Second Order Linear Ordinary Differential Equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

每一項只包含一次方的y函數，及一次方的一次或二次微分，以及一個已知x函數。

Equation of Motion is most naturally a Second Order Linear Ordinary Differential Equation

$$x'' + P(t)x' + Q(t)x = f(t)$$

We don't have general solution for these ODE but if we restrict $P(x), Q(x)$ to constants:

$$y'' + a_1 y' + a_0 y = f(x)$$

we do have a procedure using complex numner to write down the solutions.

Second Order Linear ODE with constant coefficients

Surpisingly this systemic method can be extended to Linear ODE with constant coefficients of any finite order.

$$\sum_n a_n \cdot \frac{d^n y}{dx^n} = f(x)$$

Linear second order ODE

Inhomogeneous ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

$\Longleftrightarrow$

Homogeneous ODE

$$y'' + P(x)y' + Q(x)y = 0$$

Linear second order ODE 解法

$$y'' + P(x)y' + Q(x)y = f(x)$$

Variation of parameters
Wronskian

級數法

複變函數法

Linear ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

If we find one solution of the homogeneous ODE, we can calculate the general solution.

Linear ODE with constant coefficients

$$y'' + a_1 y' + a_0 y = f(x)$$

Let's start with Homogeneous Linear ODE with constant coefficients

$$y'' + P(x)y' + Q(x)y = 0$$

It is called linear because of this 定理：

如果已找到兩個函數$y_1(x), y_2(x)$都滿足方程式： $y'' + P(x)y' + Q(x)y = 0$

那麼任一線性組合 linear combination $C_1y_1(x) + C_2y_2(x)$ 也滿足該方程式！

C_1 $y_1'' + P(x)y_1' + Q(x)y_1 = 0$ $+$ C_2 $y_2'' + P(x)y_2' + Q(x)y_2 = 0$

$$(C_1y_1 + C_2y_2)'' + P(x)(C_1y_1 + C_2y_2)' + Q(x)(C_1y_1 + C_2y_2) = 0$$

得證

如果已找到兩個函數$y_1(x), y_2(x)$都滿足方程式，我們就得到無限多個解！

微分方程式的解需要讓自己挪出足夠的空間，這樣才能滿足起始條件。

可以證明：

一定可以找到兩個函數$y_1(x), y_2(x)$都滿足方程式： $y'' + P(x)y' + Q(x)y = 0$

$y_1(x), y_2(x)$不是彼此的倍數，稱為彼此線性獨立。

所有的解都可以寫成： $C_1y_1(x) + C_2y_2(x)$

C_1, C_2可以由起始條件$y'(0), y(0)$唯一決定。

因此，加上起始條件後， $y'' + P(x)y' + Q(x)y = 0$ 只有唯一解。

There is a powerful theorem:

Arfken p362

如果已找到一個函數$y_1(x)$ 滿足方程式：

$$y'' + P(x)y' + Q(x)y = 0$$

我們可以計算出另一個線性獨立的函數$y_2(x)$，也滿足此方程式。

關鍵是一個稱為Wronskian的函數$W(x)$，它滿足很簡單的1st order ODE:

$$W(x) \equiv y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

$$\frac{dW}{dx} = y_1(x)y_2''(x) + y_1'(x)y_2'(x) - y_2(x)y_1''(x) - y_2'(x)y_1'(x)$$

$$= y_1(x)y_2''(x) - y_2(x)y_1''(x)$$

$$y_1 \times \quad y_2'' + P(x)y_2' + Q(x)y_2 = 0 \quad - \quad y_2 \times \quad y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$\frac{dW}{dx} + P(x)y_1y_2' - P(x)y_2y_1' = \frac{dW}{dx} + P(x)W = 0$$

$$W = W(0)e^{-\int_0^x P(x')\cdot dx'}$$

Józef Maria Hoene-Wroński

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}$$

If Wronskian $W(x)$ and y_1 is known, $\frac{y_2}{y_1}$ could be obtained!

$$\frac{y_2}{y_1} = C + \int dx \frac{W}{y_1^2}$$

$$y_2 = C y_1 + W(0) y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$

$$y_2 \sim y_1 \int dx \frac{\exp\left(-\int_0^x P(x') \cdot dx'\right)}{y_1^2}$$

由 y_1我們計算出另一個線性獨立的函數$y_2(x)$，也滿足此方程式。

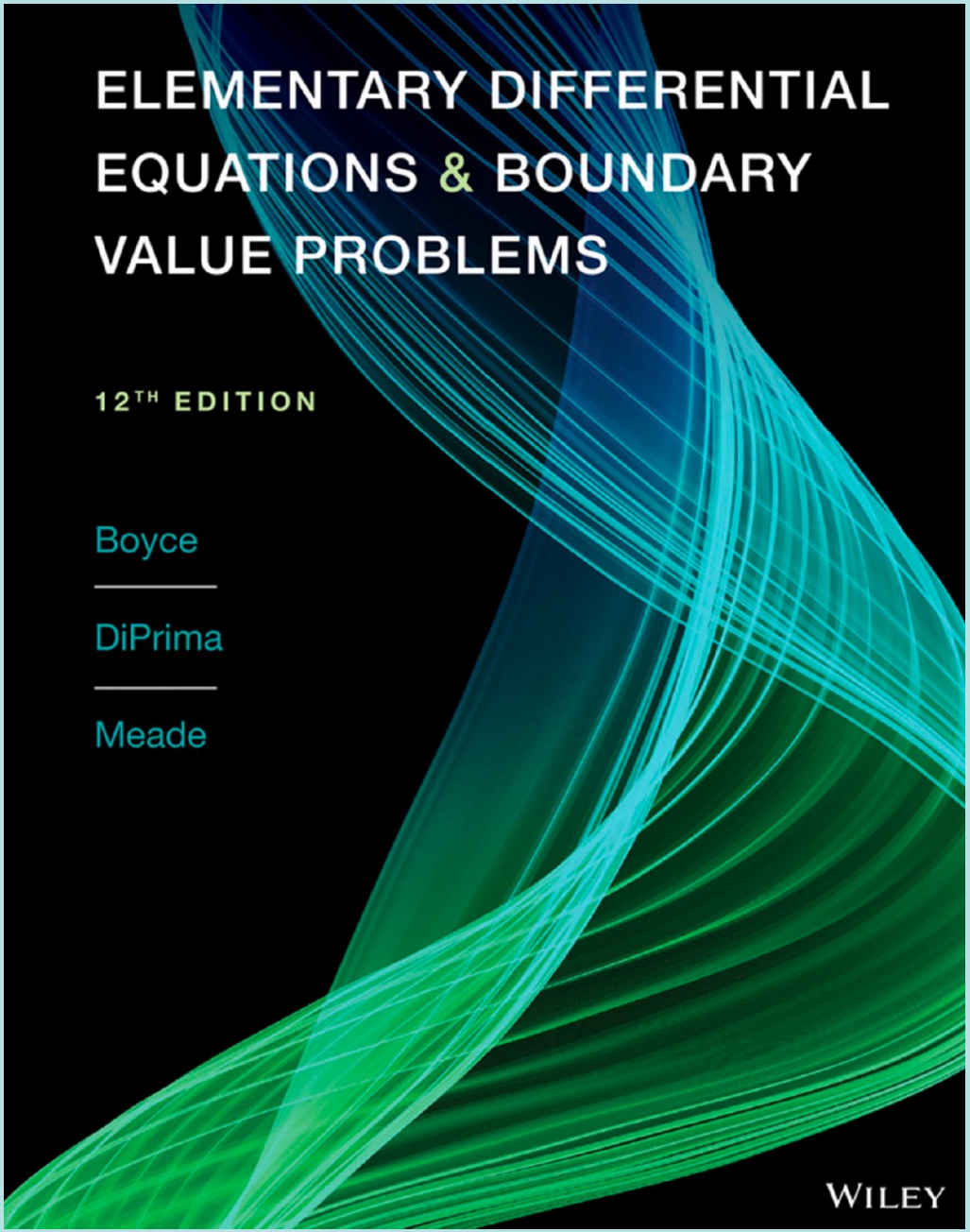
Example 1:

$$\frac{dy}{dx} - p(x)y = 0$$

$$\int \frac{1}{y} dy = \int p(x) \cdot dx + C'$$

$$\ln y = \int p(x) \cdot dx + C'$$

$$y = Ce^{\int p(x) \cdot dx}$$


ELEMENTARY DIFFERENTIAL EQUATIONS & BOUNDARY VALUE PROBLEMS
12TH EDITION
Boyce
DiPrima
Meade
WILEY

To discuss general properties of linear differential equations, it is helpful to introduce a **differential operator** notation. Let p and q be continuous functions on an open interval I — that is, for $\alpha < t < \beta$. The cases for $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

It is important to understand that the result of applying the operator L to a function ϕ is another function, which we refer to as $L[\phi]$. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin(3t)$, then

$$\begin{aligned} L[\phi](t) &= (\sin(3t))'' + t^2(\sin(3t))' + (1 + t)\sin(3t) \\ &= -9\sin(3t) + 3t^2\cos(3t) + (1 + t)\sin(3t). \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator, that is, $D[\phi] = \phi'$.

In this section we study the second-order linear homogeneous differential equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With equation (2) we associate a set of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second-order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first-order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 | Existence and Uniqueness Theorem

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.
3. The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For instance, we found in Example 3.1.1 that the initial value problem

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by equation (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

For most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first-order and second-order linear differential equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

Let us now assume that y_1 and y_2 are two solutions of equation (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0,$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 | Principle of Superposition

If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any constant multiple of a solution of equation (2) is also a solution.

To prove Theorem 3.2.2, we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{7}$$

for y in equation (2). By calculating the indicated derivatives and rearranging terms, we obtain

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ &= c_1y_1'' + c_2y_2'' + c_1p(t)y_1' + c_2p(t)y_2' + c_1q(t)y_1 + c_2q(t)y_2 \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1L(y_1) + c_2L(y_2). \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, the function y as given by equation (7) satisfies the differential equation (2), and the proof of Theorem 3.2.2 is complete.

Theorem 3.2.2 states that, beginning with only two solutions of equation (2), we can construct an infinite family of solutions by means of equation (7). The next question is whether all solutions of equation (2) are included in equation (7) or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in equation (7) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0'. \end{aligned} \tag{8}$$

The determinant of coefficients of the system (8) is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{9}$$

If $W \neq 0$, then equations (8) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. This solution is given by

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2, the linear combination $y = c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions (3) as well as the differential equation (2). Note that the denominator in the expressions for c_1 and c_2 is the nonzero determinant W.

On the other hand, if $W = 0$, then the denominators appearing in equations (10) and (11) are zero. In this case equations (8) have no solution unless y_0 and y_0' have values that also make the numerators in equations (10) and (11) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

The determinant W is called the **Wronskian**[3] **determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W[y_1, y_2](t_0)$ to stand for the expression on the right-hand side of equation (9), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument establishes the following result.

Theorem 3.2.3

Suppose that y_1 and y_2 are two solutions of equation (2)

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (3)

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (2) and the initial conditions (3) if and only if the Wronskian

$$W[y_1, y_2] = y_1y_2' - y_1'y_2$$

is not zero at t_0.

Theorem 3.2.4

Suppose that y_1 and y_2 are two solutions of the second-order linear differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the two-parameter family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of equation (2) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Let the function ϕ be any solution of equation (2). To prove the theorem, we must determine whether ϕ is included in the linear combinations $c_1 y_1 + c_2 y_2$. That is, we must determine whether there are values of the constants c_1 and c_2 that make the linear combination the same as ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; that is,

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{12}$$

The function ϕ is certainly a solution of this initial value problem. Further, because we are assuming that $W[y_1, y_2](t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 such that $y = c_1 y_1(t) + c_2 y_2(t)$ is also a solution of the initial value problem (12). In fact, the proper values of c_1 and c_2 are given by equations (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t), \tag{13}$$

and therefore ϕ is included in the family of functions $c_1 y_1 + c_2 y_2$. Finally, since ϕ is an *arbitrary* solution of equation (2), it follows that *every* solution of this equation is included in this family.

Now suppose that there is no point t_0 where the Wronskian is nonzero. Thus

Now suppose that there is no point t_0 where the Wronskian is nonzero. Thus $W[y_1, y_2](t_0) = 0$ for every point t_0. Then (by Theorem 3.2.3) there are values of y_0 and y_0' such that no values of c_1 and c_2 satisfy the system (8). Select a pair of such values for y_0 and y_0' and choose the solution $\phi(t)$ of equation (2) that satisfies the initial condition (3). Observe that this initial value problem is guaranteed to have a solution by Theorem 3.2.1. However, this solution is not included in the family $y = c_1y_1 + c_2y_2$. Thus, in cases where $W[y_1, y_2](t_0) = 0$ for every t_0, the linear combinations of y_1 and y_2 do not include all solutions of equation (2). This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that the Wronskian of y_1 and y_2 is not everywhere zero if and only if the linear combination $c_1y_1 + c_2y_2$ contains all solutions of equation (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of equation (2). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of equation (2) if and only if their Wronskian is nonzero.

We can restate the result of Theorem 3.2.4 in slightly different language: to find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition.

Now that you have a little experience verifying the nonzero Wronskian condition for the examples from Section 3.1, the following example handles all second-order linear differential equations whose characteristic polynomial has two distinct real roots.

In several cases we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question arises as to whether a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 3.2.5

Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of equation (2) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of equation (2) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then y_1 and y_2 form a fundamental set of solutions of equation (2).

First observe that the *existence* of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

Note that the potentially difficult part of this proof, demonstrating the existence of a pair of solutions, is taken care of by reference to Theorem 3.2.1. Note also that Theorem 3.2.5 does not address the question of how to find the solutions y_1 and y_2 by solving the specified initial value problems. Nevertheless, it may be reassuring to know that a fundamental set of solutions always exists.

Now let us examine further the properties of the Wronskian of two solutions of a second-order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 3.2.7 | Abel's Theorem[4]

If y_1 and y_2 are solutions of the second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right), \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem, we start by noting that y_1 and y_2 satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0,$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Niels Henrik Abel 1802-1829

If we multiply the first equation by $-y_2$, multiply the second by y_1, and add the resulting equations, we obtain

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0. \tag{25}$$

Next, we let $W(t) = W[y_1, y_2](t)$ and observe that

$$W' = y_1 y_2'' - y_1'' y_2. \tag{26}$$

Then we can write equation (25) in the form

$$W' + p(t)W = 0. \tag{27}$$

Equation (27) can be solved immediately since it is both a first-order linear differential equation (Section 2.1) and a separable differential equation (Section 2.2). Thus

$$W(t) = c \exp\left(-\int p(t)\,dt\right), \tag{28}$$

where c is a constant.

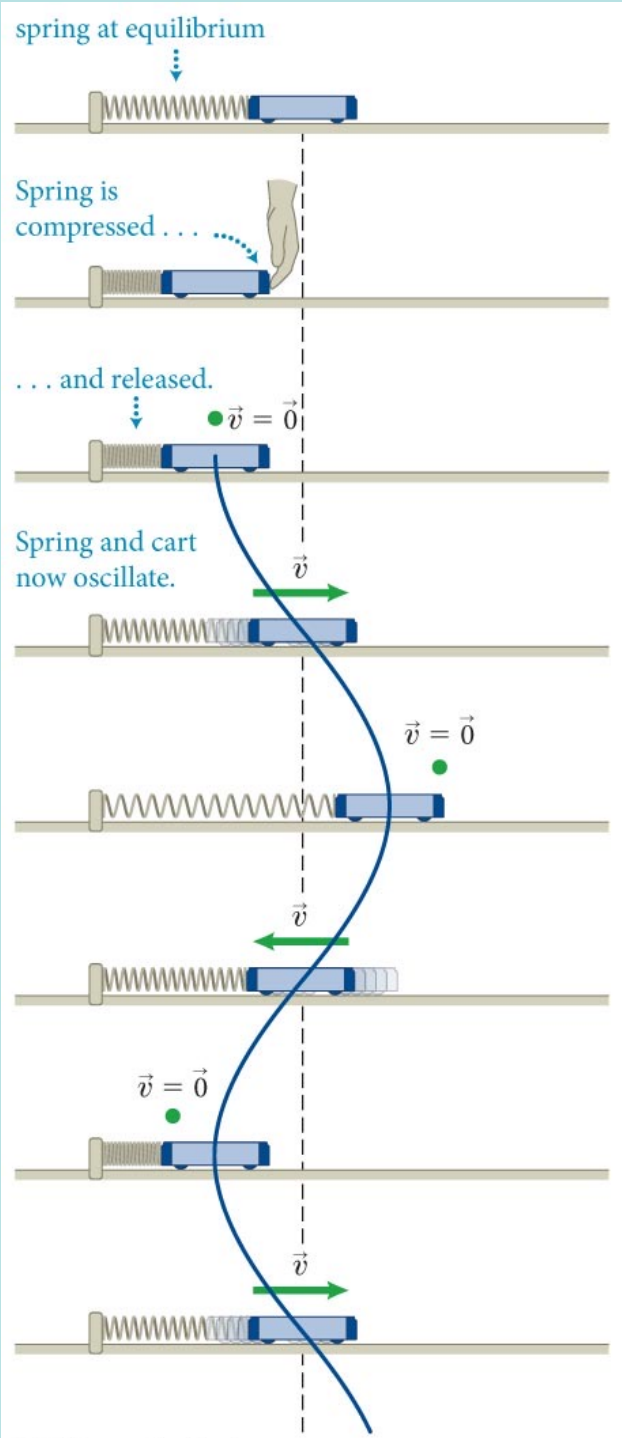
The value of c depends on which pair of solutions of equation (22) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t. This completes the proof of Theorem 3.2.7.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

彈簧的簡諧運動

Simple Harmonic Motion SHM

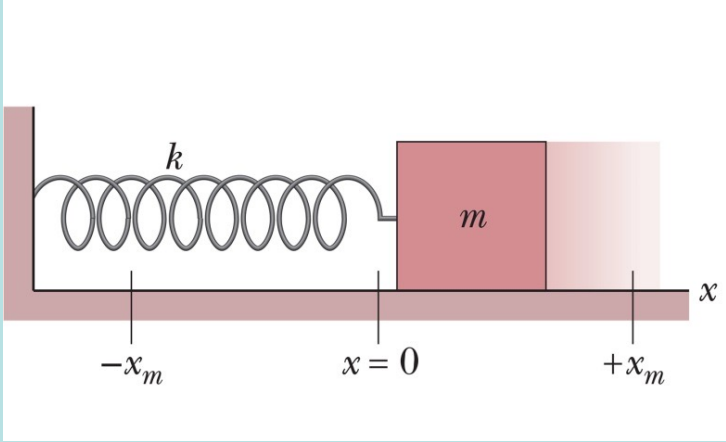
簡諧運動是週期運動。



簡諧運動　　最簡單的週期運動

Simple Harmonic Motion　　Periodic Motion

Simple Harmonic Oscillation　簡諧振盪

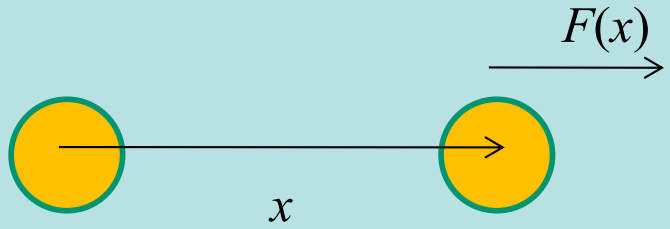


$$F = -kx$$

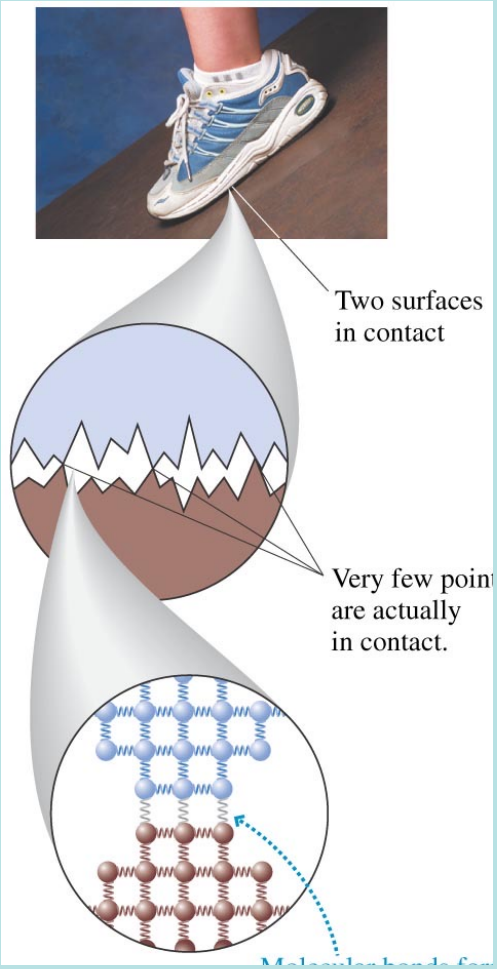
這是受力只與位置有關的粒子的運動方程式。

任一個在平衡點附近的小範圍周期運動，都是簡諧運動！

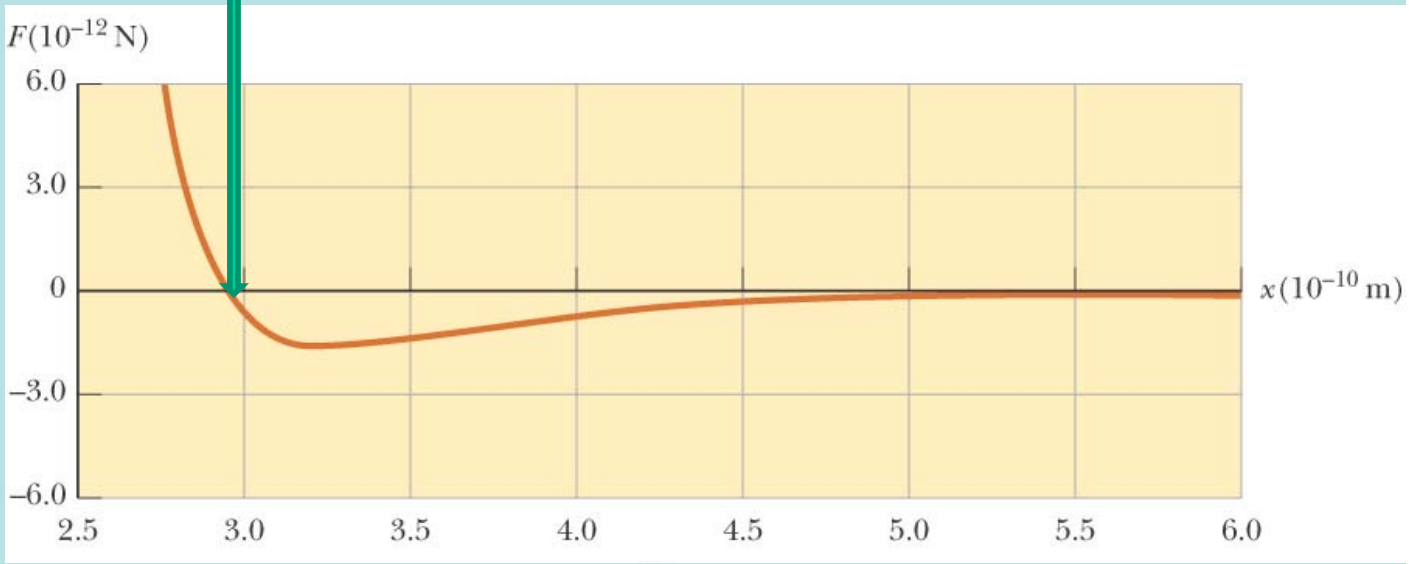
原子力



正力為排斥力，力為負則為吸引力



原子力有一平衡點，力剛好為零：

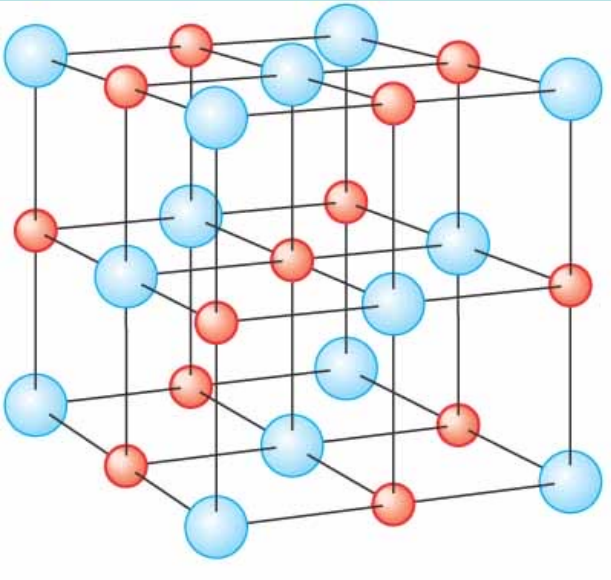


摩擦力的根源是原子力

如果把原子安排使得其距離恰在原子力的平衡點上，

這些原子可以組成有大規模秩序的晶格，此晶格會是平衡的，

晶格的大小與形狀處於平衡穩定狀態，這就是固態！

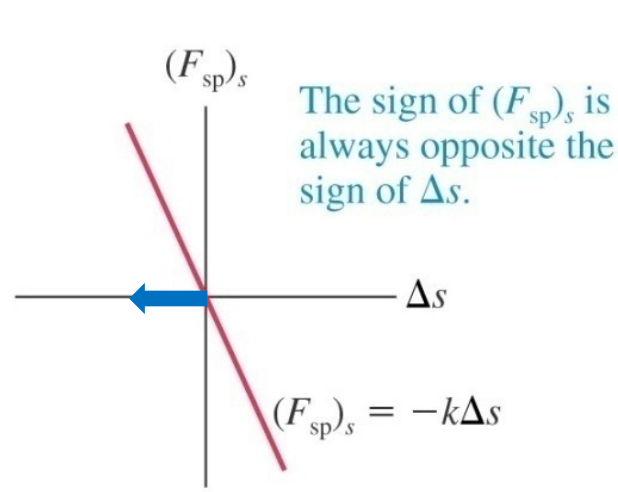


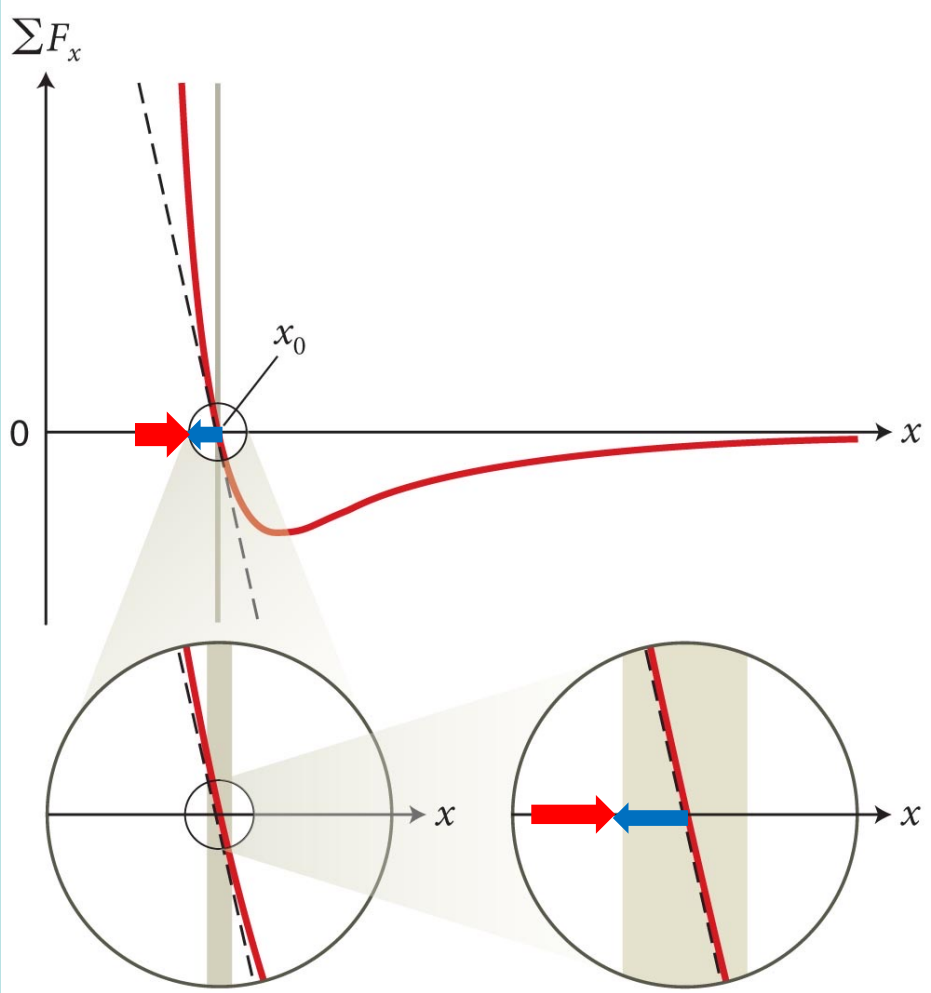
晶格會是穩定狀態的嗎？在小擾動作用下會回復原狀稱為穩定。

設想若固體中的原子向內離開平衡點，

正的原子排斥力會將原子推回平衡點。

定性上會如彈簧一樣。





在相當接近平衡點的附近，原子力的曲線大致上可以用一條直線來近似。

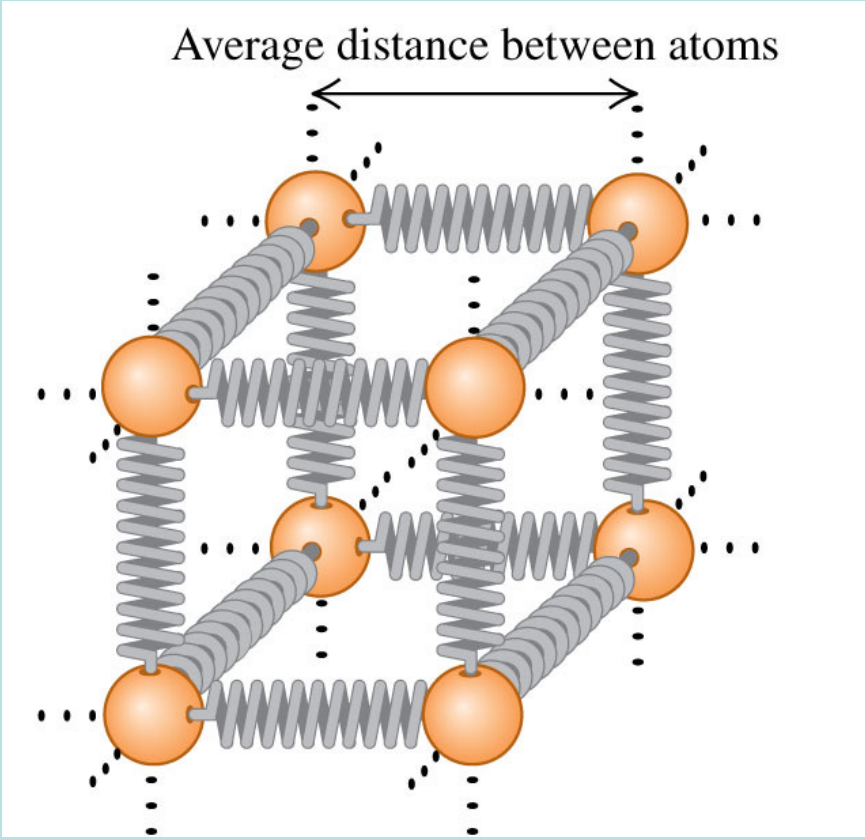
力正比於平衡點算起的位移，定量上也與與彈簧是相同的： $F = -k\Delta x$

