System of 2nd order Linear ODE with constant coefficients

$$-\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X}$$

Matrix $\boldsymbol{A}$ is what determines the evolution of the system.

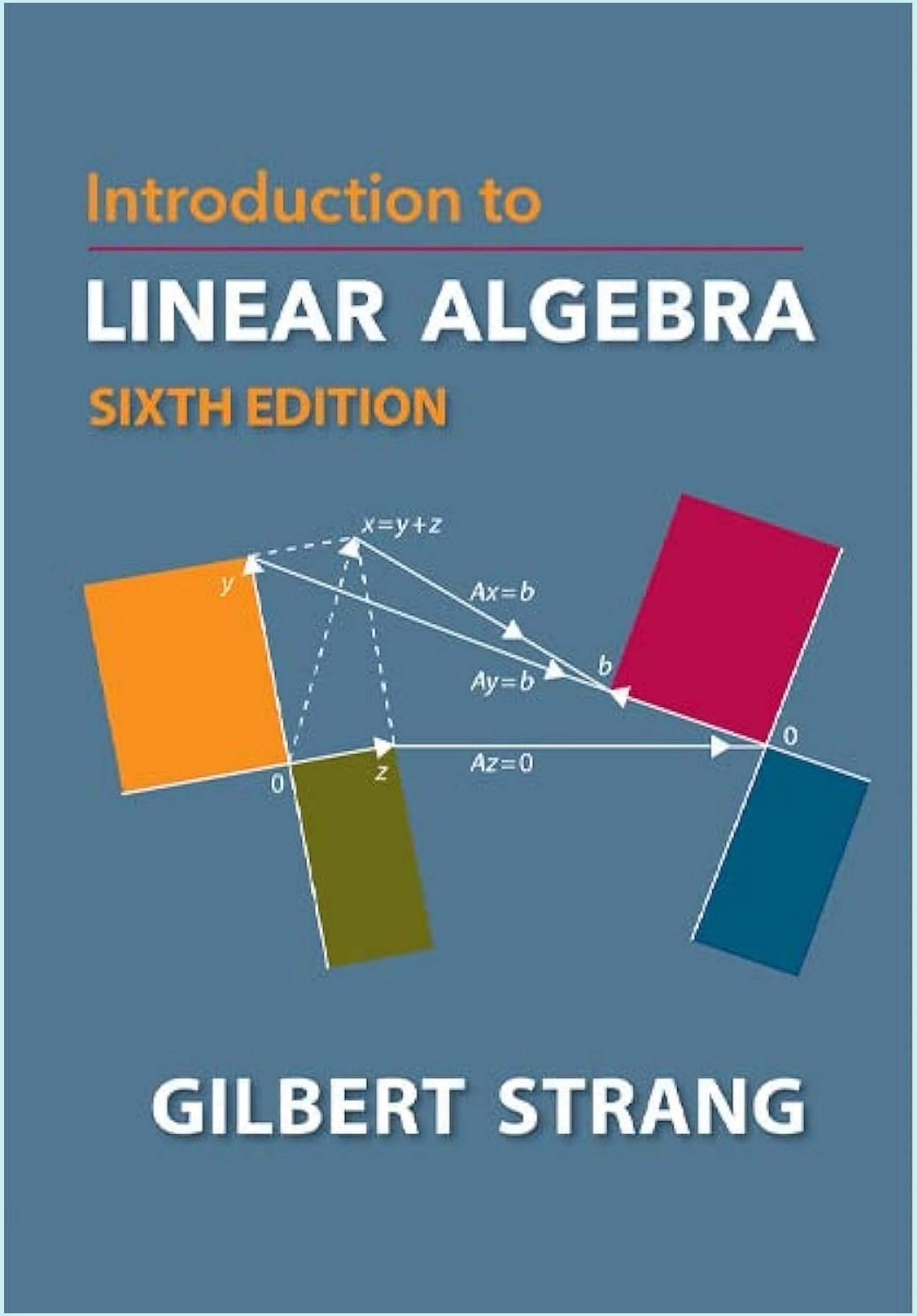
例如耦合振盪 $\boldsymbol{A} \equiv \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

↓ 複變函數法 $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

$$\frac{d^2\boldsymbol{a}e^{i\omega t}}{dt^2} = \omega^2\boldsymbol{a}e^{i\omega t} = \boldsymbol{A}\cdot\boldsymbol{a}e^{i\omega t}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

這是矩陣的本徵問題：線性代數Linear Algebra的中心問題。


Introduction to
LINEAR ALGEBRA
SIXTH EDITION
x=y+z
Ax=b
Ay=b
Az=0
y
b
0
z
0
GILBERT STRANG

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues: $Ax = \lambda x$

6.2 Diagonalizing a Matrix

6.3 Symmetric Positive Definite Matrices

6.4 Complex Numbers and Vectors and Matrices

6.5 Solving Linear Differential Equations

向量或行向量 vector or column vector

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

兩個數($\rightarrow N$個數，實數或複數)：$a_i, i = 1{,}2$，組成的一組數學量。

空間向量就是例子，但行向量不一定是空間向量，可以是彈簧組的位移。

可以是電子自旋的狀態！

But like 3D vectors, Vectors can be multiplied by a number, and two vectors can add up.

Linear combinations of vectors are still vectors. Vector space is also called linear space.

$$\boldsymbol{v} = c_1\boldsymbol{a} + c_2\boldsymbol{b}$$

線性組合或線性疊加

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + 2.0 \cdot \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

If the components of two vectors are proportional, we say they are in the same direction.

$$\boldsymbol{a} = c\boldsymbol{b}$$

$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ and $\begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$ are in the same direction.

