單體的運動方程式 → Ordinary Differential Equation

↓

多體且彼此耦合的運動方程式 → System of ODE

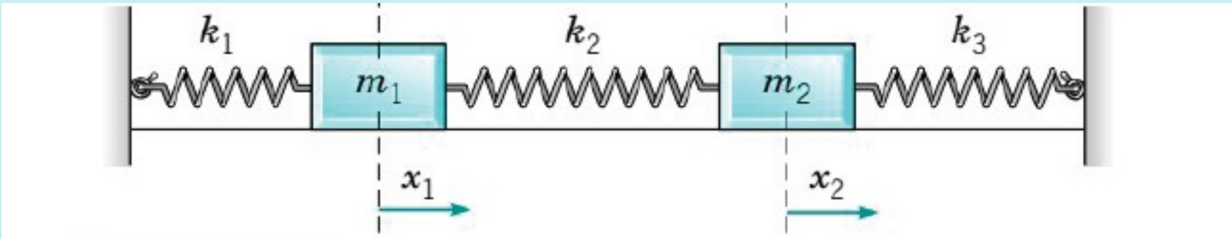
Matrix and Linear Algebra

Eigenvalue problem of Matrix

↓ 連續及大量極限

連續介質的波方程式 → Partial Differential Equation

耦合振盪 coupled oscillation



$$m_1 \frac{d^2x_1}{dt^2} = k_2(x_2 - x_1) - k_1 x_1 = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) - k_3 x_2 = k_2 x_1 - (k_2 + k_3)x_2$$

$$\frac{d^2x_1}{dt^2} = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

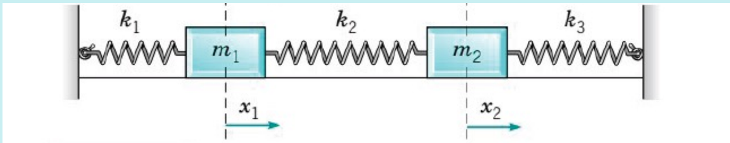
$$\frac{d^2x_2}{dt^2} = \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2$$

為簡化討論，假設： $k_1 = k_2 = k$ $m_1 = m_2 = m$

$$\frac{d^2x_1}{dt^2} = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$$

$$\frac{d^2x_2}{dt^2} = \frac{k}{m} x_1 - \frac{2k}{m} x_2$$

All terms are linear in x's. Systems of 2nd order linear ODE.



線性方程式組的右手邊可以以矩陣及行向量的乘積來簡化符號！

$$\frac{d^2x_1}{dt^2} = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\frac{d^2x_2}{dt^2} = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

$$\begin{pmatrix} \dfrac{d^2x_1}{dt^2} \\ \dfrac{d^2x_2}{dt^2} \end{pmatrix} = -\begin{pmatrix} \dfrac{2k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{2k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

行向量

$$\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boldsymbol{A} \equiv \begin{pmatrix} \dfrac{2k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{2k}{m} \end{pmatrix}$$

矩陣

$$\frac{d^2}{dt^2}\boldsymbol{x} = \begin{pmatrix} \dfrac{d^2x_1}{dt^2} \\ \dfrac{d^2x_2}{dt^2} \end{pmatrix}$$

$$\boldsymbol{A} \cdot \boldsymbol{x} \equiv \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$\frac{d^2}{dt^2}\boldsymbol{x} = -\boldsymbol{A} \cdot \boldsymbol{x}$$

矩陣乘行向量得一行向量：

$$(\boldsymbol{Sa})_i = \sum_{j=1}^{2} S_{ij}a_j = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$

此規則保證此乘積是線性的 $\boldsymbol{S}(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2) = c_1\boldsymbol{Sa}_1 + c_2\boldsymbol{Sa}_2$

列向量乘矩陣得一列向量：

$$\left(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{S}\right)_j = \sum_{i=1}^{2} a_iS_{ij} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} a_1S_{11} + a_2S_{21} & a_1S_{12} + a_2S_{22} \end{pmatrix}$$

矩陣乘矩陣還是矩陣：

$$(\boldsymbol{SA})_{mn} = \sum_{j=1}^{2} S_{mj}A_{jn} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S_{11}A_{11} + S_{12}A_{21} & S_{11}A_{12} + S_{12}A_{22} \\ S_{21}A_{11} + S_{22}A_{21} & S_{21}A_{12} + S_{22}A_{22} \end{pmatrix}$$

列向量乘矩陣乘行向量，等於列向量乘行向量，得到一個數：

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{Sa} = \sum_{i,j=1}^{2} b_iS_{ij}a_j$$

Arfken p96-99

$$\frac{d^2x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2 \qquad \frac{d^2x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2$$

先設$x_{1,2}$為複數，猜想其解如簡諧運動也可以寫成虛數的指數函數：

$$x_1 = a_1 \cdot e^{i\omega t} \qquad x_2 = a_2 \cdot e^{i\omega t} \qquad \boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

注意此猜想中，$x_{1,2}$是以同樣的方式一起隨時間振盪：$e^{i\omega t}$。

可以想見此解要成立，ω須是特定的值，$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$必須是特定的行向量。

要找到它們，就要將解代入方程式：

$$\omega^2 e^{i\omega t} a_1 = 2\omega_0^2 e^{i\omega t} a_1 - \omega_0^2 e^{i\omega t} a_2$$

$$\omega^2 e^{i\omega t} a_2 = -\omega_0^2 e^{i\omega t} a_1 + 2\omega_0^2 e^{i\omega t} a_2$$

➡

$$2\omega_0^2 a_1 - \omega_0^2 a_2 = \omega^2 a_1$$

$$-\omega_0^2 a_1 + 2\omega_0^2 a_2 = \omega^2 a_2$$

微分方程組被轉化為a_1, a_2的一次代數方程組以及一個未知數ω。

這個a_1, a_2的一次代數方程組也可以以矩陣及行向量表示：

以矩陣及行向量的語言來寫： $\boldsymbol{A} \cdot \boldsymbol{a} = \omega^2 \boldsymbol{a}$ $\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

這稱為矩陣$\boldsymbol{A}$的本徵值ω問題，$\boldsymbol{a}$稱為本徵向量。

$$2\omega_0^2 a_1 - \omega_0^2 a_2 = \omega^2 a_1$$

$$-\omega_0^2 a_1 + 2\omega_0^2 a_2 = \omega^2 a_2$$

這稱為矩陣$\boldsymbol{A}$的本徵值ω問題，$\boldsymbol{a}$稱為本徵向量。

$$(2\omega_0^2 - \omega^2)a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 + (2\omega_0^2 - \omega^2)a_2 = 0$$

因為右式為零，此方程組只有在行列式為零時才有解：

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0 \qquad (\omega^2 - 2\omega_0^2)^2 - (\omega_0^2)^2 = 0$$

未知數ω^2應該只有兩個解！

$$\omega^2 = \omega_0^2, 3\omega_0^2 \qquad \omega_1 \equiv \omega_0 \qquad \omega_2 \equiv \sqrt{3}\omega_0$$

對特定的ω，我們可以解出對應的a_1, a_2，

$$\omega_1 = \omega_0$$

$$(2\omega_0^2 - \omega_0^2)a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 + (2\omega_0^2 - \omega_0^2)a_2 = 0$$

➡

$$\omega_0^2 a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 + \omega_0^2 a_2 = 0$$

$$a_1 = a_2 \equiv C_1 = A_1 e^{i\phi_1}$$

The solution is:

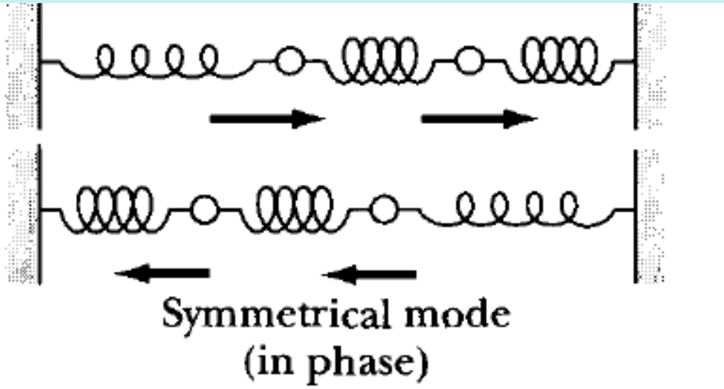
$$x_1 = C_1 \cdot e^{i\omega_0 t} = A_1 e^{i(\omega_0 t + \phi_1)} \qquad x_2 = C_1 \cdot e^{i\omega_0 t} = A_1 e^{i(\omega_0 t + \phi_1)}$$

Now taking the real part would give us the solution.