所以在平衡點附近的小範圍運動，都近似是一個簡諧運動！

如果嘗試改變固體大小，固體會反彈如同彈簧！

固體中原子彼此之間，如同以彈簧連接。

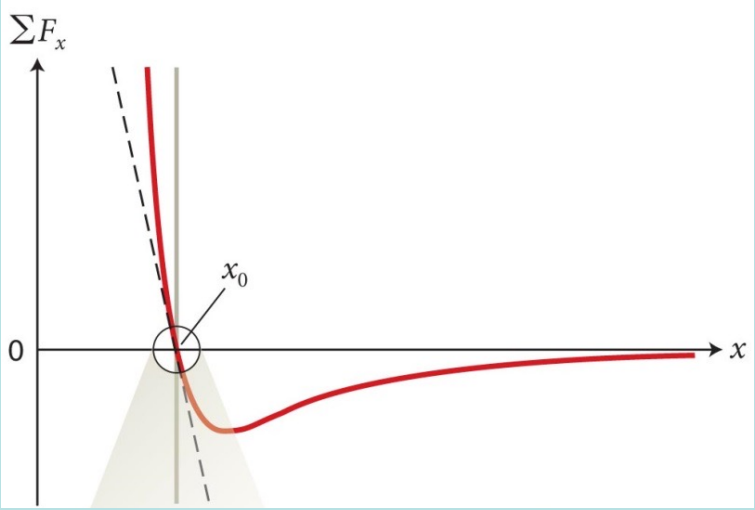
固體的晶格會是穩定的！

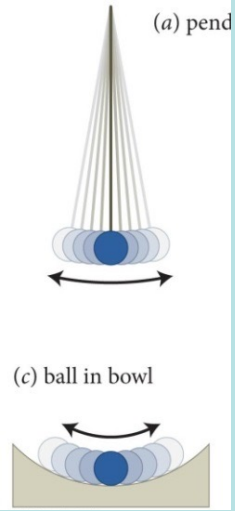


這是固體的彈性（變形回復力）的來源。

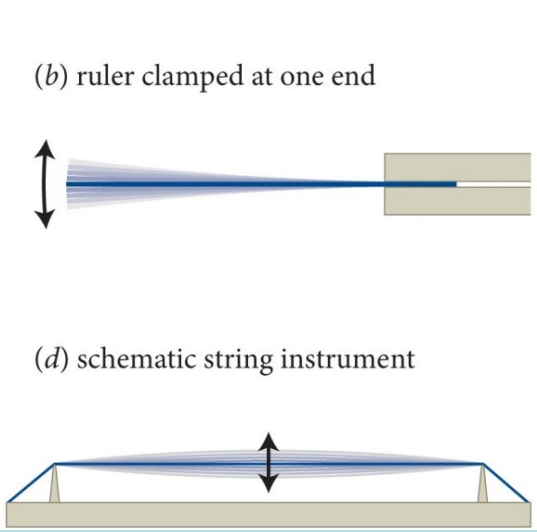
任一個有穩定平衡點的力，小範圍在平衡點附近，力的函數都可以以直線近似！

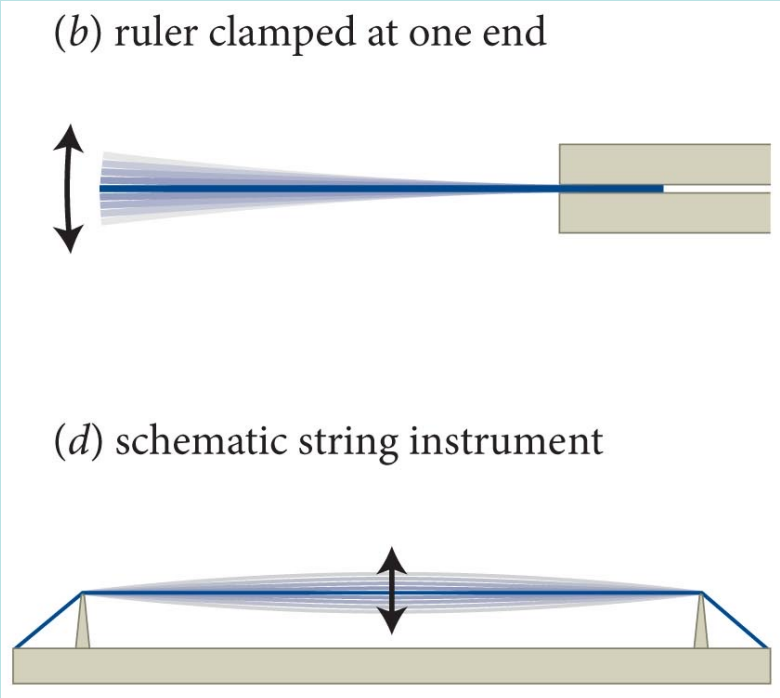
因此任一個小範圍在穩定平衡點附近的運動，都可以以簡諧運動近似！





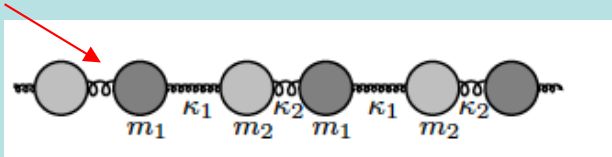
固體的原始形狀就是一平衡點，因此固體的變形運動，近似地是一個簡諧運動。

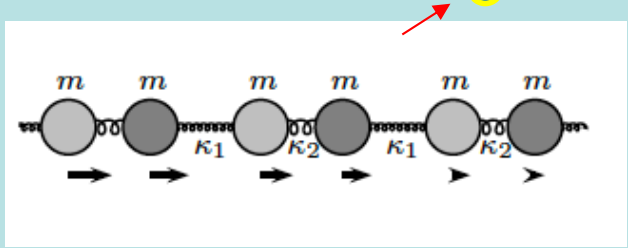




這些結構體都有簡諧運動。

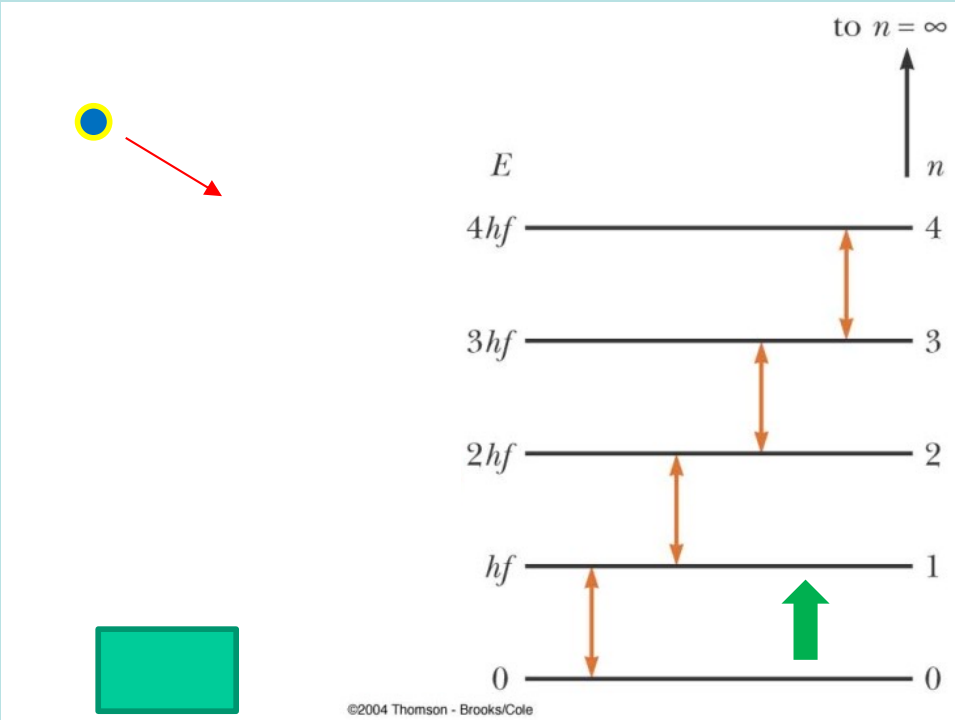
電子碰撞晶體中的原子鏈

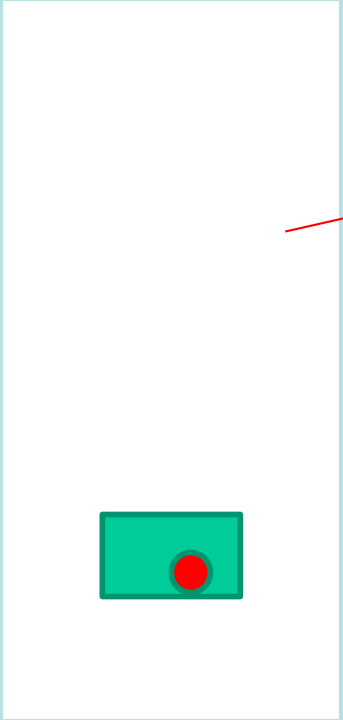




等同

電子碰撞激發某個模式k由$n = 1$定態躍遷到$n = 2$定態。



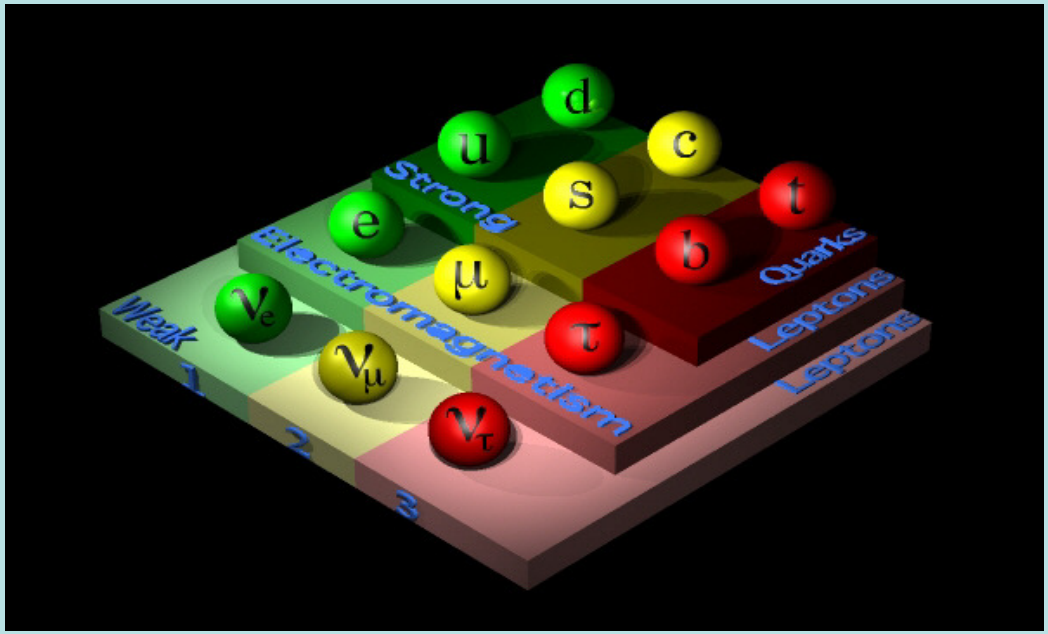


等同

電子移動中放出一個角波數為k的聲子

聲子好像一個基本粒子！

每一個基本粒子都對應一個量子場

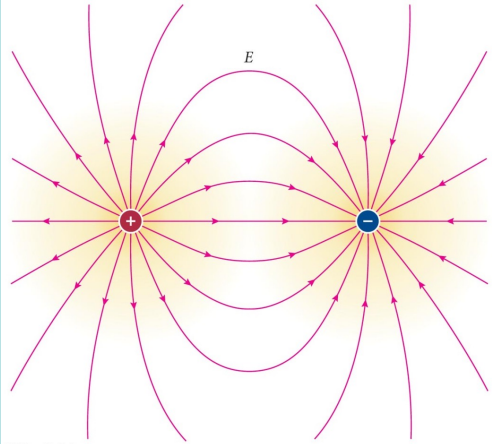


電子 ⟺ 電子場

u夸克 ⟺ u夸克場

描述可產生可消滅的粒子的理論：量子場論！

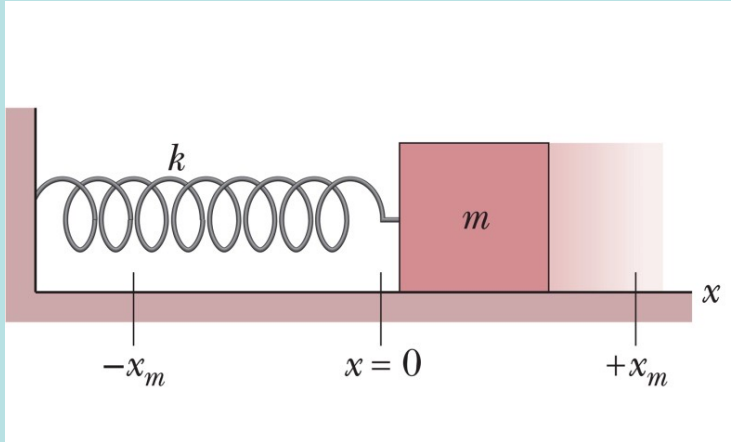
但我們早已熟悉、真空中場的擾動，也可看成一個一個的彈簧的組合！



如果場也量子化了，這就稱為量子場，激發態就會是一個個基本粒子。

我們的宇宙是由彈簧組成的

簡諧運動的運動方程式



$$F = -kx$$

$$\frac{d^2x}{dt^2} = -\omega^2 x = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

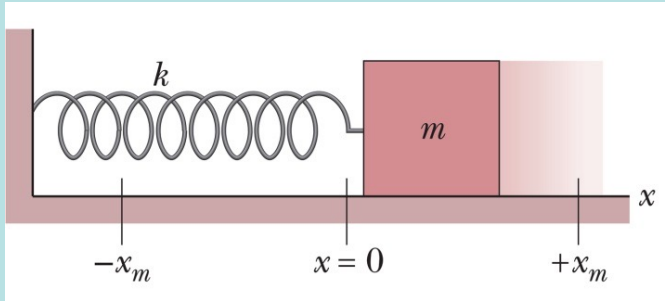
一個簡諧運動，完全由一個特徵常數ω（角頻率）決定！

具有相同的ω的簡諧運動，運動方程式的解就完全一樣。

This is a Homogeneous 2nd order Linear ODE with constant coefficients.

$$y'' + a_1 y' + a_0 y = f(x)$$

簡諧運動的運動方程式求解



$$\frac{d^2x}{dt^2} = -\omega^2 x$$

位置函數的兩次微分與自己成正比：多項式不符合。

因為式中的負號，指數函數也不行！

三角函數正好具有這樣的性質！

$$\frac{d(\cos\omega t)}{dt} = \frac{d\cos\omega t}{d(\omega t)} \cdot \frac{d}{dt}(\omega t) = -\omega\sin\omega t$$

$$-\omega\frac{d(\sin\omega t)}{dt} = -\omega^2\cos\omega t$$

$$\frac{d^2(\cos\omega t)}{dt^2} = -\omega^2\cos\omega t$$

$$\frac{d^2(\sin\omega t)}{dt^2} = -\omega^2\sin\omega t$$

找到兩個解

定理

如果已找到兩個函數$x_1(t), x_2(t)$都滿足方程式： $\frac{d^2x}{dt^2} + \omega^2 x = 0$

那麼任一線性組合 linear combination $ax_1(t) + bx_2(t)$ 也滿足該方程式！

$$a \quad \frac{d^2x_1}{dt^2} + \omega^2 x_1 = 0 \quad + \quad b \quad \frac{d^2x_2}{dt^2} + \omega^2 x_2 = 0$$

$$\frac{d^2(ax_1 + bx_2)}{dt^2} + \omega^2(ax_1 + bx_2) = 0$$

得證

定理

如果已找到兩個函數$x_1(t), x_2(t)$都滿足齊次 Homogeneous方程式：

$$\sum_n a_n \cdot \frac{d^n x}{dt^n} = 0$$

完全由函數微分的一次項組成的微分方程式。

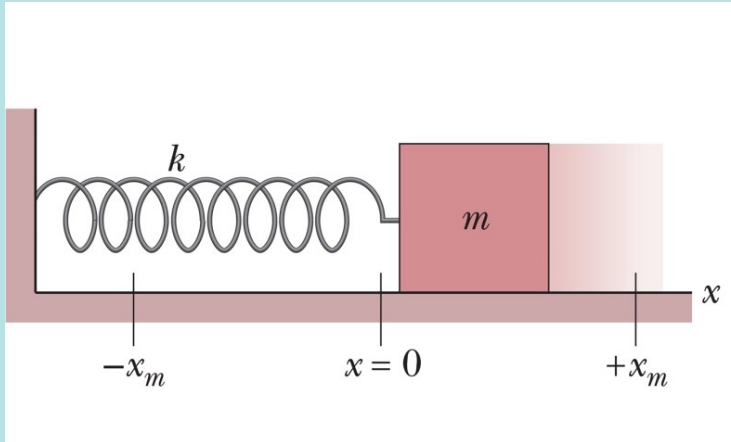
那麼任一線性組合 linear combination $ax_1(t) + bx_2(t)$ 也滿足該方程式！

$$a \sum_n a_n \cdot \frac{d^n x_1}{dt^n} = 0 \quad + \quad b \sum_n a_n \cdot \frac{d^n x_2}{dt^n} = 0$$

$$\sum_n a_n \cdot \frac{d^n (ax_1 + bx_2)}{dt^n} = 0$$

得證

簡諧運動的解



$$\frac{d^2x}{dt^2} = -\omega^2 x$$

正弦函數與餘弦函數的兩次微分都和負的自己成正比！

因此很容易就找到兩個解 $x_1 = \sin \omega t$ $x_2 = \cos \omega t$

那麼任一線性組合 $x = a \cos \omega t + b \sin \omega t$ 也是解！

我們得到無限多個解！

微分方程式的解需要讓自己挪出足夠的空間，這樣才能滿足起始條件。

$$x = a\cos\omega t + b\sin\omega t$$

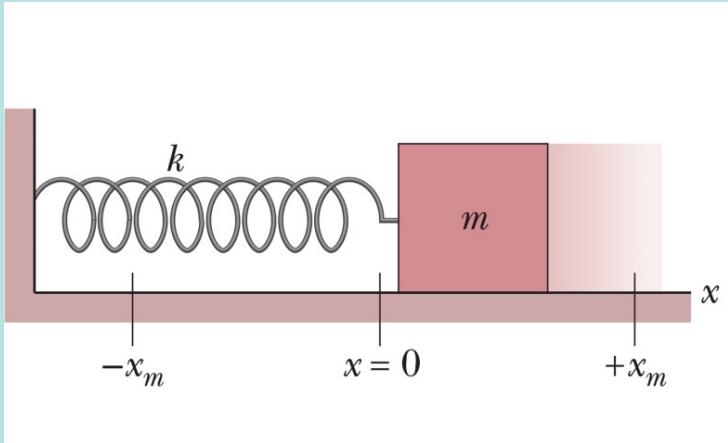
$$v = -\omega a\sin\omega t + \omega b\cos\omega t$$

a, b由起始條件決定

$$x(0) = a = x_0$$

$$v(0) = \omega b = v_0$$

$$b = \frac{v_0}{\omega}$$



$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

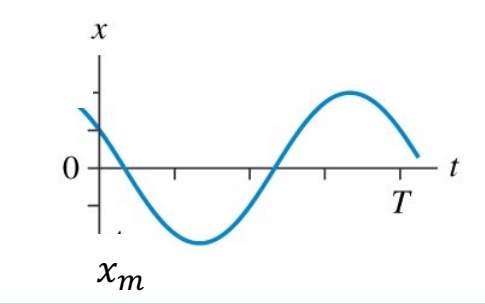
這個函數同時滿足運動方程式以及兩個起始條件，因此是唯一的解！

不用再找了！

$$x = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

這個式子較適合運動方程式求解。

較容易明瞭其物理意義的表示式是：



$$x = x_m \cos(\omega t + \phi)$$

這兩個數學式是一樣的，因為：

$$x = x_m \cos(\omega t + \phi) = x_m \cos \phi \cos \omega t - x_m \sin \phi \sin \omega t$$

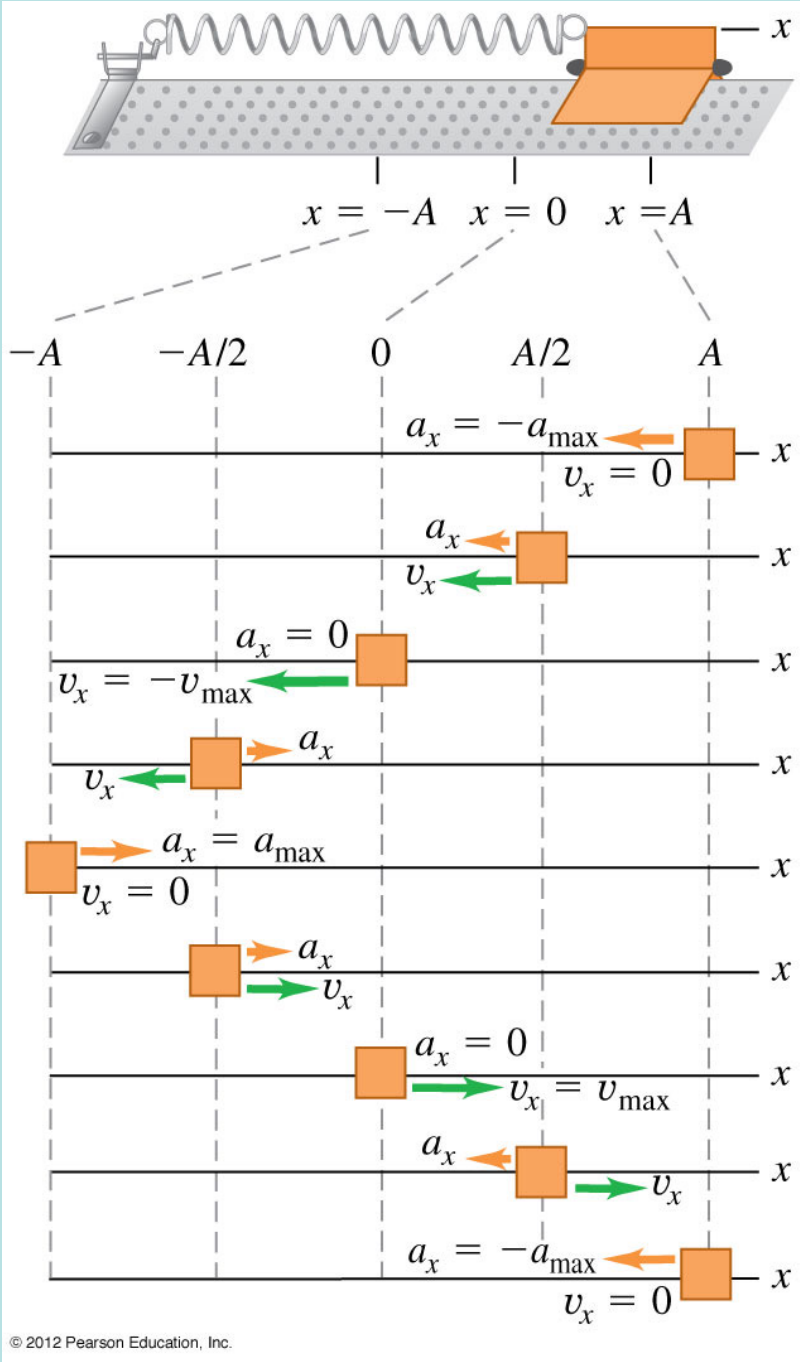
兩組常數之間的關係：

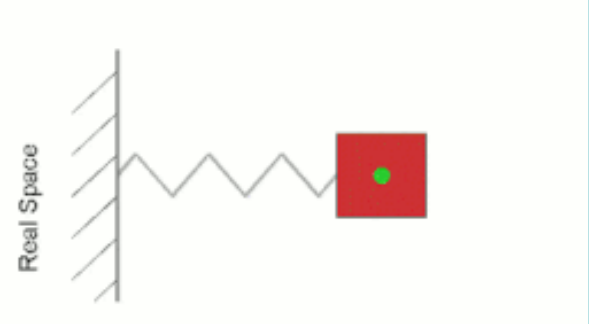
$$x_0 = x_m \cos \phi \,, \frac{v_0}{\omega} = -x_m \sin \phi$$

$$x_m = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

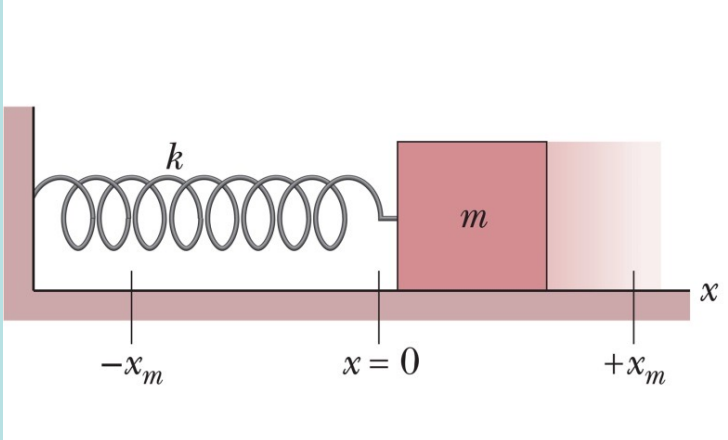
$$\tan \phi = -\frac{v_0}{\omega x_0}$$

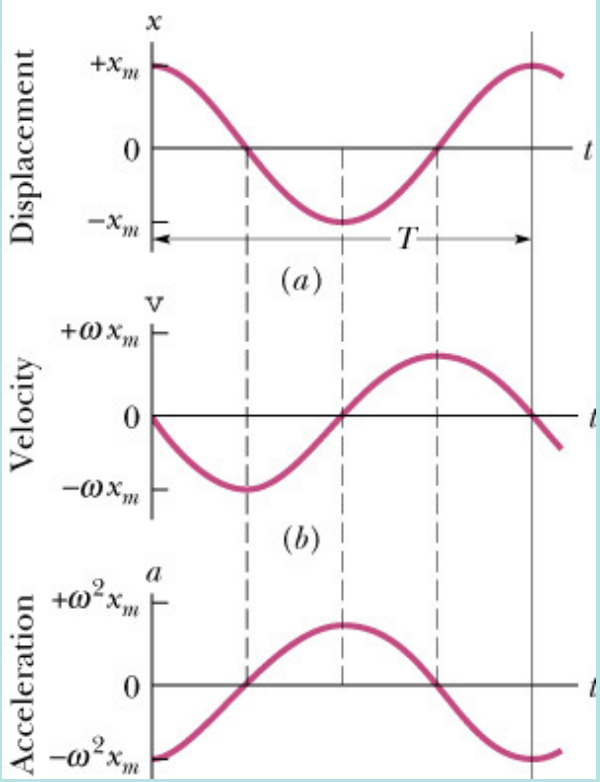
簡諧運動的解就是時間平移過的一個三角函數！





動畫 Simulation





$$x = x_m \cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi)$$

x_m 及 ϕ 兩個常數是由起始條件決定：

對個別的運動，所取的值不同。

$$x_m = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

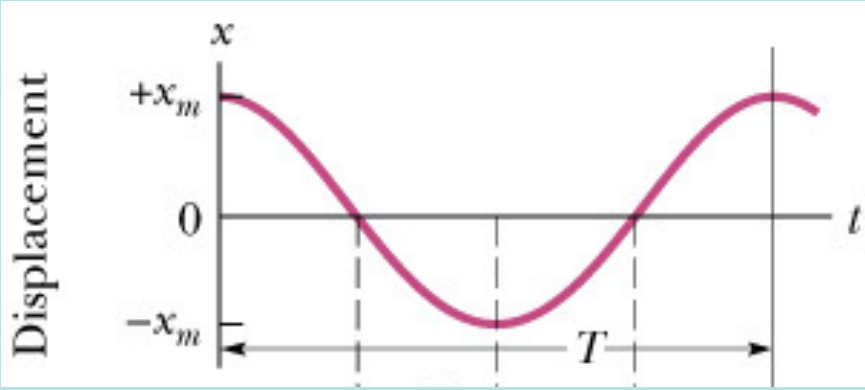
ω是由簡諧振盪子的性質決定：

同一個彈簧組，只有一個值。

$$\omega = \sqrt{\frac{k}{m}}$$

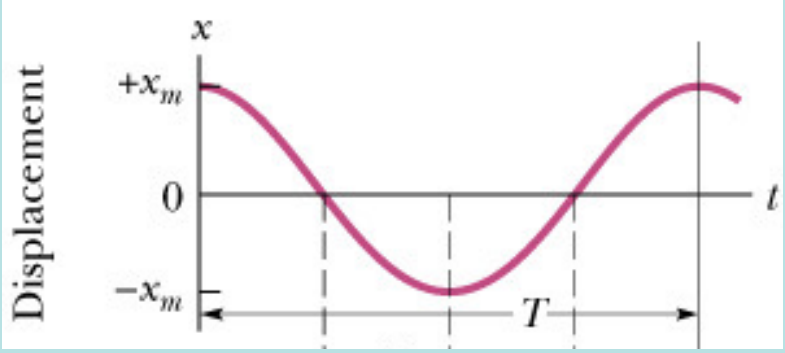
$$x = x_m \cos(\omega t + \phi)$$

常數的物理意義：



x_m振幅 Amplitude 是振動的極大值

$x = x_m \cos(\omega t + \phi)$ 三角函數是一個週期函數。 ω 決定了振盪的週期 T。



$x(t) = x(t + T)$ $\rightarrow \cos(\omega t + \phi) = \cos(\omega t + \omega T + \phi)$

三角函數一個週期後，角度增加2π。 $\omega T = 2\pi$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

ω稱為角頻率 Angular Frequency。

簡諧運動的週期 T 等於

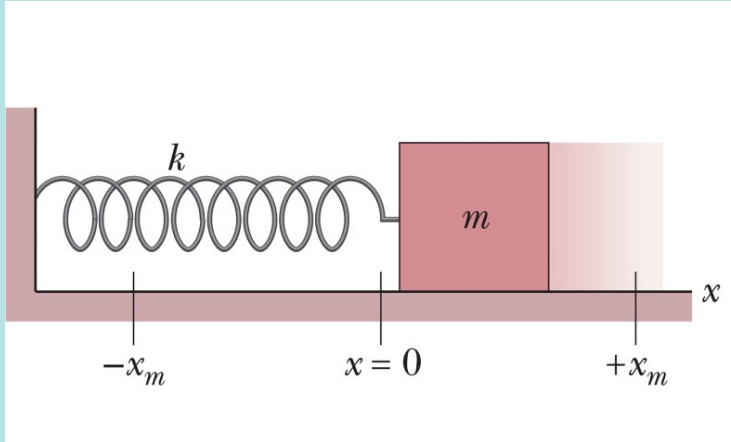
$$T = 2\pi\sqrt{\frac{m}{k}}$$

頻率Frequency: $f = 1/T$ 每秒進行了幾個周期。單位 s^{-1} = Hz。

$$\omega = \frac{2\pi}{T} = 2\pi f$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

簡諧運動的運動方程式



$$F = -kx$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

一個簡諧運動，只由一個特徵常數 ω（角頻率）決定！

簡諧運動內在的特質 ω 決定了振盪的週期 T。

簡諧運動的週期與振盪的振幅無關！

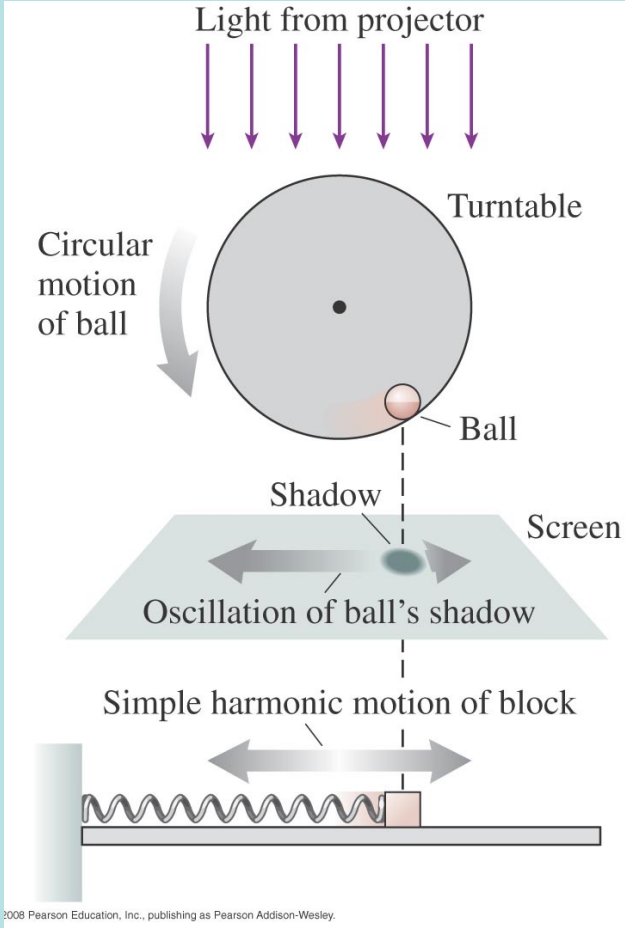
$$x = x_m \cos(\omega t + \phi)$$

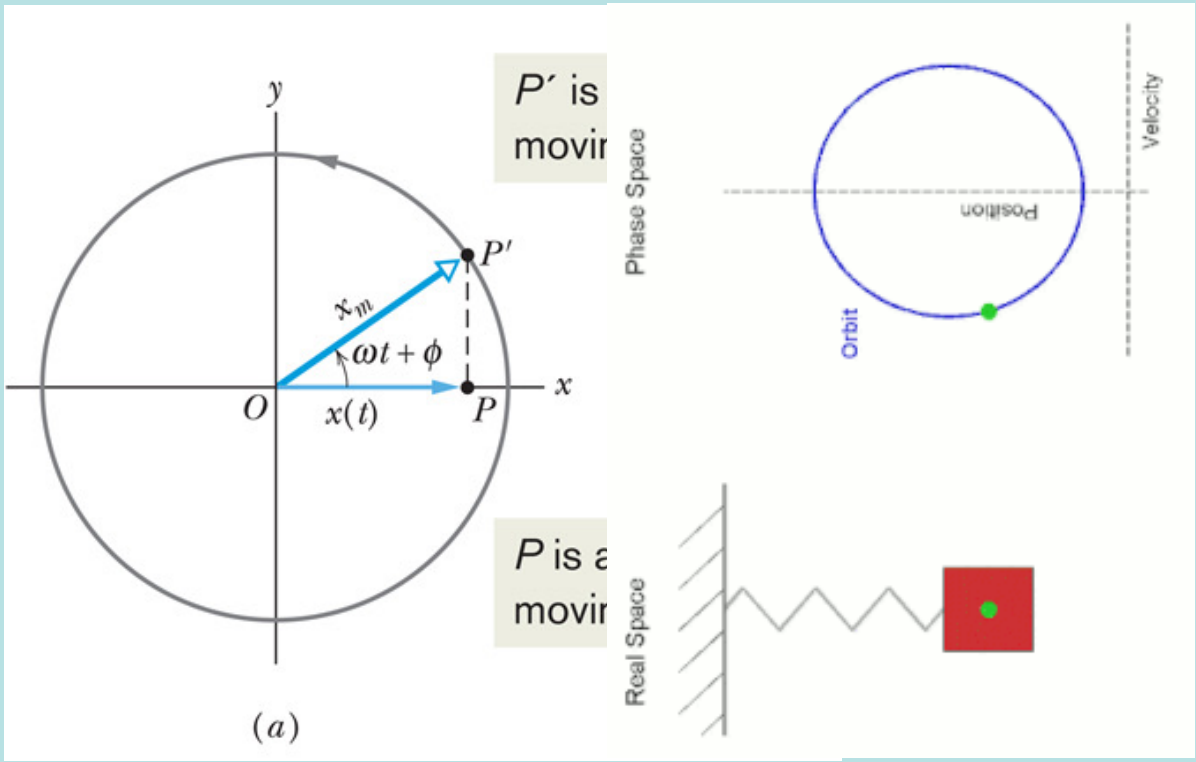
$$\omega T = 2\pi$$

正好是一個半徑為x_m的等速圓周運動的水平分量。

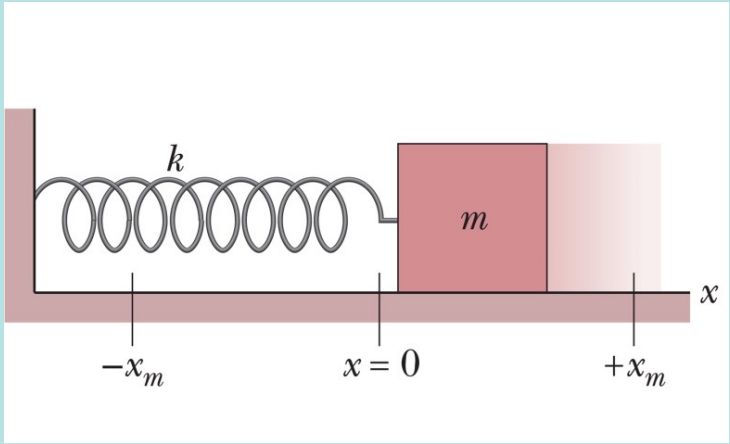
此等速圓周運動的速度與加速度的水平分量也好與簡諧運動相等。

一個等速圓周運動位置的水平投影就等於一個簡諧運動。





假想圓是一個很好的工具，但不是真的有一個圓！



$$x = x_m \cos(\omega t + \phi)$$

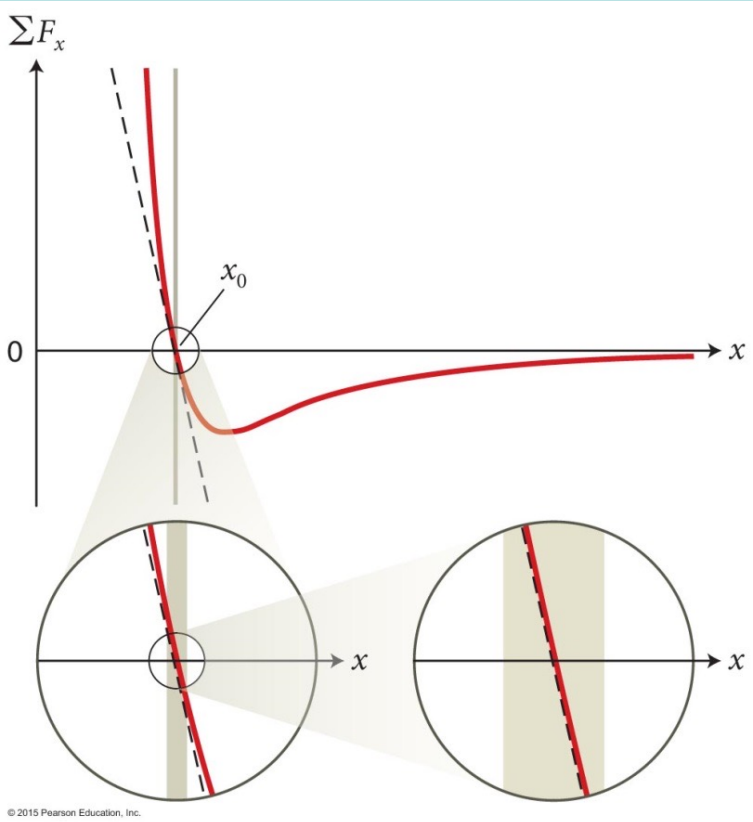
$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