向量vector可以定義長度：　$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$　$\boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_1^2}$$

從此延伸，可以定義兩個向量的內積inner product：

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 \qquad |\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$$

行向量的transpose(轉置)稱為列row向量：　$\boldsymbol{a}^{\mathrm{T}} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$

內積可以用行向量與列向量的矩陣乘積來寫：

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}$$

兩個向量的內積為零，這兩個向量就彼此正交perpendicular：

$$\boldsymbol{a}^{\mathrm{T}} \boldsymbol{b} = 0$$

2×2 Matrix矩陣是兩組行向量(4個實數或複數)：$a_{ij}, i, j = 1{,}2$，組成的數學量。

$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ 也可以看成是兩組列向量。兩種看法都有用。

矩陣乘行向量得一行向量：

$$(\boldsymbol{Sa})_i = \sum_{j=1}^{2} S_{ij} a_j = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$

此規則保證此乘積是線性的 $\boldsymbol{S}(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2) = c_1\boldsymbol{Sa}_1 + c_2\boldsymbol{Sa}_2$

列向量乘矩陣得一列向量：

$$(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{S})_j = \sum_{i=1}^{2} a_i S_{ij} = (a_1 \quad a_2) \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = (a_1S_{11} + a_2S_{21} \quad a_1S_{12} + a_2S_{22})$$

矩陣乘矩陣還是矩陣：

$$(\boldsymbol{SA})_{mn} = \sum_{j=1}^{2} S_{mj} A_{jn} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S_{11}A_{11} + S_{12}A_{21} & S_{11}A_{12} + S_{12}A_{22} \\ S_{21}A_{11} + S_{22}A_{21} & S_{21}A_{12} + S_{22}A_{22} \end{pmatrix}$$

列向量乘矩陣乘行向量，等於列向量乘行向量，得到一個數：

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{Sa} = \sum_{i,j=1}^{2} b_i S_{ij} a_j$$

Arfken p96-99

1.4 Matrix Multiplication AB and CR

1 To multiply AB we need *row length for* A = *column length for* B.

2 The number in row i, column j of AB is **(row i of A) · (column j of B)**.

3 By columns: **A times column j of B produces column j of AB.**

4 Usually AB is different from BA. But always $(AB)C = A(BC)$.

5 If A has r independent columns in C, then $A = CR = (m \times r)(r \times n)$.

We know how to multiply a matrix A times a column vector x or b. This section moves to matrix-matrix multiplication: **a matrix A times a matrix B**. The new rule builds on the old one, when the matrix B has columns $b_1, b_2, \ldots, b_p$. We just multiply A times each of those p columns of B to find the p columns of AB.

Column j of AB equals A times column j of B

$$\textbf{If } B = \begin{bmatrix} b_1 \cdots b_p \end{bmatrix} \textbf{ then } AB = \begin{bmatrix} Ab_1 \cdots Ab_p \end{bmatrix} \quad (1)$$

To see that clearly, start with a 2 by 2 "exchange matrix" for B. So B has two columns b_1 and b_2. We multiply A times each column to produce a column of AB:

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad Ab_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

For this matrix B, the result of multiplying AB is to *exchange the columns of* A.

There is more to see when we multiply the same A by a full 2 by 2 matrix B:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \text{ has } Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \text{ and } Ab_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Here is the point. We can multiply Ab_1 (matrix times vector) the *row way* or the *column way*. The row way uses dot products of b_1 with *every row of* A:

$$\begin{matrix}\textbf{Row way} \\ \textbf{Dot products}\end{matrix} \quad Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot b_1 \\ \text{row } 2 \cdot b_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix} \quad (2)$$

The column way uses a combination of the *columns* of A to find Ab_1. Same result:

$$\begin{matrix}\textbf{Column way} \\ \textbf{Combine columns}\end{matrix} \quad Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \mathbf{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mathbf{7}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix} \quad (3)$$

Both ways use the same 4 multiplications. With numbers like these, I think most people choose the row way. **To multiply AB, take the dot product of each row of A with each column of B.** When A has 2 rows and B has 2 columns, that means 4 dot products.

AB is usually different from BA

For $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, AB exchanged the columns of A. **But BA exchanges the rows of A!**

$$\boldsymbol{AB} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} \qquad \boldsymbol{BA} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \qquad (6)$$

Matrix multiplication is not commutative. In general $\boldsymbol{BA \neq AB}$. Multiply A on the left for row operations on A, and multiply on the right by B for column operations on A.

AB times $C = A$ times BC

For matrix multiplication, **this associative law is true**. We are not willing to give up this extremely useful law. We can multiply AB first or we can multiply BC first.
The matrices stay in the order A, B, C and their sizes must be right for multiplication:

$$\boldsymbol{A} \text{ is } \boldsymbol{m \times n} \quad \boldsymbol{B} \text{ is } \boldsymbol{n \times p} \quad \boldsymbol{C} \text{ is } \boldsymbol{p \times q}. \text{ Then } \boldsymbol{AB} \text{ is } \boldsymbol{m \times p} \text{ and } \boldsymbol{(AB)C} \text{ is } \boldsymbol{m \times q}.$$

We can test the law using the exchange matrix B on the rows and the columns of A:

$$(\boldsymbol{BA})\boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{B}(\boldsymbol{AB}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

So row operations on A can come *before or after* column operations on A.

