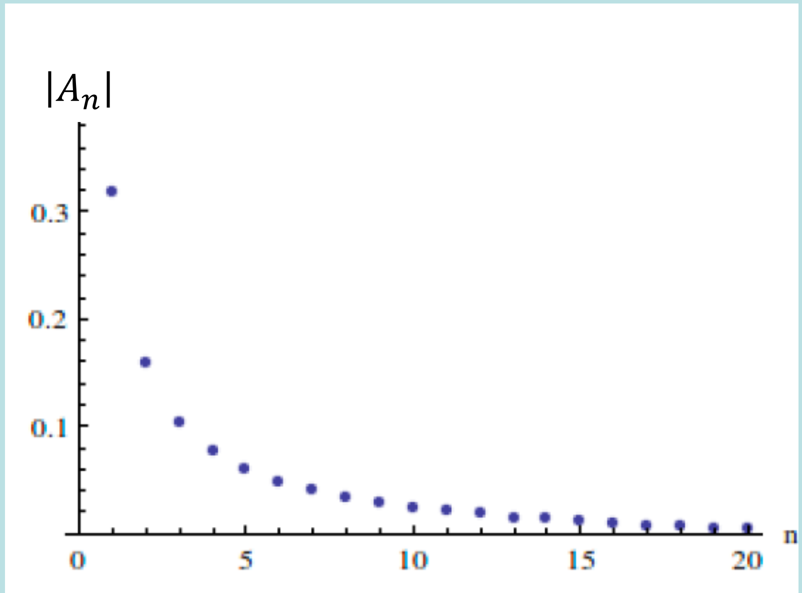
$$f(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$