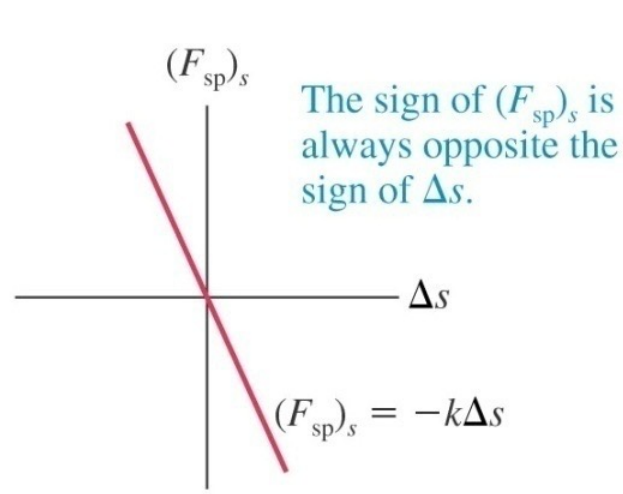
一個簡諧運動，有一個特有的內在的振動頻率 f ！

角頻率ω與振幅及相常數無關，同一個彈簧組，只有一個值。

振幅x_m及相常數ϕ由起始條件決定，

同一個彈簧組，對個別的運動，所取的值可以不同。





在力為零的平衡點附近，力的曲線大致上可以用一條直線來近似。

力與距平衡點的位移成正比，即簡諧運動！

$$F \sim F'(x_0)\cdot(x - x_0) \sim -k\Delta x$$

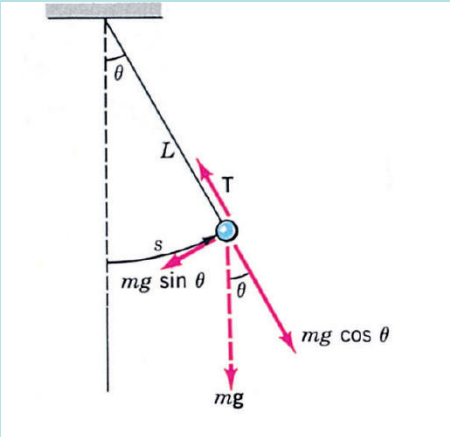
$$k = -F'(x_0) = -\left.\frac{dF}{dx}\right|_{x=x_0}$$

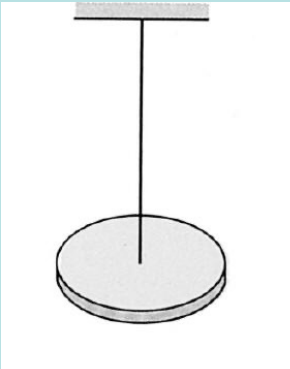
彈力常數即是通過平衡點的切線斜率乘−1：

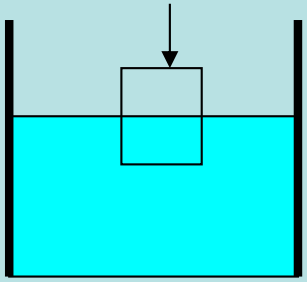
有了平衡點附近的力曲線即可得到對應的彈力常數！

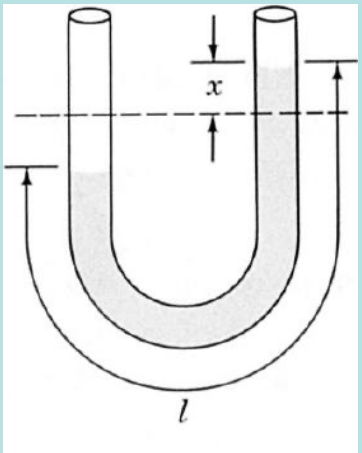
一個粒子在平衡點附近的小規模運動都是簡諧運動！

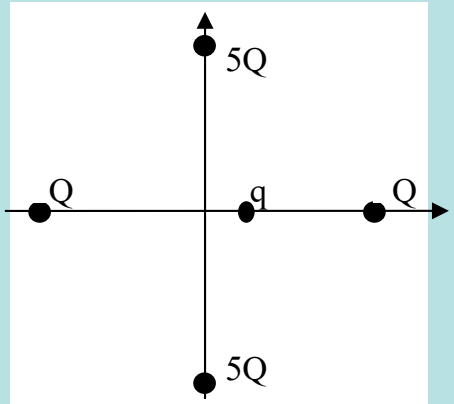
對彈簧的討論結果可以適用所有在平衡點附近的小規模運動。

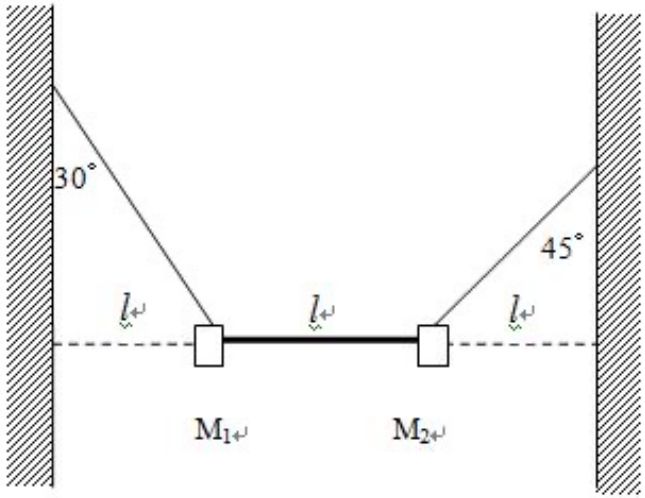


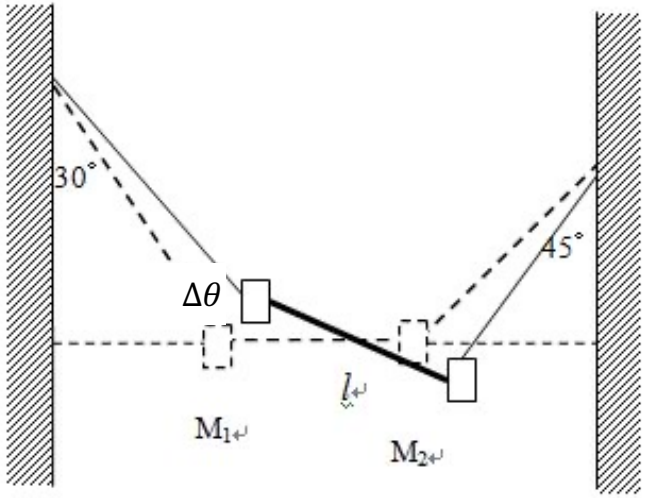









30°
45°
l
l
l
M1
M2


30°
45°
Δθ
l
M1
M2

1. 考慮一半徑為R的鐵環，在地面附近，如圖垂直豎起，鐵環上串有一質量為m的珠子，可以自由無摩擦、沿著鐵環滑動。鐵環靜止時，珠子靜止於正下方。將鐵環慢慢開始繞著通過環心的垂直軸轉動，直到角速度為ω時，便保持等角速度旋轉。此時珠子已沿著環，慢慢向上滑到與垂直線夾角為45°時停止，如圖示。

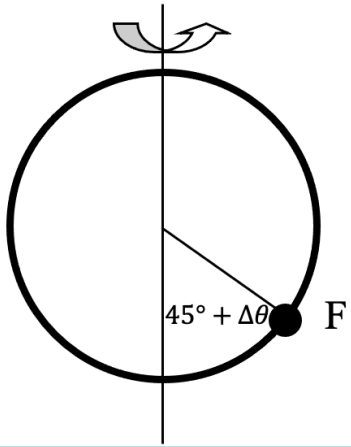
A. 計算ω。以m, R, g表示。

B. 用手將珠子向上輕撥，至夾角略大於45°處，然後放手。珠子會下滑，超過原來位置，再反彈回到起點，接著繼續在起點前後作簡諧運動。計算此簡諧運動的週期。

提示：$\sin\left(\frac{\pi}{4}+\Delta\theta\right) \sim \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\Delta\theta$，$\cos\left(\frac{\pi}{4}+\Delta\theta\right)=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\Delta\theta$，

Screenshot

此式$\Delta\theta$是以徑度表示，$\Delta\theta \ll 1$，公式對正負$\Delta\theta$都對。



有彈性的物體當發生變形時，有一回復力。

而此物體的原始形狀，就是回復力為零的平衡點，

因此物體的變形也近似是一個簡諧運動。

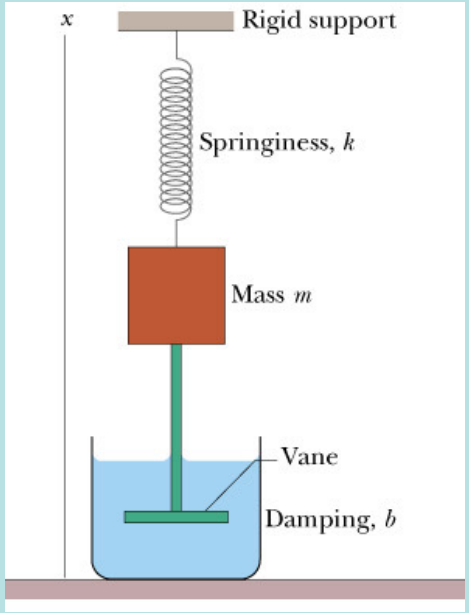
但自然界的振盪很少一直持續，能量總是會漸漸消耗掉。

這是來自振盪器內的阻力。

阻力通常與振動的速度成正相關。

近似引進一個與速率成正比的阻力：

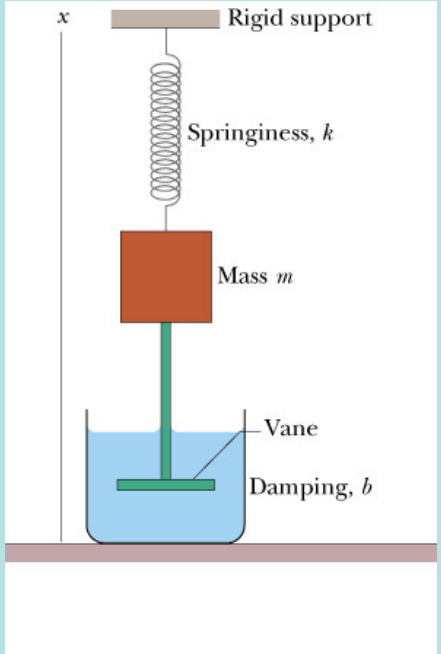
$$f_d = -bv$$



運動方程式會多一個項：

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

稱為Damped Oscillation 阻尼振盪。



Damped Oscillation 阻尼震盪

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

This is a Homogeneous 2nd order Linear ODE with constant coefficients.

單一個正弦或餘弦函數似乎不能滿足此式。

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

Damped oscillation 應該還是會振盪，但振幅會隨時間變化。

大膽的猜想：

$$x(t) = f(t) \cdot \cos(\omega' t + \phi)$$

$$-\omega^2 f \cos(\omega' t + \phi)$$

$$-\frac{b}{m} \cdot (f' \cdot \cos(\omega' t + \phi) - \omega' f \sin(\omega' t + \phi))$$

$$(-f'' + \omega'^2 f) \cdot \cos(\omega' t + \phi) + 2\omega' f' \sin(\omega' t + \phi)$$

要求$\sin(\omega' t + \phi)$的係數為零：

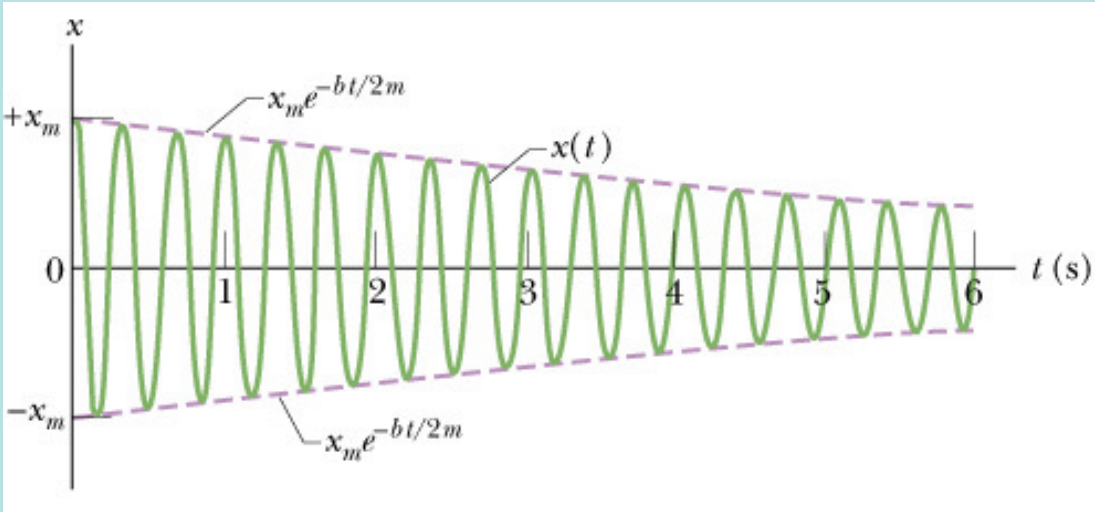
$$2f' = -\frac{b}{m} f$$

$$f = x_m \cdot e^{-\frac{b}{2m}t}$$

要求$\cos(\omega' t + \phi)$的係數為零：

$$\omega'^2 - \omega^2 + \frac{b^2}{4m^2} = 0$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

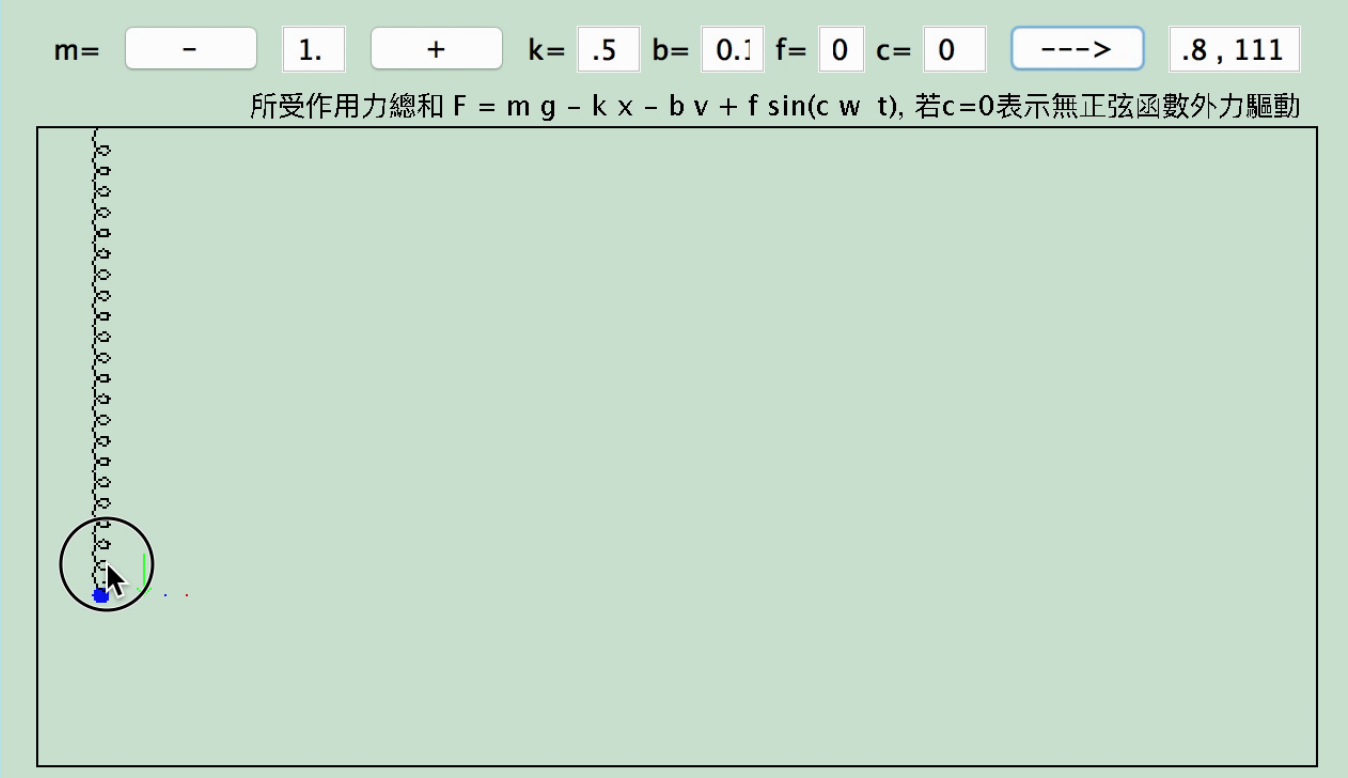


$$x(t) = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

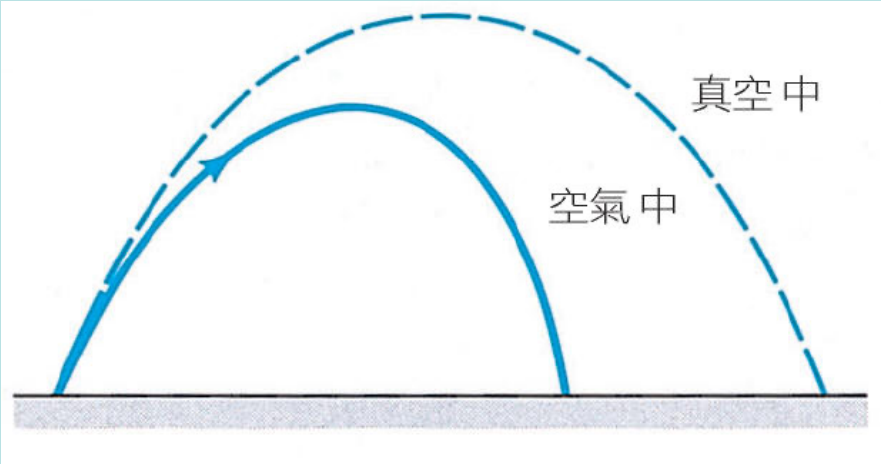
振幅對時間呈現指數衰減！ $x_m \sim x_m e^{-\frac{b}{2m}t}$

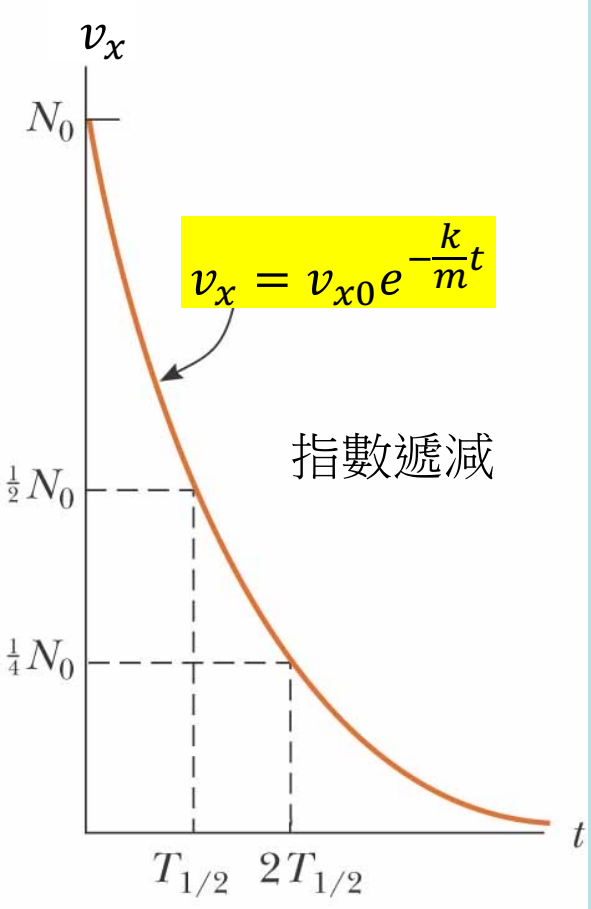
角頻率會比無阻力時稍低。

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

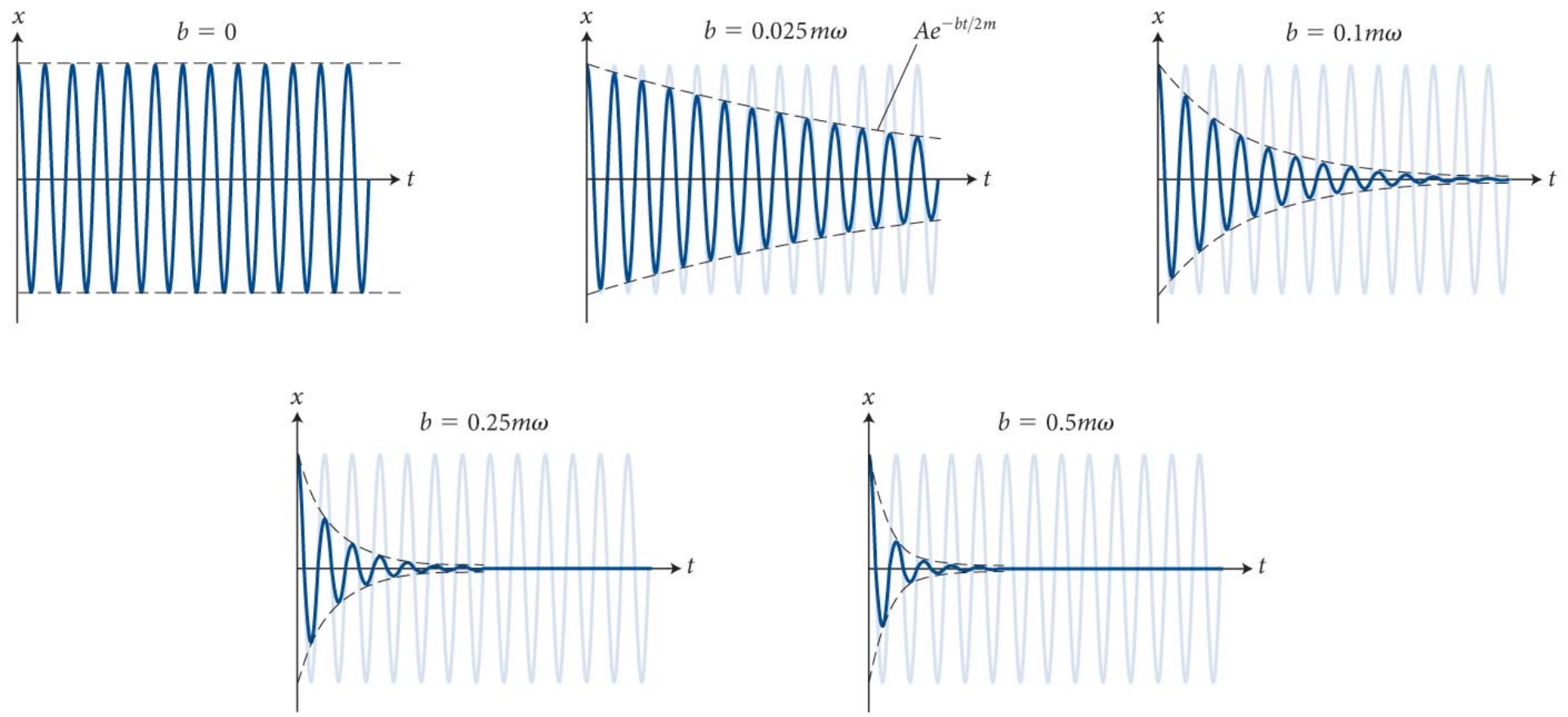

m=
-
1.
+
k=
.5
b=
0.1
f=
0
c=
0
--->
.8 , 111
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

$$x_m \sim x_m e^{-\frac{b}{2m}t}$$

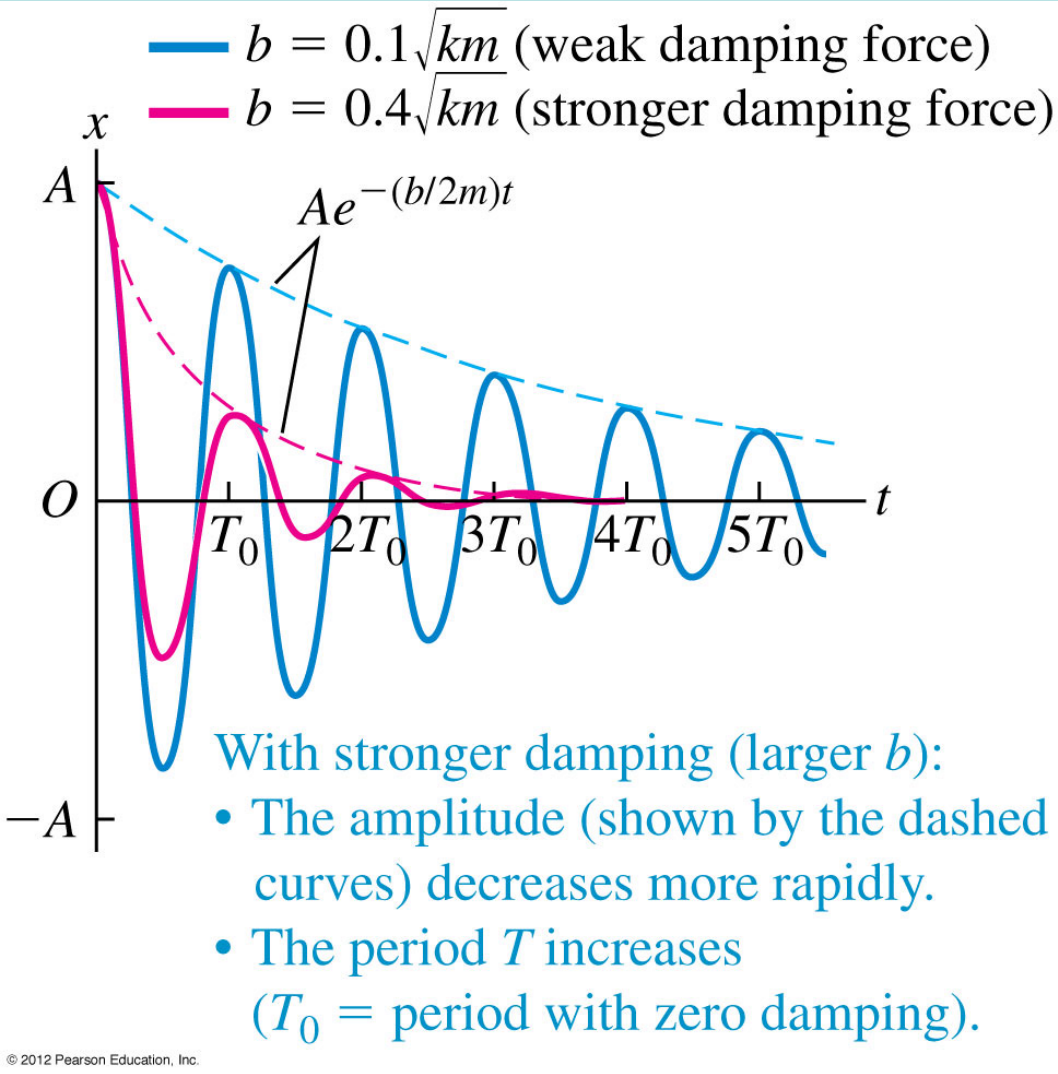




振幅的減小如同空氣阻力下的運動！



阻力常數b越大，振幅的減少也越快！



振動角頻率減小！周期增加。 $\omega' = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$

振動頻率減小！阻力越大減少越多！

$$x(t) = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

若 $\frac{b^2}{4m^2} > \omega^2$

那就根本沒有振動了！以上的式子就不對了。

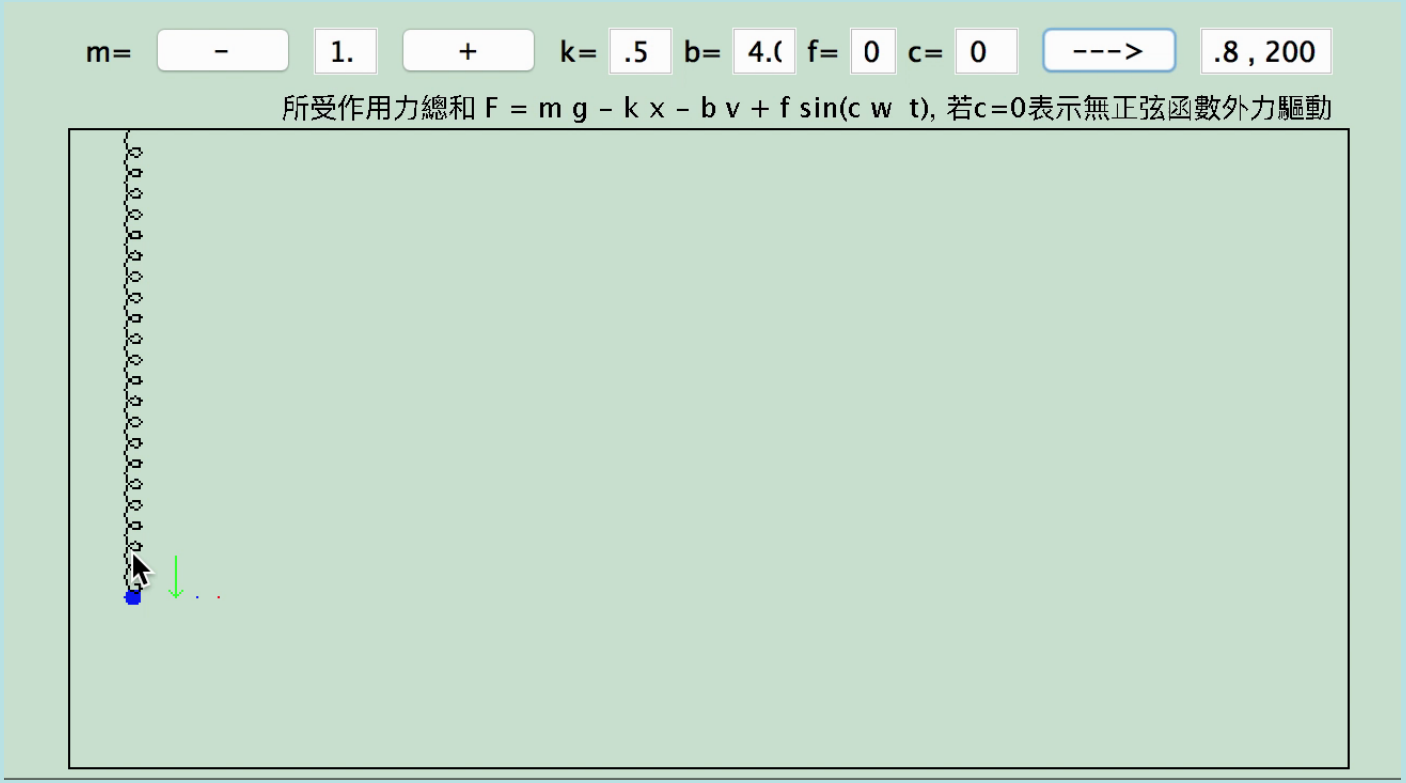
這時的解會是一個隨時間指數遞減的函數！

http://www.phy.ntnu.edu.tw/moodle/mod/resource/view.php?id=124

m= - 1. + k= .5 b= 1.5 f= 0 c= 0 ---> 2.6 , 98

所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

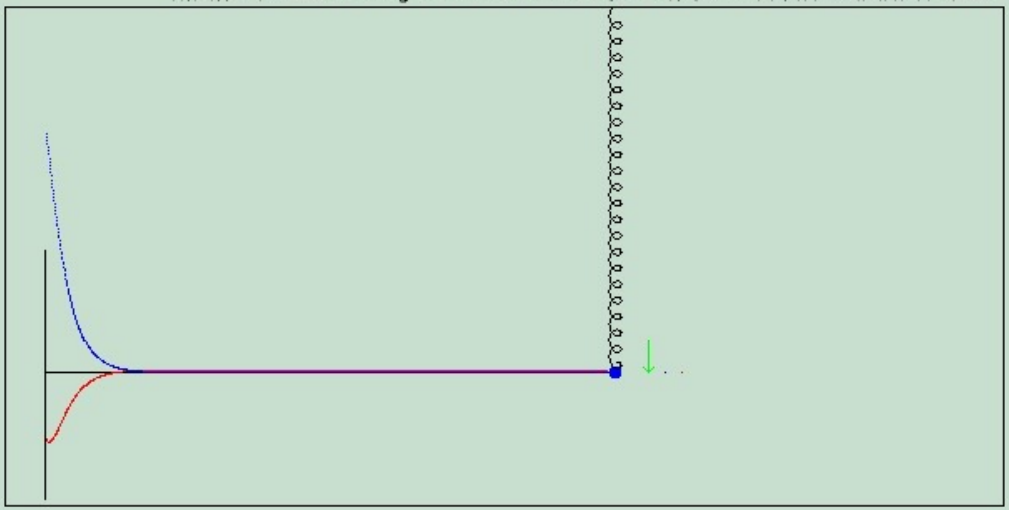
Over-damping



物體所受合力 $\mathbf{F} = m\,\mathbf{g} - k\,\mathbf{x} - b\,\mathbf{v} + \mathbf{f}_o \sin(\mathrm{cwt})$

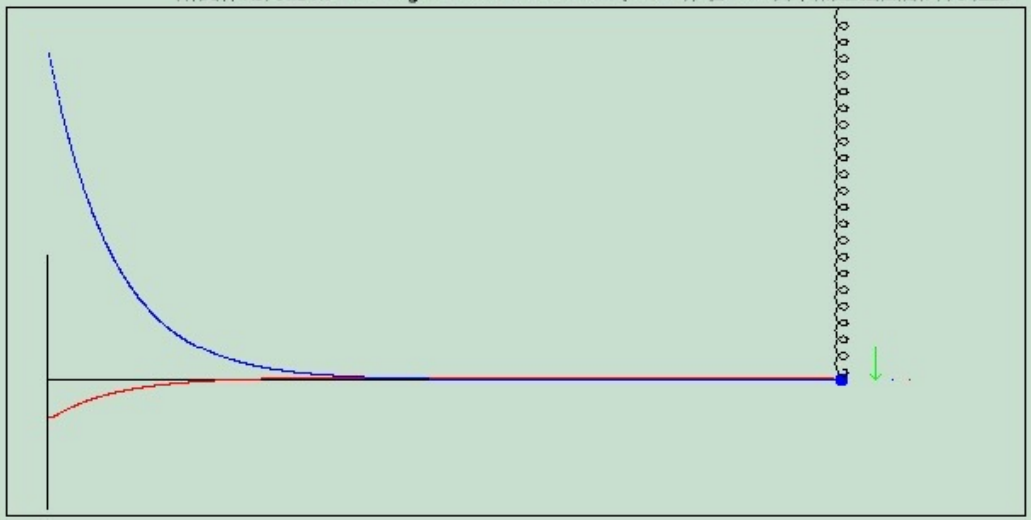
m= - 1. + k= .5 b= 1.5 f= 0 c= 0 重置 .4 , 220