Notice the meaning of $(AB)C = A(BC)$ when C is just a column vector $\boldsymbol{x}$. If that vector $\boldsymbol{x}$ has a single 1 in component j, then the associative law is $(AB)\boldsymbol{x} = A(B\boldsymbol{x})$. This tells us how to multiply matrices! The left side is **column $\boldsymbol{j}$ of $\boldsymbol{AB}$**. The right side is **$\boldsymbol{A}$ times column $\boldsymbol{j}$ of $\boldsymbol{B}$**. So their equality is exactly the rule for matrix multiplication that we saw in equation (1). It is simply the right rule.

Let me bring together the important facts about ABC and also A times $B + C$:

$$\boxed{\textbf{Associative} \quad (\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC}) \text{ and } \textbf{Distributive} \quad \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}} \qquad (7)$$

Inverse matrix

$$\boldsymbol{S}^{-1} = \frac{1}{\det \boldsymbol{S}} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

Transpose of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad \boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix} \quad \boldsymbol{A}_{ij}^{\mathrm{T}} = \boldsymbol{A}_{ji}$$

例如 $\boldsymbol{A}_{12}^{\mathrm{T}} = \boldsymbol{A}_{21}$

Symmetric Matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 \\ 4 & 9 \end{pmatrix} \quad \boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 8 & 9 \end{pmatrix}$$

Not symmetric

$$\boldsymbol{S} = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = \boldsymbol{S}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$$

Symmetric $\boldsymbol{S}_{ij} = \boldsymbol{S}_{ij}^{\mathrm{T}} = \boldsymbol{S}_{ji}$

例如 $\boldsymbol{S}_{12} = \boldsymbol{S}_{12}^{\mathrm{T}} = \boldsymbol{S}_{21}$

Arfken p99, 104-107

System of 2nd order Linear ODE with constant coefficients

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{A}\cdot\boldsymbol{X}$$

Matrix $\boldsymbol{A}$ is what determines the evolution of the system.

複變函數法 $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

Eigenvalue problem of matrix

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

這是線性代數Linear Algebra的中心問題。

Eigenvalue problem of matrix

$$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$$

這代表矩陣$\boldsymbol{A}$乘在行向量$\boldsymbol{a}$，與一常數乘法沒有差異，並未改變$\boldsymbol{a}$的方向。

任一矩陣$\boldsymbol{A}$有特定的行向量$\boldsymbol{a}$，及對應的常數λ滿足上式。

行向量$\boldsymbol{a}$稱為矩陣$\boldsymbol{A}$的本徵向量eigenvector，常數λ稱為本徵值eigenvalue。

For example, take $\boldsymbol{A}$ as a 2×2 matrix.

$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

In general, $\boldsymbol{A}$ times a vector will produce another vector not in the same direction.

$$\boldsymbol{A}\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \neq c \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$$

But there are two special vectors that matrix multiplication is just a number product.

For them, $\boldsymbol{A}$ times the vector will produce a vector in the same direction.

$\boldsymbol{A}$ times the eigenvector will produce a vector in the same direction.

Solving eigenvalue problem of matrix

$$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a} = 0$$

$$\boldsymbol{S}^{-1} = \frac{1}{\det \boldsymbol{S}} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$\boldsymbol{A} - \lambda \boldsymbol{I}$ is also a matrix. If it has an inverse, the vector $\boldsymbol{a}$ can only be zero:

$$\boldsymbol{a} = (\boldsymbol{A} - \lambda \boldsymbol{I})^{-1} \cdot 0 = 0$$

For λ to be an eigenvalue, the condition is $\boldsymbol{A} - \lambda \boldsymbol{I}$ has no inverse and hence the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ is zero.

$$\mathbf{det}\,(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

$$\mathbf{det}\,(\boldsymbol{A} - \lambda \boldsymbol{I}) = \mathbf{det} \begin{pmatrix} 0.8 - \lambda & 0.3 \\ 0.2 & 0.7 - \lambda \end{pmatrix} = \lambda^2 - \tfrac{3}{2}\lambda + \tfrac{1}{2} = (\lambda - 1)\left(\lambda - \tfrac{1}{2}\right) = 0$$

$\lambda = \lambda_1 = 1$ or $\lambda_2 = \frac{1}{2}$ are eigenvalues of $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

對特定的λ，我們可以解出對應的a_1, a_2

$\lambda = \lambda_1 = 1$ $\boldsymbol{a}_1 \equiv \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a}_1 = \begin{pmatrix} 0.8 - 1 & 0.3 \\ 0.2 & 0.7 - 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

$0.2a_{11} = 0.3a_{21}$

Eigenvector has one undetermined constant.

$$\boldsymbol{a}_1 = c_1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

同理：

$$\lambda = \lambda_2 = \frac{1}{2}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{a}_2 = \begin{pmatrix} 0.8 - 0.5 & 0.3 \\ 0.2 & 0.7 - 0.5 \end{pmatrix} \cdot \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

$0.3a_{12} = -0.3a_{22}$

$$\boldsymbol{a}_2 = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boldsymbol{A}$ times the eigenvector will produce a vector in the same direction.

$$\boldsymbol{A} \cdot \boldsymbol{a}_1 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = 1 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\boldsymbol{A} \cdot \boldsymbol{a}_2 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For the two special eigenvectors, matrix multiplication is just a number product.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.
3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = 0$ ***to find an eigenvector*** $\boldsymbol{x}$.