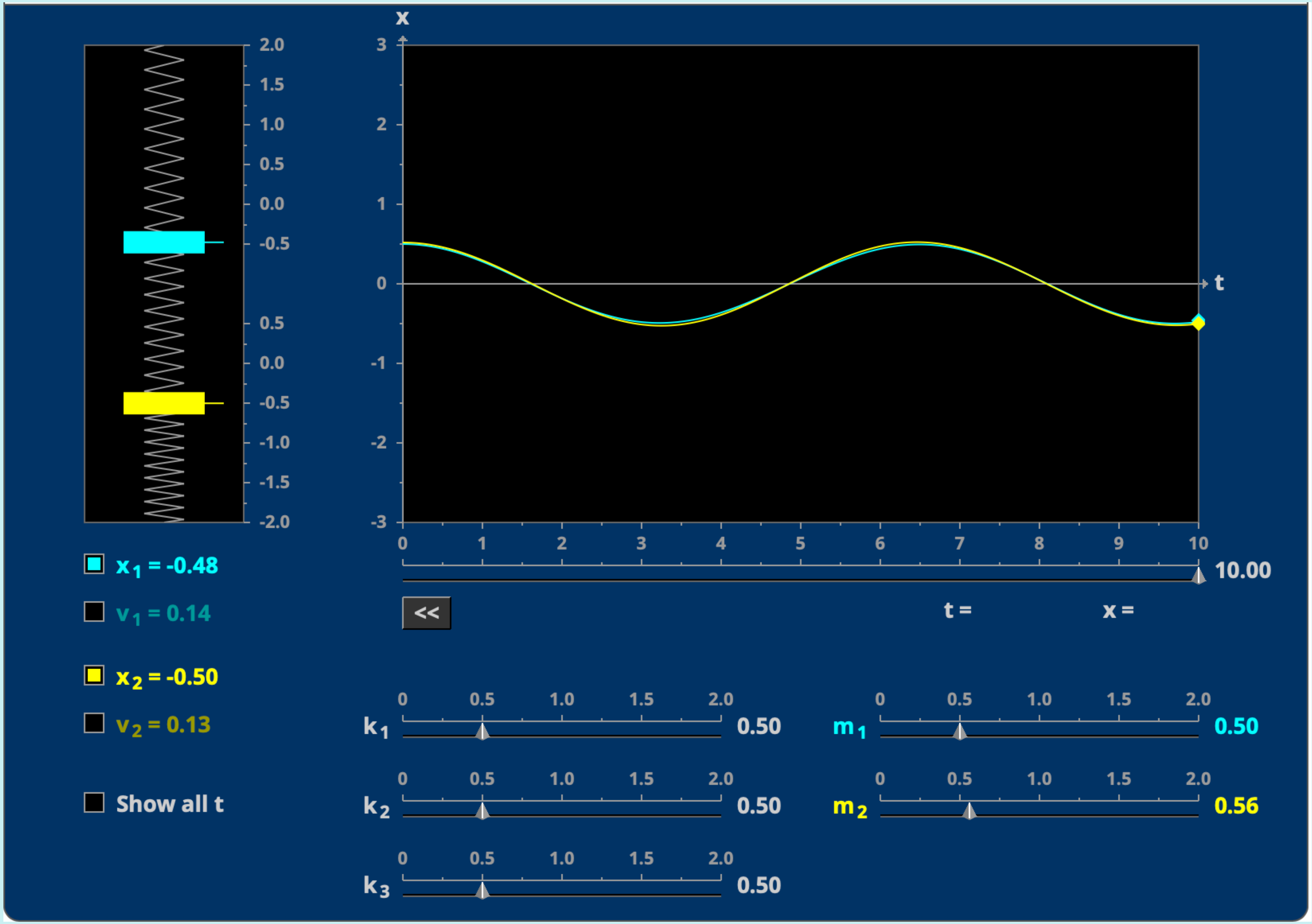
$$x_1 = x_2 = \text{Re}\, A_1 e^{i(\omega_0 t + \phi_1)} = A_1 \cos(\omega_0 t + \phi_1)$$

The two move together!

$$x_1' = x_2' = -\omega_0 A_1 \sin(\omega_0 t + \phi_1)$$



例如起始條件$x_1(0) = x_2(0) = a_m, x_1'(0) = x_2'(0) = 0$，此對稱運動模式就會進行。



x
t
x₁ = -0.48
v₁ = 0.14
x₂ = -0.50
v₂ = 0.13
Show all t
10.00
<<
t =
x =
k1 0.50
k2 0.50
k3 0.50
m1 0.50
m2 0.56

$$\omega_2 = \sqrt{3}\omega_0$$

$$(2\omega_0^2 - 3\omega_0^2)a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 + (2\omega_0^2 - 3\omega_0^2)a_2 = 0$$

➡

$$-\omega_0^2 a_1 - \omega_0^2 a_2 = 0$$

$$-\omega_0^2 a_1 - \omega_0^2 a_2 = 0$$

$$a_1 = -a_2 \equiv C_2 = A_2 e^{i\phi_2}$$

The solution is:

$$x_1 = C_2 \cdot e^{i\sqrt{3}\omega_0 t} = A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)}$$

$$x_2 = -C_2 \cdot e^{i\sqrt{3}\omega_0 t} = A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)}$$

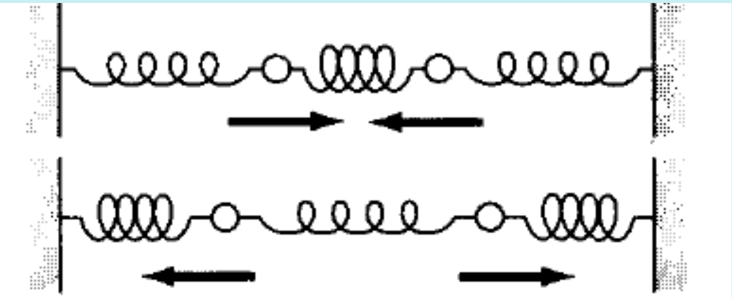
Now taking the real part would give us the solution.

$$x_1 = \mathrm{Re}\, A_2 e^{i(\sqrt{3}\omega_0 t + \phi_2)} = A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

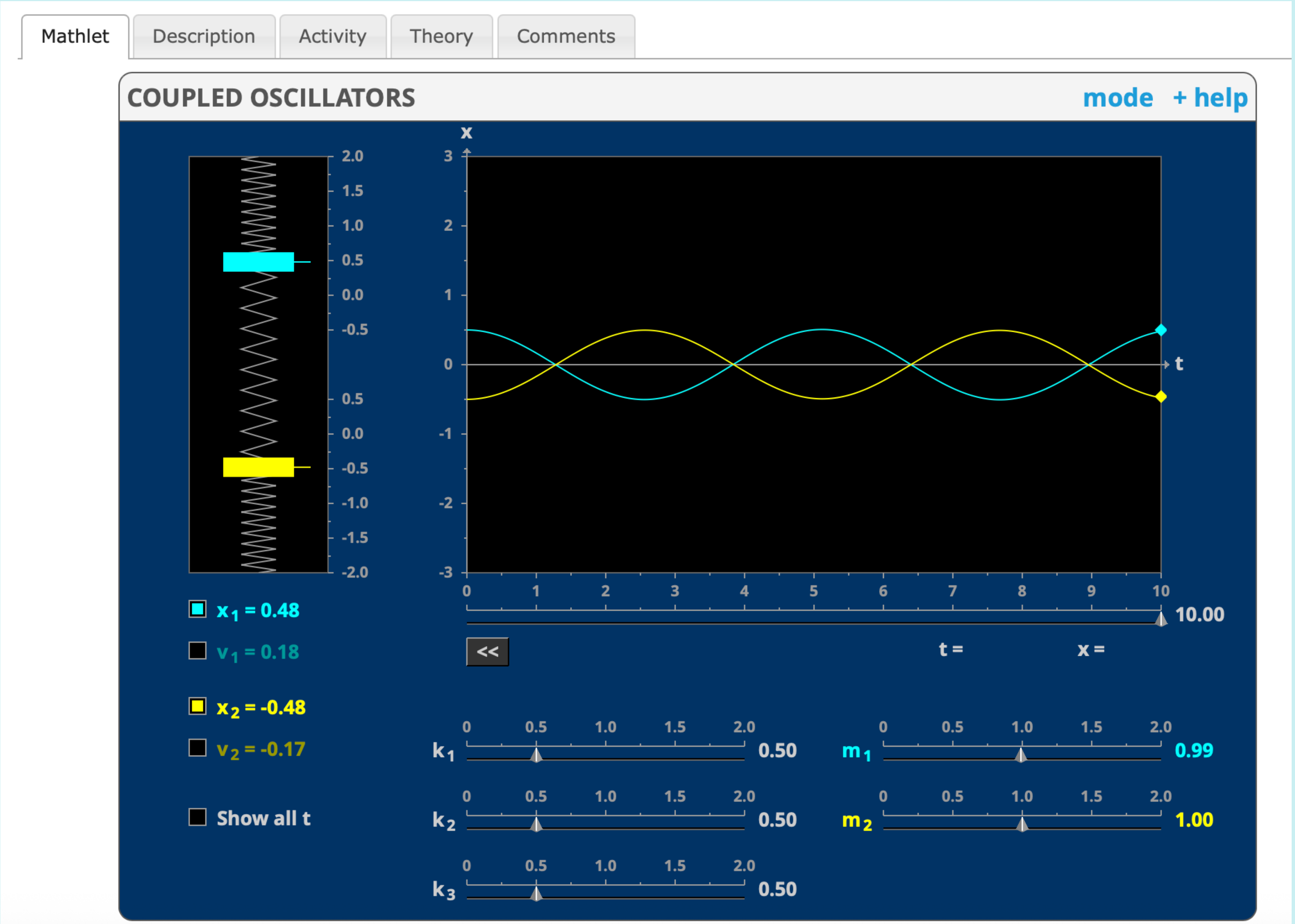
$$x_2 = -x_1 = -A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_1' = -x_2' = \sqrt{3}\omega_0 A_2 \sin(\sqrt{3}\omega_0 t + \phi_1)$$