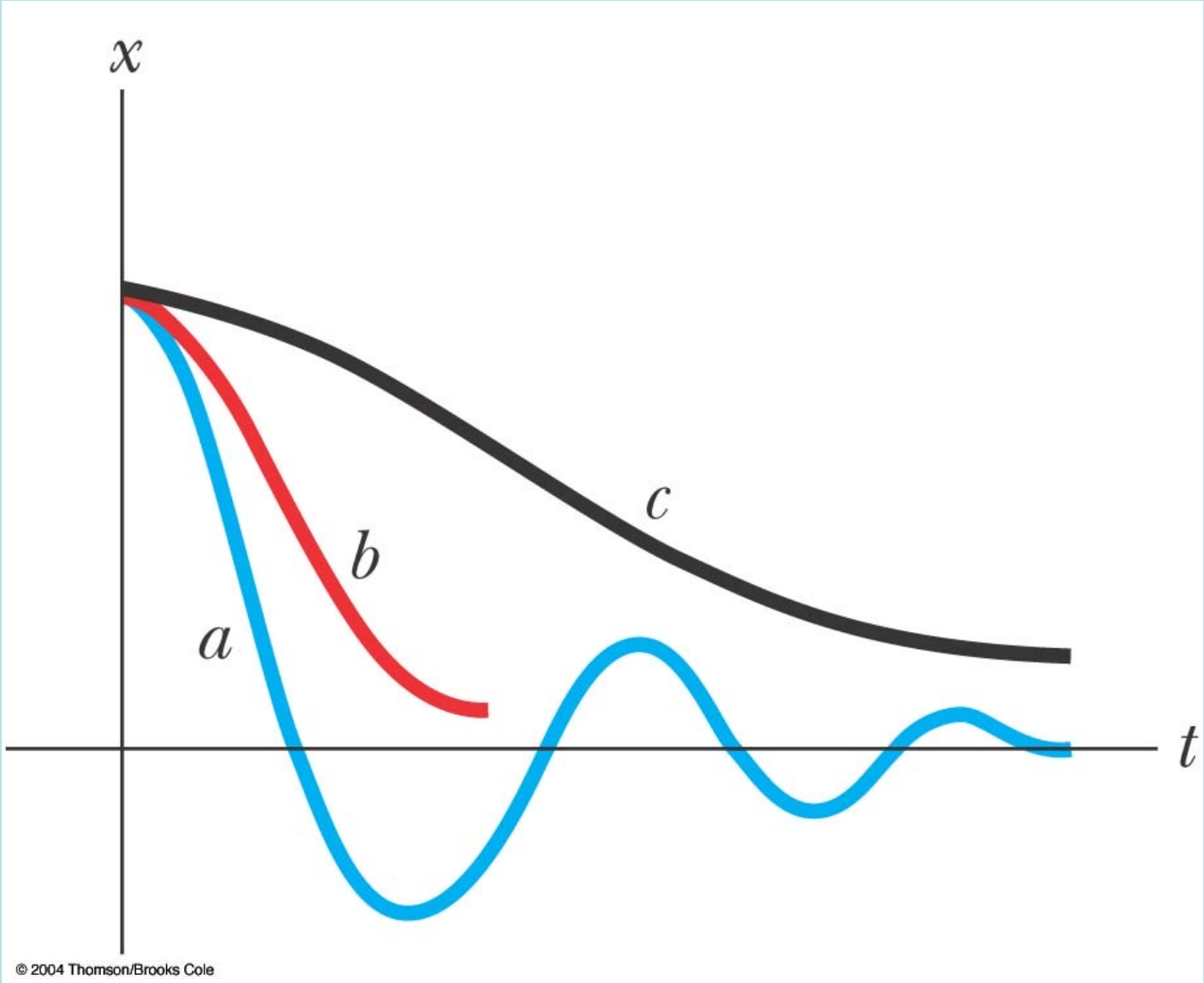
所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動



m= - 1. + k= .5 b= 4.0 f= 0 c= 0 重置 .6 , 207

所受作用力總和 F = m g – k x – b v + f sin(c w t), 若c=0表示無正弦函數外力驅動





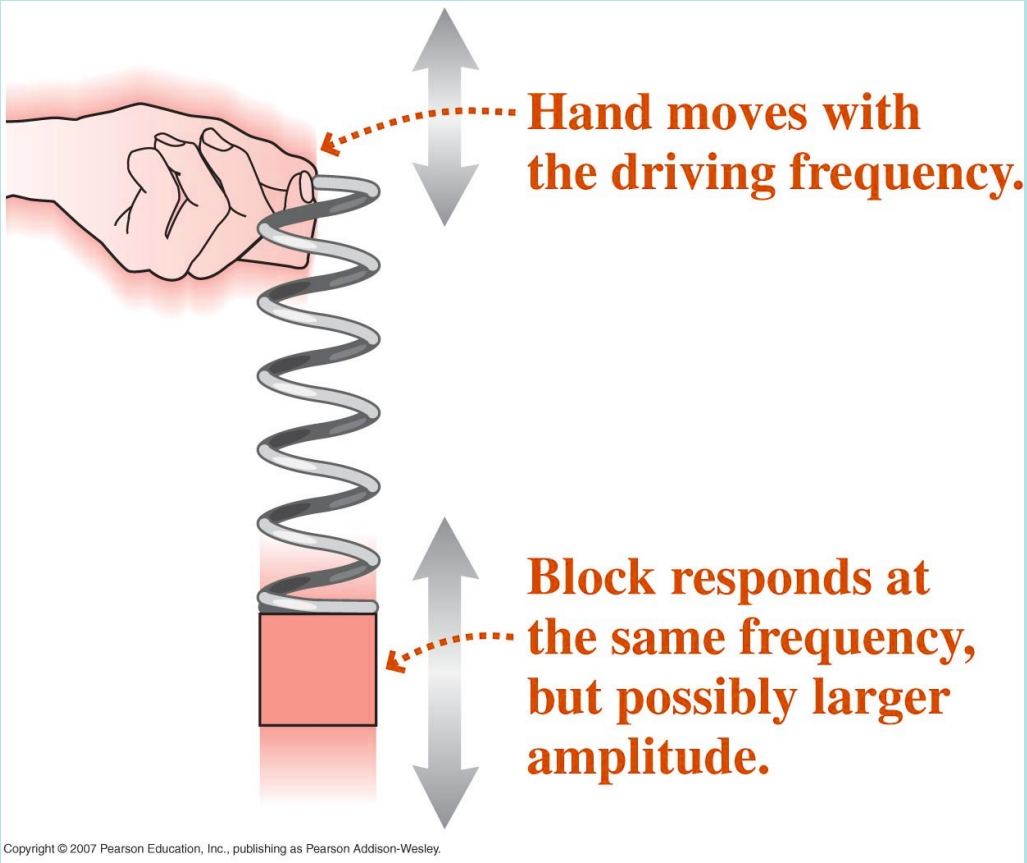
如果 $b/2m$ 大於 ω_0 (c)，那就根本沒有振動了！

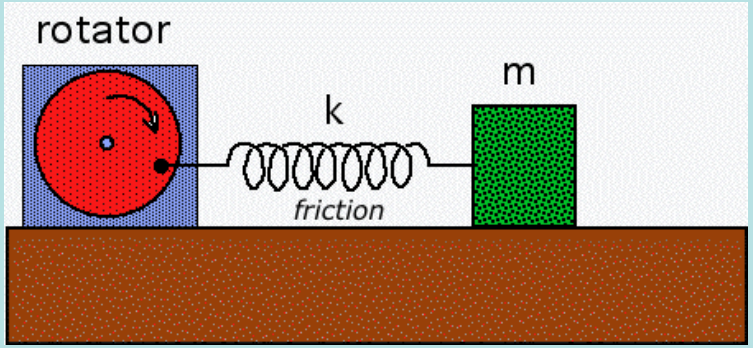
阻尼可以大到連一次震盪都未完成！

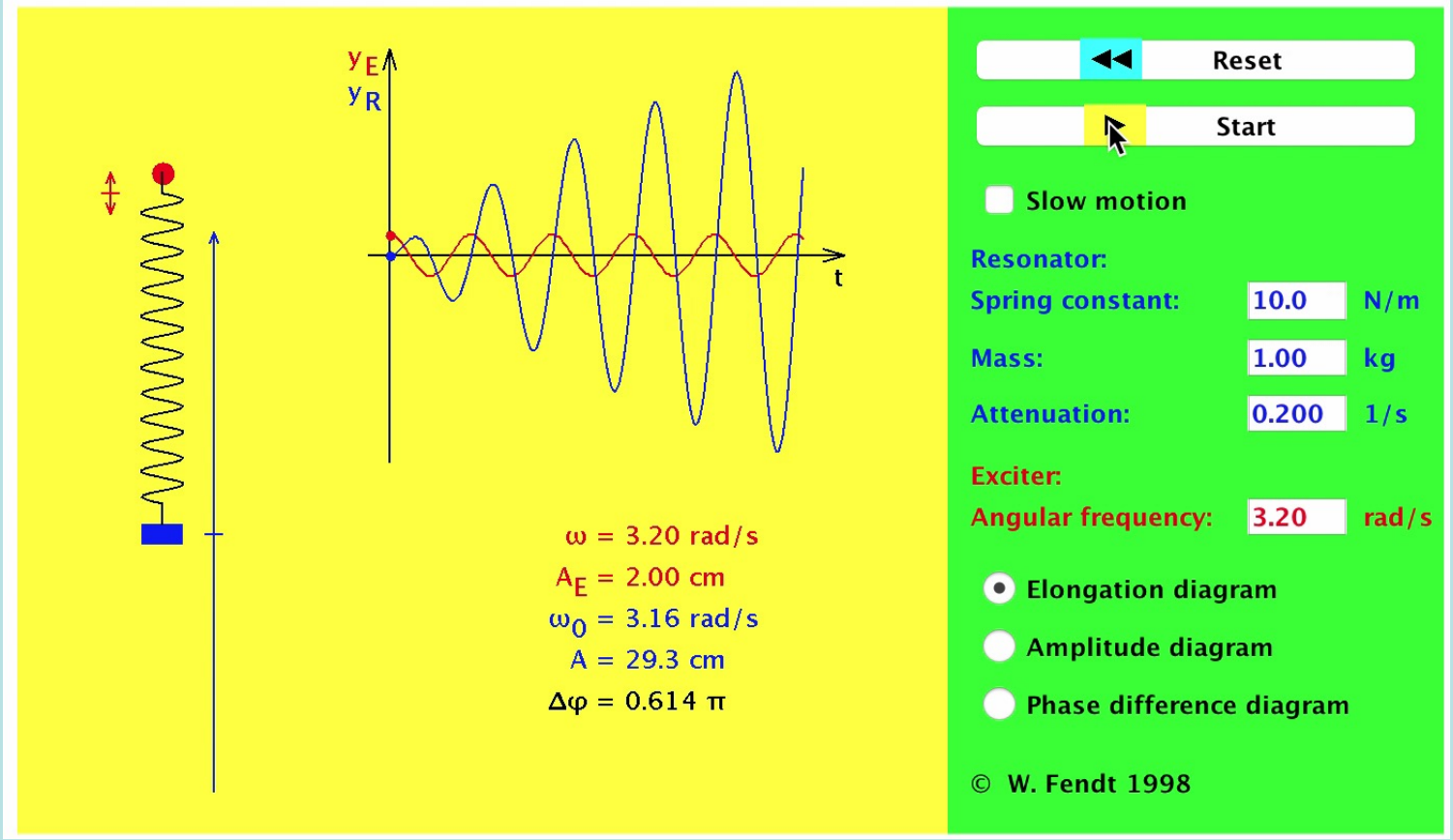
簡諧運動會因阻尼而使振幅減小，必須施力使它繼續振動

若施予一個常數力，所做的功在一個週期內會彼此抵消！

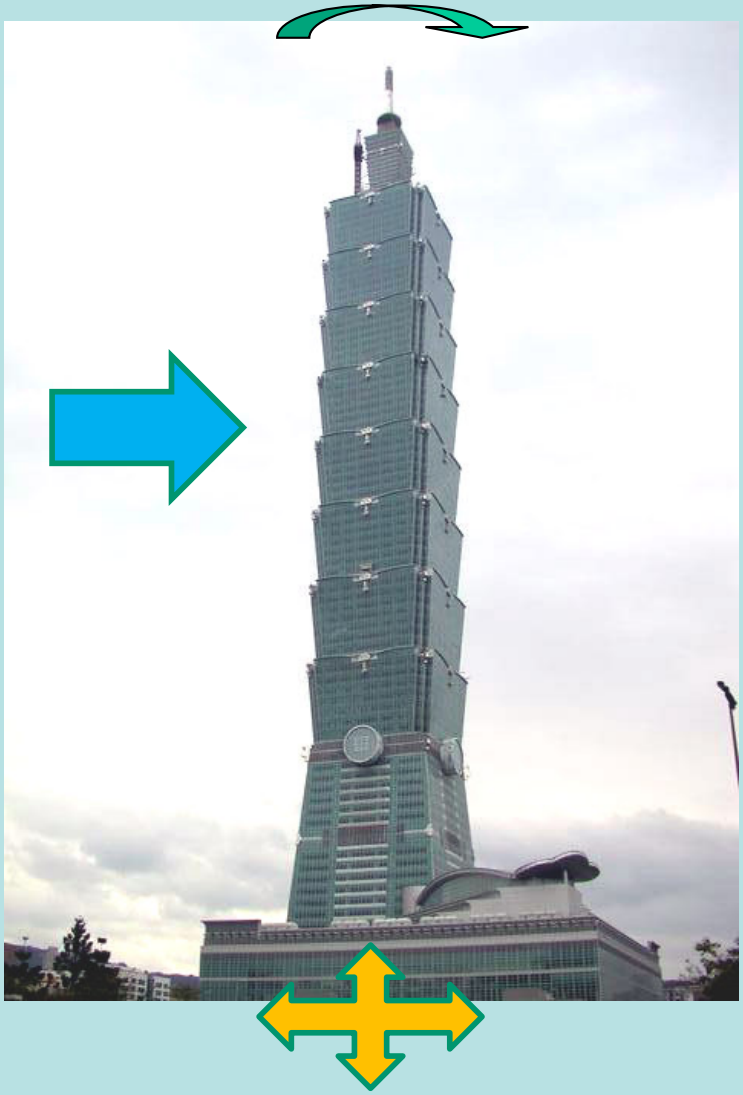
想使彈簧繼續振盪，必須施以一周期性的外力。






y_E
y_R
t
ω = 3.20 rad/s
A_E = 2.00 cm
$ω_0$ = 3.16 rad/s
A = 29.3 cm
Δφ = 0.614 π
Reset
Start
Slow motion
Resonator:
Spring constant: 10.0 N/m
Mass: 1.00 kg
Attenuation: 0.200 1/s
Exciter:
Angular frequency: 3.20 rad/s
Elongation diagram
Amplitude diagram
Phase difference diagram
© W. Fendt 1998

風吹或地震對101即是週期性的外力！



外力下的震盪 Forced Oscillation

假設所施的外力可以寫成：$F_0 \cos \omega_D t$。

運動方程式又多了一個項：

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_0 \cos \omega_D t$$

彈簧內在的角頻率

週期外力的角頻率

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos \omega_D t$$

$$y'' + a_1 y' + a_0 y = f(x)$$

This is an Inhomogeneous 2nd order Linear ODE with constant coefficients.

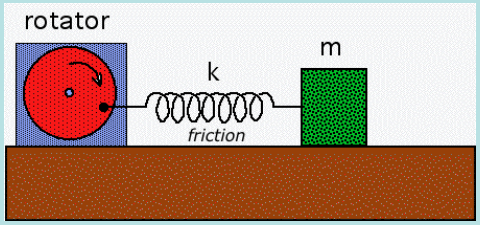
外力下的振盪由兩個頻率來決定：

為簡單起見而專注於此二頻率的影響，先忽略阻尼：

$$\frac{d^2x}{dt^2} = -\omega^2 x + \frac{F_0}{m}\cos \omega_D t$$

很容易可以猜到它的解：

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t$$



不是ω！

猜測 $x = A \cdot \cos\omega_D t$ 因為它的兩次微分還是正比於同樣的正弦函數！

代入 $$-A \cdot \omega_D^2 \cos\omega_D t + A \cdot \omega^2 \cos\omega_D t = \frac{F_0}{m}\cos\omega_D t$$

$$A = -\frac{F_0}{m(\omega^2 - \omega_D^2)}$$

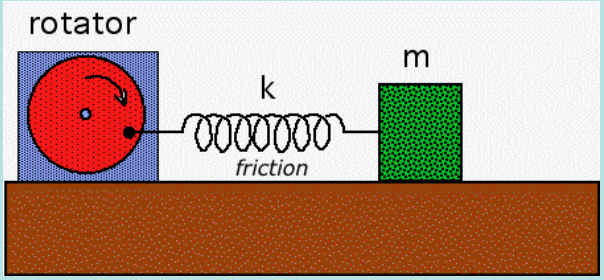
得到外力下振盪的一個解，姑稱為共振解 Resonance。

$$x_r = -\frac{F_0}{m(\omega^2 - \omega_D^2)}\cos\omega_D t$$

這解並不完整（起始條件還未放入），等一下會論證：長時間後，此解最重要。

先讓我們研究一下此共振解的性質：

$$x_r = -\frac{F_0}{m(\omega^2 - \omega_D^2)} \cdot \cos \omega_D t$$



週期外力驅動下，彈簧的反應依然是簡諧運動，

但頻率是施力的頻率 ω_{D} ，而不是彈簧的自然頻率 ω ！

此簡諧運動振幅A不再是任意，大小與施力大小成正比： $A \propto F_0$。

振幅A的大小與施力的頻率 ω_{D} 密切相關： $\omega_D \to \omega$ $A \uparrow$

施力的頻率 ω_{D} 愈接近彈簧的自然頻率 ω，反應愈強，能量的吸收愈好。

約會物理學三大定律

如同每一個彈簧有一個自然頻率，每一個人也有自己的喜好！

$\omega_D \to \omega$時猶如兩人頻率一致，心意相投，所以稱之為共振現象。

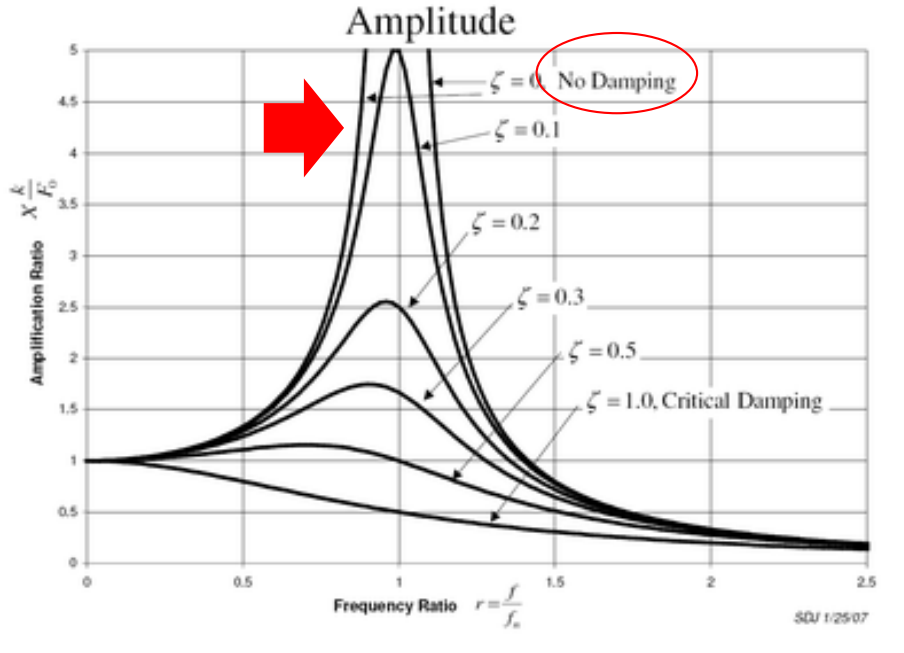
$$x_r = -\frac{F_0}{m(\omega^2 - \omega_D^2)}\cos\omega_D t$$

$$\omega_D \to \omega$$

$$\frac{F_0}{m(\omega^2 - \omega_D^2)} \uparrow$$

施力頻率ω_D越接近內在頻率ω，彈簧的反應越大！

共振 Resonance



這個共振解在 $\omega_D = \omega$ 時會趨近無限大！在自然界是不可能的。

這是因為忽略阻尼：自然界一定存在阻力！

考慮阻尼後

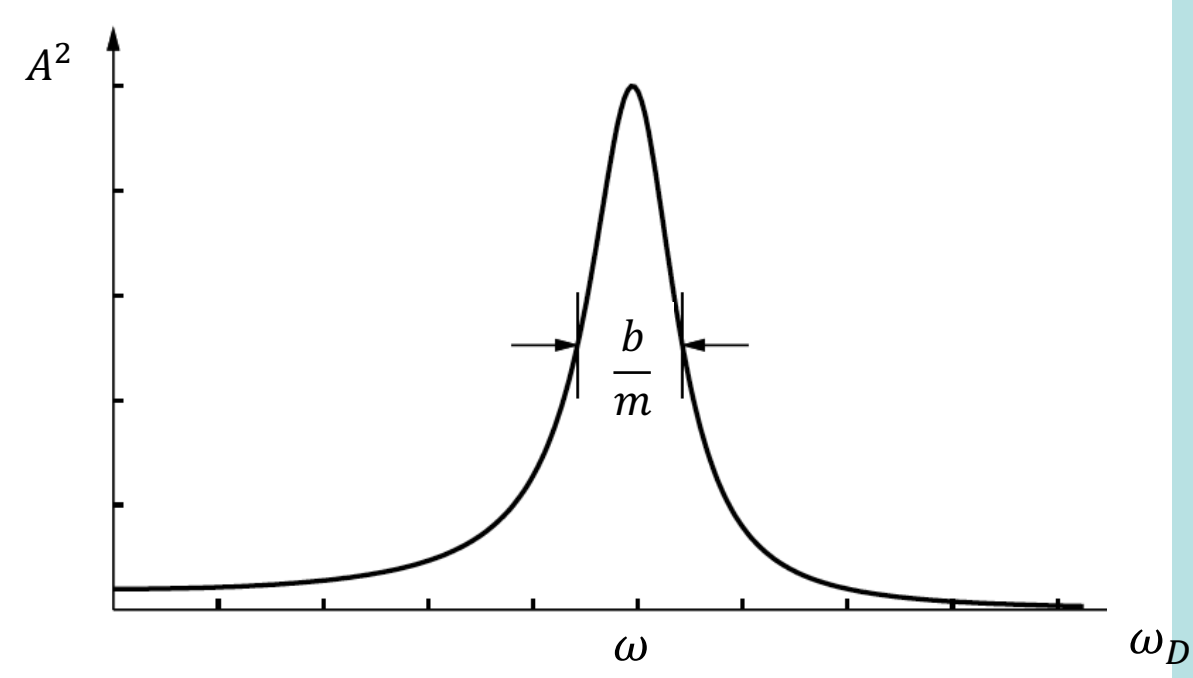
$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=\frac{F_0}{m}\cos\omega_D t$$

振幅的分母會多一個與阻力有關的項！

$$A=-\frac{F_0}{m\sqrt{(\omega^2-\omega_D^2)^2+\left(\frac{b}{m}\omega\right)^2}}$$

$$x_r=A\cos(\omega_D t+\phi)$$

振幅極大值在 $\omega_D=\omega$ 附近，但已不再是無限大！



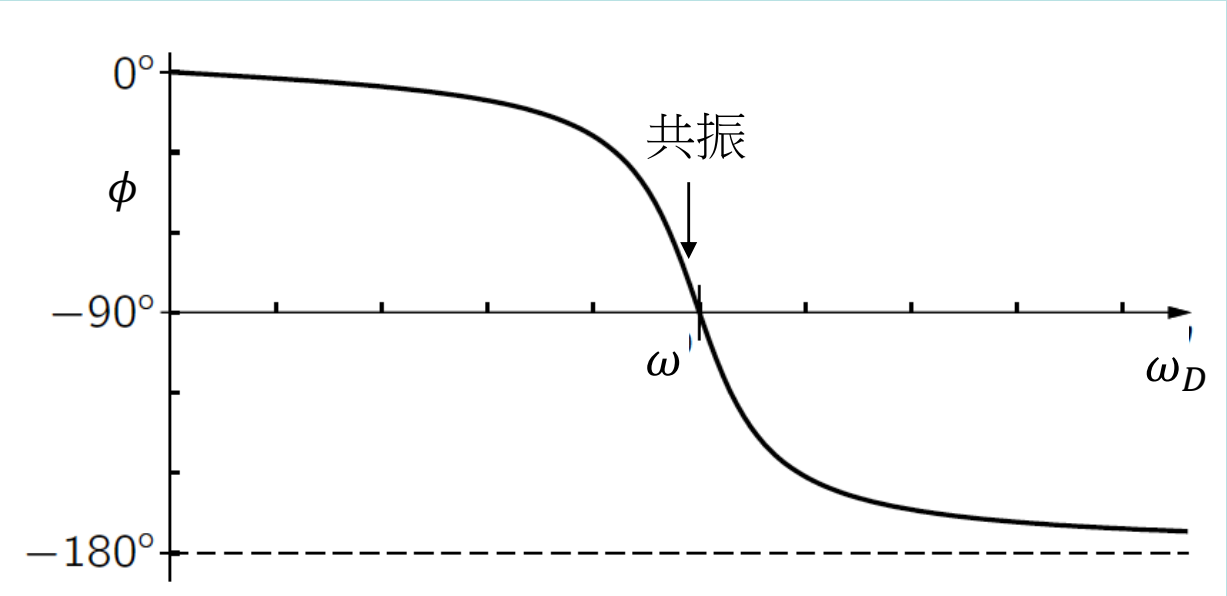
共振曲線的寬度現在與阻力大小成正比。

$$x_r = A\cos(\omega_D t + \phi)$$

所施的外力：$F_0 \cos \omega_D t$。

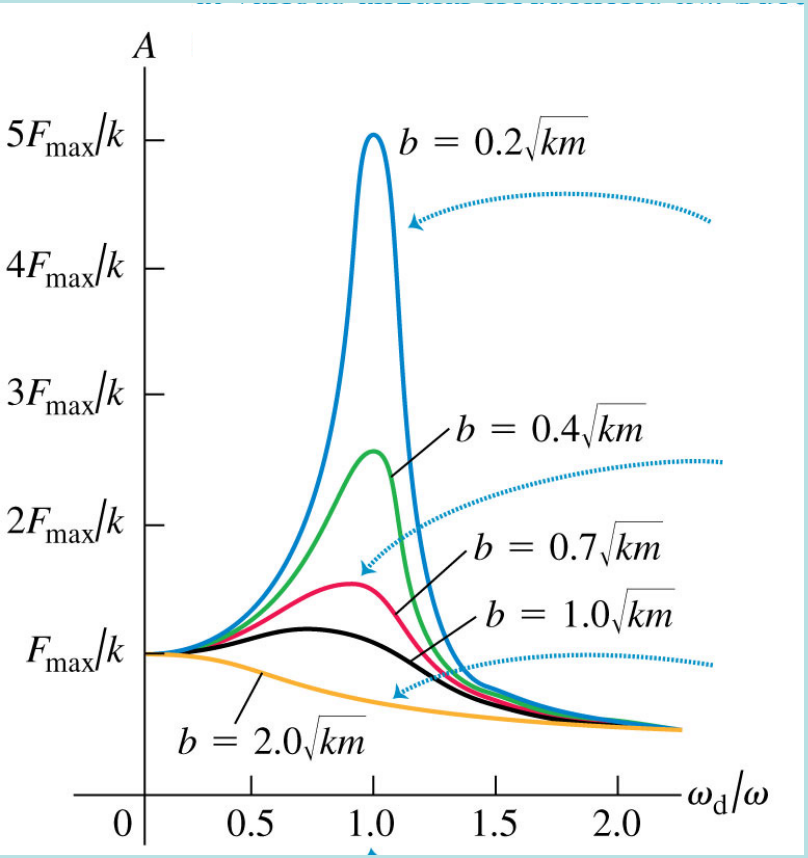
震盪多了一個相常數，反應與施力有相差、或叫時間差。

$$\tan\phi = -\frac{\frac{b}{m}\omega}{\omega^2 - \omega_D^2}$$



注意共振時$\phi = 90°$。也就是施力若是正弦，反應卻是餘弦。

在外力驅動下，簡諧振盪器的運動依舊是一個週期性振盪:



$$x_r = A\cos(\omega_D t + \phi)$$

$$A = -\frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}}$$

$$\tan\phi = -\frac{\frac{b}{m}\omega}{\omega^2 - \omega_D^2}$$

前面忽略阻尼時，得到的約會物理學三大定律還是成立。

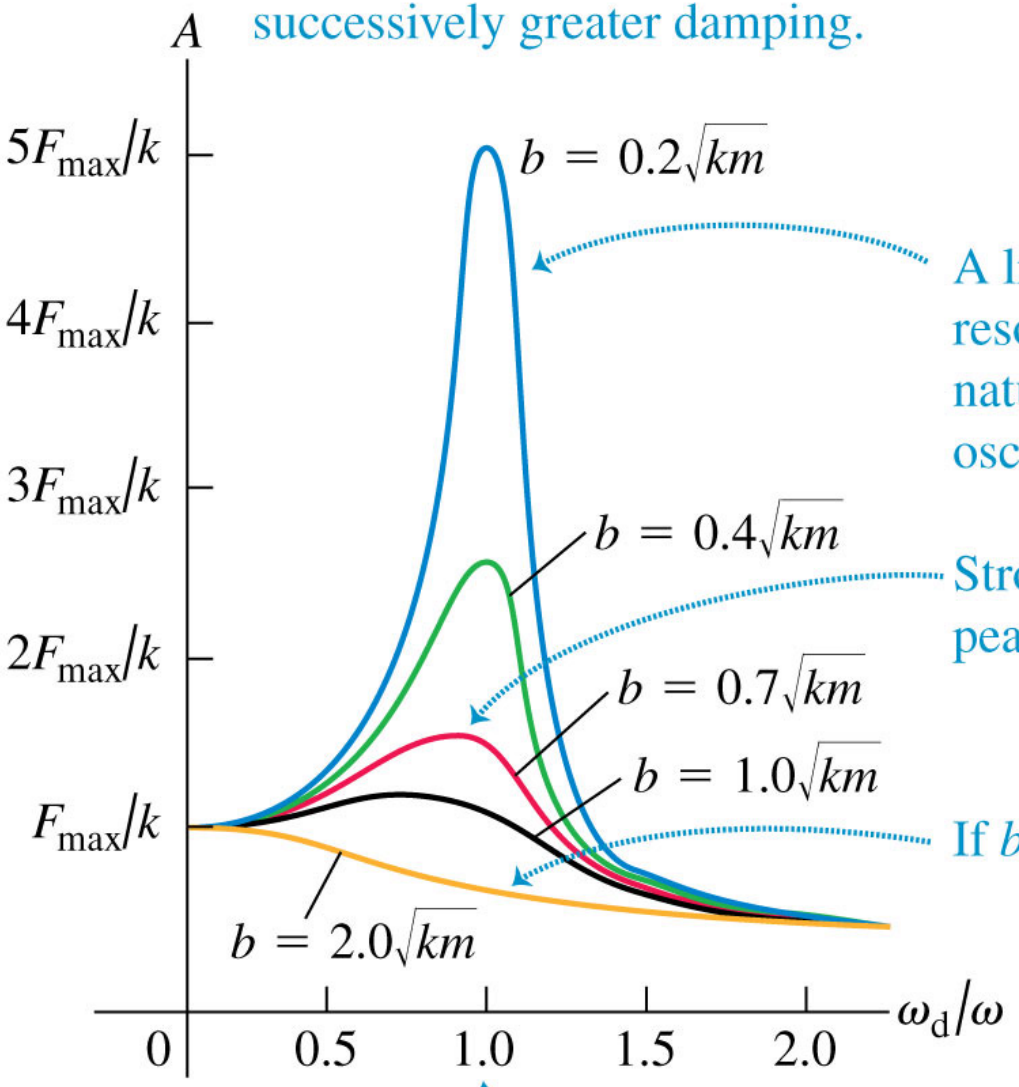
振幅與外力的大小成正比。

以外力的頻率ω_D來振盪，而不是彈簧的自然頻率ω！

外力頻率越接近彈簧的自然頻率，振盪振幅也就越大！ $\omega_D \to \omega$ $A \uparrow$

共振曲線的寬度與阻力大小成正比，阻力會削弱共振的現象！

Each curve shows the amplitude A for an oscillator subjected to a driving force at various angular frequencies ω_d. Successive curves from blue to gold represent successively greater damping.



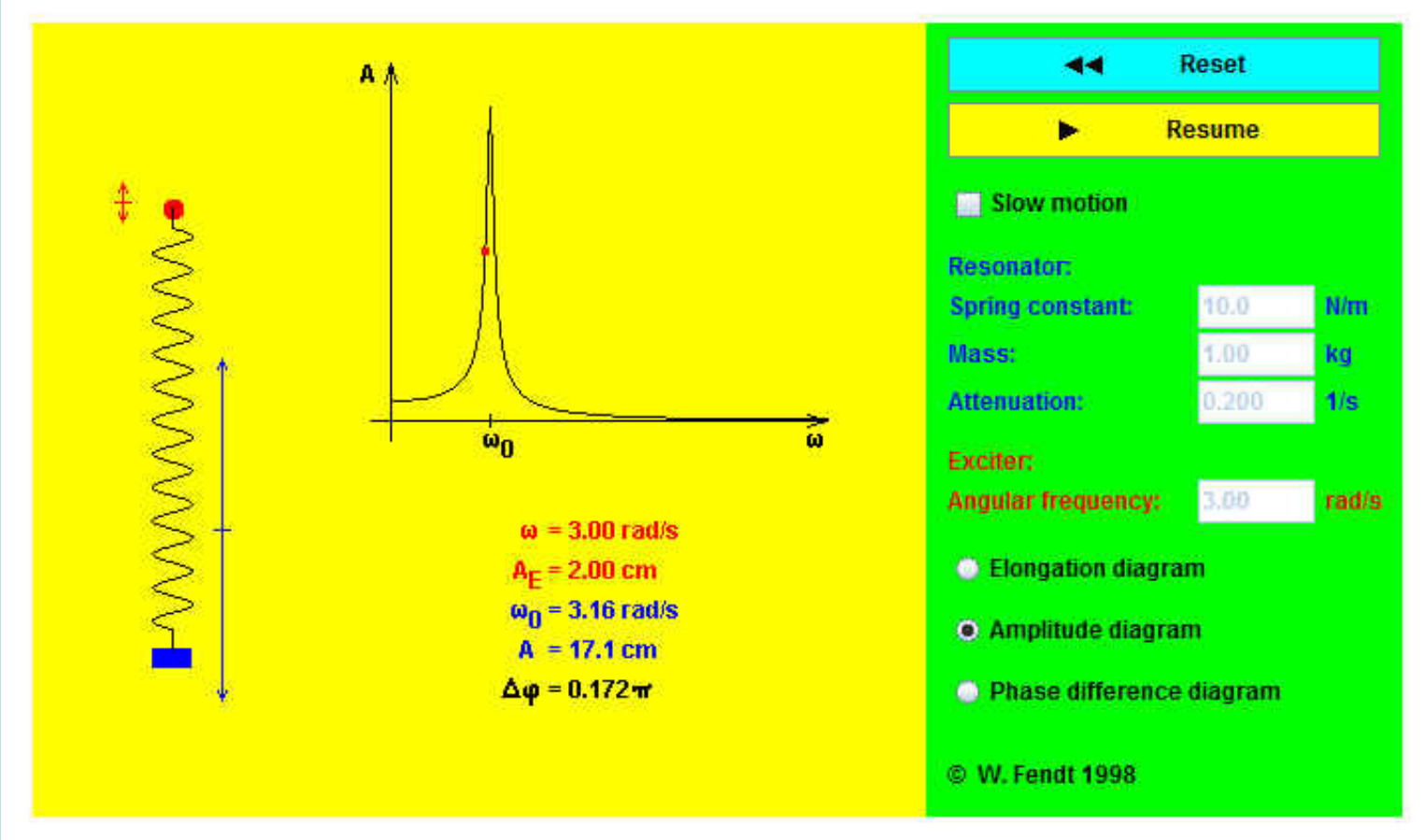
A lightly damped oscillator exhibits a sharp resonance peak when ω_d is close to ω (the natural angular frequency of an undamped oscillator).