我們再試一個矩陣，這次是一對稱矩陣！

Solving eigenvalue problem of matrix

$$\boldsymbol{S} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\boldsymbol{S} \cdot \boldsymbol{u} = \lambda \boldsymbol{u}$$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u} = 0$$

If it has an inverse, the vector $\boldsymbol{u}$ can only be zero:

$$\boldsymbol{u} = (\boldsymbol{S} - \lambda \boldsymbol{I})^{-1} \cdot 0 = 0$$

For λ to be an eigenvalue, the condition is $\boldsymbol{S} - \lambda \boldsymbol{I}$ has no inverse and hence the determinant of $\boldsymbol{S} - \lambda \boldsymbol{I}$ is zero.

$$\mathbf{det}\ (\boldsymbol{S} - \lambda \boldsymbol{I}) = 0$$

$$\mathbf{det}\ (\boldsymbol{S} - \lambda \boldsymbol{I}) = \mathbf{det} \begin{bmatrix} 9-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = \lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$

$\lambda = \lambda_1 = 0$ or $\lambda_2 = 10$ are eigenvalues.

對特定的λ，我們可以解出對應的u_1, u_2

$\lambda = \lambda_1 = 0$ $\quad$ $\boldsymbol{u_1} \equiv \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u_1} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = 0$$

$3u_{11} = -u_{21}$

Eigenvector has one undetermined constant.

$$\boldsymbol{u_1} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

同理：

$\lambda = \lambda_2 = 10$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u_2} = \begin{pmatrix} 9 - 10 & 3 \\ 3 & 1 - 10 \end{pmatrix} \cdot \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} = 0$$

$u_{12} = 3u_{22}$

$$\boldsymbol{u_2} = c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

There are two beautiful theorems that illustrate the utility of eigenvectors

Theorem: The eigenvectors $\boldsymbol{a}$ of $\boldsymbol{A}$ are also eigenvectors of $\boldsymbol{A}^n$ with the eigenvalue λ^n.

$$\boldsymbol{A}\cdot\boldsymbol{a}=\lambda\boldsymbol{a} \longrightarrow \boldsymbol{A}^n\cdot\boldsymbol{a}=\lambda^n\boldsymbol{a}$$

Theorem: All vectors can be written as the linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

$$\boldsymbol{X}=c_1\boldsymbol{a}_1+c_2\boldsymbol{a}_2$$

Then the action of $\boldsymbol{A}$ and $\boldsymbol{A}^n$ on any vector can be easily written down:

$$\boldsymbol{AX}=c_1\boldsymbol{A}\boldsymbol{a}_1+c_2\boldsymbol{A}\boldsymbol{a}_2=c_1\lambda_1\boldsymbol{a}_1+c_2\lambda_2\boldsymbol{a}_2$$

The information in the eigenvectors alone of a specific matrix can represent the matrix!

一個矩陣的本徵值與本徵向量就足夠能代表該矩陣。

$$\boldsymbol{X}=\begin{pmatrix}0.8\\0.2\end{pmatrix}=\begin{pmatrix}0.6\\0.4\end{pmatrix}+2\begin{pmatrix}0.1\\-0.1\end{pmatrix}$$

一行向量可以寫成本徵向量的線性組合

$$\boldsymbol{AX}=\begin{pmatrix}0.8 & 0.3\\0.2 & 0.7\end{pmatrix}\cdot\begin{pmatrix}0.8\\0.2\end{pmatrix}=\boldsymbol{A}\cdot\begin{pmatrix}0.6\\0.4\end{pmatrix}+2\boldsymbol{A}\cdot\begin{pmatrix}0.1\\-0.1\end{pmatrix}$$

$$=1\begin{pmatrix}0.6\\0.4\end{pmatrix}+2\cdot\frac{1}{2}\begin{pmatrix}0.1\\-0.1\end{pmatrix}=\begin{pmatrix}0.7\\0.3\end{pmatrix}$$

更厲害的是：

$$\boldsymbol{A}^n\boldsymbol{X} = c_1\boldsymbol{A}^n\boldsymbol{a}_1 + c_2\boldsymbol{A}^n\boldsymbol{a}_2 = c_1\lambda_1^n\boldsymbol{a}_1 + c_2\lambda_2^n\boldsymbol{a}_2$$

$$\boldsymbol{X} = \begin{pmatrix}0.8\\0.2\end{pmatrix} = \begin{pmatrix}0.6\\0.4\end{pmatrix} + 2\begin{pmatrix}0.1\\-0.1\end{pmatrix}$$

$$\boldsymbol{A}^{100}\boldsymbol{X} = \begin{pmatrix}0.8 & 0.3\\0.2 & 0.7\end{pmatrix}^{100} \cdot \begin{pmatrix}0.8\\0.2\end{pmatrix} = \boldsymbol{A}^{100} \cdot \begin{pmatrix}0.6\\0.4\end{pmatrix} + 2\boldsymbol{A}^{100} \cdot \begin{pmatrix}0.1\\-0.1\end{pmatrix}$$

$$= 1^{100}\begin{pmatrix}0.6\\0.4\end{pmatrix} + 2 \cdot \left(\frac{1}{2}\right)^{100}\begin{pmatrix}0.1\\-0.1\end{pmatrix} \sim \begin{pmatrix}0.6\\0.4\end{pmatrix}$$

This matrix is called a Markov matrix, with an eigenvalue = 1, the other < 1.

The repeated action of a Markov matrix will push any vector into the eigenvector with λ = 1.