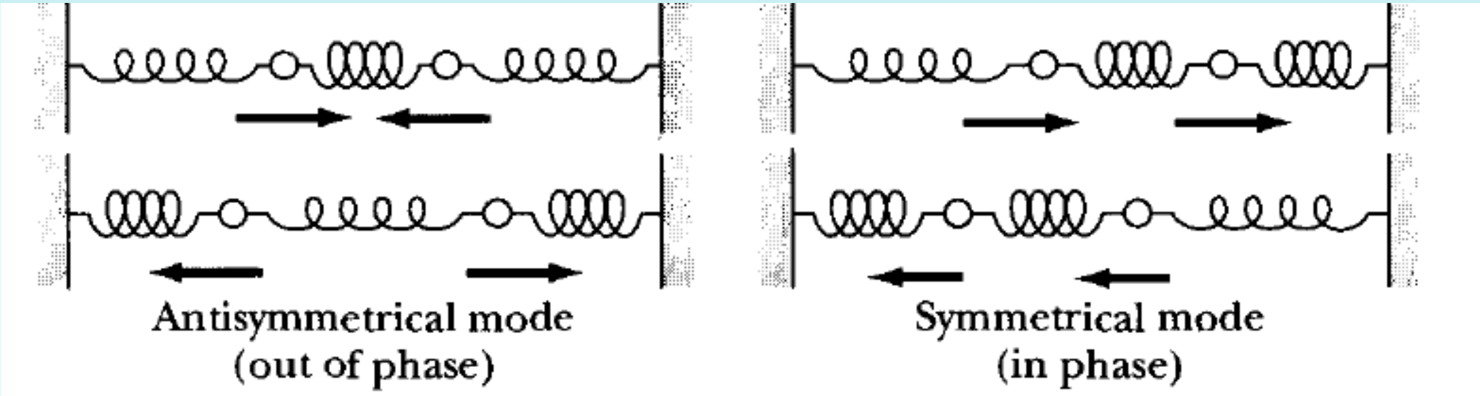
The two move opposite!

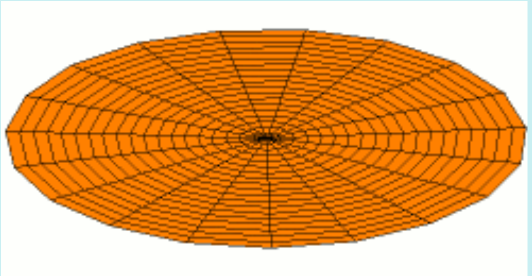
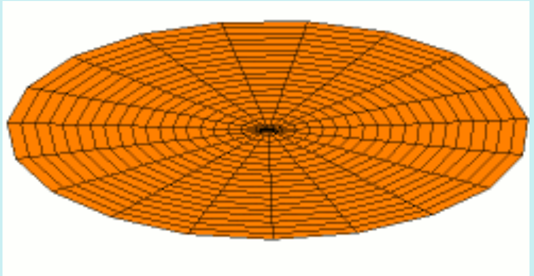
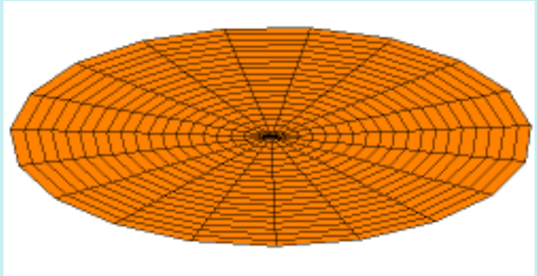
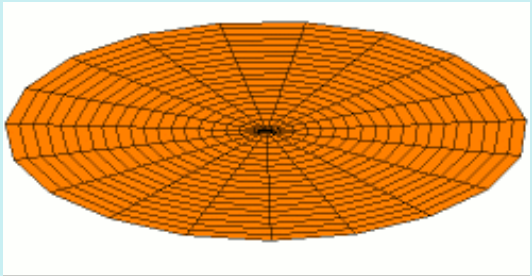
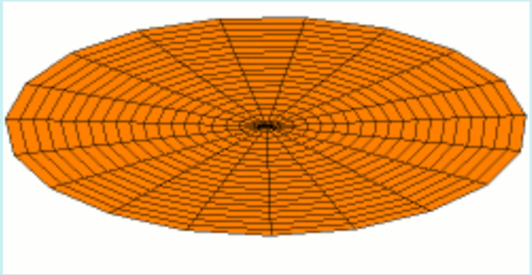
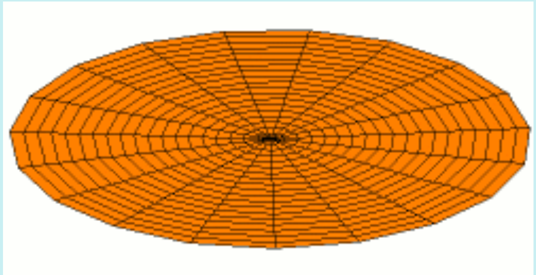

Antisymmetrical mode
(out of phase)

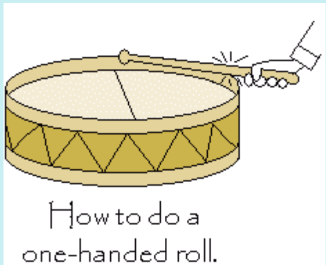
起始條件為$x_1(0) = -x_2(0) = a_m, x_1'(0) = x_2'(0) = 0$，此反對稱運動模式就會進行


Mathlet
Description
Activity
Theory
Comments
COUPLED OSCILLATORS
mode
+ help
x
t
x₁ = 0.48
v₁ = 0.18
x₂ = -0.48
v₂ = -0.17
Show all t
10.00
<<
t =
x =
k₁
0.50
k₂
0.50
k₃
0.50
m₁
0.99
m₂
1.00

在這兩個振盪模式，彈簧組作簡諧運動。





原則上物體的變形模式有無限多個，但並不是連續分布。

因此可以分離地一個一個編號。

每一個模式，如同簡諧運動，對應一個內在的特定的振動頻率！

不同模式頻率不同。一般來說，頻率越高的模式，越難激發。

物體的所有變形就是以這些模式或它們的疊加來進行！

一個物體有那些振盪模式 Norm以及對應的頻率，就是該物體的一個特徵。

A general solution is a sum of the two modes.

一般起始條件就是兩種模式的疊加！

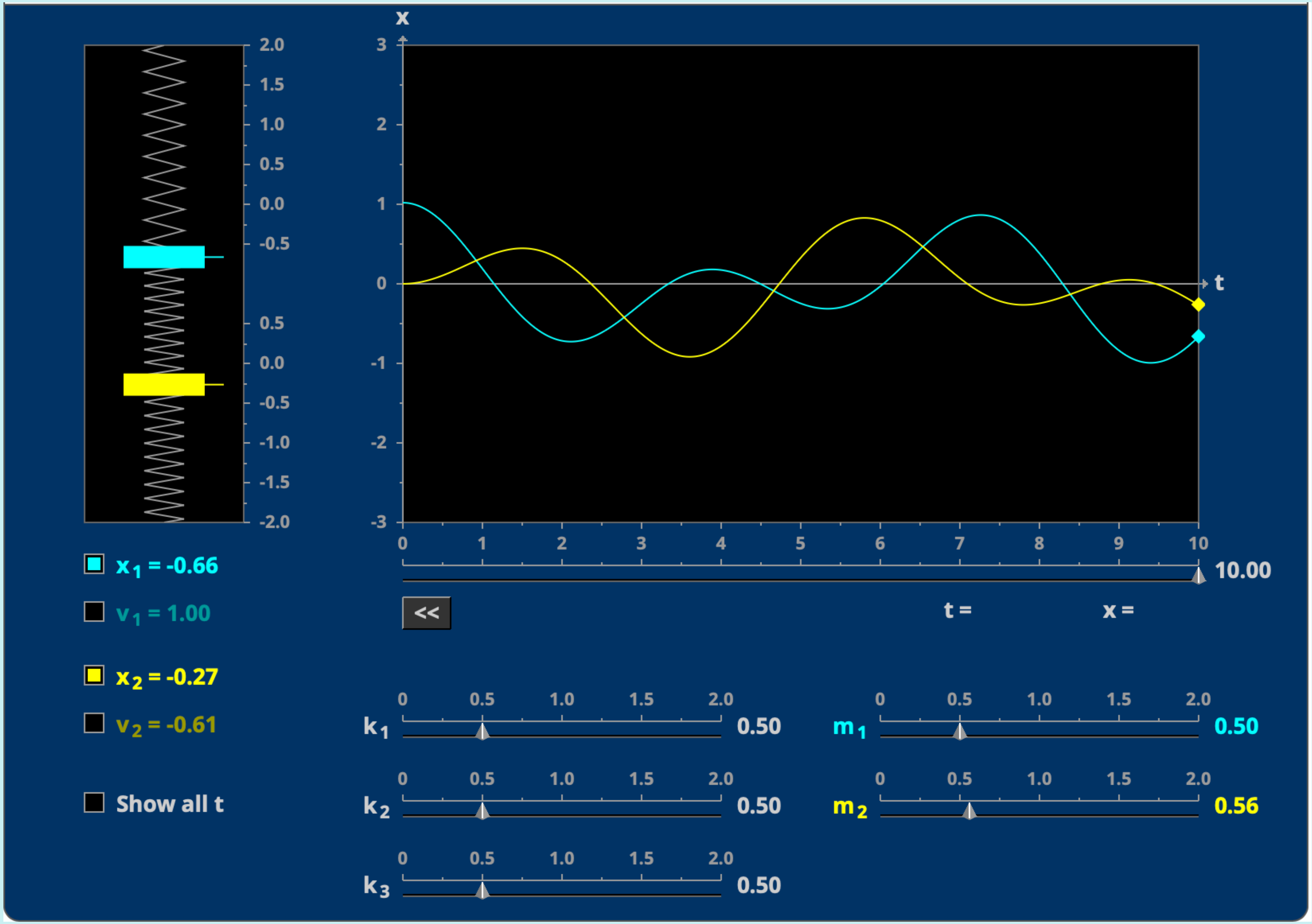
There are four undetermined constants $A_{1,2}, \phi_{1,2}$.