Stronger damping reduces and broadens the peak and shifts it to lower frequencies.

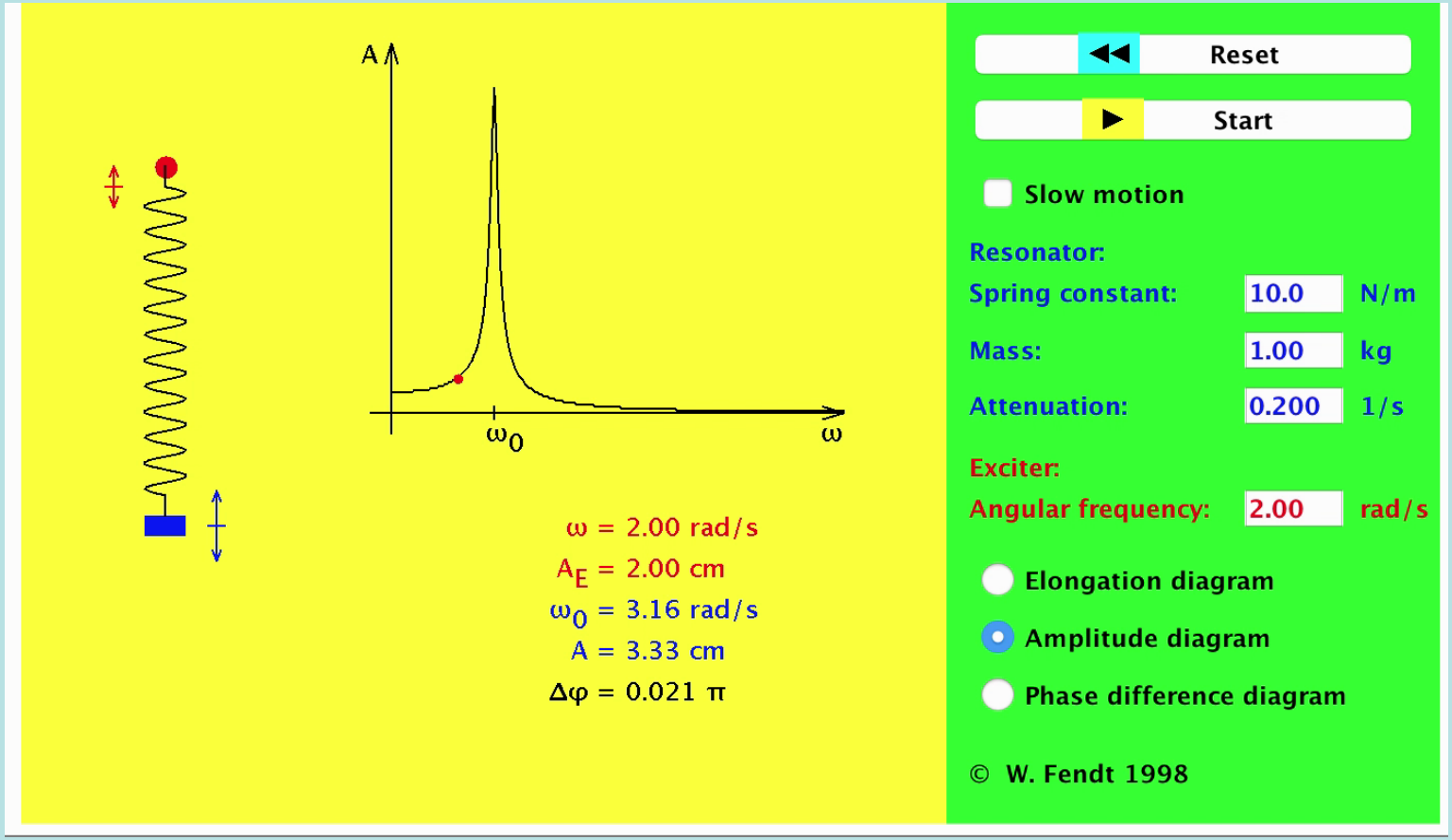
If $b \geq \sqrt{2km}$, the peak disappears completely.

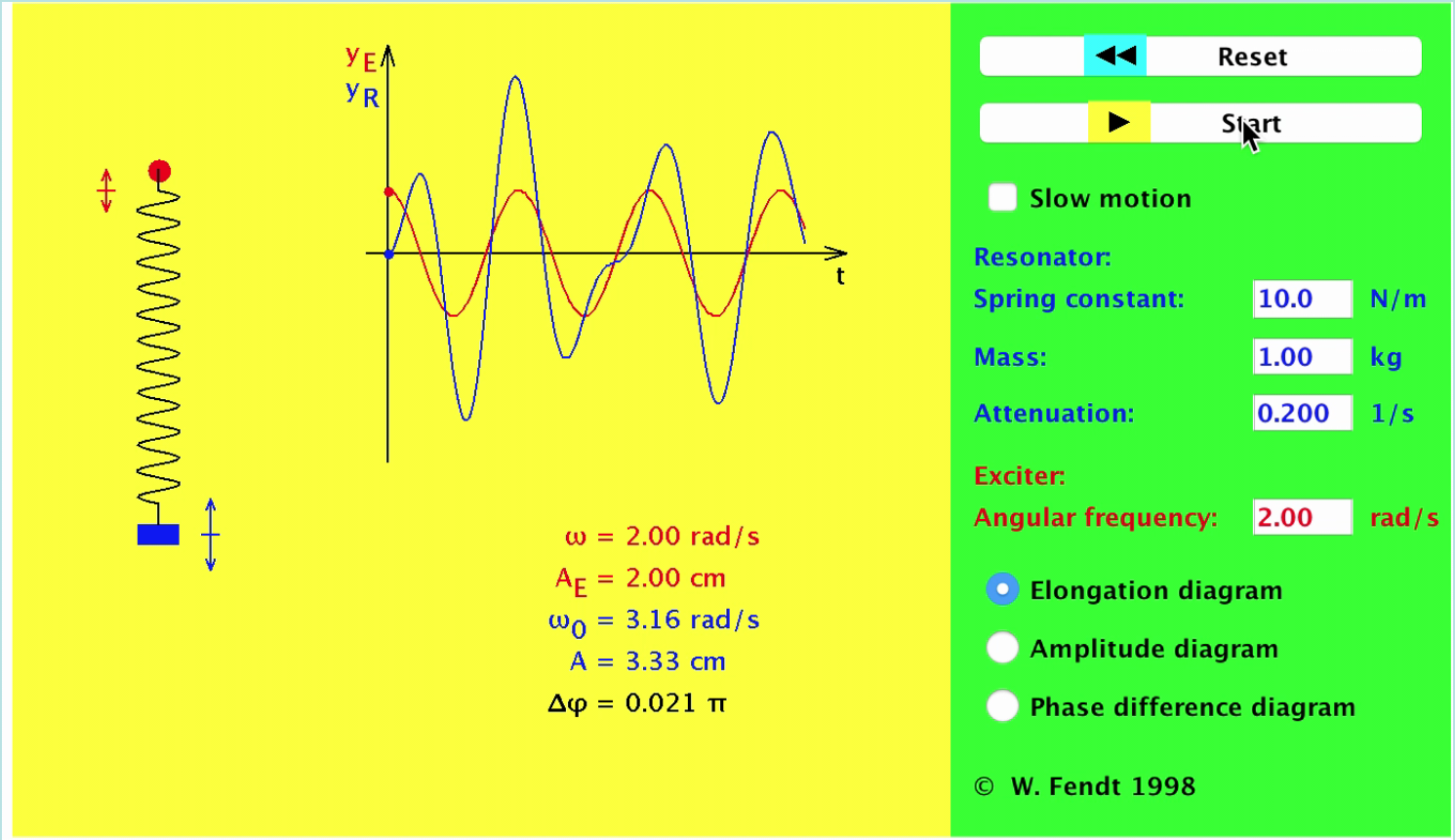
Driving frequency ω_d equals natural angular frequency ω of an undamped oscillator.

http://www.walter-fendt.de/ph14e/resonance.htm

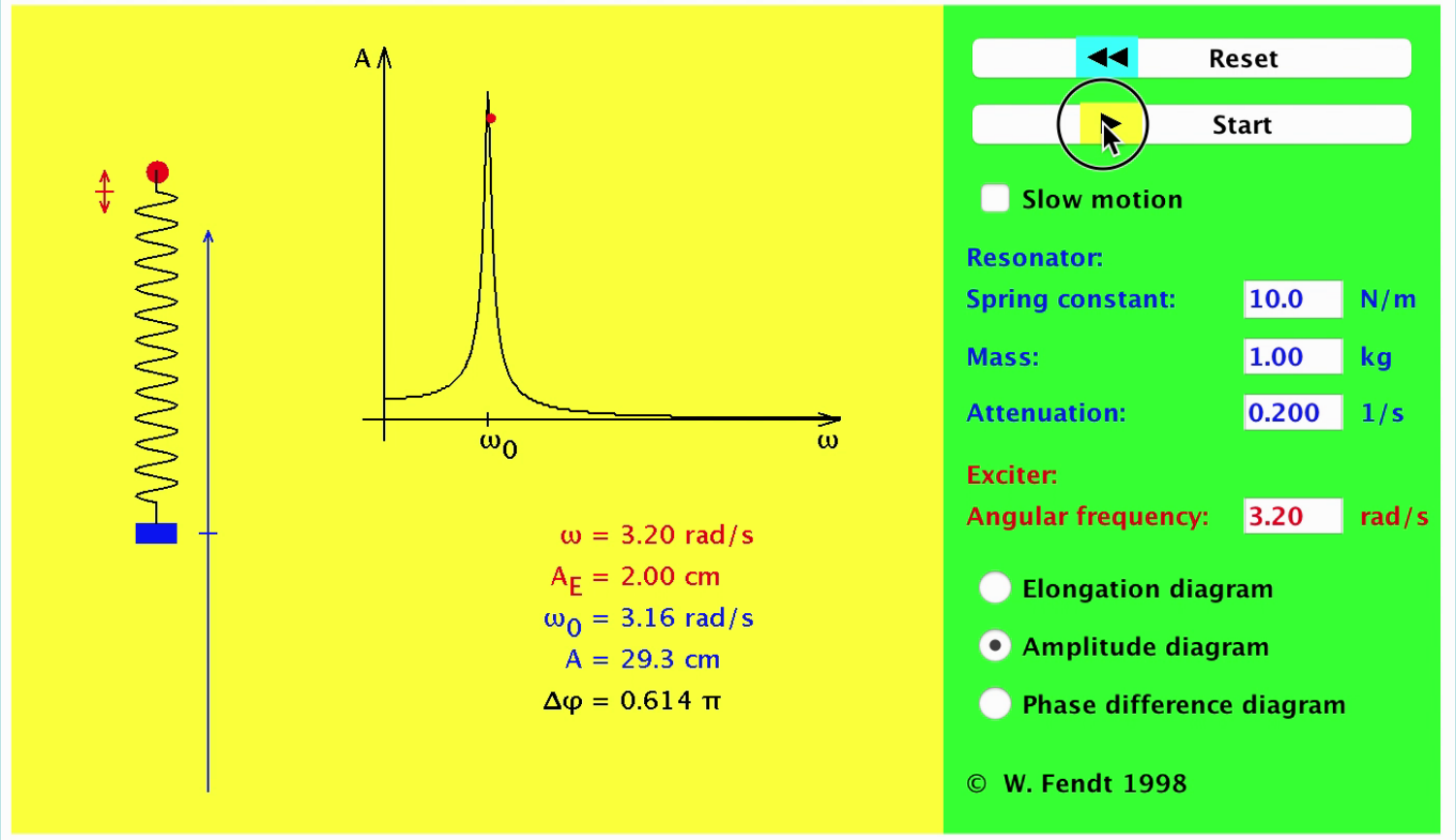


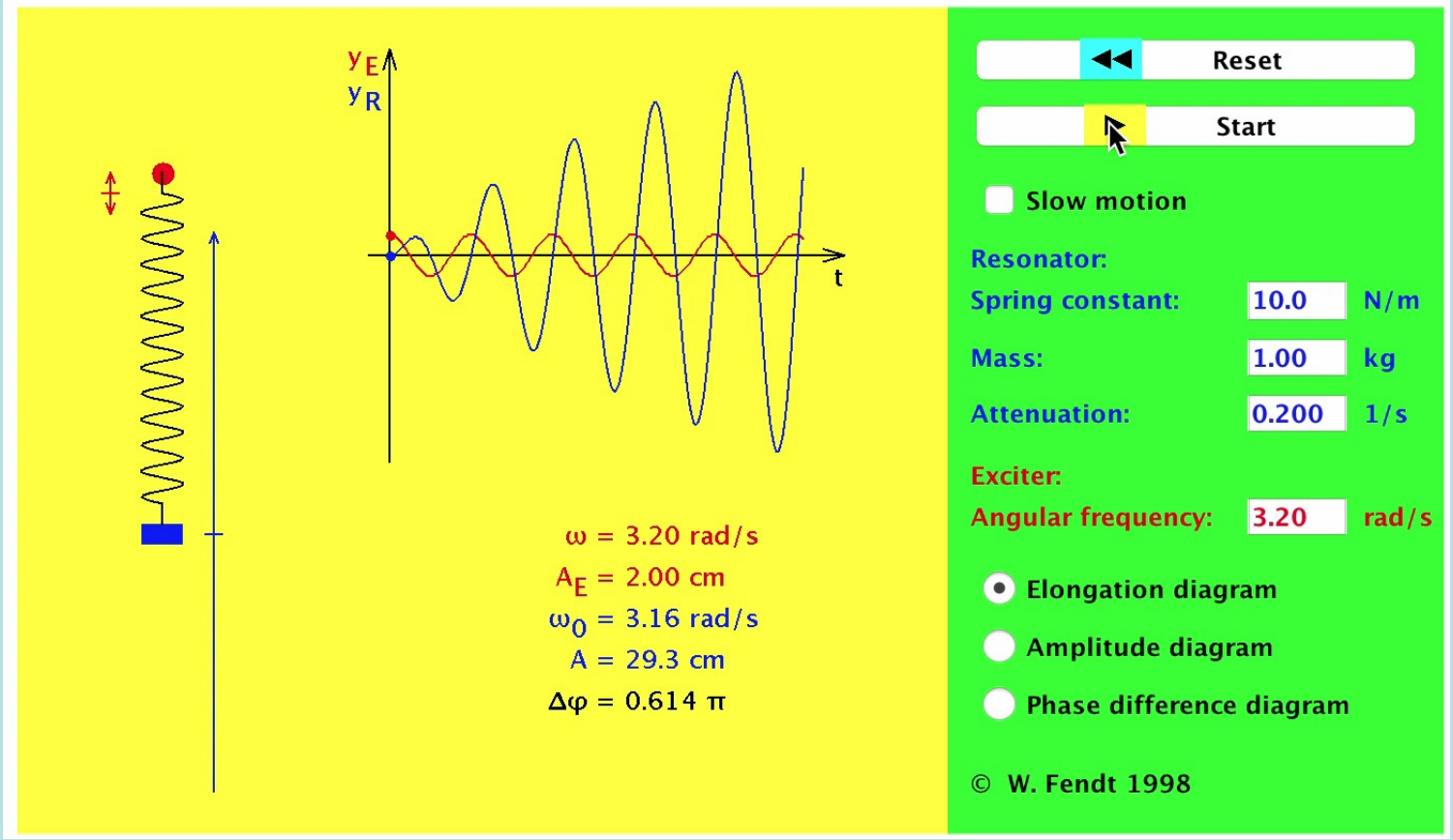
Under resonance



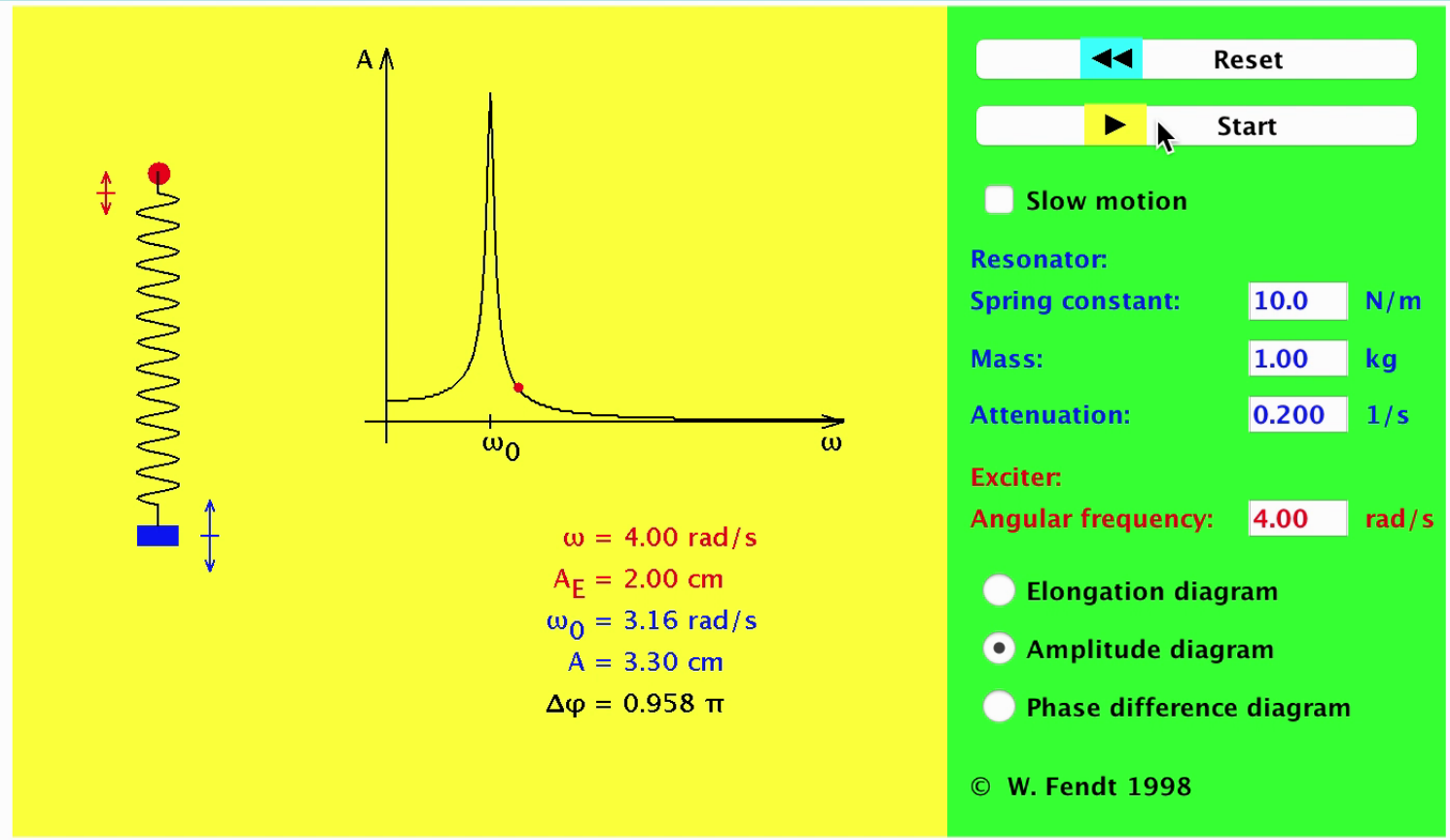

y_E
y_R
t
ω = 2.00 rad/s
A_E = 2.00 cm
$ω_0$ = 3.16 rad/s
A = 3.33 cm
Δφ = 0.021 π
Reset
Start
Slow motion
Resonator:
Spring constant: 10.0 N/m
Mass: 1.00 kg
Attenuation: 0.200 1/s
Exciter:
Angular frequency: 2.00 rad/s
Elongation diagram
Amplitude diagram
Phase difference diagram
© W. Fendt 1998

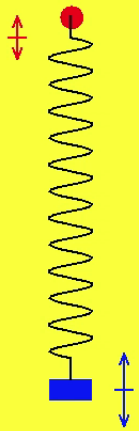
On resonance

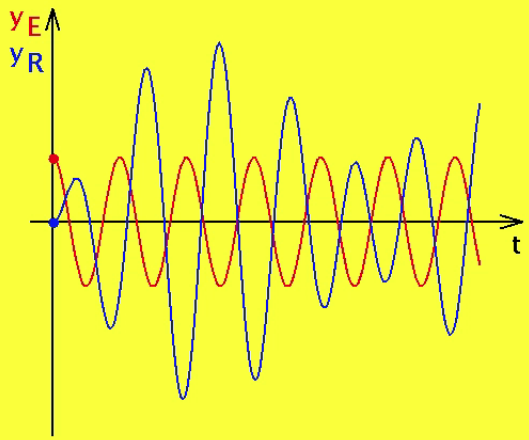



y_E
y_R
t
ω = 3.20 rad/s
A_E = 2.00 cm
$ω_0$ = 3.16 rad/s
A = 29.3 cm
Δφ = 0.614 π
Reset
Start
Slow motion
Resonator:
Spring constant: 10.0 N/m
Mass: 1.00 kg
Attenuation: 0.200 1/s
Exciter:
Angular frequency: 3.20 rad/s
Elongation diagram
Amplitude diagram
Phase difference diagram
© W. Fendt 1998

Above Resonnace







ω = 4.00 rad/s
A_E = 2.00 cm
ω_0 = 3.16 rad/s
A = 3.30 cm
$\Delta\varphi$ = 0.958 π

Reset

Start

Slow motion

Resonator:

Spring constant:	10.0	N/m
Mass:	1.00	kg
Attenuation:	0.200	1/s

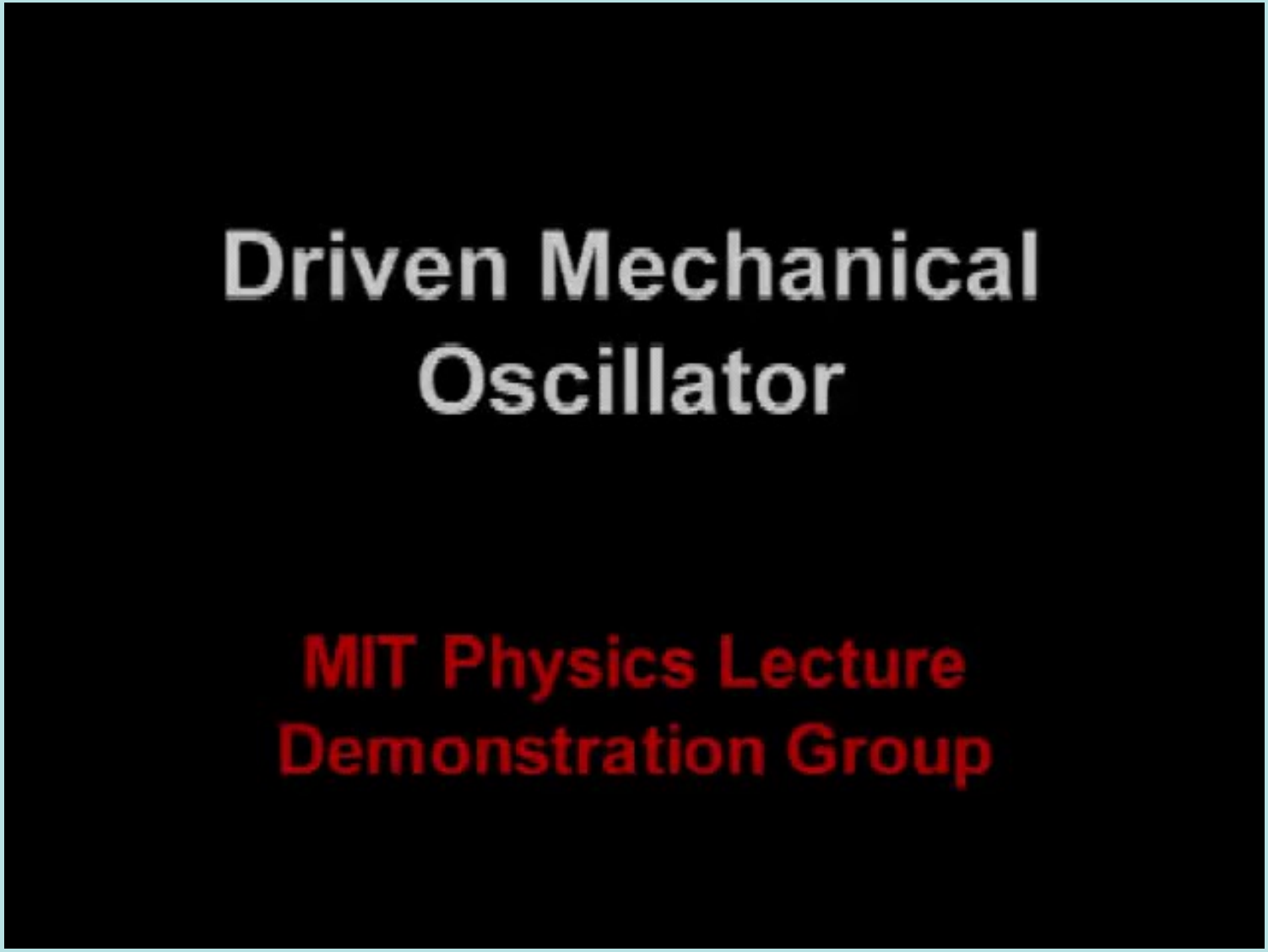
Exciter:

Angular frequency:	4.00	rad/s

- (●) Elongation diagram
- ( ) Amplitude diagram
- ( ) Phase difference diagram

https://www.youtube.com/watch?v=aZNnwQ8HJHU

http://techtv.mit.edu/collections/physicsdemos/videos/769-mit-physics-demo----driven-mechanical-oscillator



只有一個解如何滿足起始條件？

x_r共振解滿足：

$$\frac{d^2x_r}{dt^2}+\frac{b}{m}\frac{dx_r}{dt}+\omega^2x_r=\frac{F_0}{m}\cos\omega_D t$$

在共振解x_r上加一個去掉外力後的阻尼簡諧運動x_s ， x_r+x_s還是一個解：

$$\frac{d^2x_s}{dt^2}+\frac{b}{m}\frac{dx_s}{dt}+\omega^2x_s=0$$

x_s 稱為非共振解 $x_s=x_m\cdot e^{-\frac{b}{2m}t}\cdot\cos(\omega' t+\phi)$

將以上兩個微分方程式相加，可以得到：

$$\frac{d^2(x_r+x_s)}{dt^2}+\frac{b}{m}\frac{d(x_r+x_s)}{dt}+\omega^2(x_r+x_s)=\frac{F_0}{m}\cos\omega_D t$$

x_r+x_s 依舊滿足原來 x_r 所滿足的微分方程式。

x_s 非共振解有兩個未定常數x_m ,ϕ，於是我又得到無限多個解。

Solutions y of Linear inhomogeneous ODE equals one solution y_2 of the inhomogeneous ODE plus the solution y_1 of the homogeneous ODE.

The difference between any two solutions $y - y_2$ of a Linear inhomogeneous ODE equals a solution of the homogeneous ODE.

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t \quad - \quad \frac{d^2x_r}{dt^2} + \frac{b}{m}\frac{dx_r}{dt} + \omega^2 x_r = \frac{F_0}{m}\cos\omega_D t$$

$$\Downarrow$$

$$\frac{d^2(x - x_r)}{dt^2} + \frac{b}{m}\frac{d(x - x_r)}{dt} + \omega^2(x - x_r) = 0$$

$(x - x_r)$ equals a solution x_s of the homogeneous ODE.

$$x = x_r + x_s$$

定理：

x_r滿足非齊次微分方程式：

$$\sum_n a_n \cdot \frac{d^n x_r}{dt^n} = f(t)$$

右手邊的非齊次項是時間函數

在x_r上加一個去掉非齊次項的其次方程式的解x_s ，

$$\sum_n a_n \cdot \frac{d^n x_s}{dt^n} = 0$$

$x_r + x_s$ 還是一個非齊次方程式的解：

證明：

將以上兩個微分方程式相加，可以得到：

$$\sum_n a_n \cdot \frac{d^n (x_r + x_s)}{dt^n} = f(t)$$

$x_r + x_s$ 依舊滿足原來 x_r 所滿足的微分方程式。

$$x_s = x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi)$$

$$x_r = -\frac{F_0}{m\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b}{m}\omega\right)^2}} \cos(\omega_D t + \phi)$$