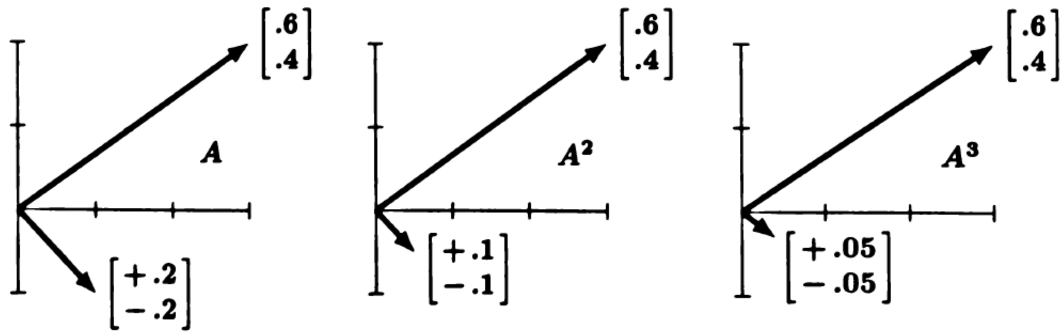


Figure 6.1: The first columns of A, A^2, A^3 are $\begin{bmatrix}.8\\.2\end{bmatrix}, \begin{bmatrix}.7\\.3\end{bmatrix}, \begin{bmatrix}.65\\.35\end{bmatrix}$ approaching $\begin{bmatrix}.6\\.4\end{bmatrix}$.

System of 2nd order Linear ODE with constant coefficients

$$-\frac{d^2\boldsymbol{X}}{dt^2}=\boldsymbol{A}\cdot\boldsymbol{X}$$

Matrix $\boldsymbol{A}$ is what determines the evolution of the system.

例如耦合振盪 $\boldsymbol{A}\equiv\frac{k}{m}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

複變函數法 $\boldsymbol{X}=\boldsymbol{a}e^{i\omega t}$

$$\frac{d^2\boldsymbol{a}e^{i\omega t}}{dt^2}=\omega^2\boldsymbol{a}e^{i\omega t}=\boldsymbol{A}\cdot\boldsymbol{a}e^{i\omega t}$$

$$\boldsymbol{A}\cdot\boldsymbol{a}=\omega^2\boldsymbol{a}$$

這是矩陣的本徵值問題：線性代數Linear Algebra的中心問題。

利用本徵向量，線性微分方程組很容易求解：

With the eigenvectors and eigenvalues, we can solve the ODE system.

$\boldsymbol{A} \cdot \boldsymbol{a} = \omega^2 \boldsymbol{a} = \lambda \boldsymbol{a}$

$\omega_i^2 = \lambda_i$ $\boldsymbol{a}_i$是對應的本徵向量。

例如： $\boldsymbol{X} = \mathrm{Re}\, C_1 \boldsymbol{a}_1 e^{i\sqrt{\lambda_1}t} = c_1 \boldsymbol{a}_1 \cos\left(\sqrt{\lambda_1}t + \phi_1\right)$

加上另一解： $\boldsymbol{X} = c_1 \boldsymbol{a}_1 \cos\left(\sqrt{\lambda_1}t + \phi_1\right) + c_2 \boldsymbol{a}_2 \cos\left(\sqrt{\lambda_2}t + \phi_2\right)$

The four initial conditions $\boldsymbol{X}(0), \boldsymbol{X}'(0)$ will determine four constants $c_{1,2}, \phi_{1,2}$.

For simplicity, set $\boldsymbol{X}'^{(0)} = 0$, hence $\phi_1 = \phi_2 = 0$.

Initial condition $\boldsymbol{X}(0)$ is a vector and can always be written as a linear combination of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ 起始條件是一個向量，可以寫成$\boldsymbol{a_1}$與$\boldsymbol{a_2}$的線性組合

$$\boldsymbol{X}(0) = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2$$

如此就決定了constants $c_{1,2}$，接下來時間演化：

$$\boldsymbol{X}(t) = c_1\boldsymbol{a}_1 \cos\sqrt{\lambda_1}t + c_2\boldsymbol{a}_2\cos\sqrt{\lambda_2}t$$

這個解，可以理解為 $\boldsymbol{a_1}$與$\boldsymbol{a_2}$ 這兩個成分，各自以簡諧運動隨時間演化發展。

如果起始時$\boldsymbol{X}(0) \propto \boldsymbol{a}_1$，彈簧組就會以$\sqrt{\lambda_1}$為角頻率作純粹簡諧運動。 $\sim\boldsymbol{a}_1\cos\sqrt{\lambda_1}t$

起始時$\boldsymbol{X}(0) \propto \boldsymbol{a}_2$，彈簧組就會以$\sqrt{\lambda_2}$為角頻率作純粹簡諧運動。 $\sim\boldsymbol{a}_2\cos\sqrt{\lambda_2}t$

如果起始時兩種成分都有，兩種模式可以分別作純粹簡諧運動演化後再疊加：

$$\boldsymbol{X}(t) = c_1\boldsymbol{a}_1 \cos\sqrt{\lambda_1}t + c_2\boldsymbol{a}_2\cos\sqrt{\lambda_2}t$$

這兩種特定模式的簡諧運動好像是獨立的！

位能下薛丁格方程式的普遍解法：

首先， 位能下薛丁格方程式會有一系列的定態解：

$\Psi_n(x,t) = u_n(x) \cdot e^{-i\frac{E_n}{\hbar}t}$ 定態解就是時間部分與空間可以分離的解！

將$t = 0$時的波函數，即起始條件，對定態解u_n展開如下：

$$\Psi(x,0) = \sum_{n=1}^{\infty} A_n u_n(x)$$

$$u_n = C \sin\left(\frac{n\pi}{a}x\right)$$

這就是傅立葉分析！

$t = 0$時此狀態可以視為定態的如上疊加，

而接下來定態隨時間的演化，就是在u_n上乘$e^{-i\frac{E_n}{\hbar}t}$，這滿足位能薛丁格方程式。

乘完之後依同樣方式疊加，整個波函數也就滿足薛丁格方程式。

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_n u_n(x) e^{-i\frac{E_n}{\hbar}t}$$