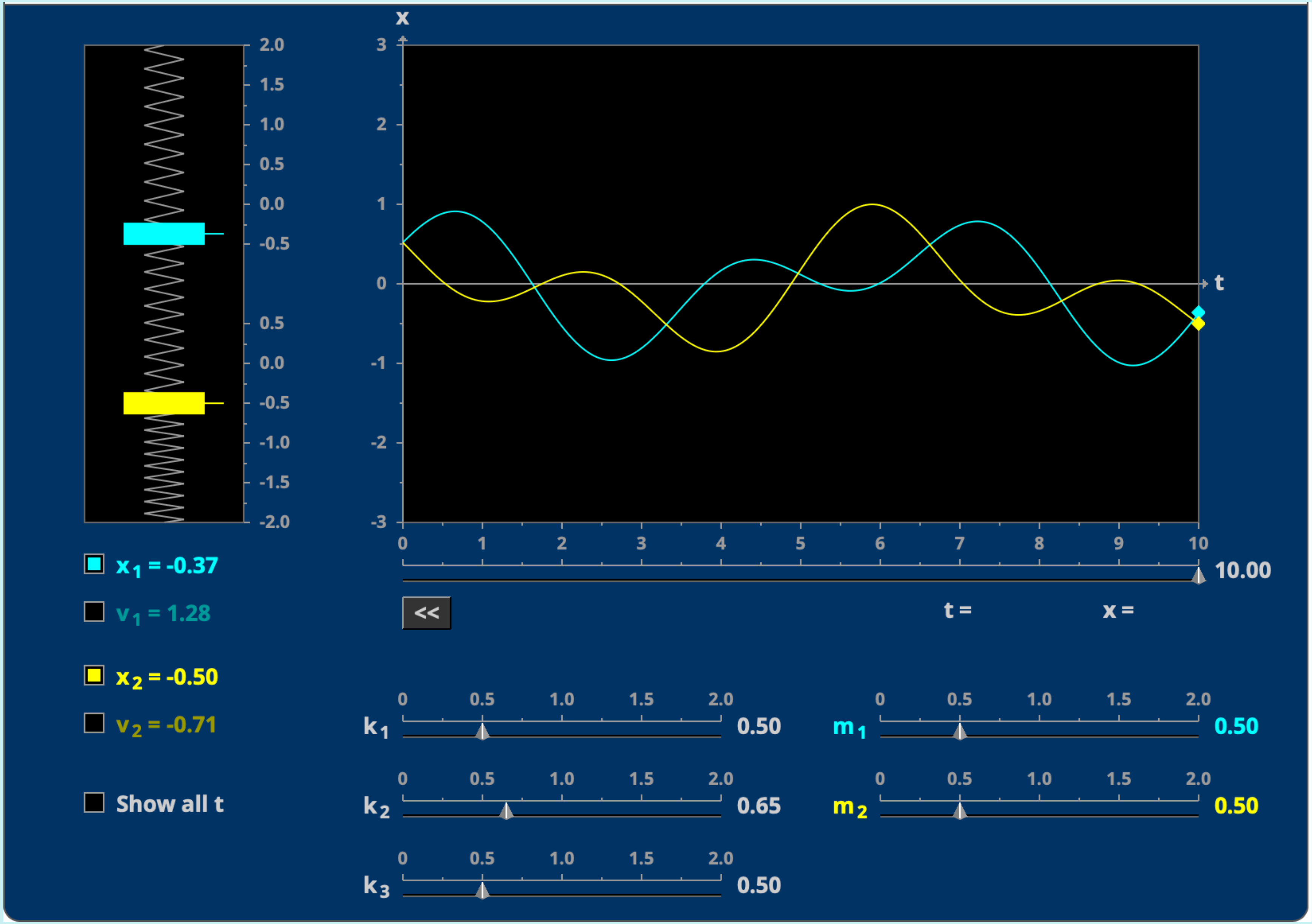
$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos\left(\sqrt{3}\omega_0 t + \phi_2\right)$$

$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos\left(\sqrt{3}\omega_0 t + \phi_2\right)$$

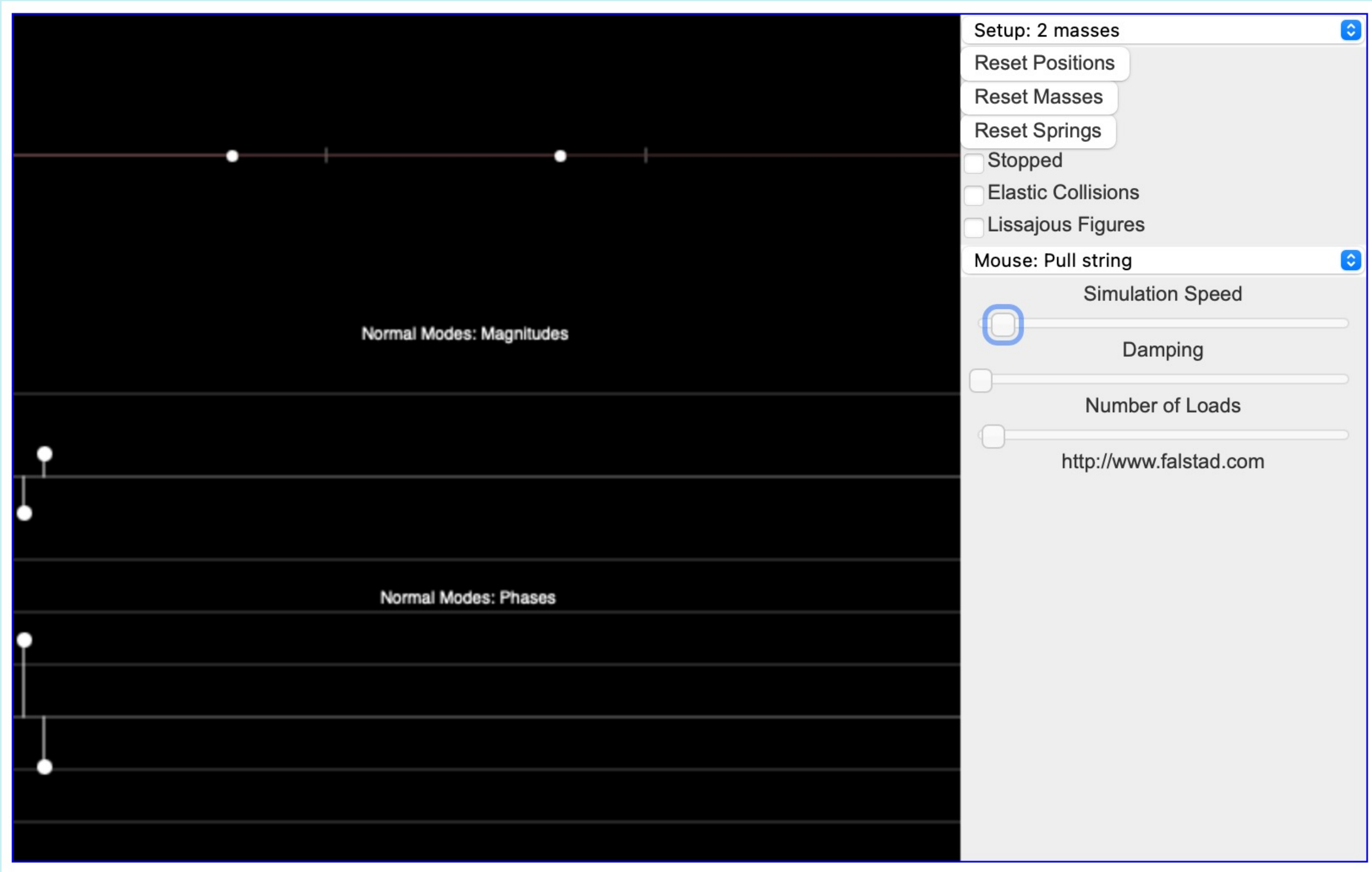
There are also four initial conditions $x_{1,2}(0), x'_{1,2}(0)$.

It would be the one unique solution.


x
t
x1 = -0.66
v1 = 1.00
x2 = -0.27
v2 = -0.61
Show all t
10.00
<<
t =
x =
k1
0.50
k2
0.50
k3
0.50
m1
0.50
m2
0.56



x
t
x_1 = -0.37
v_1 = 1.28
x_2 = -0.50
v_2 = -0.71
Show all t
10.00
<<
t =
x =
k_1 0.50
k_2 0.65
k_3 0.50
m_1 0.50
m_2 0.50



Normal Modes: Magnitudes
Normal Modes: Phases
Setup: 2 masses
Reset Positions
Reset Masses
Reset Springs
Stopped
Elastic Collisions
Lissajous Figures
Mouse: Pull string
Simulation Speed
Damping
Number of Loads
http://www.falstad.com

Method I

$$-\boldsymbol{A}\cdot\boldsymbol{x}=\frac{d^2}{dt^2}\boldsymbol{x}\qquad \boldsymbol{x}\equiv\begin{pmatrix}x_1\\x_2\end{pmatrix}=\boldsymbol{a}e^{i\omega t}\qquad \boldsymbol{a}\equiv\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

This is the eigenvalue problem of Matrix $\boldsymbol{A}$.