$$x = x_r + x_s$$

滿足原來 x_r 所滿足的外力下簡諧運動的微分方程式：

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_D t$$

調整x_s中的 x_m, ϕ 可以滿足起始條件。

$x_r + x_s$這個函數同時滿足運動方程式以及兩個起始條件，因此是唯一的解！

注意非共振解x_s是以彈簧自然頻率ω震盪，而不是ω_D。

但隨時間振幅會變小，長期來說可以忽略。

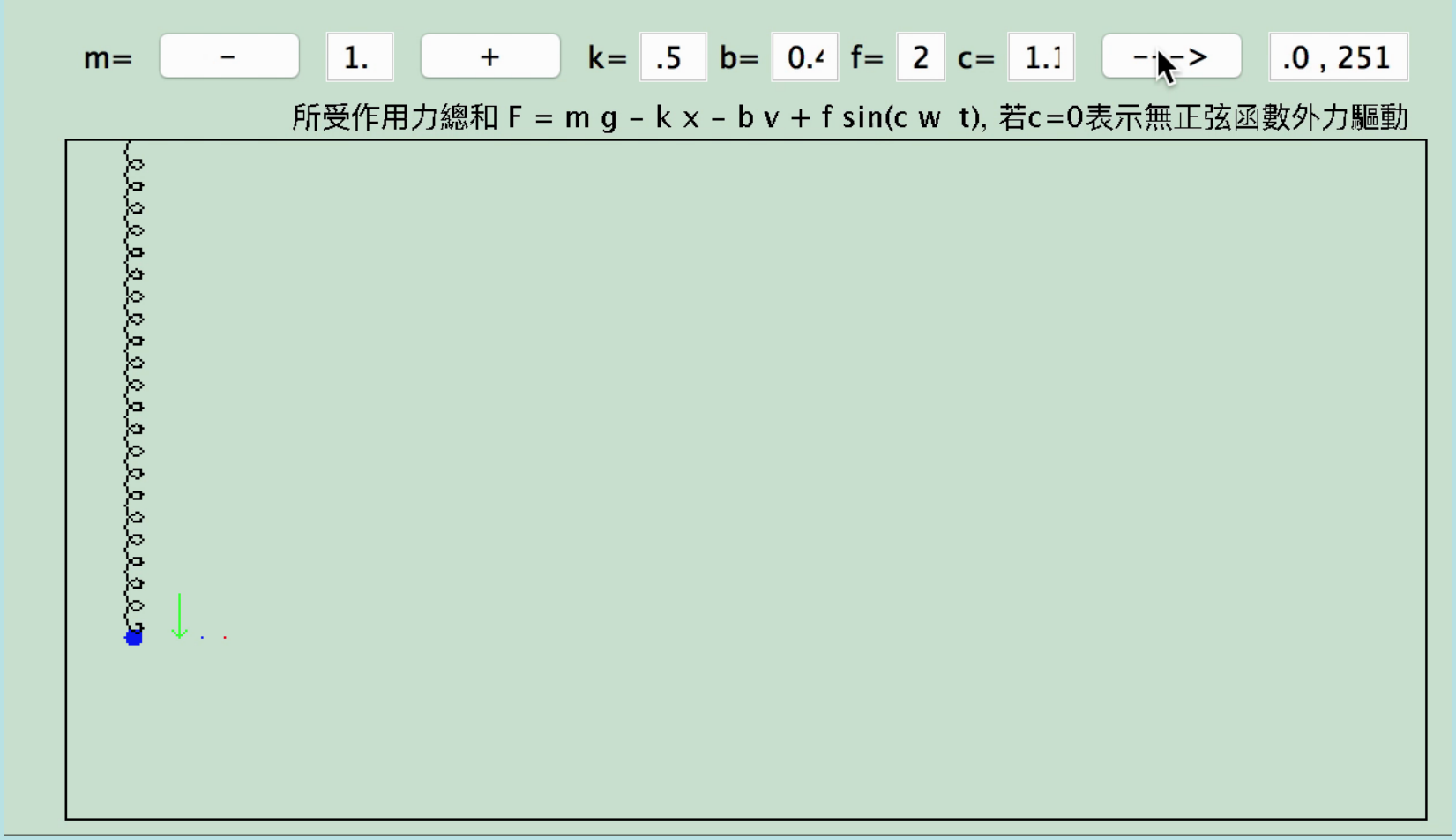
$$x = x_r + x_s \rightarrow x_r$$

長期而言，只有共振解是重要的，起始條件無關緊要

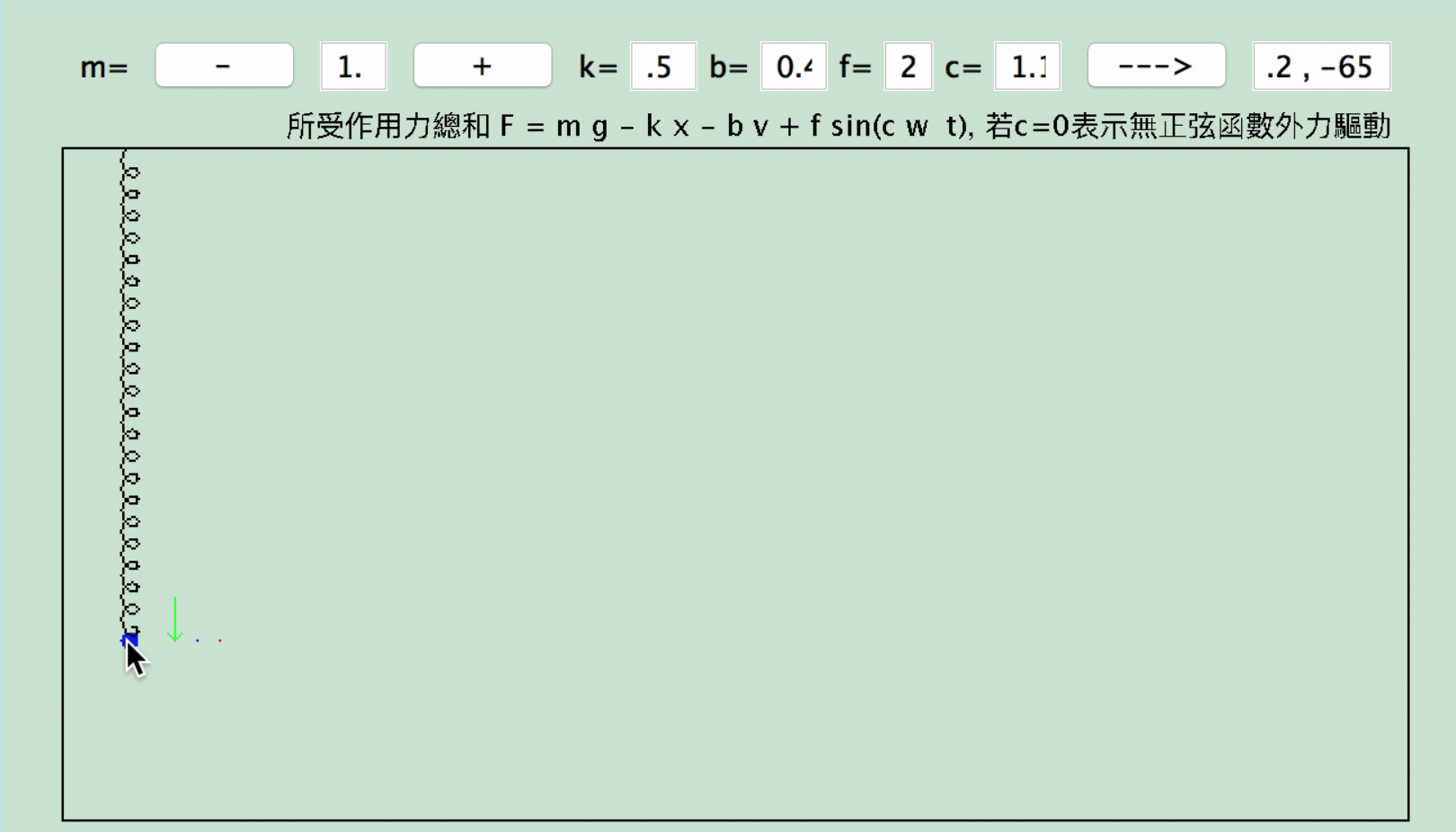
愛拼才會贏定律

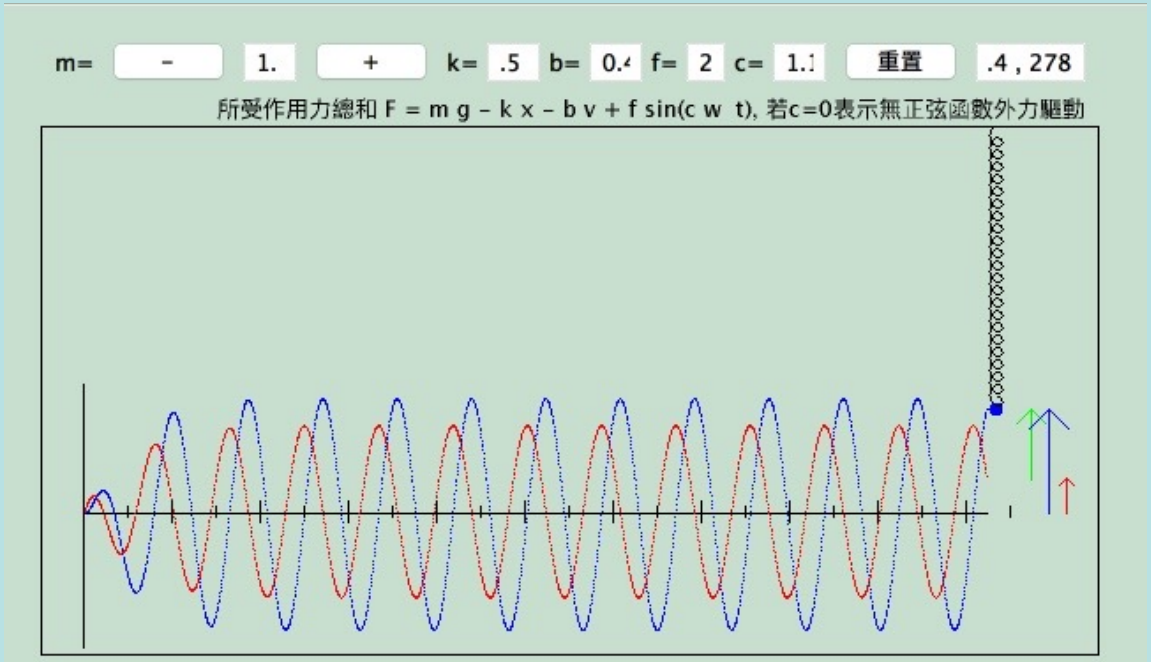
模擬

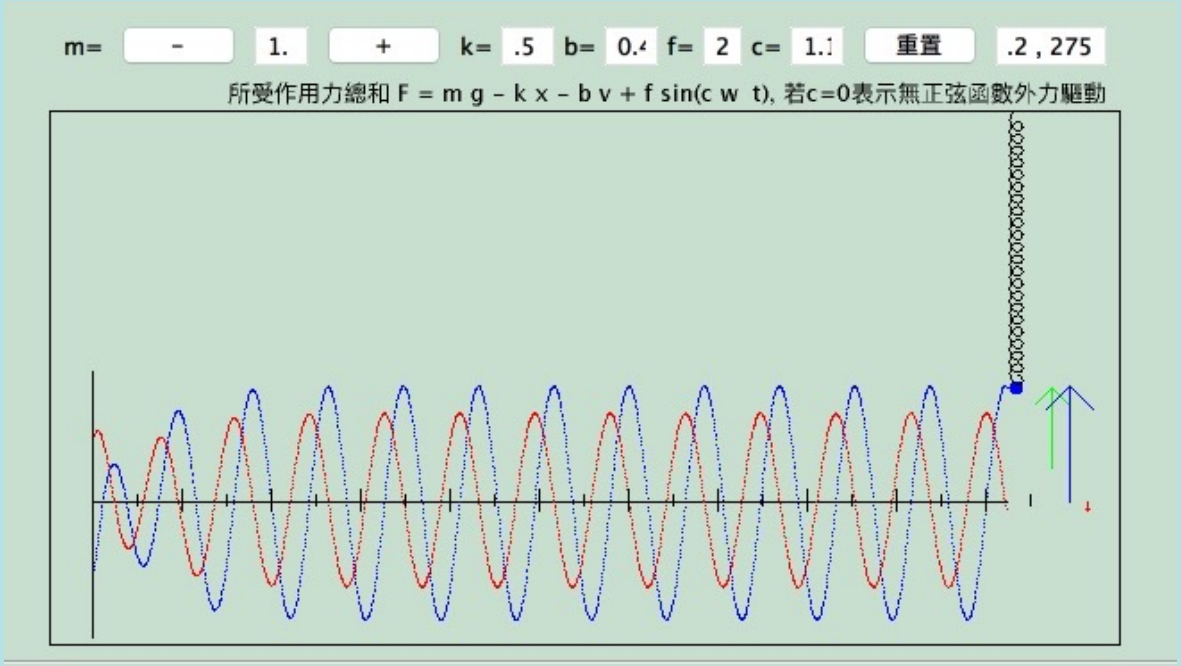
http://www.phy.ntnu.edu.tw/moodle/mod/resource/view.php?id=124

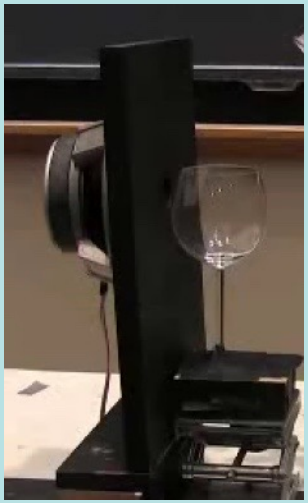


With a different initial condition:




m= - 1. + k= .5 b= 0.4 f= 2 c= 1.1 重置 .4 , 278
所受作用力總和 F = m g - k x - b v + f sin(c w t), 若c=0表示無正弦函數外力驅動


m= - 1. + k= .5 b= 0.4 f= 2 c= 1.1 重置 .2 , 275
所受作用力總和 F = m g - k x - b v + f sin(c w t), 若c=0表示無正弦函數外力驅動

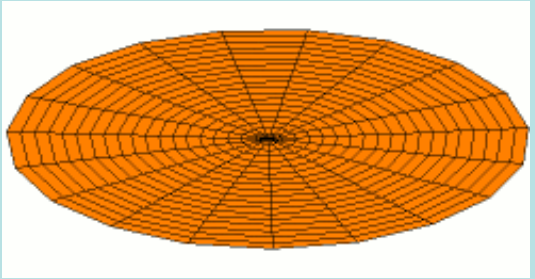


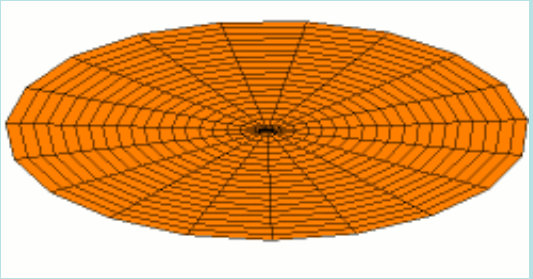
麥克風震碎玻璃杯的現象就是共振的實例。

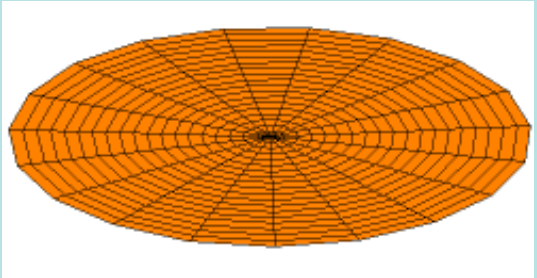
物體的變型模式有無限多個，每一個模式的振動頻率不同。

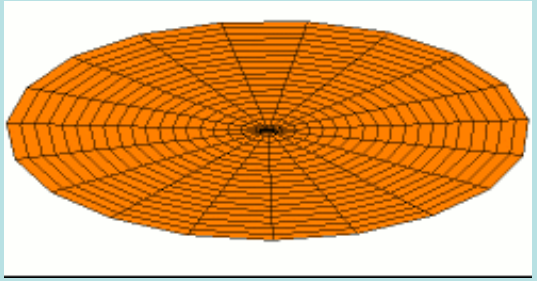
玻璃杯的變形等同於多個彈簧系統。

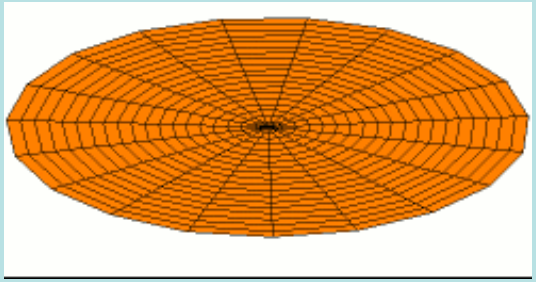
麥克風發出的聲波就是週期性外力。

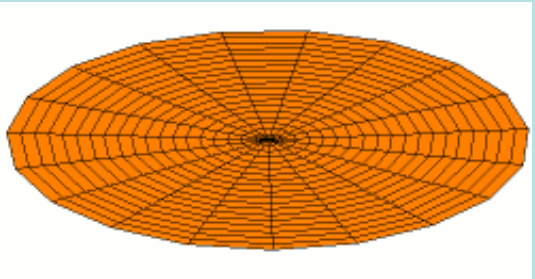






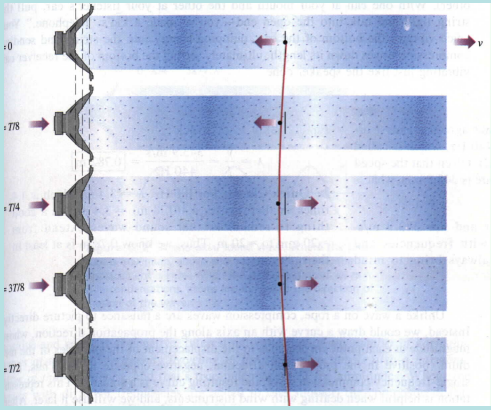


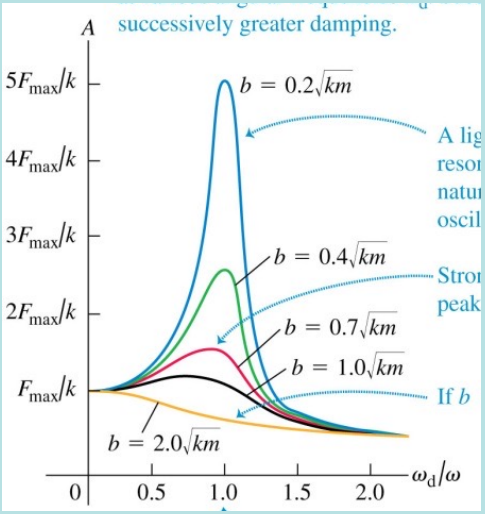




如果模式的頻率ω接近聲波頻率ω_D，週期性外力會帶動該模式開始振盪。

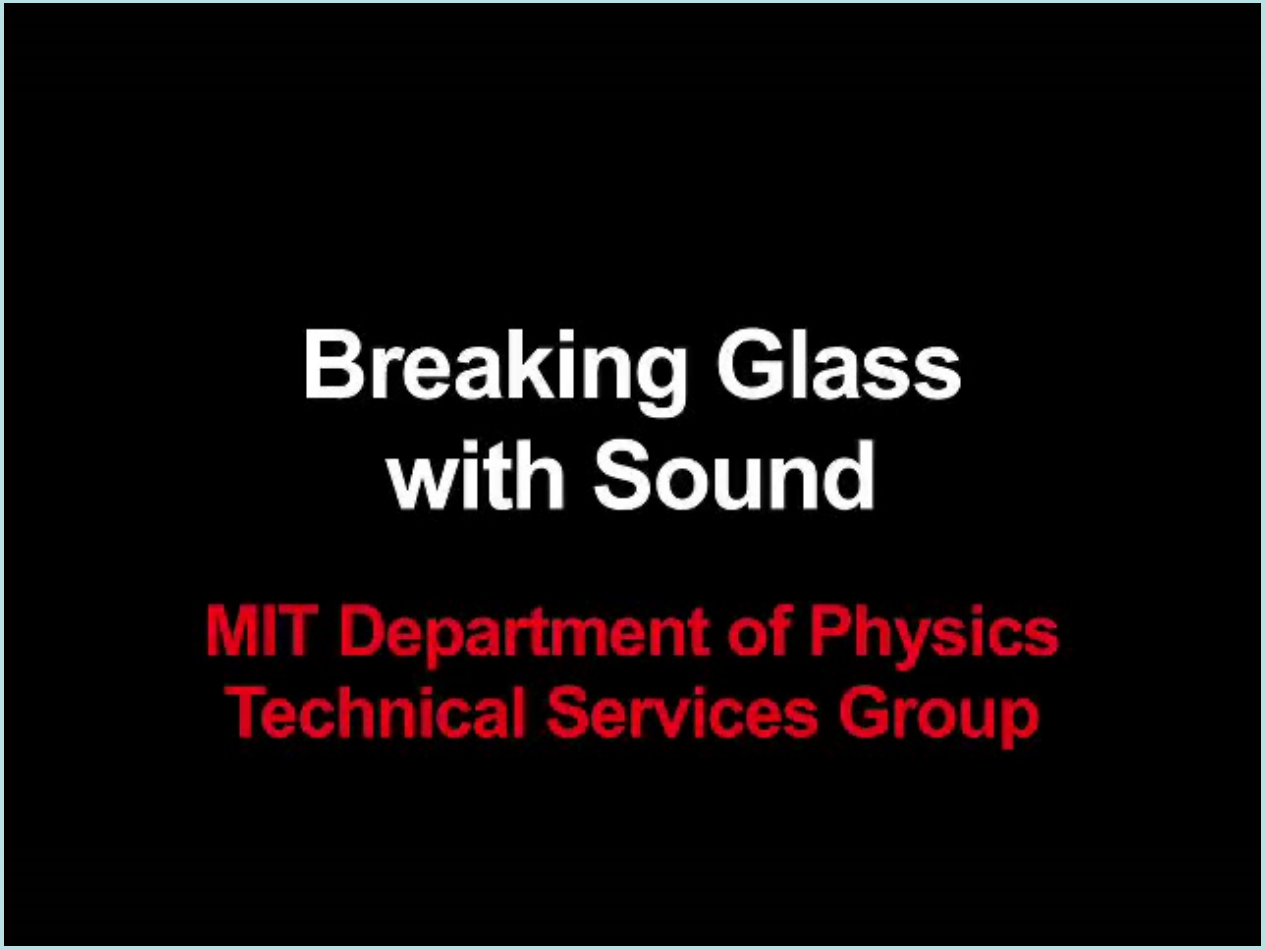
其他模式的振盪就很小。因此共振下的玻璃杯變型就如同一根彈簧！



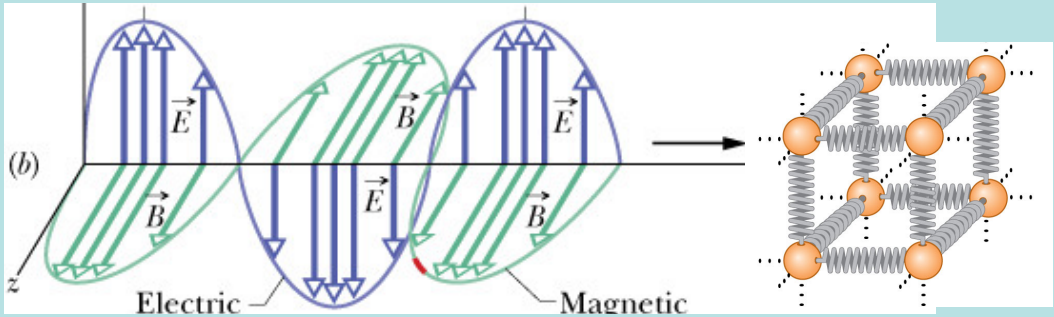


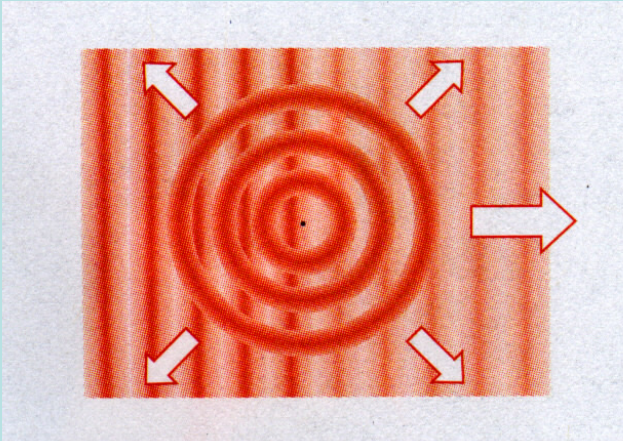
物體的振動

http://www.youtube.com/watch?v=17tqXgvCN0E&NR=1



電磁波打在一個晶體內的原子上的散射現象，也是如此。

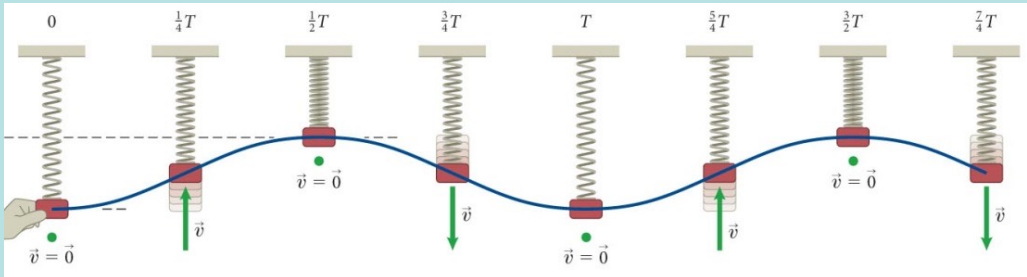




$$F = \frac{qE_0}{m}\cos\omega t$$

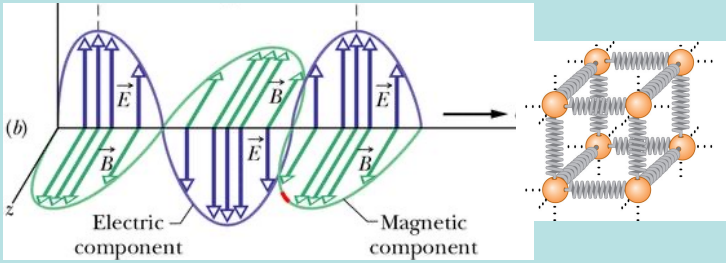
運動方程式

$$\frac{d^2y}{dt^2} = -\frac{b}{m}\frac{dy}{dt} - \omega^2 y + \frac{F_0}{m}\cos\omega_D t$$



電磁波頻率ω_D越接近分子間彈簧的振動頻率ω，彈簧振動越大，

越多電磁波能量被晶體內原子吸收！



將電磁波的吸收率對其頻率ω_D(或波長)作圖：

吸收光譜的共振頻率即是分子彈簧的自然內在頻率ω！

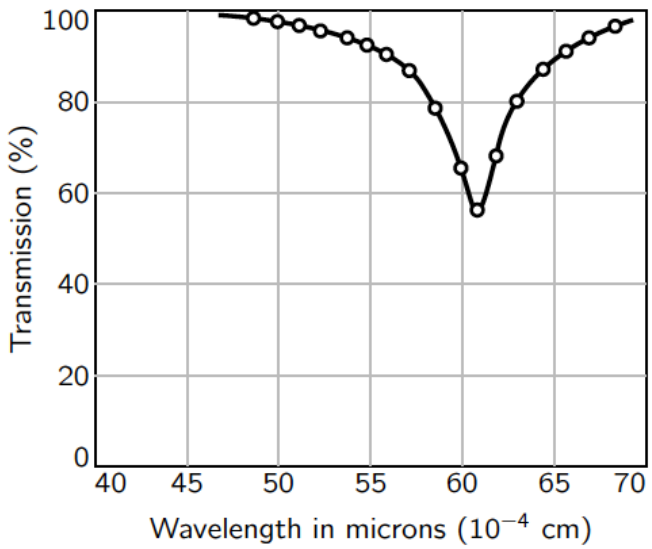


Fig. 23-7. Transmission of infrared radiation through a thin (0.17 μ) sodium chloride film. [After R. B. Barnes, *Z. Physik* **75**, 723 (1932). Kittel, *Introduction to Solid State Physics*, Wiley, 1956.]

建築必須避免其自然頻率ω，與環境外力頻率ω_D接近的狀況。

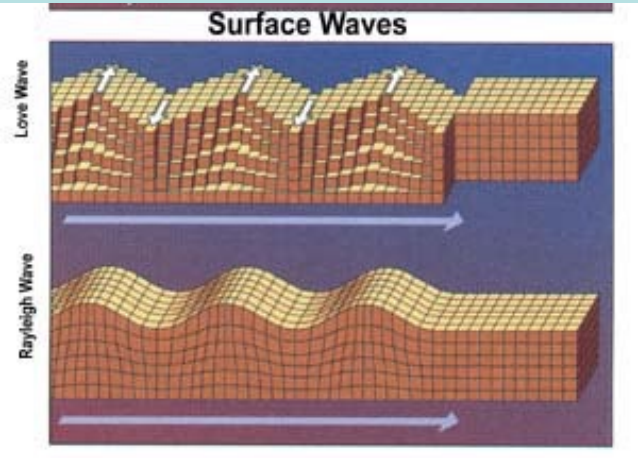
1985墨西哥市七級地震：

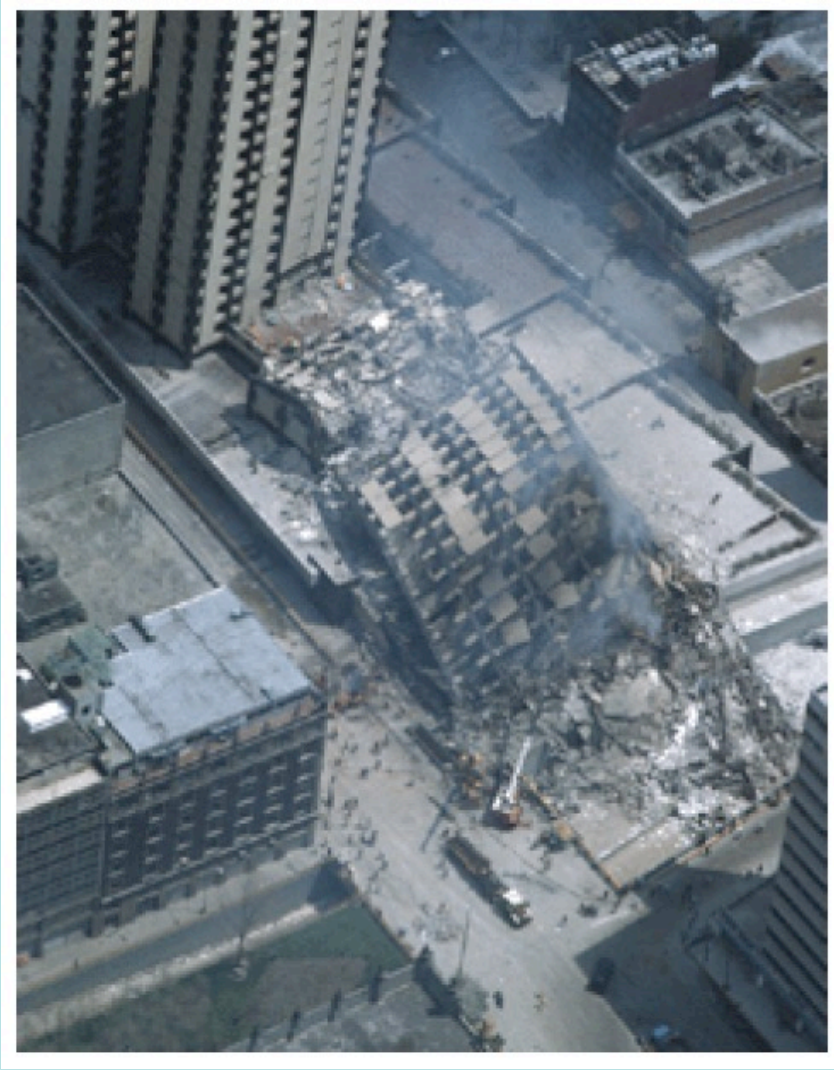
鬆軟土地面有一個自然的振盪頻率：

$f \sim 0.5$ Hz

這個頻率與淺層地震波正好接近。

共振現象使得墨西哥市地面振盪振幅大。





而建築物也有自己的自然頻率。

六至十樓中等高度建築正好有同樣自然頻率：

$f_{\text{中樓層}} \sim 0.5\ \text{Hz}$

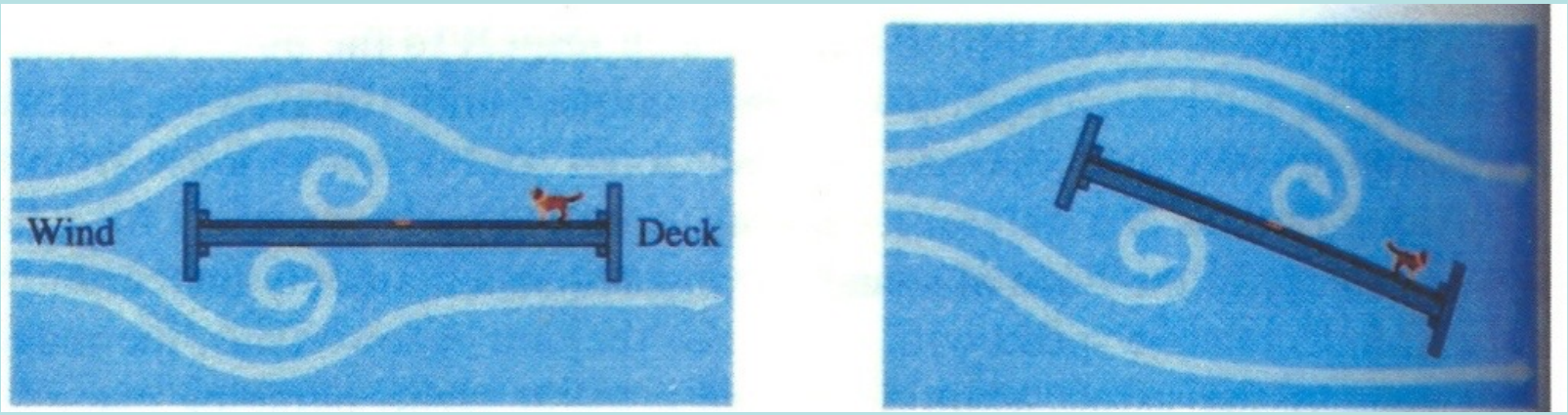
$$f \approx \sqrt{\frac{k}{m}}$$

$h \downarrow, k \uparrow, f \uparrow$

低樓層的老建築自然頻率較高。

在軟土地面上反而沒有共振。

Tacoma Narrows Bridge



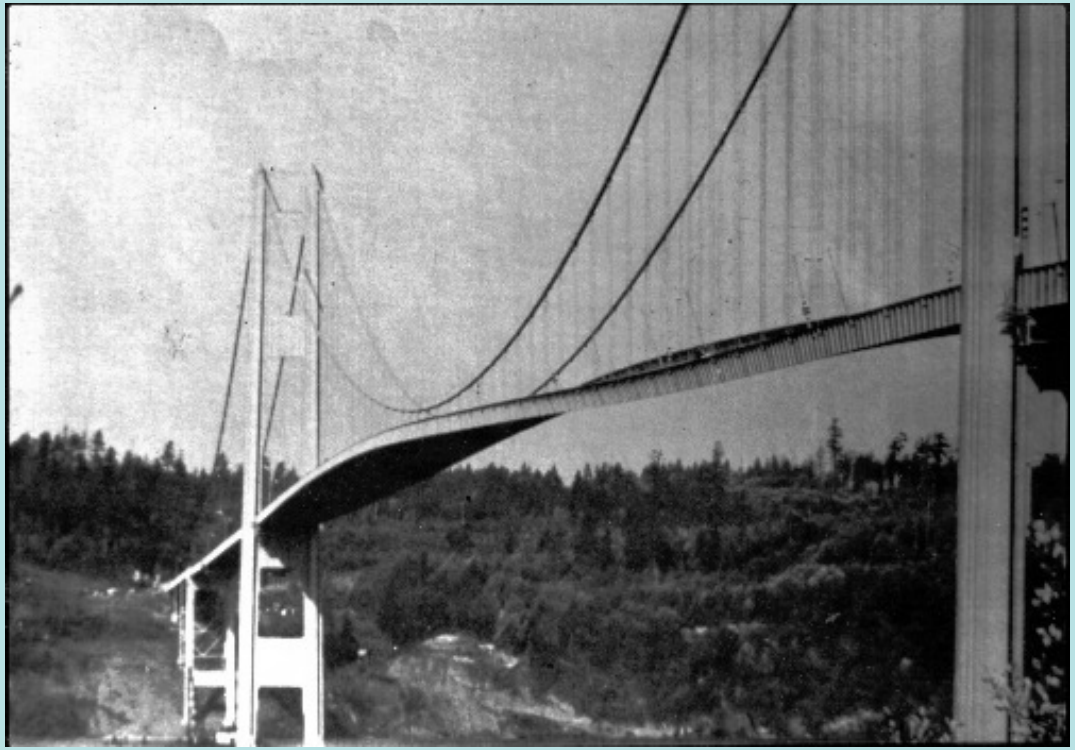
當橋未扭曲振動時，風力的力矩是平衡的，

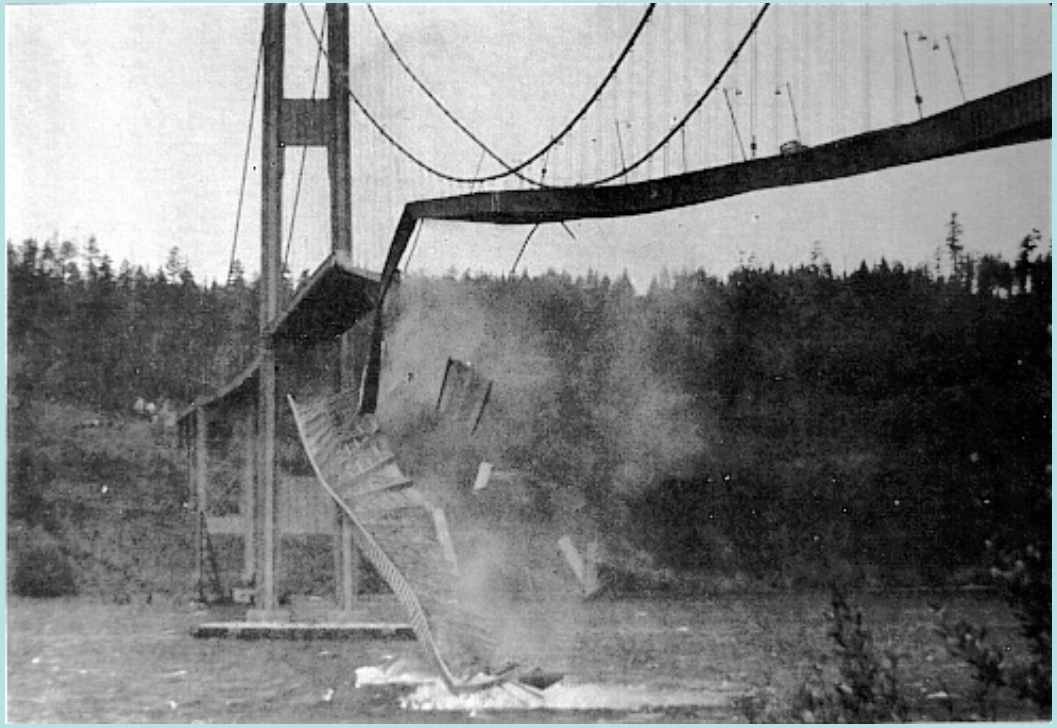
當橋開始些許某一模式的扭曲振動後，橋面傾斜，風力對橋面施予一淨力矩，

關鍵是：風力造成的力矩的頻率，自然與該扭曲振盪模式的頻率相等！

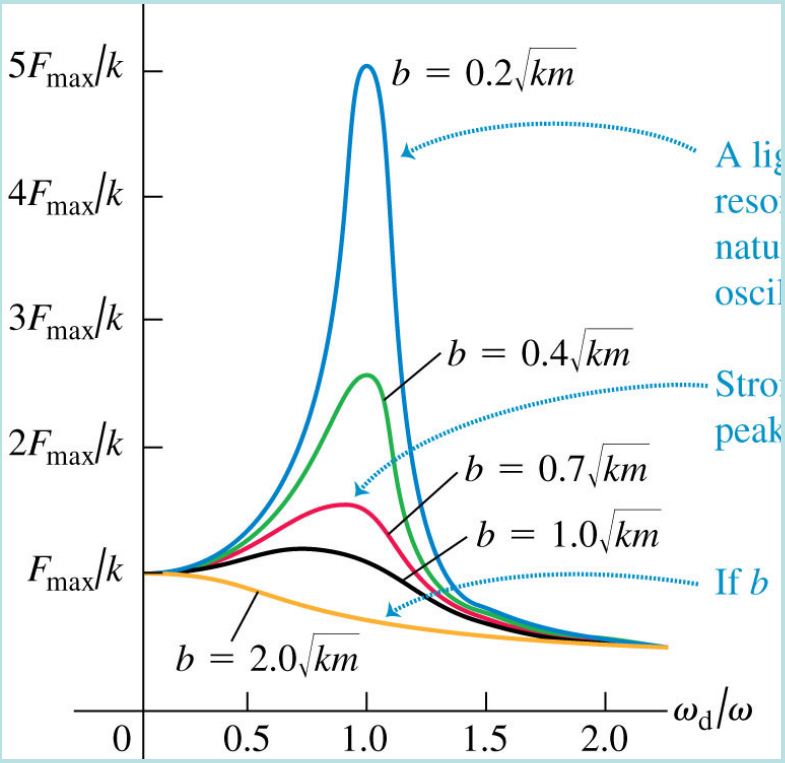
https://www.youtube.com/watch?v=lXyG68_caV4