我們已經在自由薛丁格方程式用了這樣的策略！當時的正弦波就是定態。

這個程序原則上適用於任何位能。

Diagonalizing a matrix 對角化

$\boldsymbol{A}\cdot\boldsymbol{a}_1=\lambda_1\boldsymbol{a}_1$ $\boldsymbol{A}\cdot\boldsymbol{a}_2=\lambda_2\boldsymbol{a}_2$

$\boldsymbol{a}_1,\boldsymbol{a}_2$ are column vectors and we can use them to form a matrix.

$$\boldsymbol{U}\equiv(\boldsymbol{a}_1\quad\boldsymbol{a}_2)=\left(\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\quad\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\right)=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$$

$$\boldsymbol{AU}=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\left(\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\quad\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\right)=\left(\boldsymbol{A}\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\quad\boldsymbol{A}\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\right)$$

$$=\begin{pmatrix}\lambda_1a_{11}&\lambda_2a_{12}\\\lambda_1a_{21}&\lambda_2a_{22}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}=\boldsymbol{U}\cdot\boldsymbol{\Lambda}$$

$\boldsymbol{AU}=\boldsymbol{U\Lambda}$ 兩邊乘$\boldsymbol{U}$的反矩陣$\boldsymbol{U}^{-1}$：

$\boldsymbol{U}^{-1}\cdot\boldsymbol{A}\cdot\boldsymbol{U}=\boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$ is diagonal with eigenvalues as diagonal elements.

$\boldsymbol{U}^{-1}\boldsymbol{AU}$是一個對角矩陣，矩陣元素就是本徵值！

$\boldsymbol{A}=\boldsymbol{U\Lambda U}^{-1}=\boldsymbol{U}\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}\boldsymbol{U}^{-1}$ Any Matrix can be decomposed into 3 factors, involving just eigenvalues and eigenvectors.

Info. of eigenvalues Info. of eigenvectors

$$\begin{bmatrix}0.8&0.3\\0.2&0.7\end{bmatrix}=\begin{bmatrix}0.6&1\\0.4&-1\end{bmatrix}\begin{bmatrix}1&0\\0&0.5\end{bmatrix}\begin{bmatrix}1&1\\0.4&-0.6\end{bmatrix}$$

一個矩陣的本徵值與本徵向量就足夠能代表該矩陣。

$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$ This formula is useful for the following calculation!

$$\boldsymbol{A}^k = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} \cdot \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} \cdot \cdots \cdot \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}^k\boldsymbol{U}^{-1} = \boldsymbol{U}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}\boldsymbol{U}^{-1}$$

矩陣的指數函數很容易定義與計算。

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \cdots$$

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} + \frac{\boldsymbol{U}\boldsymbol{\Lambda}^1\boldsymbol{U}^{-1}}{2!} + \frac{\boldsymbol{U}\boldsymbol{\Lambda}^2\boldsymbol{U}^{-1}}{3!} + \cdots = \boldsymbol{U}\left(\boldsymbol{I} + \boldsymbol{\Lambda} + \frac{\boldsymbol{\Lambda}^2}{2!} + \frac{\boldsymbol{\Lambda}^3}{3!} + \cdots\right)\boldsymbol{U}^{-1}$$

$$= \boldsymbol{U}e^{\boldsymbol{\Lambda}}\boldsymbol{U}^{-1} = \boldsymbol{U}\begin{pmatrix} \boldsymbol{e}^{\lambda_1} & 0 \\ 0 & \boldsymbol{e}^{\lambda_2} \end{pmatrix}\boldsymbol{U}^{-1}$$

利用對角化後的矩陣，線性微分方程組很容易求解：

$$-\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X} \longleftarrow \qquad \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$$

$$-\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}\cdot\boldsymbol{X}$$

兩邊乘$\boldsymbol{U}^{-1}$

$$-\frac{d^2\boldsymbol{U}^{-1}\boldsymbol{X}}{dt^2} = \boldsymbol{\Lambda}\cdot\boldsymbol{U}^{-1}\boldsymbol{X} \qquad \boldsymbol{\Lambda} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

選擇Generalized Coordinates 廣義座標y_1, y_2

$$\boldsymbol{Y} = \begin{pmatrix}y_1\\ y_2\end{pmatrix} = \boldsymbol{U}^{-1}\boldsymbol{X}$$

他們滿足的微分方程非常簡單：

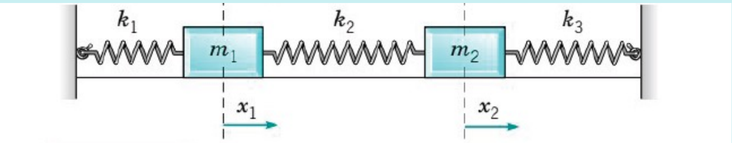
$$\frac{d^2\boldsymbol{Y}}{dt^2} = \boldsymbol{\Lambda}\cdot\boldsymbol{Y} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\boldsymbol{Y} \qquad \frac{d^2}{dt^2}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$\frac{d^2 y_1}{dt^2} = \lambda_1 y_1 \qquad \frac{d^2 y_2}{dt^2} = \lambda_2 y_2$$

負責廣義座標的時間演化的矩陣是對角，因此無耦合，此廣義座標彼此獨立。

這樣的廣義座標y_1, y_2稱為Normal Coordinates，獨立作簡諧運動！

$$y_1 = c_1\cos\left(\sqrt{\lambda_1}t + \phi_1\right) \qquad y_2 = c_2\cos\left(\sqrt{\lambda_2}t + \phi_2\right)$$