微分方程組被轉化為矩陣$\boldsymbol{A}$的本徵值ω問題，$\boldsymbol{a}$稱為本徵向量 。

$$\boldsymbol{A}\cdot\boldsymbol{a}=\omega^2\boldsymbol{a}\qquad \boldsymbol{A}\equiv\begin{pmatrix}2\omega_0^2 & -\omega_0^2\\ -\omega_0^2 & 2\omega_0^2\end{pmatrix}$$

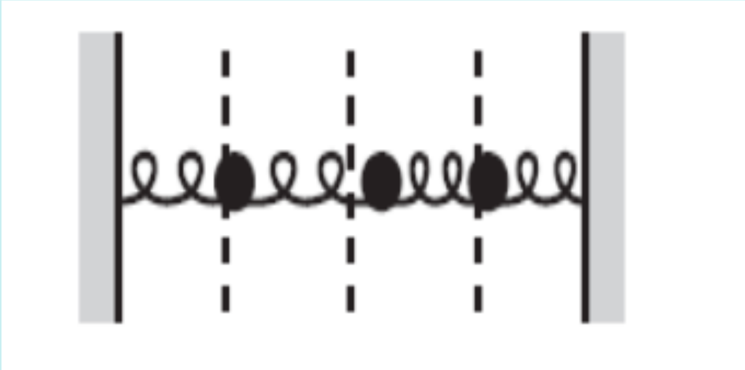
本徵值ω^2有兩個，各自對應一本徵向量$\boldsymbol{a}$。

$$\omega_1\equiv\omega_0\qquad \begin{pmatrix}a_1\\a_2\end{pmatrix}=c_1\begin{pmatrix}1\\1\end{pmatrix}$$

$$\omega_2\equiv\sqrt{3}\omega_0\qquad \begin{pmatrix}a_1\\a_2\end{pmatrix}=c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=c_1\begin{pmatrix}1\\1\end{pmatrix}\cos\omega_0 t+c_2\begin{pmatrix}1\\-1\end{pmatrix}\cos\sqrt{3}\omega_0 t$$

This formalism could be easily extended to more than two masses.



(29)

The equations of motion for this system will be of the form

$$m_1\ddot{x}_1 = k_{11}\,x_1 + k_{12}\,x_2 + k_{13}x_3 \tag{30}$$

$$m_2\ddot{x}_3 = k_{21}\,x_1 + k_{22}\,x_2 + k_{23}x_3 \tag{31}$$

$$m_3\ddot{x}_3 = k_{31}\,x_1 + k_{32}\,x_2 + k_{33}x_3 \tag{32}$$

Some of these k_{ij} are probably zero, but we don't care. Writing $x_1 = c_1e^{i\omega t}$, $x_2 = c_2e^{i\omega t}$ and $x_3 = c_3e^{i\omega t}$, these equations become algebraic:

$$-\omega^2 c_1 = \frac{k_{11}}{m_1}c_1 + \frac{k_{12}}{m_1}c_2 + \frac{k_{13}}{m_1}c_3$$

$$-\omega^2 c_2 = \frac{k_{21}}{m_2}c_1 + \frac{k_{22}}{m_2}c_2 + \frac{k_{23}}{m_2}c_3$$

$$-\omega^2 c_3 = \frac{k_{31}}{m_3}c_1 + \frac{k_{32}}{m_3}c_2 + \frac{k_{33}}{m_3}c_3$$

In other words,

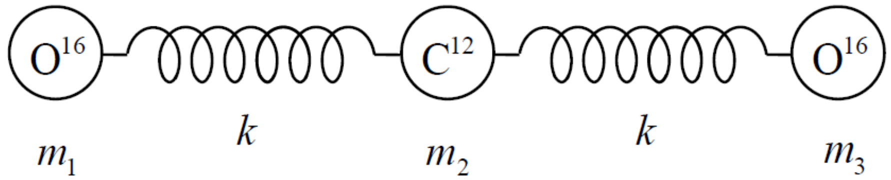
$$\boldsymbol{A} \equiv \begin{pmatrix} \frac{k_{11}}{m_1} & \frac{k_{12}}{m_1} & \frac{k_{13}}{m_1} \\ \frac{k_{21}}{m_2} & \frac{k_{22}}{m_2} & \frac{k_{23}}{m_2} \\ \frac{k_{31}}{m_3} & \frac{k_{32}}{m_3} & \frac{k_{33}}{m_3} \end{pmatrix}$$

$$-\boldsymbol{A}\cdot\boldsymbol{x} = \frac{d^2}{dt^2}\boldsymbol{x}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

【例 1】

CO_2 分子的振動情形可用圖十一所示的耦合系統來模擬。圖中的耦合彈簧相同，其力常數為 k。



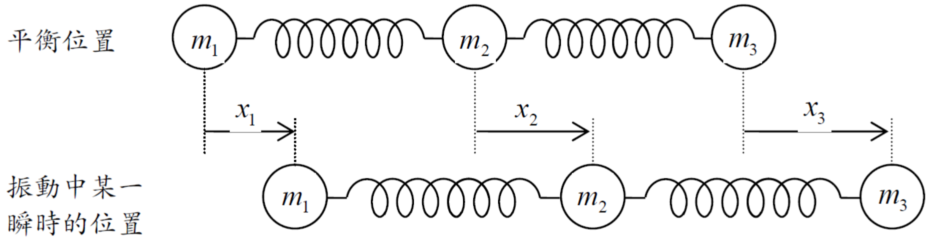
圖十一

(1)若僅考慮各原子的縱向振動（即沿原子中心連線的方向），試寫出其運動方程式。

(2)氧和碳的原子量分別為 16 和 12，試求此分子縱向振動的簡正模頻率的比值。

解：

(1)參考圖十二，各原子偏離其平衡位置的位移分別為 x_1，x_2 和 x_3。



圖十二

各原子的運動方程式如下：

$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = -k(x_1 - x_2) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \quad \text{.......(2-7)} \\ m_3 \dfrac{d^2 x_3}{dt^2} = -k(x_3 - x_2) \end{cases}$$

$$x_n = a_n \cos(\omega_n t + \phi_n)$$

簡正模頻率，則

$$x_n = a_n \cdot e^{i(\omega_n t + \phi_n)}$$

$$x_n = a_n \cos \omega t \text{，} n = 1,2,3 \quad \text{.......(2-8)}$$

將(2-8)代入(2-7)可得

$$\begin{cases} \left(\omega^2 - \dfrac{k}{m_1}\right) a_1 + \left(\dfrac{k}{m_1}\right) a_2 = 0 \\ \left(\dfrac{k}{m_2}\right) a_1 + \left(\omega^2 - \dfrac{2k}{m_2}\right) a_2 + \left(\dfrac{k}{m_2}\right) a_3 = 0 \quad \text{.......(2-9)} \\ \left(\dfrac{k}{m_3}\right) a_2 + \left(\omega^2 - \dfrac{k}{m_3}\right) a_3 = 0 \end{cases}$$

各原子的運動方程式如下：

$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = -k(x_1 - x_2) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \\ m_3 \dfrac{d^2 x_3}{dt^2} = -k(x_3 - x_2) \end{cases} \qquad \text{(2-7)}$$

(2)設 ω 為可能的簡正模頻率，則

$$x_n = a_n \cos \omega t \text{，} n = 1,2,3 \qquad \text{(2-8)}$$

將(2-8)代入(2-7)可得

$$\begin{cases} \left(\omega^2 - \dfrac{k}{m_1}\right)a_1 + \left(\dfrac{k}{m_1}\right)a_2 = 0 \\ \left(\dfrac{k}{m_2}\right)a_1 + \left(\omega^2 - \dfrac{2k}{m_2}\right)a_2 + \left(\dfrac{k}{m_2}\right)a_3 = 0 \\ \left(\dfrac{k}{m_3}\right)a_2 + \left(\omega^2 - \dfrac{k}{m_3}\right)a_3 = 0 \end{cases} \qquad \text{(2-9)}$$

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\frac{d^2\boldsymbol{X}}{dt^2} = \begin{pmatrix} -\dfrac{k}{m_1} & \dfrac{k}{m_1} & 0 \\ \dfrac{k}{m_2} & -\dfrac{2k}{m_2} & \dfrac{k}{m_2} \\ 0 & \dfrac{k}{m_3} & -\dfrac{k}{m_3} \end{pmatrix} \cdot \boldsymbol{X} \equiv \boldsymbol{A} \cdot \boldsymbol{X}$$

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i\omega t} \equiv \boldsymbol{a} e^{i\omega t}$$

代入 $\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A} \cdot \boldsymbol{X}$

$\boldsymbol{A} \cdot \boldsymbol{a} = -\omega^2 \boldsymbol{a}$ 微分方程式被轉化為本徵值方程式：

ω^2 及 a 分別為矩陣 A 的Eigenvalues本徵值及Eigenvectors本徵向量。

$$\begin{pmatrix} -\frac{k}{m_1} + \omega^2 & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -\frac{2k}{m_2} + \omega^2 & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} + \omega^2 \end{pmatrix} \cdot \boldsymbol{a} = 0$$

a 只有當矩陣的行列式為零時有非零解：，

$$\begin{vmatrix} -\frac{k}{m_1} + \omega^2 & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -\frac{2k}{m_2} + \omega^2 & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} + \omega^2 \end{vmatrix} = 0$$

ω^2 有三個解Eigenvalues。每一個解對應一個向量 a：eigenvectors

取 $m_1 = m_3 = 16m_0$，$m_2 = 12m_0$，且令 $\omega_0^2 \equiv \dfrac{k}{m_0}$，則(2-9)可改寫為

$$\begin{cases} \left(\omega^2 - \dfrac{\omega_0^2}{16}\right)a_1 + \left(\dfrac{\omega_0^2}{16}\right)a_2 = 0 \\ \left(\dfrac{\omega_0^2}{12}\right)a_1 + \left(\omega^2 - \dfrac{\omega_0^2}{6}\right)a_2 + \left(\dfrac{\omega_0^2}{12}\right)a_3 = 0 \\ \left(\dfrac{\omega_0^2}{16}\right)a_2 + \left(\omega^2 - \dfrac{\omega_0^2}{16}\right)a_3 = 0 \end{cases} \quad \text{......................(2-10)}$$