線段內的連續函數等價於$n \to \infty$維向量$R^{n=\infty}$！

$$f(x) \sim (A_1, A_2 \cdots A_j \cdots) \in R^{n=\infty}$$

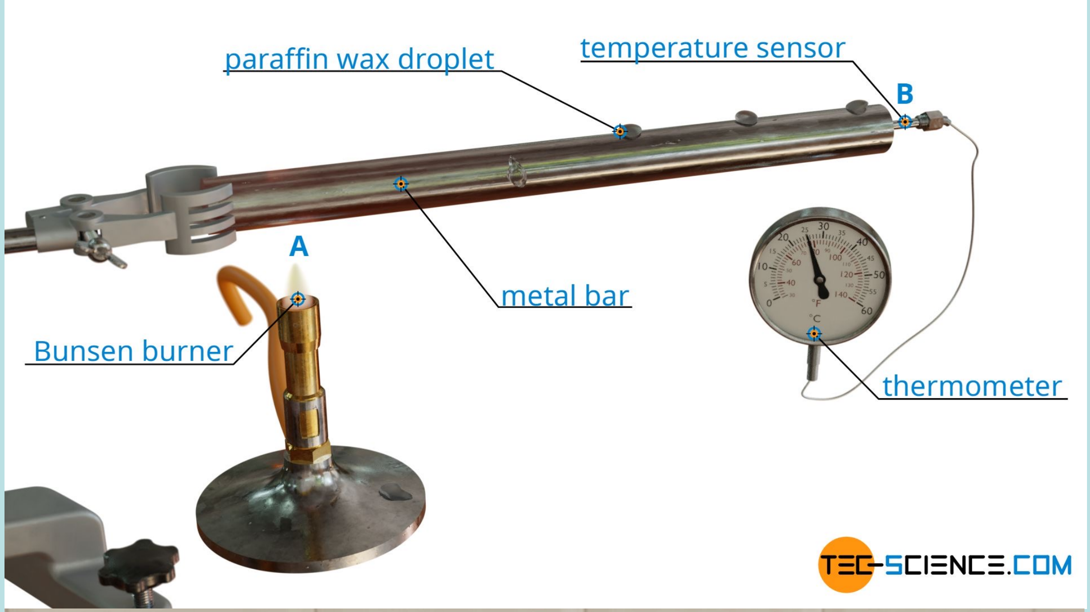
函數空間等價於無限但可數維度的向量空間。



Jean-Baptiste Joseph Fourier

1768-1830

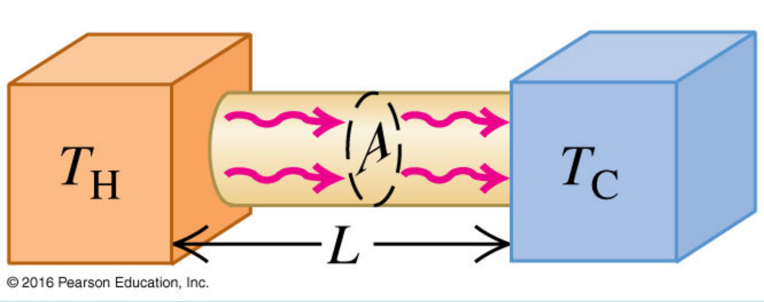
傅立葉分析熱傳導棒的溫度分佈

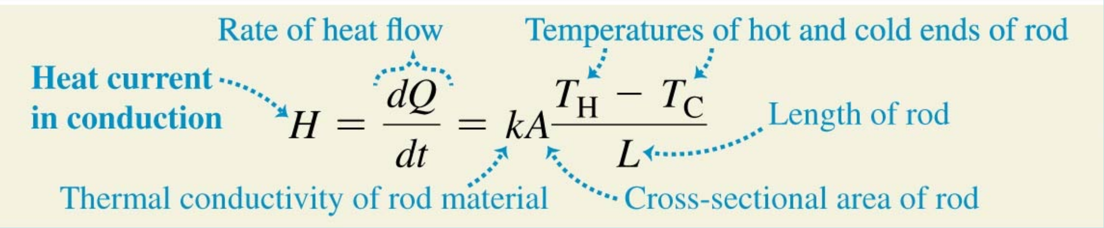


沿電位相等的兩點間沒有梯度電場。

溫度相等的兩點間則沒有傳導熱量流動。

兩相近點之間的溫度差驅動熱量的流動。



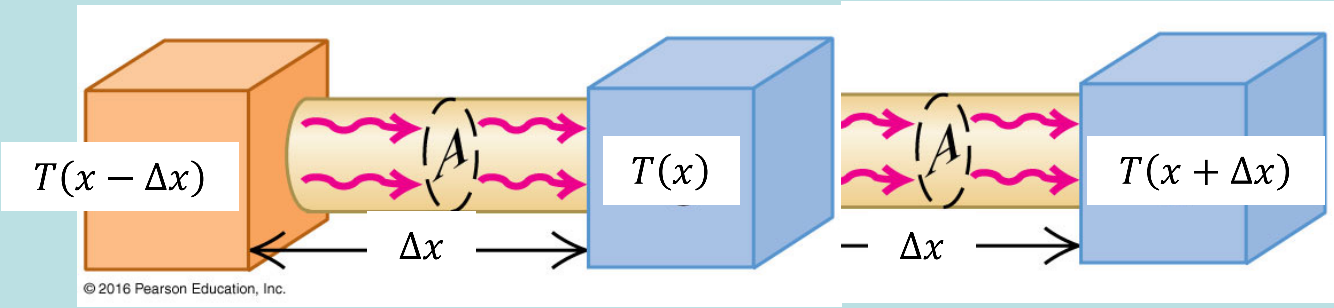


當溫度$T(x)$不同兩點非常靠近時：

$$H = \frac{dQ}{dt} = h\frac{\Delta T}{\Delta x} \to h\frac{dT}{dx}$$

$$H = h\frac{\Delta T}{\Delta x} \Leftrightarrow E = \frac{\Delta V}{\Delta x}$$

將一傳導棒視為一系列非常靠近的溫度$T(x)$不同的點：



中間這一點的熱量變化，正比於溫度變化，就等於兩邊熱量流動的差！

$$C\Delta x\frac{dT}{dt} = h\frac{dT}{dx}(x + \Delta x) - h\frac{dT}{dx}(x) = h\Delta x\frac{d^2T}{dx^2}$$

$$\frac{\partial T}{\partial t} \propto \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

The Heated Ring

So far, we have not paid any attention to boundary conditions. As noted above, these will eliminate nonphysical eigensolutions and thereby reduce the collection to a manageable, albeit still infinite, number. In this subsection, we will discuss a particularly important case, which, following Fourier's line of reasoning, leads us directly into the heart of Fourier series.