以耦合振盪為例：

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A}\cdot\boldsymbol{x}$$

$$\boldsymbol{A} \equiv \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

本徵值有兩個，各自對應一本徵向量。

$$\omega_1 \equiv \sqrt{\frac{k}{m}} \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 \equiv \sqrt{3}\sqrt{\frac{k}{m}} \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

代入前面的解：

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_1 t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\omega_2 t$$

如果用對角化矩陣來思考：

$$\boldsymbol{U} \equiv (\boldsymbol{a}_1 \quad \boldsymbol{a}_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

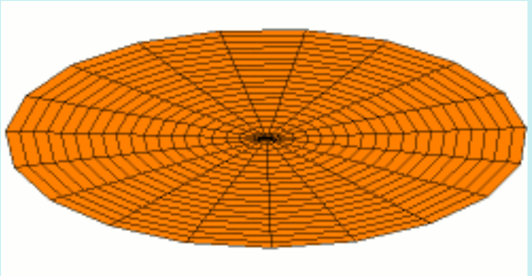
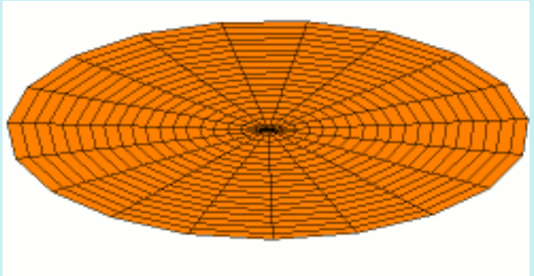
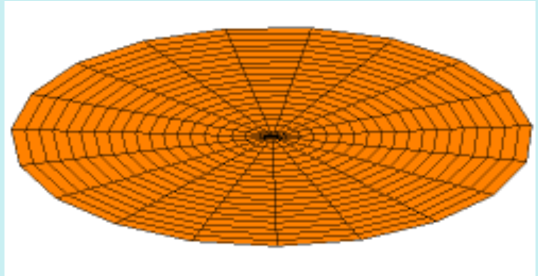
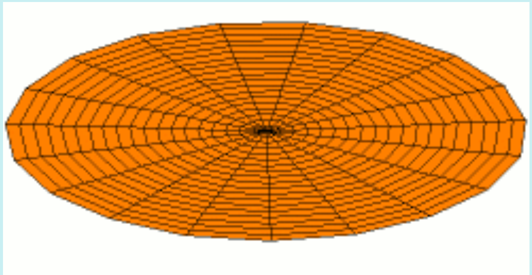
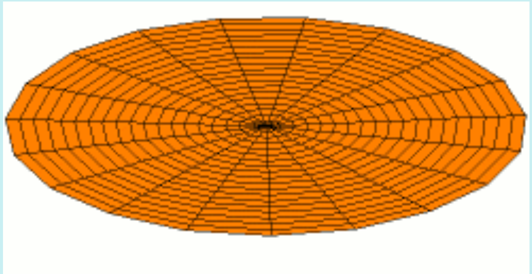
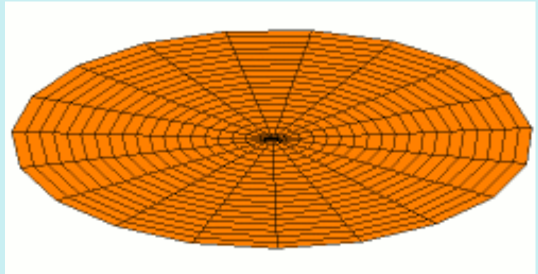
$$\boldsymbol{U}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

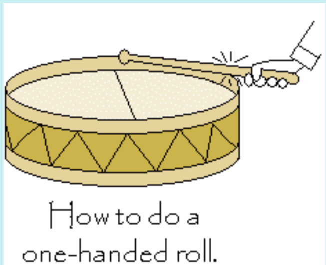
$$\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \boldsymbol{U}^{-1}\boldsymbol{X} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

這樣的廣義座標y_1, y_2稱為Normal Coordinates，獨立作簡諧運動！

$$y_1 = \frac{1}{2}(x_1 + x_2) = x_+$$

$$y_2 = \frac{1}{2}(x_1 - x_2) = x_-$$



原則上物體的變形模式有無限多個，但並不是連續分布。

因此可以分離地一個一個編號。

每一個模式，如同簡諧運動，對應一個內在的特定的振動頻率！

不同模式頻率不同。一般來說，頻率越高的模式，越難激發。

物體的所有變形就是以這些模式或它們的疊加來進行！

一個物體有那些振盪模式 Norm以及對應的頻率，就是該物體的一個特徵。

一個矩陣會有實數的本徵值嗎？

定理：一個對稱矩陣的本徵值都是實數，且不同本徵值的本徵向量彼此正交。

All n eigenvalues λ of a symmetric matrix S are real.

The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

反例：$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

$\boldsymbol{a_1} = c_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ $\boldsymbol{a_2} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boldsymbol{a_1} \cdot \boldsymbol{a_2} \neq \boldsymbol{0}$

All n eigenvalues λ of a symmetric matrix S are real.

Proof:

$\boldsymbol{Su} = \lambda\boldsymbol{u}$

Take a inner product of the both sides with the complex conjugate of eigenvector $\boldsymbol{u}^*$.

$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su} = \lambda\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{u}$

$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{u} = u_1^* u_1 + u_2^* u_2$ is real.