(2-10)有非零解之條件為係數行列式等於零，即

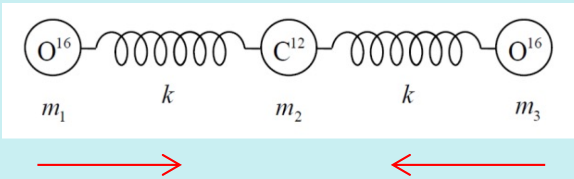
$$\begin{vmatrix} \omega^2 - \dfrac{\omega_0^2}{16} & \dfrac{\omega_0^2}{16} & 0 \\ \dfrac{\omega_0^2}{12} & \omega^2 - \dfrac{\omega_0^2}{6} & \dfrac{\omega_0^2}{12} \\ 0 & \dfrac{\omega_0^2}{16} & \omega^2 - \dfrac{\omega_0^2}{16} \end{vmatrix} = 0$$

解得 $\omega = \dfrac{\omega_0}{4}$，$\sqrt{\dfrac{11}{48}}\omega_0$，0（即為處於平衡狀態，不振動）

兩振動中之簡正模頻率之比值為 $\sqrt{\dfrac{11}{3}}$。

三個頻率分別對應三個不同的(a_1, a_2, a_3)模式mode

例如$\omega = \frac{\omega_0}{4}$時，$a = (1,0,-1)$

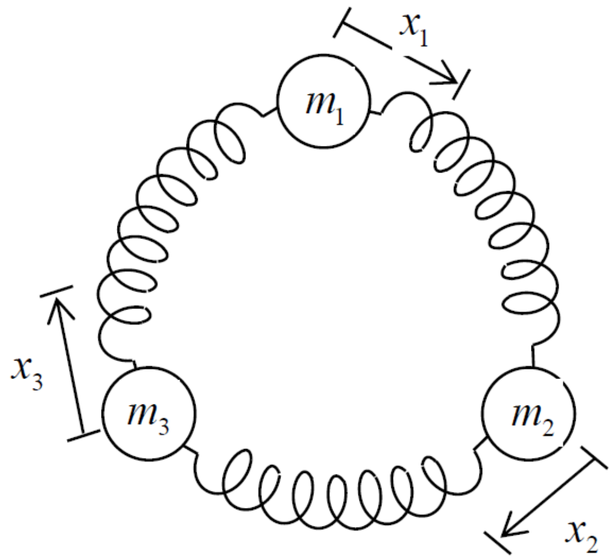


【例2】

如圖十三所示，三個質量各為 m 的質點由三條相同的輕彈簧聯接在一起，彈簧的力常數為 k。當此系統處於平衡狀態時，各彈簧的長度為其自然長度。當處於運動狀態時，各質點被限制在同一圓周上運動。

(1) 若此三質點偏離其平衡位置的小位移，分別為 x_1，x_2 和 x_3，試寫出每一質點的運動方程式。

(2) 試求出此系統可能的簡正模頻率。



圖十三

解：

(1)

$$\begin{cases} m\dfrac{d^2x_1}{dt^2}=-k(x_1-x_2)-k(x_1-x_3) \\ m\dfrac{d^2x_2}{dt^2}=-k(x_2-x_1)-k(x_2-x_3) \\ m\dfrac{d^2x_3}{dt^2}=-k(x_3-x_1)-k(x_3-x_2) \end{cases} \qquad \text{.......}(2\text{-}11)$$

(2)設ω為可能的簡正模頻率，則

$$x_n=a_n\cos\omega t \text{ ， } n=1,2,3 \qquad \text{.......}(2\text{-}12)$$

將(2-12)代入(2-11)可得

$$\begin{cases} \left(\dfrac{\omega^2}{\omega_0^2}-2\right)a_1+a_2+a_3=0 \\ a_1+\left(\dfrac{\omega^2}{\omega_0^2}-2\right)a_2+a_3=0 \\ a_1+a_2+\left(\dfrac{\omega^2}{\omega_0^2}-2\right)a_3=0 \end{cases} \qquad \text{.......}(2\text{-}13)$$

其中$\omega_0\equiv\sqrt{\dfrac{k}{m}}$

(2-13)式有非零解之條件為係數行列式等於零，即

$$\begin{vmatrix} \dfrac{\omega^2}{\omega_0^2}-2 & 1 & 1 \\ 1 & \dfrac{\omega^2}{\omega_0^2}-2 & 1 \\ 1 & 1 & \dfrac{\omega^2}{\omega_0^2}-2 \end{vmatrix}=0$$

(2-13)式有非零解之條件為係數行列式等於零，即

$$\begin{vmatrix} \frac{\omega^2}{\omega_0^2}-2 & 1 & 1 \\ 1 & \frac{\omega^2}{\omega_0^2}-2 & 1 \\ 1 & 1 & \frac{\omega^2}{\omega_0^2}-2 \end{vmatrix} = 0$$

解得 $\left(\frac{\omega^2}{\omega_0^2}\right)\left(\frac{\omega^2}{\omega_0^2}-3\right)^2 = 0$

因此振動中之簡正模頻率為 $\sqrt{3}\omega_0 = \sqrt{\frac{3k}{m}}$

There is another way to look at this solution. Method 2

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

$$x_+ \equiv x_1 + x_2 = 2A_1 \cos(\omega_0 t + \phi_1)$$

Generalized Coordinates 廣義座標

$$x_- \equiv x_1 - x_2 = 2A_2 \cos(\sqrt{3}\omega_0 t + \phi_2)$$

廣義座標可以完全決定真實座標！

x_+, x_- are pure simple harmonic oscillators with respective angular frequencies.

從廣義座標x_+, x_-滿足的運動方程式的角度來看：

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2 \quad + \quad \frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2 \;\Rightarrow\; \frac{d^2(x_1 + x_2)}{dt^2} = -\omega_0^2 x_1 - \omega_0^2 x_2$$

$$\frac{d^2 x_+}{dt^2} = -\omega_0^2 x_+$$

$$\frac{d^2 x_1}{dt^2} = -2\omega_0^2 x_1 + \omega_0^2 x_2 \quad - \quad \frac{d^2 x_2}{dt^2} = \omega_0^2 x_1 - 2\omega_0^2 x_2 \;\Rightarrow\; \frac{d^2 x_-}{dt^2} = -3\omega_0^2 x_-$$

x_+, x_- are called Normal Coordinates. 模式座標

$$\boldsymbol{x} \equiv \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \qquad -\boldsymbol{A} \cdot \boldsymbol{x} = \frac{d^2}{dt^2}\boldsymbol{x} \qquad \boldsymbol{A} \equiv \begin{pmatrix} \omega_0^2 & 0 \\ 0 & 3\omega_0^2 \end{pmatrix}$$

For normal coordinates, $\boldsymbol{A}$ is now diagonal. Off-diagonal elements means coupling.

x_+, x_- are uncoupled, independent SHM.