Consider the heat equation on the interval $-\pi \leq x \leq \pi$, subject to the *periodic boundary conditions*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(t,-\pi) = u(t,\pi), \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi). \tag{3.21}$$

HEAT CONDUCTIVITY RING - WOOD BASE

A$19.25

Code: **TCRING**

Share

這個問題很難實施固定端邊界條件。自由端較方便。

週期性邊界條件：

$$T(t,0) = T(t,2\pi)$$

$$\frac{\partial T}{\partial x}(t,0) = \frac{\partial T}{\partial x}(t,2\pi)$$

Solving heat Equation: and Schrodiner Wave Equation

8.2 THE METHOD OF SEPARATION OF VARIABLES

Since the additional conditions imposed on $u(x, t)$ in our string problem fall into two groups, (a) those involving x (boundary conditions) and (b) those involving t (initial conditions), it may be reasonable to seek solutions of the PDE in the form

$$u(x, t) = X(x)T(t),$$

首先尋找時間部分與空間部分，可以分離為x, t兩變數個別的函數的模式解：

$$\phi(x,t) = X(x) \cdot T(t)$$

波方程式是線性方程式，一系列這樣的模式解，作線性組合，就得到一般解。

要得到線性組合的配方係數， 將起始條件分解為$X(x)$的線性組合(永遠可以)。

讓組合中的$X(x)$各自作時間演化$T(t)$(簡諧振盪)後，到要求的時間t再組合！

$$\phi(x,t) = \sum_i c_i X_i(x) \cdot T_i(t)$$

這樣的波函數既滿足波方程式，又滿足起始條件，就是唯一解。

Solving Wave Equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ 幾乎就是 Schrodiner Wave Equation

8.2 THE METHOD OF SEPARATION OF VARIABLES

首先尋找時間部分與空間部分，可以分離為x, t兩變數個別的函數的模式解：

$$u(x,t) = X(x) \cdot T(t)$$

波方程式是線性方程式，一系列這樣的模式解，作線性組合，就得到一般解。

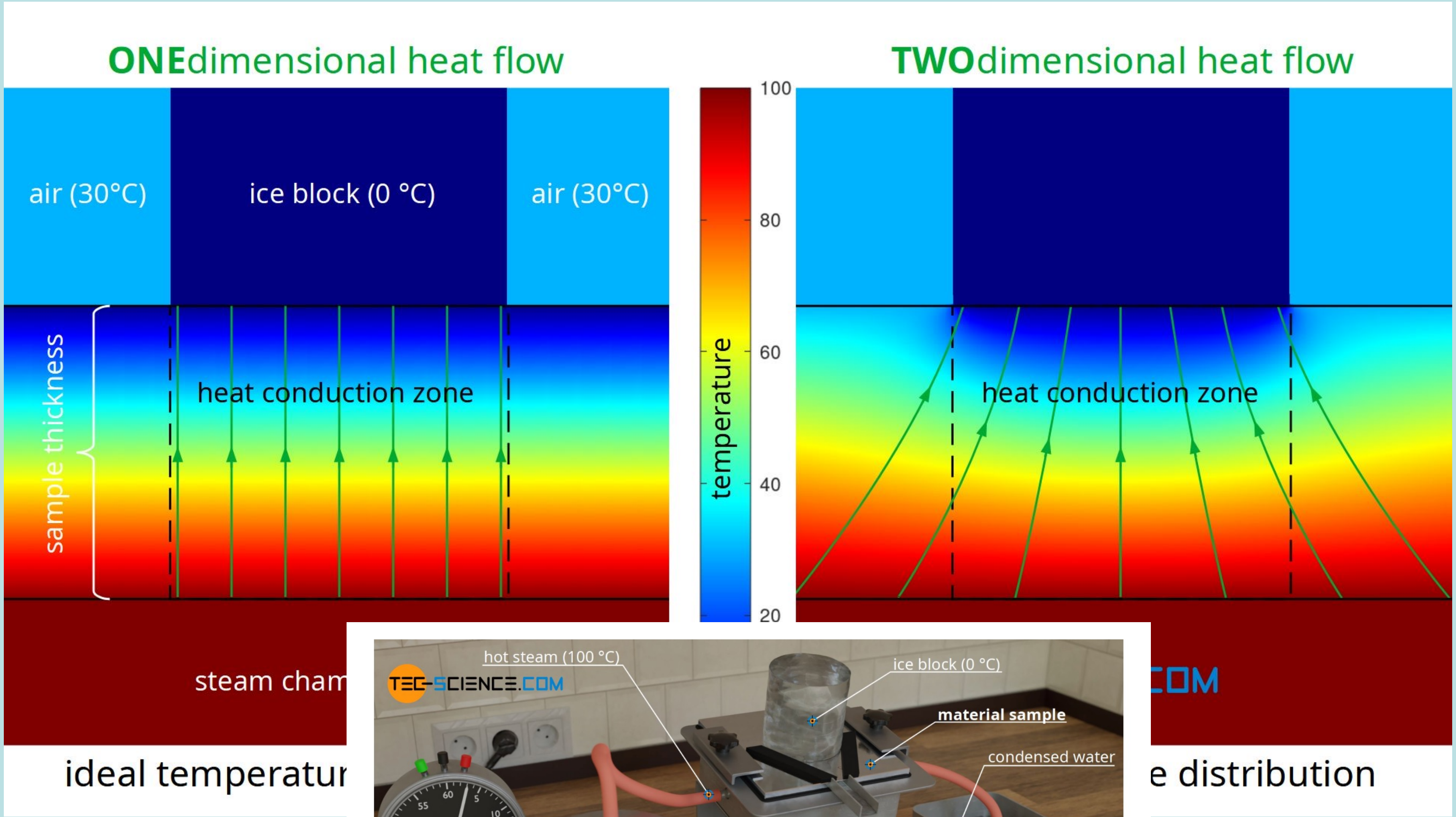
要得到線性組合的配方係數， 將起始條件分解為$X(x)$的線性組合(永遠可以)。