http://www.youtube.com/watch?v=3mclp9QmCGs





要降低共振的大小，可以增加 阻尼damping b。



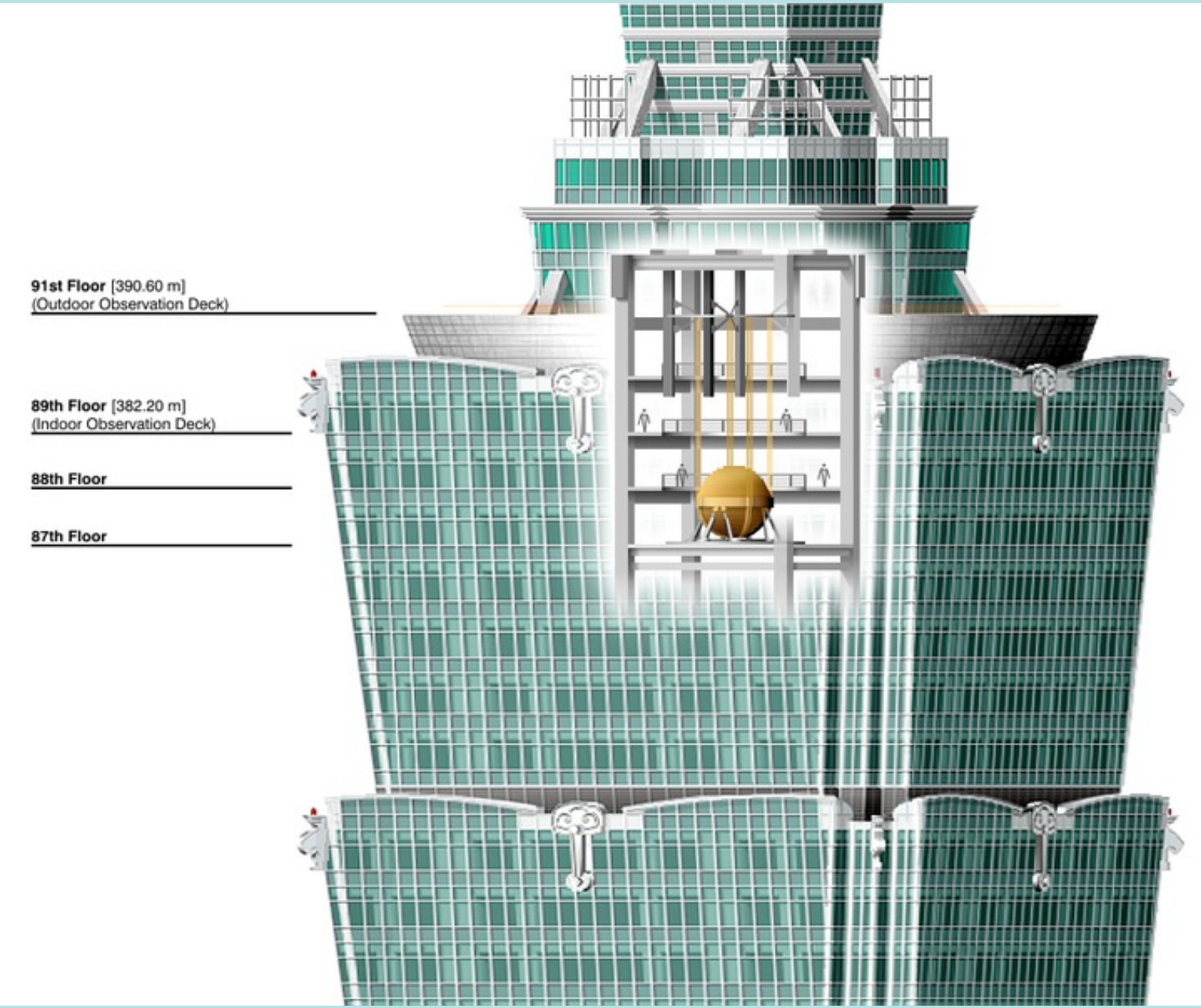
Fluid Damper

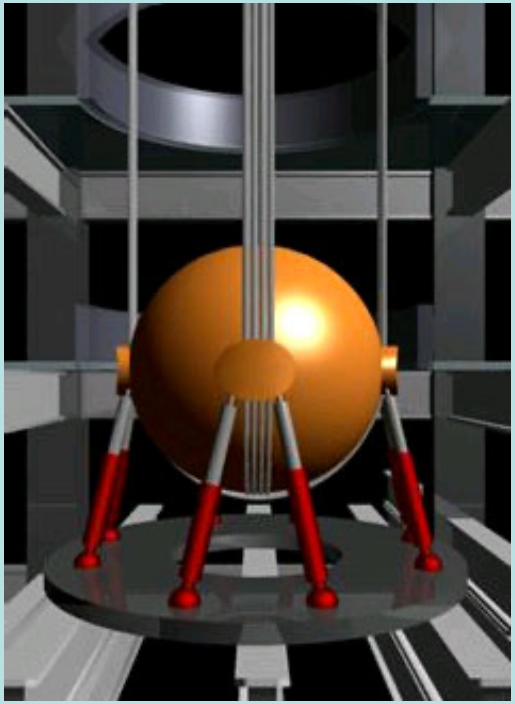
Tune Mass Damper

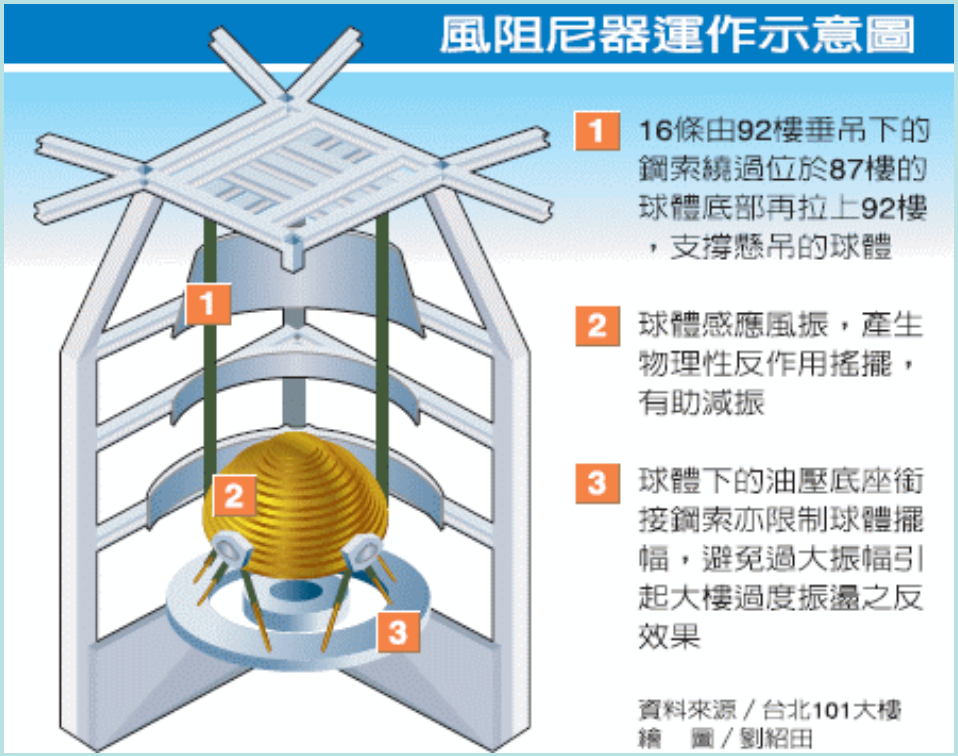
101五十噸的阻尼球

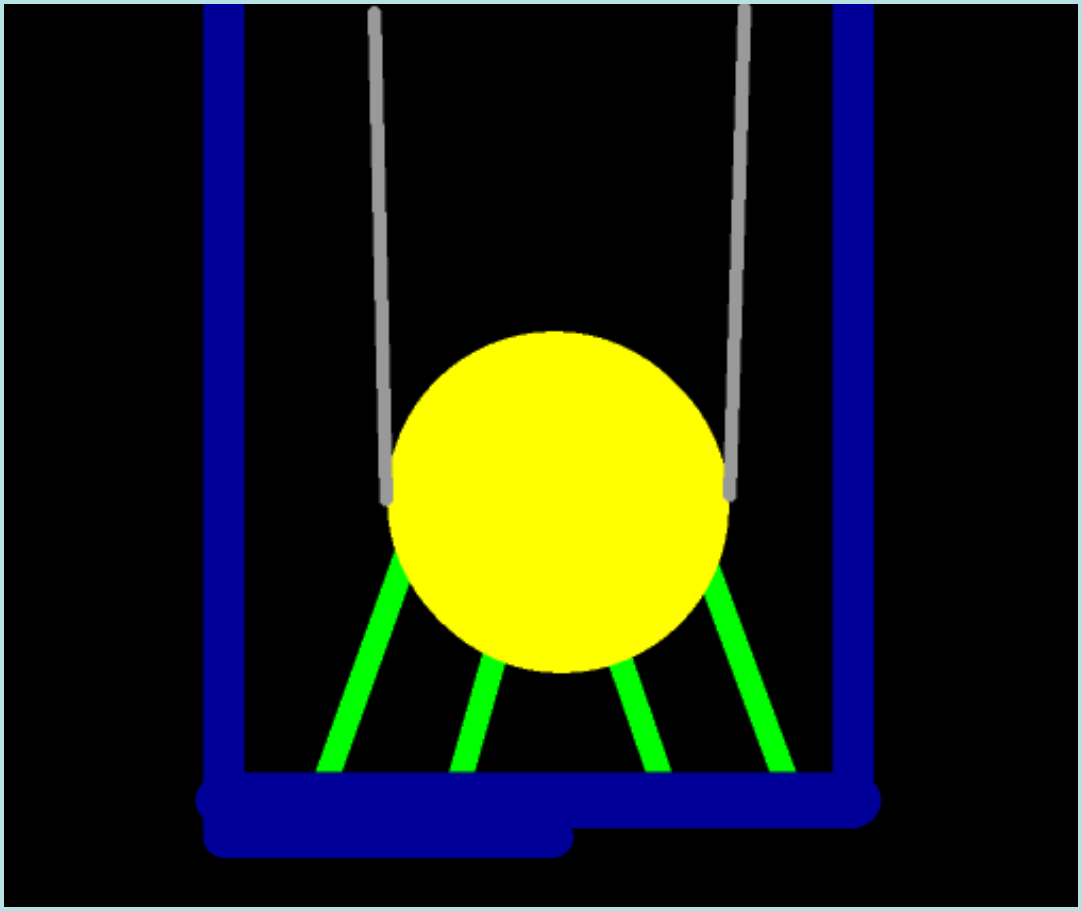
15 Oscillations

If a tall building sways slowly in a wind, the occupants may not even notice the motion, but if the swaying repeats more than 10 times per second, it becomes annoying and may even cause motion sickness. One reason is that when a person is standing, the head tends to sway even more than the feet, setting off motion sensors in the balancing region of the inner ear. Various mechanisms are employed to decrease a building's sway. For example, the large ball (5.4×10^5 kg) seen in this photograph hangs on the 92nd floor of one of the world's tallest buildings.


91st Floor [390.60 m]
(Outdoor Observation Deck)
89th Floor [382.20 m]
(Indoor Observation Deck)
88th Floor
87th Floor




風阻尼器運作示意圖
1
2
3
1 16條由92樓垂吊下的鋼索繞過位於87樓的球體底部再拉上92樓，支撐懸吊的球體
2 球體感應風振，產生物理性反作用搖擺，有助減振
3 球體下的油壓底座銜接鋼索亦限制球體擺幅，避免過大振幅引起大樓過度振盪之反效果
資料來源／台北101大樓
繪　圖／劉紹田



Linear second order ODE 解法

$$y'' + P(x)y' + Q(x)y = f(x)$$

Variation of parameters

級數法

複變函數法

Linear ODE

$$y'' + P(x)y' + Q(x)y = f(x)$$

If we find one solution of the homogeneous ODE, we can calculate the general solution.

Linear ODE with constant coefficients

$$y'' + a_1 y' + a_0 y = f(x)$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

Linear ODE

Variation of Parameters
Wronskian

If we find one solution of the homogeneous ODE, we can calculate the general solution.

$$y'' + P(x)y' + Q(x)y = 0$$

已經證明如果已找到一個函數$y_1(x)$ 滿足homogeneous方程式：

我們可以計算出另一個線性獨立的函數$y_2(x)$，也滿足此homogeneous方程式。

我們猜測，inhomogeneous 有一個解可以寫成： $y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$

這是模仿homogeneous方程式的一般解，將常數係數改寫成函數：

$$C_1y_1(x) + C_2y_2(x)$$

Variation of Parameters

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

將它代入要滿足的方程式：

$$y'' + P(x)y' + Q(x)y = f(x)$$

將$y(x)$微分：

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

選擇$v_{1,2}$滿足以下條件：

$$v_1'y_1 + v_2'y_2 = 0$$

$y'(x)$可簡化：

$$y' = v_1y_1' + v_2y_2'$$

將$y'(x)$微分：

$$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$$

代入要滿足的方程式： $y'' + P(x)y' + Q(x)y = f(x)$

$$v_1'y_1' + v_2'y_2' + v_1[y_1'' + Py_1' + Qy_1] + v_2[y_2'' + Py_2' + Qy_2] = f(x)$$

$y_{1,2}$滿足homogeneous方程式。

$$v_1'y_1' + v_2'y_2' = f(x)$$

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = f(x)$$

於是我們有兩個條件，兩個未知函數$v_{1,2}'$。正好都能解出來。

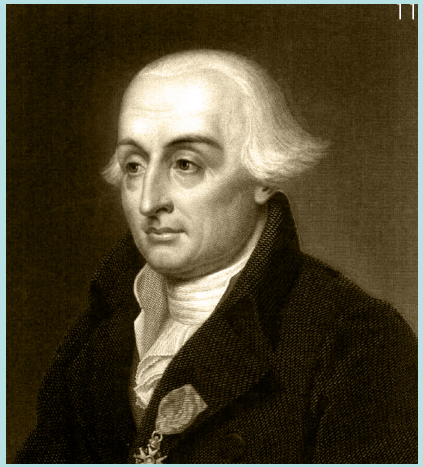
解的時候用的行列式，就是找y_2時用的Wronskian $W(x)$。

$$v_1' = -\frac{y_2 f}{W} \qquad v_2' = \frac{y_1 f}{W} \qquad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

簡單積分：

$$v_1 = -\int dx \frac{y_2 f}{W} \qquad v_2 = \int dx \frac{y_1 f}{W}$$

If we find one solution of the homogeneous ODE, we can calculate the general solution of the inhomogeneous ODE.



Joseph-Louis Lagrange 1736-1813

以複數解簡諧運動的運動方程式

Ordinary Differential Equation with constant coefficients

$$\sum_n a_n \cdot \frac{d^n x}{dt^n} = f(t)$$

Arfken 7.3

齊次微分方程式，原子核衰變、簡諧運動與阻尼振盪都是特例！

$$\sum_n a_n \cdot \frac{d^n x}{dt^n} = 0$$

$$\frac{d^2 x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

這個方程式的解不會是多項式，因為微分會降一個冪次。

指數函數的所有微分都與自己成正比，但比例常數是恆正的。

三角函數的偶次微分與自己成正比，且比例常數為負。

而阻尼振盪的解是三角函數與指數函數的乘積。

有可能將三角函數與指數函數統合在一起嗎？

指數函數與三角函數有許多類似之處：

$$\frac{de^{b\theta}}{d\theta} = be^{b\theta}$$

$$\frac{d^2e^{b\theta}}{d\theta^2} = b^2e^{b\theta}$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta$$

$$\frac{d^2\sin\theta}{d\theta^2} = \frac{d(\cos\theta)}{d\theta} = -\sin\theta$$

如果只看二次微分，大膽假設： $b^2 = -1, b = \sqrt{-1} = i$

指數函數與正弦函數似乎可以被統合成一種函數。

$$\sin\theta \sim e^{i\theta}$$

但此虛數的指數函數的定義，對一次微分不成立： $\sin\theta$的微分不是$\sin\theta$。

而且：

$$\frac{de^{i\theta}}{d\theta} = ie^{i\theta}$$

因此$e^{i\theta}$必須是複數，有實數部及虛數部！

$e^{i\theta}$ 的一次微分是自己乘上 i ，也就是將實數部及虛數部互換

三角函數的一次微分將 cos 與 sin 互換

$$\frac{d(\cos\theta)}{dt} = -\sin\theta$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta$$

何不就假設 $e^{i\theta}$ 的實數部與虛數部分別是正弦與餘弦？

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

Euler's Formula　　虛數的指數函數

Carl Friedrich Gauss (1777–1855)

$$\frac{d}{d\theta}e^{i\theta} = -\sin\theta + i\cos\theta = ie^{i\theta}$$

$$\frac{d^2}{d\theta^2}e^{i\theta} = -\cos\theta - i\sin\theta = -e^{i\theta} = i^2e^{i\theta}$$

正好是我們期待指數函數必須滿足的微分關係。

$$\frac{d^n}{dx^n}e^{i\alpha x} = (i\alpha)^n \cdot e^{i\alpha x}$$

$$e^{i\alpha} \cdot e^{i\beta} = (\cos\alpha + i\sin\alpha) \cdot (\cos\beta + i\sin\beta) =$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) =$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$$

正好是我們期待指數函數必須滿足的乘積關係。

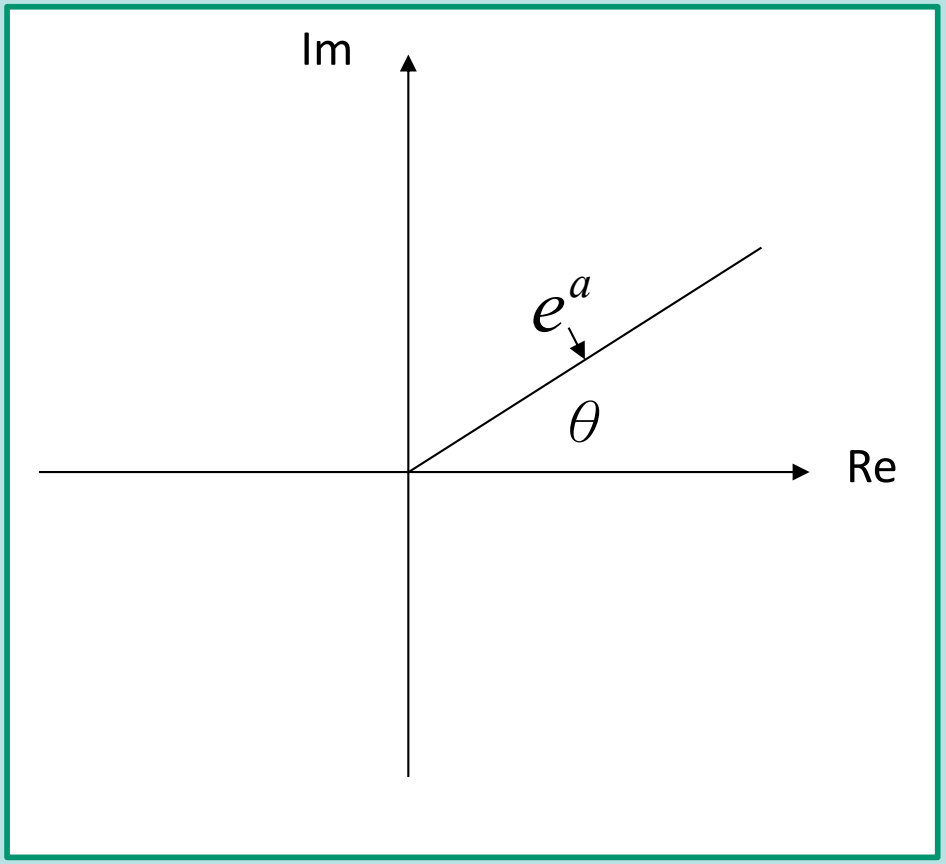
此定義滿足指數函數所有重要性質！

我們可以更進一步定義複數$\alpha = a + i\theta$的指數函數：

$$e^{\alpha} = e^{a+i\theta} = e^{a}e^{i\theta} = e^{a}(\cos\theta + i\sin\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

在複數平面上表示，a 決定絕對值，θ 決定幅角



$$\left|e^{i\theta}\right| = 1$$

$$\left|e^{a+i\theta}\right| = e^{a}$$

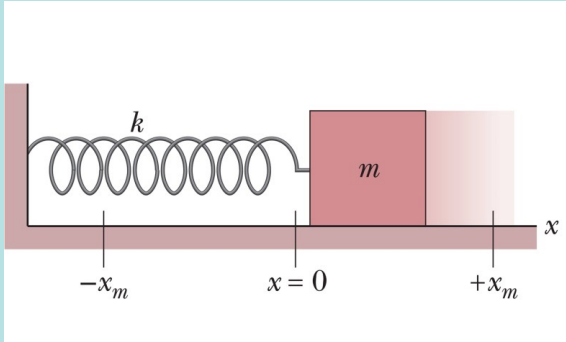
$$|mn| = |m| \cdot |n|$$

$$\frac{d^n}{dx^n}e^{\alpha x} = (\alpha)^n \cdot e^{\alpha x}$$

所以，複數的指數函數，所有的微分都與自己成正比！

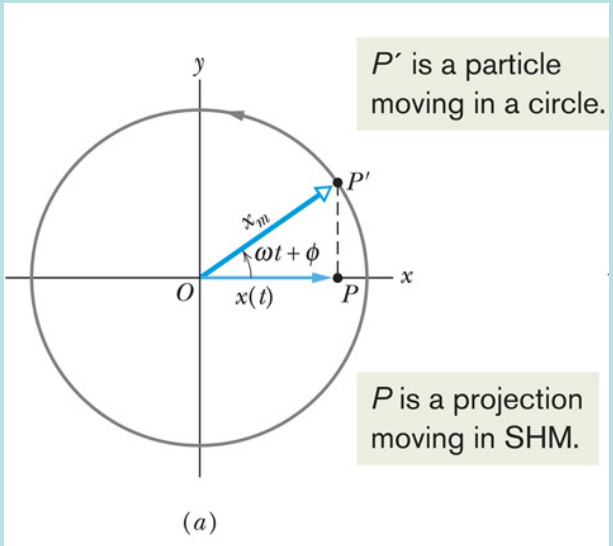
$$e^{i\theta} = \cos\theta + i\sin\theta$$

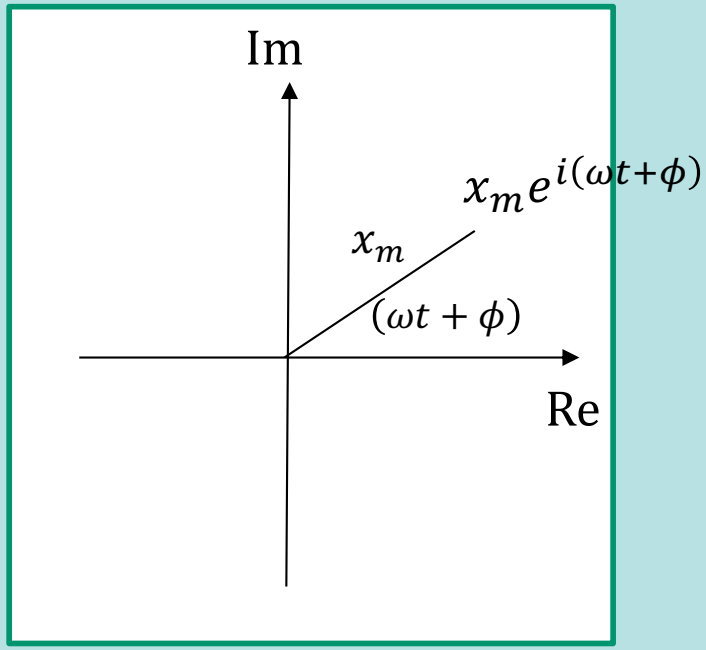
如此，三角函數可以寫成虛數指數函數的實數部。



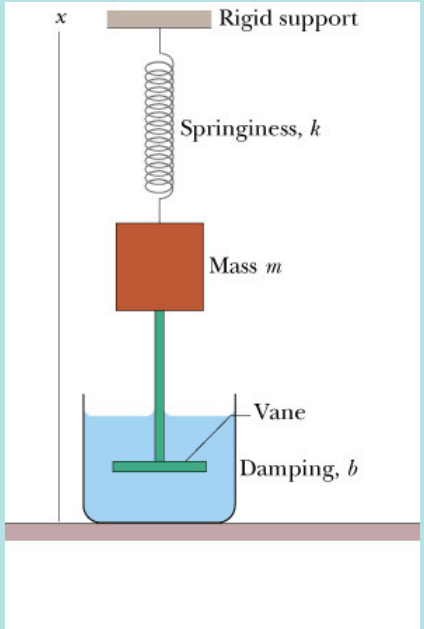
$$x_m \cos(\omega t + \phi) = \mathrm{Re}\left[x_m e^{i(\omega t+\phi)}\right]$$

如此簡諧運動，就是虛數指數函數的實數部！





假想圓就是$x_m e^{i(\omega t+\phi)}$在其複數平面上的表現！



阻尼簡諧運動，也是複數指數函數的實數部！

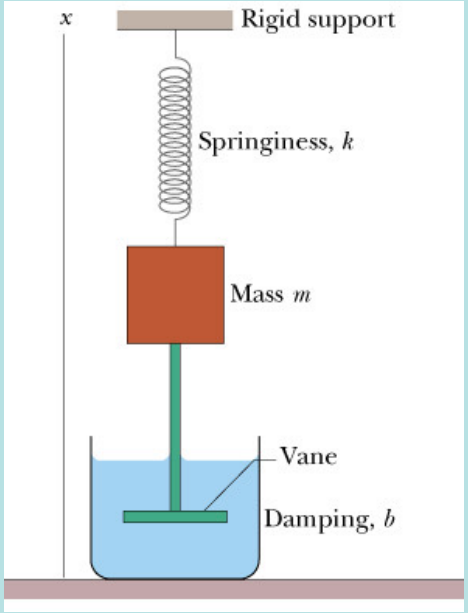
$$x_m \cdot e^{-\frac{b}{2m}t} \cdot \cos(\omega' t + \phi) = x_m e^{-\frac{b}{2m}t} \cdot \mathrm{Re}\left[e^{i(\omega' t+\phi)}\right] = \mathrm{Re}\left[x_m e^{-\frac{b}{2m}t+i(\omega t+\phi)}\right]$$

是不是所有線性微分方程式的解都是指數函數？

有阻力的簡諧振盪器，以複數方法求解：

這方法對一般簡諧振盪器自然也成立：

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=0$$



首先將這個式子裡的 x 推廣為一個複數 z。

$$\frac{d^2z}{dt^2}+\frac{b}{m}\frac{dz}{dt}+\omega^2z=0$$

注意複數 z 包含實數部Re z與虛數部Im z。

$$\frac{d^2(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})}{dt^2}+\frac{b}{m}\frac{d(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})}{dt}+\omega^2(\mathrm{Re}\,z+i\mathrm{Im}\,\mathrm{z})=0+i0$$

那麼 z 的實數部與虛數部都同時滿足原來x滿足的方程式！

因此如果能解出複數 z，再取其實數部，即可得到原方程式的實數解 x。

$$\frac{d^2(\mathrm{Re}\,z)}{dt^2}+\frac{b}{m}\frac{d(\mathrm{Re}\,z)}{dt}+\omega^2(\mathrm{Re}\,z)=0$$

看起來似乎比較複雜的複數微分方程，原來反而比較容易解！

複數指數函數所有的微分都正比於此指數函數。

$$\frac{d^2z}{dt^2} + \frac{b}{m}\frac{dz}{dt} + \omega^2 z = 0$$

$$z = z_0 e^{\alpha t}$$

$$\frac{d^n}{dt^n} z = \alpha^n z$$

如果z是一個複數指數函數，上式所有項都正比於z。

我們可以大膽地猜，其解正比於一個複數指數函數：

$$\alpha^2 z + \frac{b}{m}\alpha z + \omega^2 z = 0$$

$$\alpha^2 + \frac{b}{m}\alpha + \omega^2 = 0$$

This is called Characteristic Equation.

原來的微分方程式現在一項對一項地轉化為代數方程式。

對時間的微分對應α 的冪次。

α 即可被解出：

$$\alpha_{\pm} = -\frac{b}{2m} \pm i\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$$

These are called Eigenvalues of the SHO.

若阻力不大 $\omega > \frac{b}{2m}$ 我們得到兩個複數解 $\omega' \equiv \sqrt{\omega^2 - \frac{b^2}{4m^2}}$

$$z_\pm \equiv e^{\alpha_\pm t} = e^{-\frac{b}{2m}t} \cdot e^{\pm i\omega' t} = e^{-\frac{b}{2m}t} \cdot [\cos\omega' t \pm i \sin\omega' t]$$

因為微分方程式是齊次，我們可以將這兩個複數解各乘上一個複數常數再相加（線性組合），所得到的依舊會是一個解：

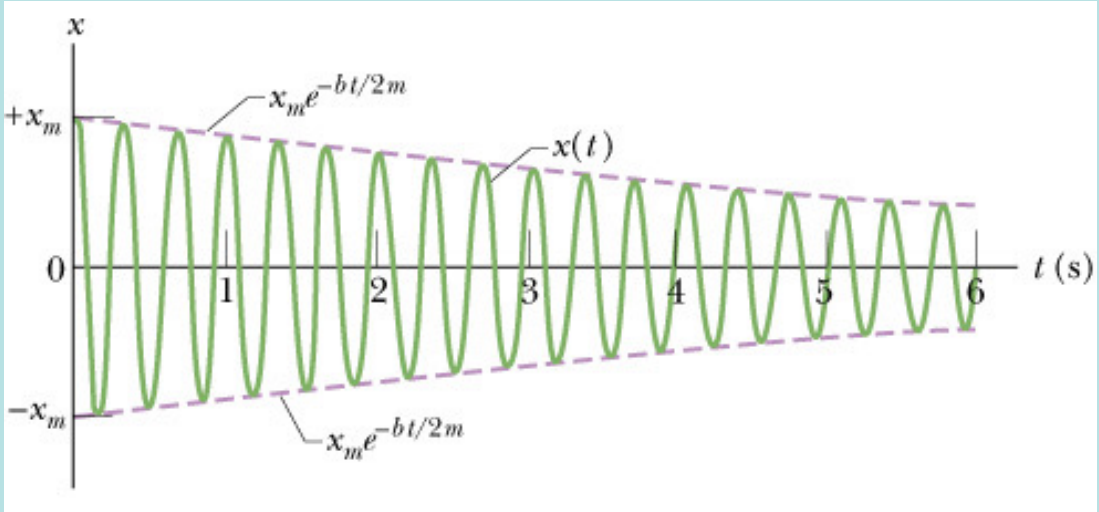
$$z = c_+ z_+ + c_- z_-$$
$$= c_+ e^{-\frac{b}{2m}t} \cdot [\cos\omega' t + i \sin\omega' t] + c_- e^{-\frac{b}{2m}t} \cdot [\cos\omega' t - i \sin\omega' t]$$

實數部就是我們原來尋找的實數解

$$x = \mathrm{Re}\, z = \mathrm{Re}\,(c_+ z_+ + c_- z_-)$$
$$= (\mathrm{Re}\, c_+ + \mathrm{Re}\, c_-) \cdot e^{-\frac{b}{2m}t} [\cos\omega' t] + (-\mathrm{Im}\, c_+ + \mathrm{Im}\, c_-) \cdot e^{-\frac{b}{2m}t} [\sin\omega' t]$$
$$= e^{-\frac{b}{2m}t} \cdot (a \cos\omega' t + b \sin\omega' t) = x_m \cdot e^{-\frac{b}{2m}t} [\cos(\omega' t + \phi)]$$

$$x = x_m \cdot e^{-\frac{b}{2m}t}\,[\cos(\omega' t + \phi)]$$

$$\omega' = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$



振幅呈現指數衰減！

振動頻率減小！

若阻力很大 $\omega < \frac{b}{2m}$ α得到兩個小於零的實數解

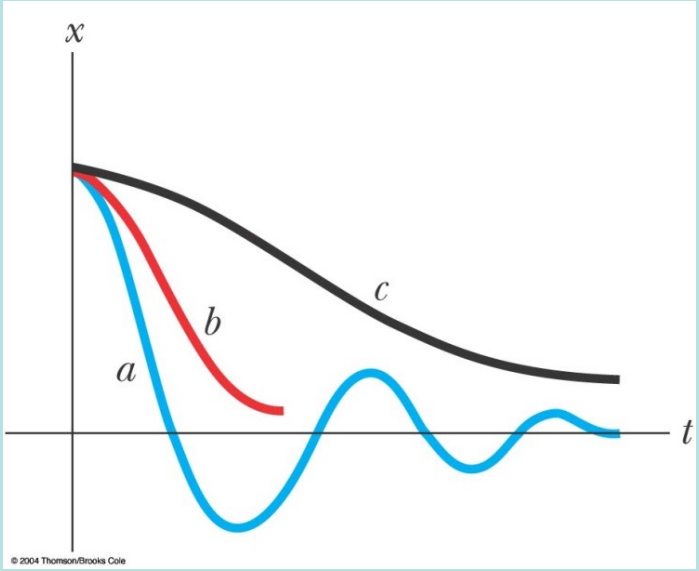
$$\alpha_{\pm} = -\frac{b}{2m} \pm i\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} \Rightarrow \alpha_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}$$

兩個解都是隨時間指數遞減，而沒有震盪！

$$z_{\pm} \equiv e^{\alpha_{\pm} t} = e^{-\left[\frac{b}{2m} \mp \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}\right]t}$$

恆正

解都是指數遞減函數！



如果 $\frac{b}{2m}$大於ω，那就根本沒有振動了！

阻尼可以大到連一次震盪都未完成！

以上的解法很容易地就可以推廣到任意的齊次微分方程式

$$\sum_{n=0}^{N} a_n \cdot \frac{d^n x}{dt^n} = 0$$

先推廣為複數

$$\sum_{n=0}^{N} a_n \cdot \frac{d^n z}{dt^n} = 0$$

猜解並代入

$$z = z_0 e^{\alpha t}$$

將微分方程式轉化為代數方程式

$$\sum_{n=0}^{N} a_n \cdot \alpha^n z = 0 \quad \Rightarrow \quad \sum_{n=0}^{N} a_n \cdot \alpha^n = 0$$