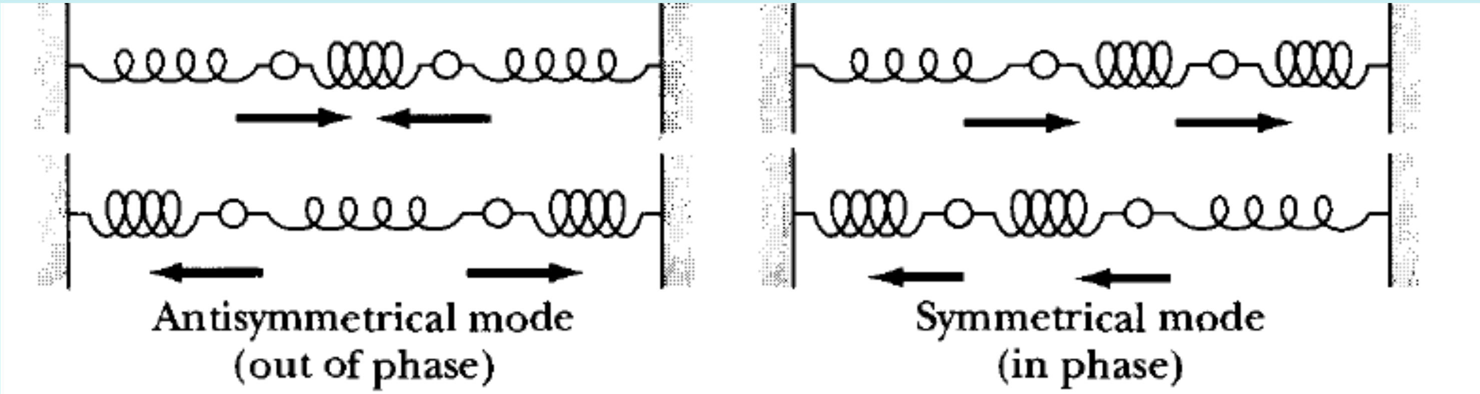
在這兩個模式，廣義座標非常簡單。

$$x_1 = -x_2 = A_2 \cos(\omega_0 t + \phi_1)$$

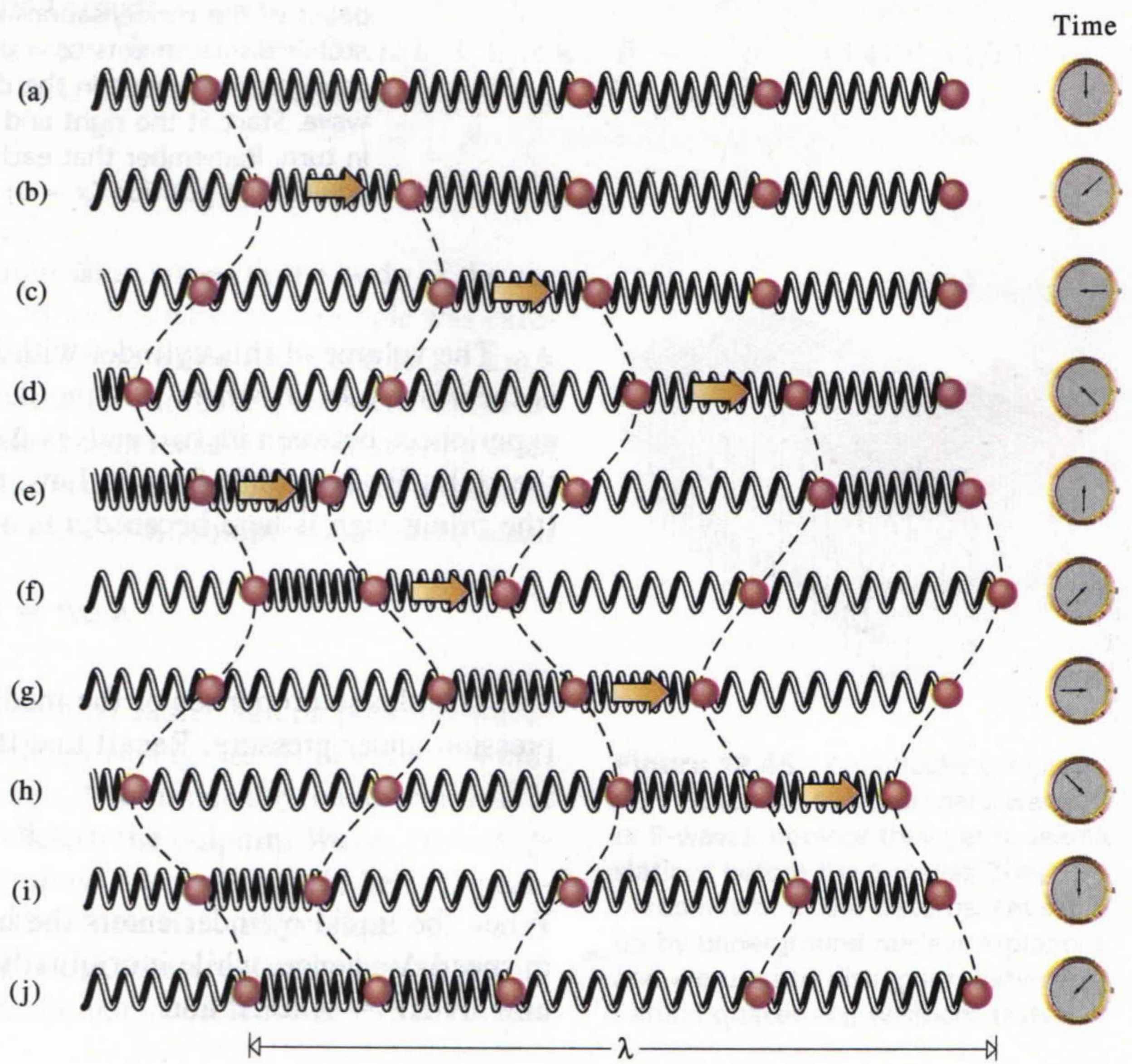
$$x_1 = x_2 = A_1 \cos(\omega_0 t + \phi_1)$$

$$x_+ = 0, x_- = 2A_2 \cos\left(\sqrt{3}\omega_0 t + \phi_2\right)$$

$$x_+ = A_1 \cos(\omega_0 t + \phi_1), x_- = 0$$



可以說這兩個模式座標就是來標定這兩個模式的。


Time
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
λ

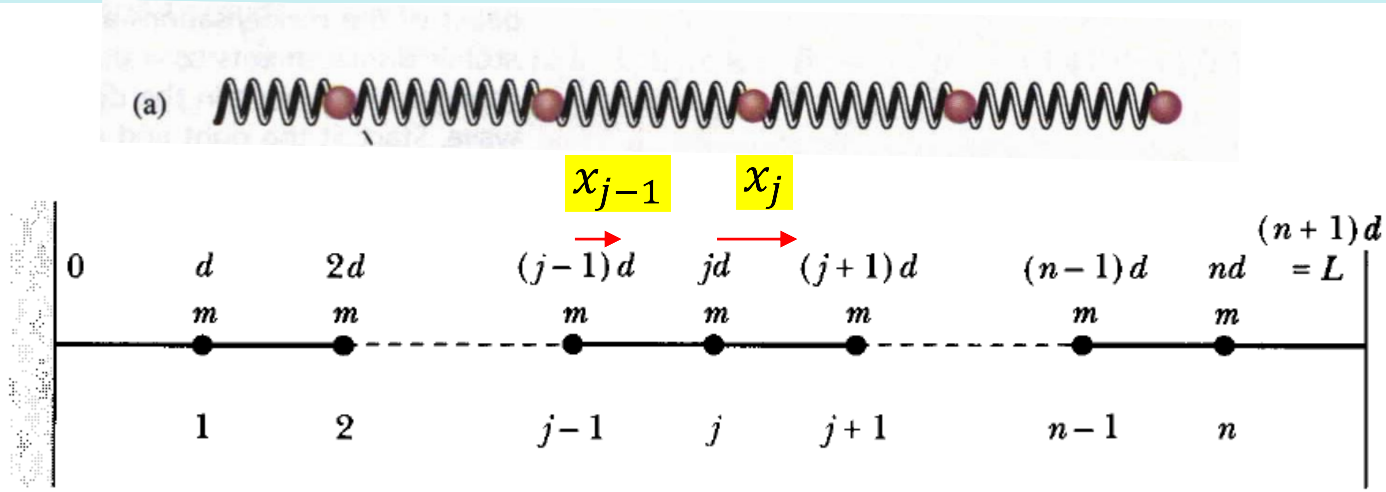


FIGURE 12-9 A schematic of the loaded string. In equilibrium, identical masses are spaced equidistantly. The ends of the string are fixed.

$$m\frac{d^2x_j}{dt^2} = -k(x_j - x_{j-1}) + k(x_{j+1} - x_j)$$

$$\frac{d^2x_j}{dt^2} = \frac{k}{m}x_{j-1} - \frac{2k}{m}x_j + \frac{k}{m}x_{j+1}$$

$$\omega_0^2 = \frac{k}{m}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix} = \omega_0^2 \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

$$\boldsymbol{X} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{pmatrix}$$

$$\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A} \cdot \boldsymbol{X}$$

Consider the ends fixed, called boundary condition. $x_0 = x_{n+1} = 0$

先設x_i為複數，如下猜解並代入：

$$\boldsymbol{X} = \boldsymbol{a}e^{i\omega t} \qquad \frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X} \quad \Rightarrow \quad -\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_{n-1} \end{pmatrix}$$

微分方程組被轉化為矩陣的本徵值問題。

$$-\omega_0^2\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}\cdot\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix} = \omega^2\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}$$

在此本徵向量可以猜出來，下一頁將驗證：

$$a_j \sim e^{-ik\cdot j} \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_{n-1} \end{pmatrix} \sim \begin{pmatrix} e^{-ik} \\ \vdots \\ e^{-ik\cdot j} \\ \vdots \\ e^{-ik\cdot n} \end{pmatrix}$$

本徵方程式：

$$\omega_0^2\cdot\begin{pmatrix}-2 & \cdots & 0 & 0 & 0 & \cdots\\ \vdots & \ddots & 1 & 0 & 0 & \cdots\\ 0 & 1 & -2 & 1 & 0 & \cdots\\ 0 & 0 & 1 & -2 & 1 & \cdots\\ 0 & 0 & 0 & 1 & -2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}e^{-ik}\\ \vdots\\ e^{-i(j-1)k}\\ e^{-ijk}\\ e^{-i(j+1)k}\\ \vdots\\ e^{-ik\cdot n}\end{pmatrix}=\omega^2\begin{pmatrix}e^{-ik}\\ \vdots\\ e^{-i(j-1)k}\\ e^{-ijk}\\ e^{-i(j+1)k}\\ \vdots\\ e^{-ik\cdot n}\end{pmatrix}$$

此本徵向量的特徵是等比數列： $a_j \sim e^{-ijk}$

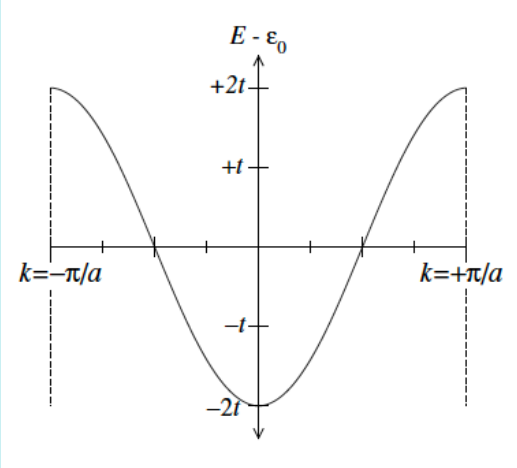
$$\frac{a_{j+1}}{a_j}\sim e^{-ik}$$

代入本徵向量方程式： 第j個元素的等式為：

$$-\omega_0^2\left[e^{-(j-1)ik}-2e^{-jik}+e^{-(j+1)ik}\right]=\omega^2 e^{-jik}$$