讓組合中的$X(x)$各自作時間演化$T(t)$(指數遞減)後，到要求的時間t再組合！

$$\phi(x,t) = \sum_i c_i X_i(x) \cdot T_i(t)$$

這樣的波函數既滿足波方程式，又滿足起始條件，就是唯一解。

3D Heat Conduction



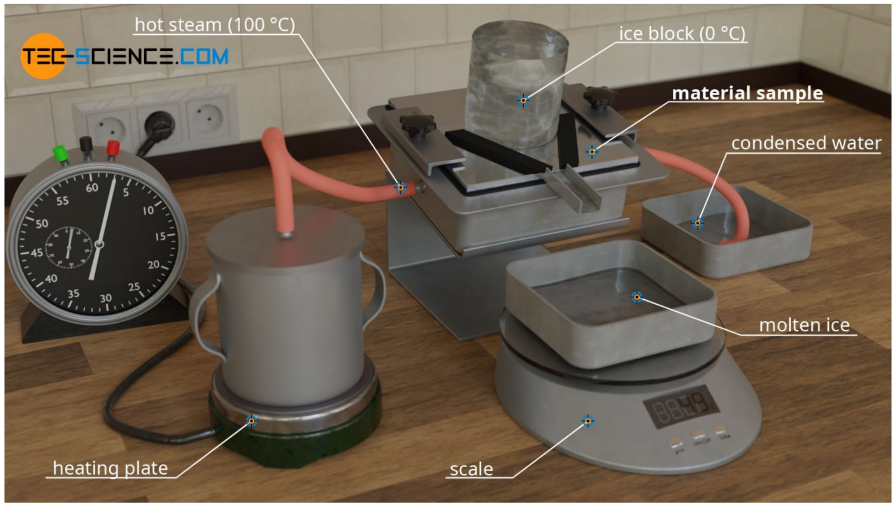
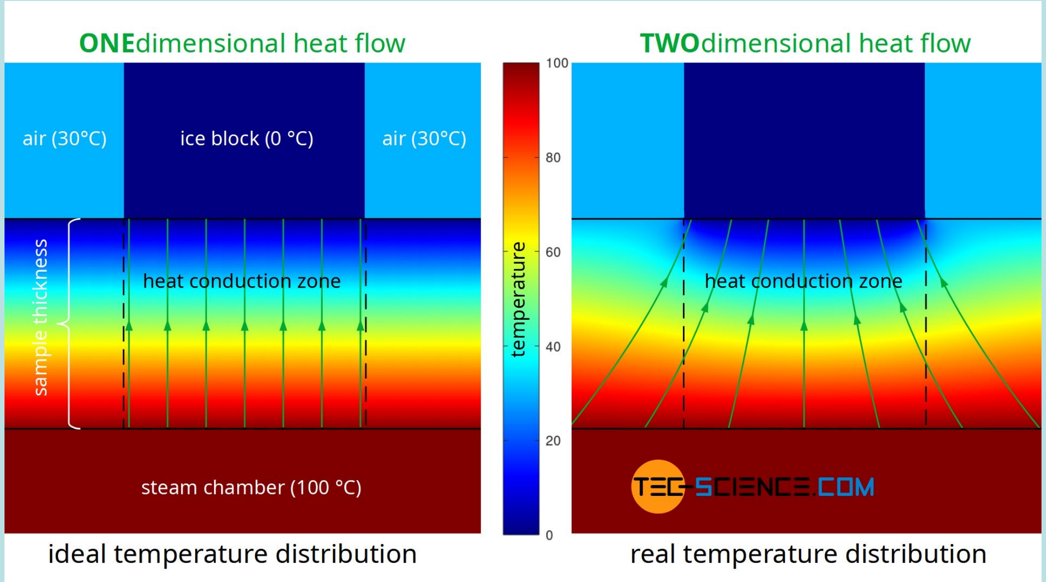
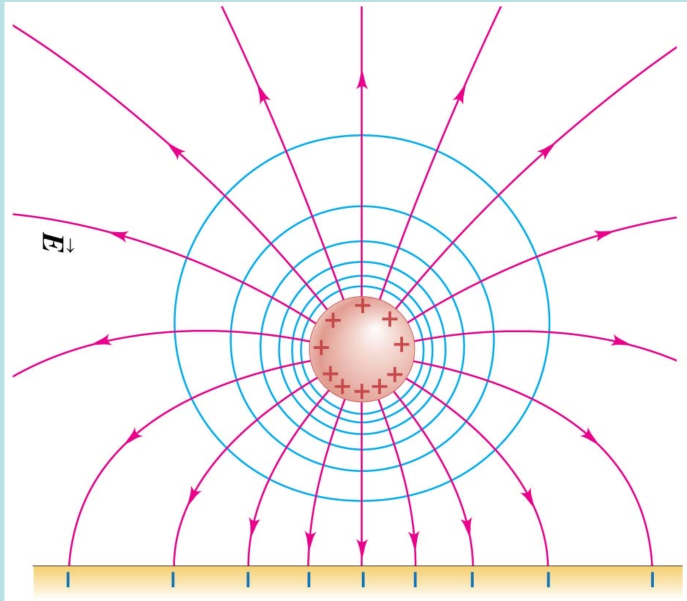


Figure: Experimental setup for the measurement of thermal conductivity

$$H = h\frac{\Delta T}{\Delta x} \quad \Leftrightarrow \quad E = -\frac{\Delta V}{\Delta x}$$





$$\vec{H} = -\vec{\nabla}T \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla}V$$

兩相近點之間的溫度差驅動熱量的流動。

溫度函數的梯度就正比於熱量流動向量。

首先尋找時間部分與空間部分，可以分離為x, t兩變數個別的函數的模式解：

$$u(x,t) = X(x) \cdot T(t)$$

Method of Separation of Variables

代入波方程式：

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

得到：

$$X \frac{\partial T}{\partial t} = T \frac{\partial^2 X}{\partial x^2}$$

一般會把 v^2 移到左邊，同時左右邊都除以 $X(x) \cdot T(t)$ 。偏微分可以寫成常微分：

$$\frac{1}{v^2} \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

現在左邊只與x有關，右邊只與t有關，但兩者是獨立變數！相等並不可能。

唯一的例外：左右兩式都與各自的變數無關，是一常數。設此常數為λ。

$$\frac{1}{v^2} \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} \equiv \lambda$$

我們就得到空間部分X與時間部分T各自需要滿足的常微分方程式。

將偏微分方程式分解Reduce為常微分方程式，這是常見的作法。