要確認$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$是否是實數，取他的Complex conjugate

$$(\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su})^* = \left(\sum_{i,j=1}^{2} u_i^* S_{ij} u_j\right)^* = \sum_{i,j=1}^{2} u_i S_{ij}^* u_j^* = \sum_{i,j=1}^{2} u_j^* S_{ij} u_i = \sum_{i,j=1}^{2} u_j^* S_{ji} u_i = \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$$

對稱

確認$\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{Su}$是實數，因此$\lambda$一定是實數。

The n eigenvectors $\boldsymbol{u}$ can be chosen to be orthogonal.

Proof:

Consider two eigenvectors with different eigenvalues:

$\boldsymbol{S}\boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_1$ $\boldsymbol{S}\boldsymbol{u}_2 = \lambda_2 \boldsymbol{u}_2$

將第一式與$\boldsymbol{u}_2$作內積，可得：

$\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{S}\boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1$

左邊可以寫成: $S_{ij} = S_{ji}$ $\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{S}\boldsymbol{u}_1 = \left(\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{S}\boldsymbol{u}_1\right)^{\mathrm{T}} = \boldsymbol{u}_1^{\mathrm{T}} \cdot \boldsymbol{S}^{\mathrm{T}}\boldsymbol{u}_2$

$$\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{S}\boldsymbol{u}_1 = \sum_{i,j=1}^{2} u_{2i}S_{ij}u_{1j} = \sum_{i,j=1}^{2} u_{1j}S_{ji}u_{2i} = \boldsymbol{u}_1^{\mathrm{T}} \cdot \boldsymbol{S}\boldsymbol{u}_2 = \boldsymbol{u}_1^{\mathrm{T}}\lambda_2\boldsymbol{u}_2 = \lambda_2\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1$$

得：

$\lambda_1\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1 = \lambda_2\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1$

$\boldsymbol{0} = (\lambda_1 - \lambda_2)\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1$ 兩式相減：

右式兩本徵向量的內積必須為零，彼此正交！

$\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$

可以選本徵向量的長度都為1，

$\boldsymbol{u}_m^{\mathrm{T}}\boldsymbol{u}_n = \boldsymbol{u}_m \cdot \boldsymbol{u}_n = \delta_{mn}$ 本徵向量的正交性

$$\boldsymbol{S} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

這就是對稱矩陣。

$$\lambda = \lambda_1 = 0$$

$$(\boldsymbol{S} - \lambda \boldsymbol{I}) \cdot \boldsymbol{u_1} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$3u_{11} = u_{21}$$

$$\boldsymbol{u_1} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

可以選常數c_1使向量長度為1:

$$\boldsymbol{u_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda = \lambda_2 = 10$$

$$\boldsymbol{u_2} = c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

很容易確認這兩個本徵向量是彼此正交。

$$\boldsymbol{u}_2^{\mathrm{T}} \boldsymbol{u}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$$

$$\boldsymbol{u}_m^{\mathrm{T}} \boldsymbol{u}_n = \boldsymbol{u}_m \cdot \boldsymbol{u}_n = \delta_{mn}$$

這兩個向量稱為orthonormal.

展開定理

若有兩個orthonormal的向量，任何向量都可展開成他們的線性組合，

且係數很容易寫下！

給定任一向量：$\boldsymbol{v}$，若展開可以如下：

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$$

取向量$\boldsymbol{v}$與本徵向量$\boldsymbol{u_1}$的內積：

$$\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_1^{\mathrm{T}}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}) = c_1\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u_1} + c_2\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u_2}$$

代入正交關係：　$\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u_2} = 0$　　$\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u_1} = 1$

$$\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_1 \cdot \boldsymbol{v} = c_1$$

同理：

$$\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_2 \cdot \boldsymbol{v} = c_2$$

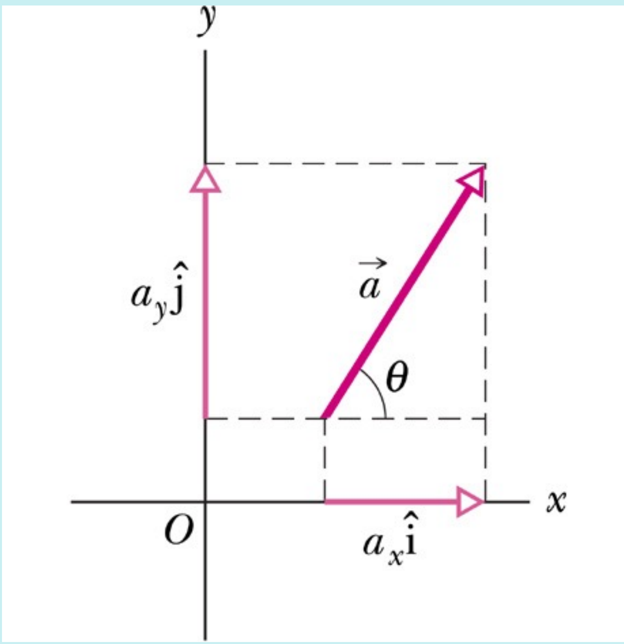
展開定理

給定任一向量：$\boldsymbol{v}$，可以展開如下：

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$$

$$\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_1 \cdot \boldsymbol{v} = c_1$$

$$\boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_2 \cdot \boldsymbol{v} = c_2$$



向量可以用它的分量表示！

把一向量投影在選取的座標軸上即是它的分量！

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (a_x, a_y)$ 分量法

$a_x = \hat{\imath}\cdot\vec{a}, \qquad a_y = \hat{\jmath}\cdot\vec{a}$

這兩個orthonormal的本徵向量$\boldsymbol{u}_1, \boldsymbol{u}_2$，類似組成一組座標軸的單位向量$\hat{\imath}, \hat{\jmath}$ ！