整式除以e^{-jik}，得： $-\omega_0^2\left[e^{ik}-2+e^{-ik}\right]=\omega^2$

化簡就可以得到：

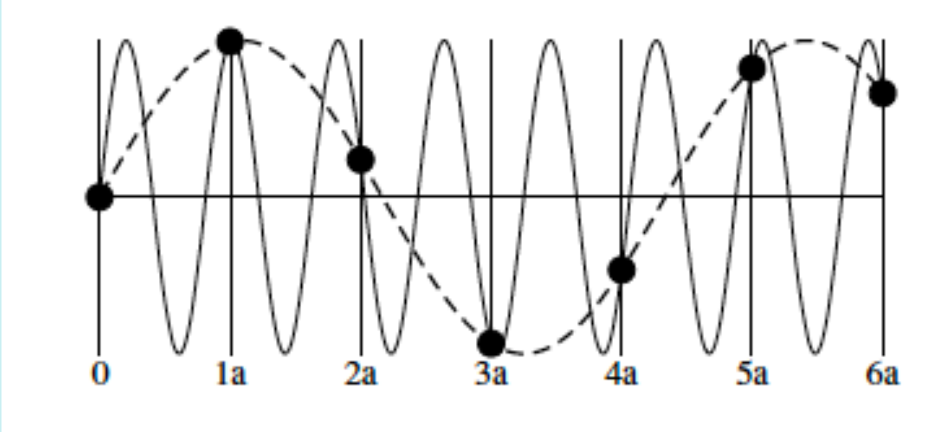


$\omega^2=\omega_0^2(2-2\cos k)$ 一個k對應一特定的本徵值。

本徵向量可以取其實數部或虛數部都滿足本徵向量方程式：

取其虛數部都滿足本徵向量方程式：

$$\boldsymbol{a} = \begin{pmatrix} e^{-ik} \\ \vdots \\ e^{-i(j-1)k} \\ e^{-ijk} \\ e^{-i(j+1)k} \\ \vdots \\ e^{-ik\cdot n} \end{pmatrix}$$



因矩陣$\boldsymbol{A}$是實數矩陣，本徵向量$\boldsymbol{a}$可取其實數部或虛數部都滿足本徵向量方程式：

取其虛數部則可以直接滿足左側的邊界條件： $x_0 = x_{n+1} = 0$

$a_j \sim e^{-ikj}$ ➡ $a_j = \sin kj$ ➡ $a_0 = 0$

若要滿足右側的邊界條件： $a_{n+1} = 0$

$$k \cdot (n+1) = s\pi$$

只能取左式這些值， $s = n+1$對應$a_j = 0$。

$$k = \frac{s\pi}{n+1} \qquad s = 1,2,3 \cdots n$$

$s = n+2$與$s = 1$對應的k差π。 對應相似的a_j。

共n個，代入色散關係：

$$\omega_s^2 = \omega_0^2 \left(2 - 2\cos\frac{s\pi}{n+1} a\right)$$

如預期有n個本徵值。

$$a_j = \sin\frac{s\pi}{n+1} \cdot j$$

Continuum limit

$$m\frac{d^2x_j}{dt^2} = -k(x_j - x_{j-1}) + k(x_{j+1} - x_j)$$

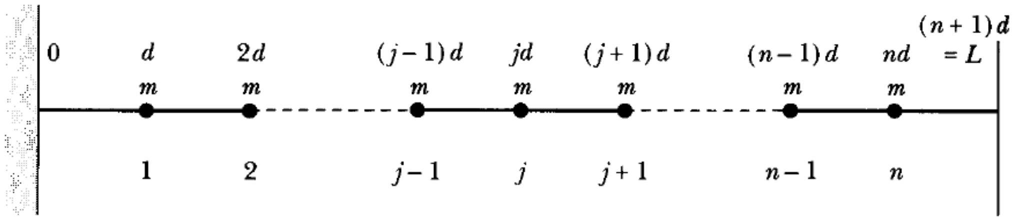
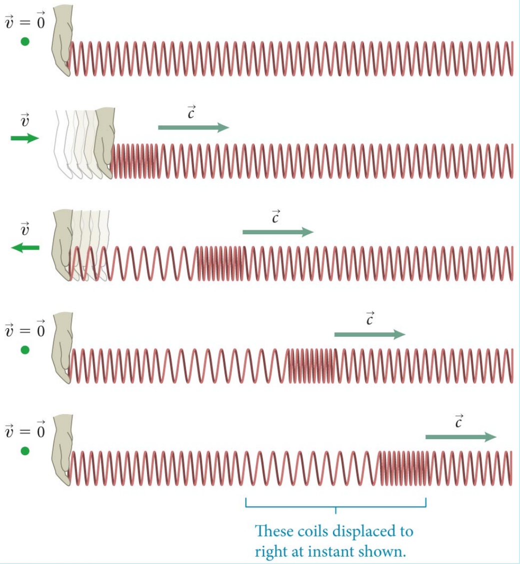
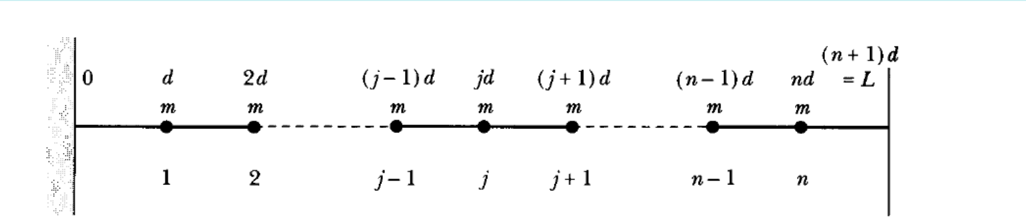


FIGURE 12-9 A schematic of the loaded string. In equilibrium, identical masses are spaced equidistantly. The ends of the string are fixed.

Wave





$$m\frac{d^2x_j}{dt^2} = -k\left(x_j - x_{j-1}\right) + k\left(x_{j+1} - x_j\right)$$

當粒子數量太龐大，用行向量就不是那麼方便，反而選一個函數$\phi(x)$較好。
函數$\phi(x)$只有在有粒子處 $x = dj$ 有定義，

$$\phi(dj) \equiv x_j$$

但當$N \to \infty, d \to 0$，有粒子處的位置就近乎是連續的變數：

$$dj \to x$$

$$\phi(dj) \to \phi(x)$$

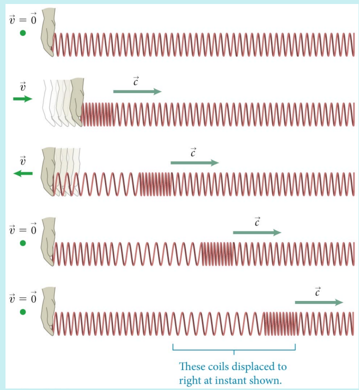
$$m\frac{d^2\phi(dj)}{dt^2} = +k\big(\phi(dj + d) - \phi(dj)\big) - k\big(\phi(dj) - \phi(dj - d)\big)$$

$$= +kd\frac{\phi(dj + d) - \phi(dj)}{d} - kd\frac{\phi(dj) - \phi(dj - d)}{d}$$

$$\to kd\left[\frac{d\phi}{dx}(dj) - \frac{d\phi}{dx}(dj - d)\right]$$

$$\to kd\frac{d^2\phi}{dx^2}(dj)$$

$$\frac{d^2\phi}{dt^2}(x) = \frac{kd}{m}\frac{d^2\phi}{dx^2}(x)$$



$$\phi(dj) \equiv x_j \qquad \phi(dj,t) \equiv x_j(t)$$

$$\phi(dj,t) \rightarrow \phi(x,t)$$

$$\frac{d^2\phi}{dt^2}(x) = \frac{kd}{m}\frac{d^2\phi}{dx^2}(x) \quad \Rightarrow \quad \frac{\partial^2\phi}{\partial t^2}(x,t) = \frac{kd}{m}\frac{\partial^2\phi}{\partial x^2}(x,t)$$

$$\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X} \qquad \frac{\partial^2\phi}{\partial t^2}(x,t) = \frac{kd}{m}\frac{\partial^2\phi}{\partial x^2}(x,t)$$

$$\boldsymbol{A}\cdot\boldsymbol{X} \quad \Longleftrightarrow \quad \frac{kd}{m}\frac{\partial^2\phi}{\partial x^2}(x,t)$$

$$\boldsymbol{X} = \boldsymbol{a}e^{i\omega t} \qquad \phi(x,t) = \tilde{\phi}(x)e^{i\omega t}$$

$$-\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a} \qquad -\frac{kd}{m}\frac{\partial^2\tilde{\phi}}{\partial x^2} = \omega^2\tilde{\phi}$$

System of 2nd order Linear ODE with constant coefficients

$$\frac{d^2\boldsymbol{X}}{dt^2} = \boldsymbol{A}\cdot\boldsymbol{X}$$

複變函數法 $\boldsymbol{X} = \boldsymbol{a}e^{i\omega t}$

Eigenvalue problem of matrix

$$\boldsymbol{A}\cdot\boldsymbol{a} = \omega^2\boldsymbol{a}$$

System of 1st order Linear ODE with constant coefficients

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{A} \cdot \boldsymbol{X}$$

複變函數法 $\boldsymbol{X} = \boldsymbol{a}e^{\lambda t}$

Identical Eigenvalue problem of matrix

$$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$$

Eigenvalue problem of matrix

$$\boldsymbol{A} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$$