代數方程式中的未知數α有n個解

$$z = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} + \cdots c_N e^{\alpha_N t}$$

最後取實數部即可得實數解 $x = \mathrm{Re}\, z$

有N個解就有N個未知數，因此就需要N個起始條件，

一般就是起始值及起始N次以下微分。

周期外力下，有阻力的簡諧振盪器，以複數方法求解。

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=\frac{F_0}{m}\cos\omega_D t$$

非齊次微分方程式

將它先推廣為複數的微分方程式，注意右手邊的技巧：

$$\frac{d^2z}{dt^2}+\frac{b}{m}\frac{dz}{dt}+\omega^2z=\frac{F_0}{m}e^{i\omega_D t}$$

$$e^{i\omega_D t}=\cos\omega_D t+i\sin\omega_D t$$

取此複數的微分方程式的實數部，果然$\mathrm{Re}(z)$滿足原來的方程式：

$$\frac{d^2\mathrm{Re}(z)}{dt^2}+\frac{b}{m}\frac{d\mathrm{Re}(z)}{dt}+\omega^2\mathrm{Re}(z)=\frac{F_0}{m}\mathrm{Re}\left(e^{i\omega_D t}\right)=\frac{F_0}{m}\cos\omega_D t$$

$$x=\mathrm{Re}(z)$$

$$\frac{d^2z}{dt^2}+\frac{b}{m}\frac{dz}{dt}+\omega^2 z=\frac{F_0}{m}e^{i\omega_D t}$$

我們可以依舊猜解為 $z=z_0e^{\alpha t}$ 代入上式

$$\left(\alpha^2+\frac{b}{m}\alpha+\omega^2\right)z_0e^{\alpha t}=\frac{F_0}{m}e^{i\omega_D t}$$

未知數α只有一種可能 $\alpha=i\omega_D$ 此解與外力以同樣頻率震盪！

z_0也只有一種可能 $\left(-\omega_D^2+\frac{ib}{m}\omega_D+\omega^2\right)z_0e^{-i\omega_D t}=\frac{F_0}{m}e^{-i\omega_D t}$

$$z_0=\frac{F_0}{m}\frac{1}{\left(\omega^2-\omega_D^2+\frac{ib}{m}\omega_D\right)}$$

以絕對值及幅角表示z_0最為方便:

$$z_0 = \frac{F_0}{m}\frac{1}{\left(\omega^2 - \omega_D^2 + \frac{ib}{m}\omega_D\right)} \equiv Ae^{i\phi}$$

$$A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

$$\tan\phi = -\frac{\frac{b\omega_D}{m}}{(\omega^2 - \omega_D^2)}$$

$$z = z_0 e^{-i\omega_D t} = Ae^{-i(\omega_D t + \phi)}$$

其實數部即原方程式實數解

$$\mathrm{Re}\, z = \mathrm{Re}\, Ae^{-i(\omega_D t + \phi)} = A\cos(\omega_D t + \phi) \equiv x_r$$

這個就稱為共振解，並沒有任何自由度可以附合起始條件。

對於非齊次方程式，當我們猜到一個解之後，可以在其上加上對應的齊次方程式的解，而仍然滿足一樣的非齊次方程式

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=\frac{F_0}{m}\cos\omega_D t \quad \Rightarrow \quad \frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega^2x=0$$

$$x_s=x_m\cdot e^{-\frac{b}{2m}t}\,[\cos(\omega' t+\phi)]$$

$$x=x_r+x_s$$

$$x_r=A\cos(\omega_D t+\phi)$$

$$x_s=x_m\cdot e^{-\frac{b}{2m}t}\,[\cos(\omega' t+\phi)]$$

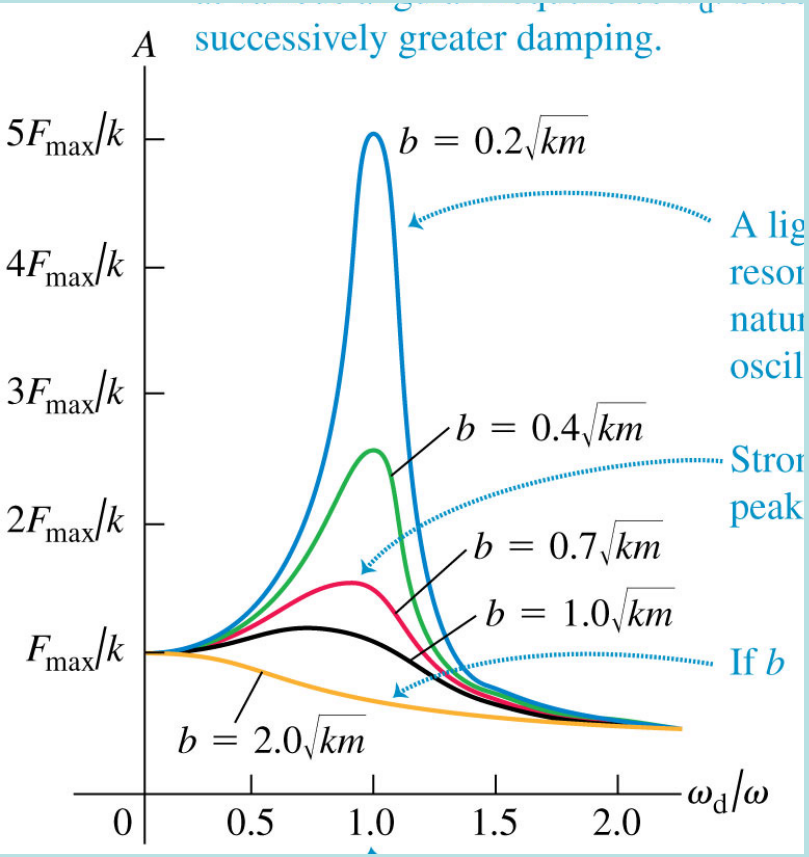
有兩個未知常數可以附合起始條件

這個解滿足微分方程式，而且可以滿足起始條件，
根據微分方程式基本定理，它必然是唯一的解。

非共振解會逐漸減小，而共振解則不會，因此過了一段時間，非共振解即可忽略，起始條件的影響會消失，只有共振解的效應必須考慮。

$$x=x_r+x_s\rightarrow x_r$$

$$x\rightarrow A\cos(\omega_D t+\phi)$$



$$x_r = A\cos(\omega_D t + \phi)$$

$$A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega^2 - \omega_D^2)^2 + \left(\frac{b\omega_D}{m}\right)^2}}$$

震盪的振幅與外力的振幅成正比

以外力的頻率來震盪，而不是彈簧的自然頻率！

外力頻率越接近彈簧的自然頻率，振動也就越大！ $\omega_D \to \omega$ $A \uparrow$

共振曲線的寬度與阻力大小成正比

傅立葉分析Fourier Series

任一週期為T的週期函數可以分解成一系列簡諧運動函數： $A_n \cdot \cos(\omega_n t + \phi_n)$

$$x(t) = \sum_{n=1}^{\infty} [A_n \cdot \cos(\omega_n t + \phi_n)]$$

A_n即是各個成分的強度

$$\omega_n = n\omega = n\omega f \qquad f_n = nf = \frac{n}{T} \qquad T_n = \frac{T}{n}$$

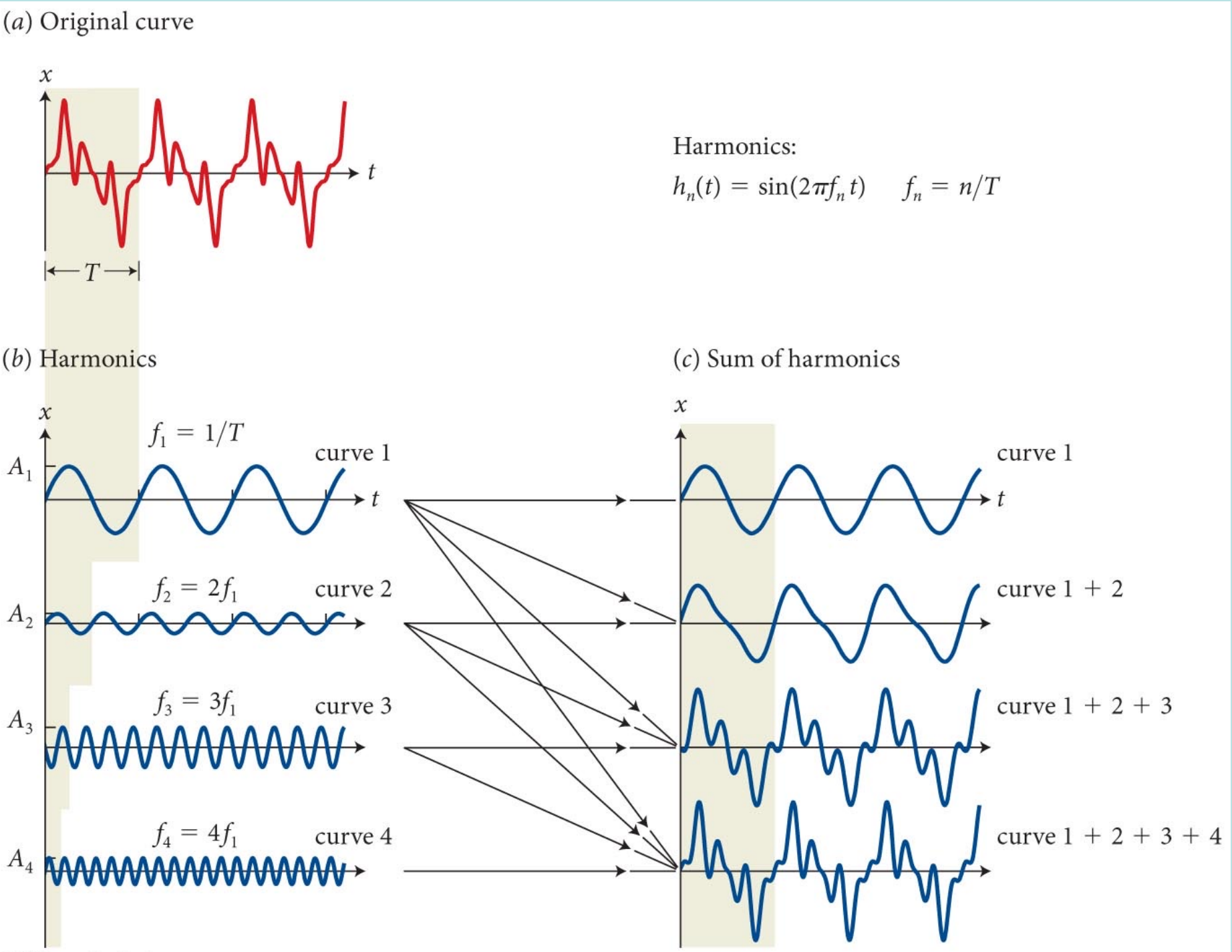
各成分的頻率f_n是函數頻率f的整數倍！

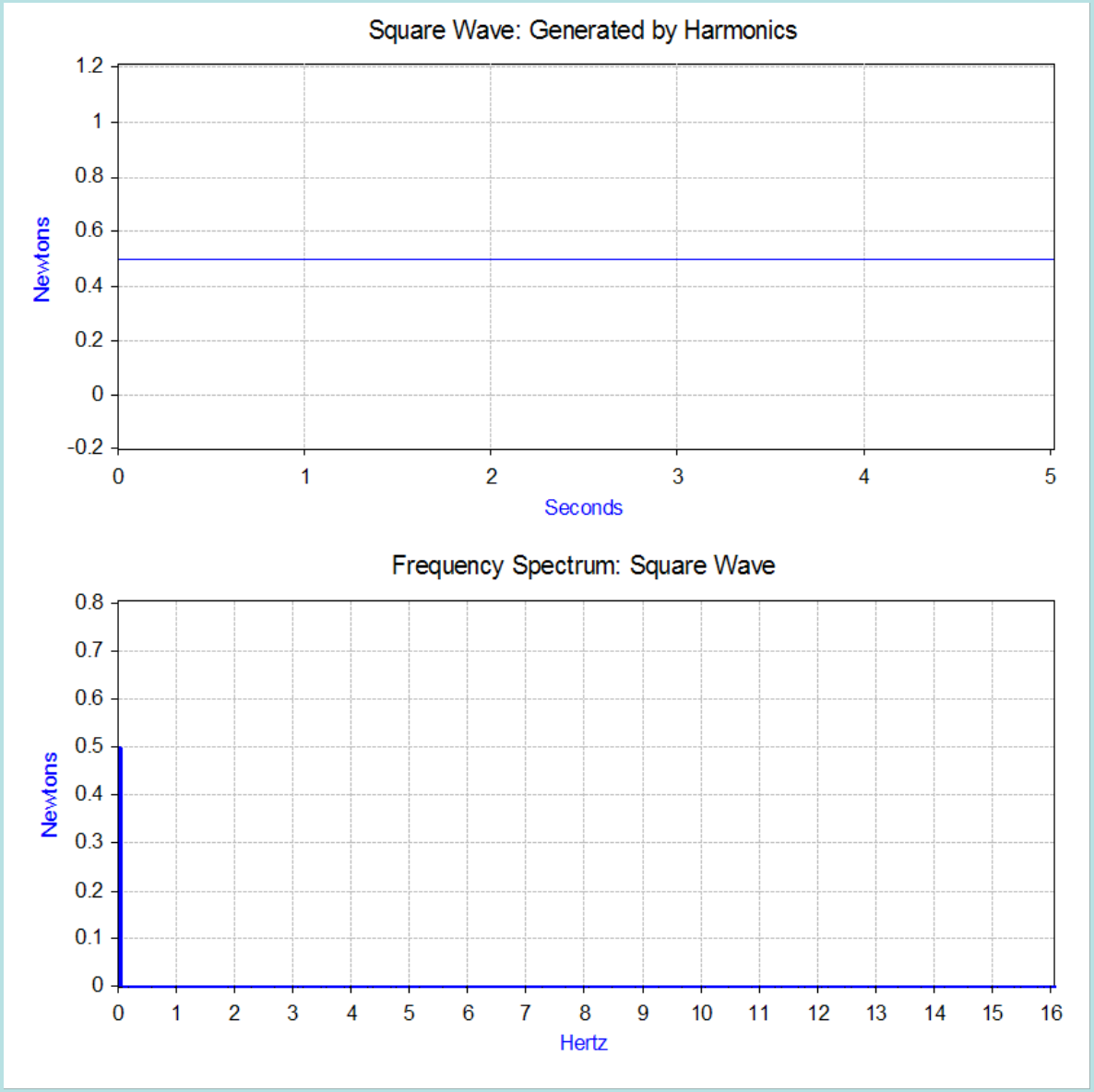
各成分的週期$T_n = \frac{T}{n}$是函數週期的整數分之一！

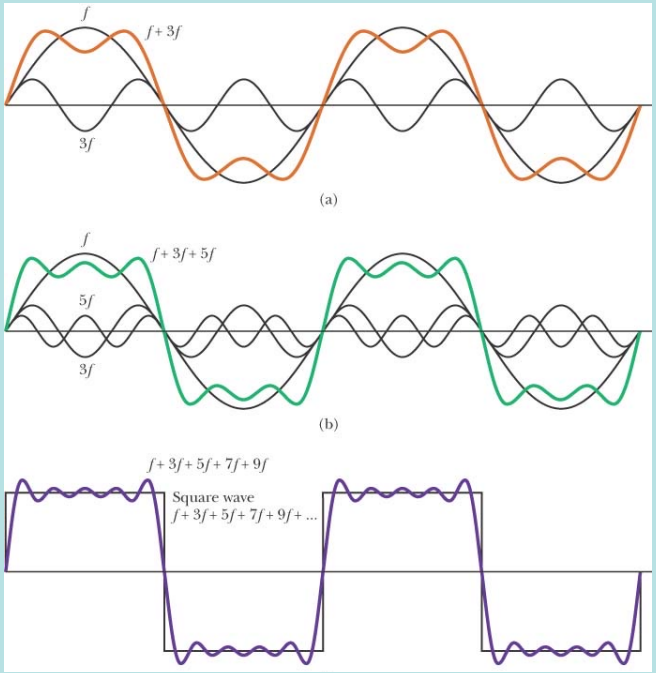
研究簡諧運動就等於研究所有週期函數。

Jean-Baptiste Joseph Fourier

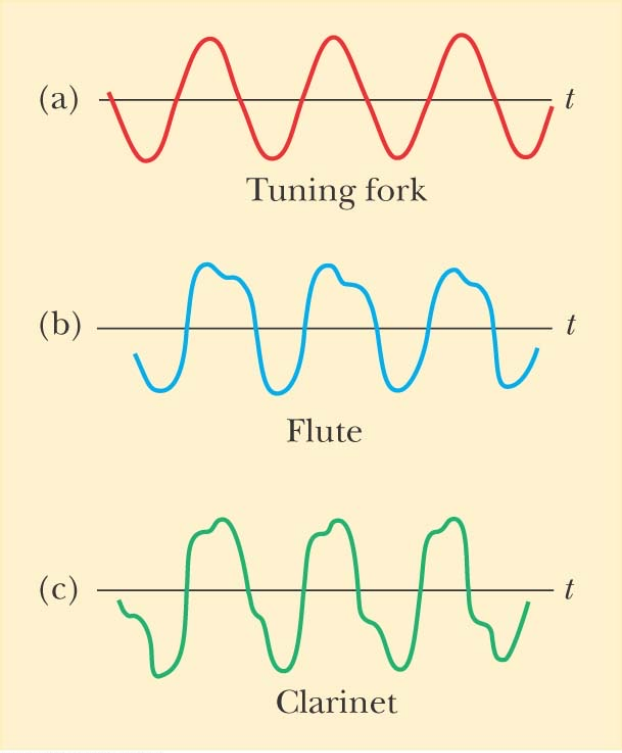
1768-1830

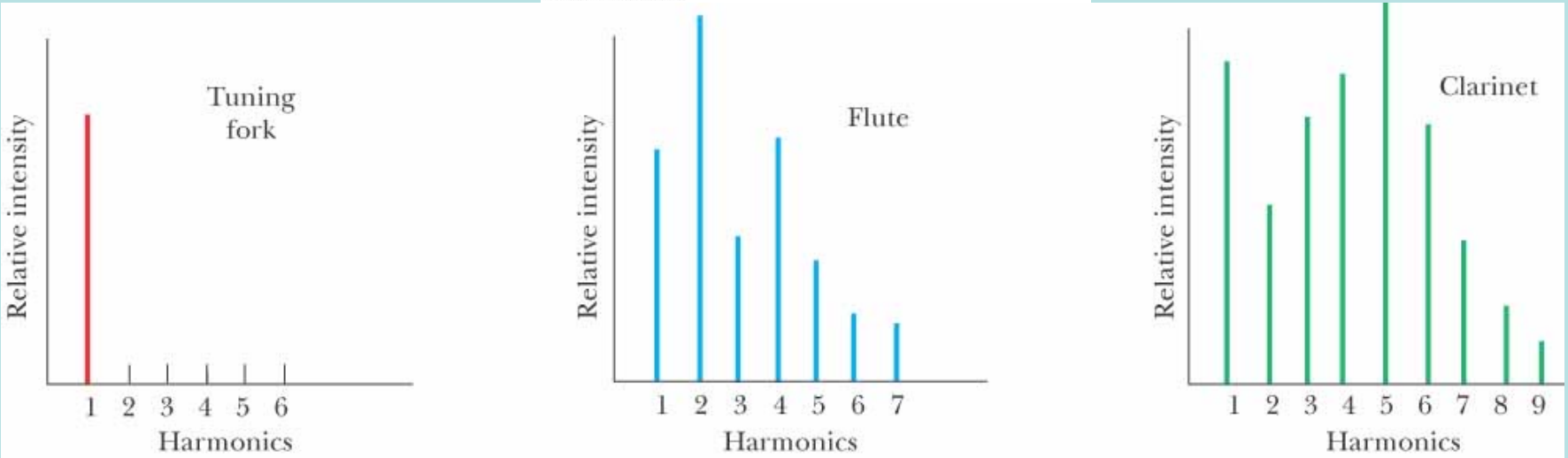

(a) Original curve
x
t
T
Harmonics:
$h_n(t) = \sin(2\pi f_n t)$ $f_n = n/T$
(b) Harmonics
x
$f_1 = 1/T$
curve 1
A_1
t
$f_2 = 2f_1$
curve 2
A_2
$f_3 = 3f_1$
curve 3
A_3
$f_4 = 4f_1$
curve 4
A_4
(c) Sum of harmonics
x
curve 1
t
curve 1 + 2
curve 1 + 2 + 3
curve 1 + 2 + 3 + 4





一個連續的運動被分解成一系列分離的數



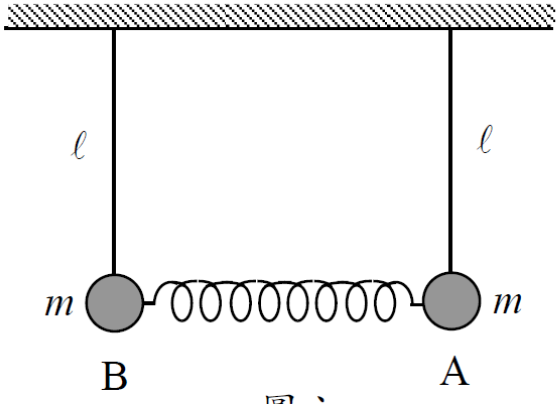


一般來說，n 越大，強度越小，所以通常只要有限個成分就能近似一個周期運動！

Eigenvalue problem of Matrix

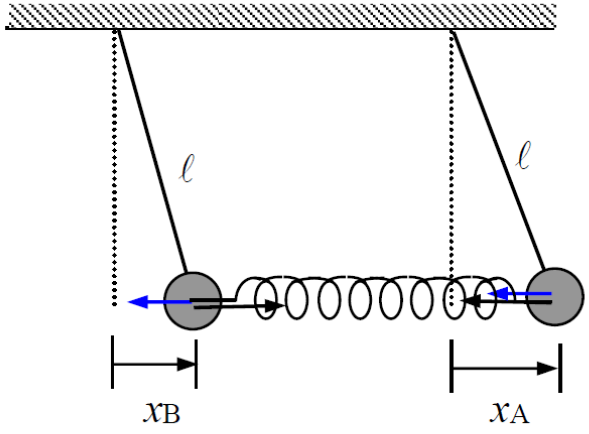
二、耦合振動

圖六所示為一簡單的耦合擺；兩個相同的單擺 A 和 B 鉛直懸掛，其間以一輕彈簧聯結。ℓ 為擺長，k 為彈簧的力常數。現把 B 固定，而將 A 拉開一小位移 A_c，然後一起自由地釋放，則 A 被彈簧的恢復力拉回時，B 也受到同樣的彈簧恢復力作用而被拉動。B 所獲得的動能，即表示 A 能量的減少。A 的能量逐漸地轉移到 B。B 振動的振幅逐漸變大，而 A 則逐漸減少，終致靜止。這時，B 開始擔任原先 A 所作的角色，逐漸把能量回轉給 A。像這樣，把能量往復傳送的運動，稱為耦合振動。



圖六

圖七所示為在某一時刻時，耦合擺各部分的位置。A 和 B 同時各受有擺繩的張力，重力，和彈簧的恢復力。x_A 和 x_B 代表 A 和 B 偏離平衡位置的位移。考慮 A 和 B 的受力情形，其運動方程分別為：



圖七

$$\begin{cases} m\dfrac{d^2x_A}{dt^2} = -\left(\dfrac{mg}{\ell}\right)x_A - k(x_A - x_B) \\ m\dfrac{d^2x_B}{dt^2} = -\left(\dfrac{mg}{\ell}\right)x_B + k(x_A - x_B) \end{cases} \quad \text{.....} \quad (2\text{-}1)$$

$$\begin{cases} m\dfrac{d^2x_A}{dt^2} = -\left(\dfrac{mg}{\ell}\right)x_A - k(x_A - x_B) \\ m\dfrac{d^2x_B}{dt^2} = -\left(\dfrac{mg}{\ell}\right)x_B + k(x_A - x_B) \end{cases} \quad \text{.....} (2\text{-}1)$$

上式右邊第一項為擺繩張力的水平分量，第二項為彈簧的恢復力。

令 $\omega_0 \equiv \sqrt{\dfrac{g}{\ell}}$ ， $\omega_c \equiv \sqrt{\dfrac{k}{m}}$

則(2-1)式可改寫為

$$\begin{cases} \dfrac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B = 0 \\ \dfrac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A = 0 \end{cases} \quad \text{.....}(2\text{-}2)$$

把(2-2)式的兩式分別相加和相減，可得

$$\begin{cases} \dfrac{d^2}{dt^2}(x_A + x_B) + \omega_0^2(x_A + x_B) = 0 \\ \dfrac{d^2}{dt^2}(x_A - x_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) = 0 \end{cases}$$

取廣義座標以去耦合decouple

令 $q_1 = x_A + x_B$ ， $q_2 = x_A - x_B$

$$\begin{cases} \dfrac{d^2q_1}{dt^2} + \omega_0^2 q_1 = 0 \\ \dfrac{d^2q_2}{dt^2} + \omega'^2 q_2 = 0 \end{cases} \quad \text{.....}(2\text{-}3)$$

式中 $\omega' = \sqrt{\omega_0^2 + 2\omega_c^2}$

則上兩式可認出為兩個各自獨立的簡諧振動方程式：

ω_0 和 ω' 稱為此耦合擺的簡正模頻率(Normal Mode Frequency)。

(2-3)式的一般解為

$$q_1 = x_A + x_B = C\cos(\omega_0 t + \phi)$$

$$q_2 = x_A - x_B = D\cos(\omega' t + \phi') \quad \text{.....(2-4)}$$

由於起始時（即 $t=0$ 時），A 和 B 皆自靜止狀態自由釋放，所以 $\dot{x}_A = \dot{x}_B = 0$。

$$\begin{cases} \dot{x}_A + \dot{x}_B = -C\omega_0 \sin(\omega_0 t + \phi) \\ \dot{x}_A - \dot{x}_B = -D\omega' \sin(\omega' t + \phi') \end{cases}$$

當 $t=0$ 時，得

$$\begin{cases} 0 = -C\omega_0 \sin\phi \\ 0 = -D\omega' \sin\phi' \end{cases} \Rightarrow \begin{cases} \phi = 0 \\ \phi' = 0 \end{cases}$$

由(2-4)式可得

$$\begin{cases} x_A = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}C\cos\omega_0 t + \frac{1}{2}D\cos\omega' t \\ x_B = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}C\cos\omega_0 t - \frac{1}{2}D\cos\omega' t \end{cases} \quad \text{.....(2-5)}$$

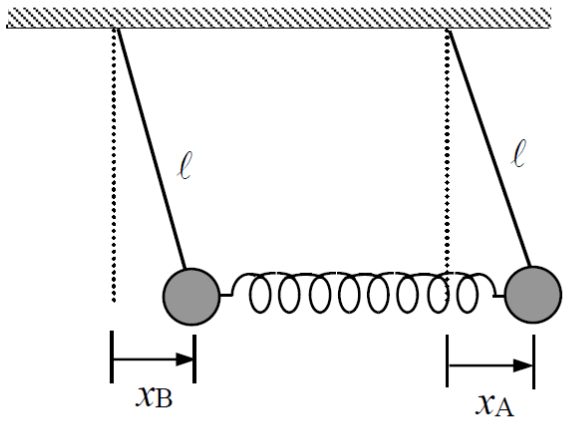
$$\begin{cases} x_A = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}C\cos\omega_0 t + \frac{1}{2}D\cos\omega' t \\ x_B = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}C\cos\omega_0 t - \frac{1}{2}D\cos\omega' t \end{cases} \quad \text{.....(2-5)}$$

下面討論三種不同的起始狀況：

(1)若起始時，$x_A = x_B$，即兩單擺同幅同相，則 D = 0。

$$x_A = x_B = \frac{1}{2}C\cos\omega_0 t$$

即 A 和 B 以同一頻率 $\omega_0 = \sqrt{\frac{g}{\ell}}$ 做同幅且同相的簡諧振動，如圖八所示，兩者擺幅和相位角相同，因此居間聯繫的彈簧既不伸長也

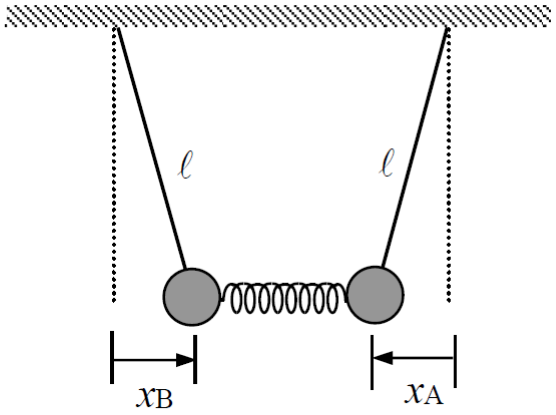


圖八　兩單擺同幅且同相擺動

(2)若起始時，$x_A = -x_B$，即兩單擺同幅但反相，則 C = 0。

$$x_A = -x_B = \frac{1}{2}D\cos\omega' t$$

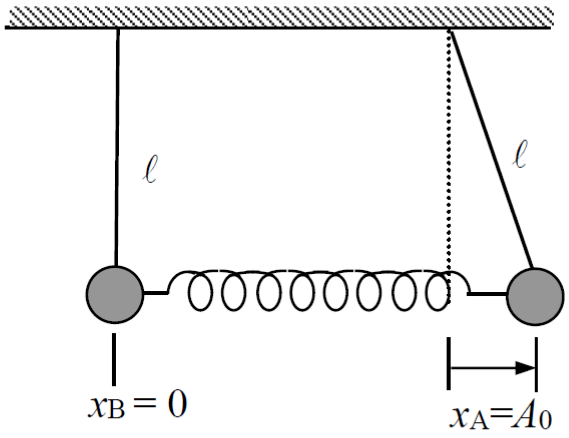
即 A 和 B 仍以同一頻率 $\omega' = \sqrt{\frac{g}{\ell} + \frac{2k}{m}}$ 做簡諧振動，兩者同幅但反相，如圖九所示。



圖九　兩單擺同幅但反相擺動

由上述的討論可知，簡正模就是指耦合系統內各成員以同一頻率振動時的運動模式。此頻率即稱為簡正模頻率。

(3)若起始時，$x_A = A_0$，$x_B = 0$，代入(2-5)式，則



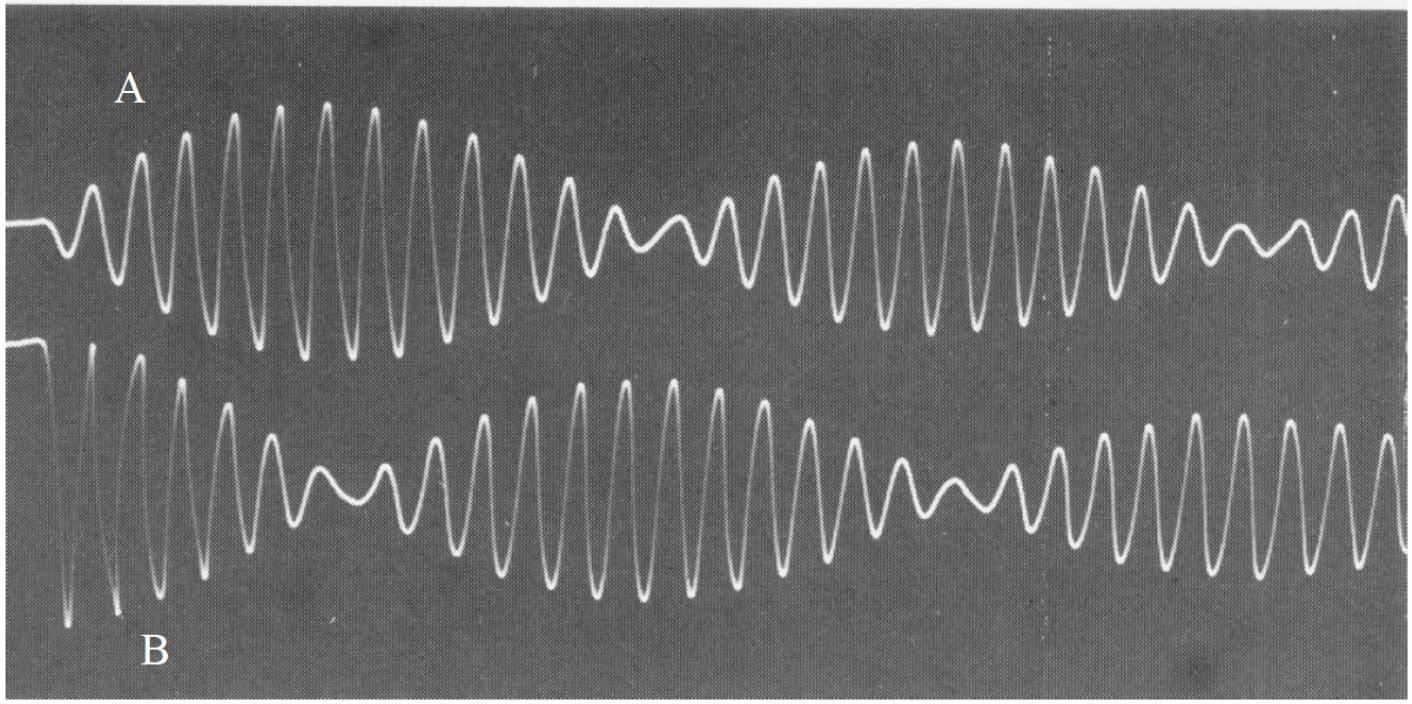
$$\begin{cases} A_0 = \frac{1}{2}C + \frac{1}{2}D \\ 0 = \frac{1}{2}C - \frac{1}{2}D \end{cases}$$

解得 $C = D = A_0$，故

$$\begin{cases} x_A = \frac{1}{2}A_0(\cos\omega_0 t + \cos\omega' t) \\ x_B = \frac{1}{2}A_0(\cos\omega_0 t - \cos\omega' t) \end{cases}$$

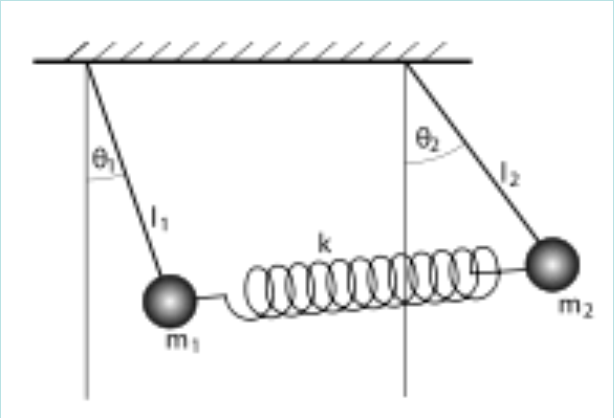
$$\Rightarrow \begin{cases} x_A = A_0\cos\left(\frac{\omega' - \omega_0}{2}t\right)\cos\left(\frac{\omega' + \omega_0}{2}t\right) \\ x_B = A_0\sin\left(\frac{\omega' - \omega_0}{2}t\right)\sin\left(\frac{\omega' + \omega_0}{2}t\right) \end{cases} \quad \text{.....} (2\text{-}6)$$

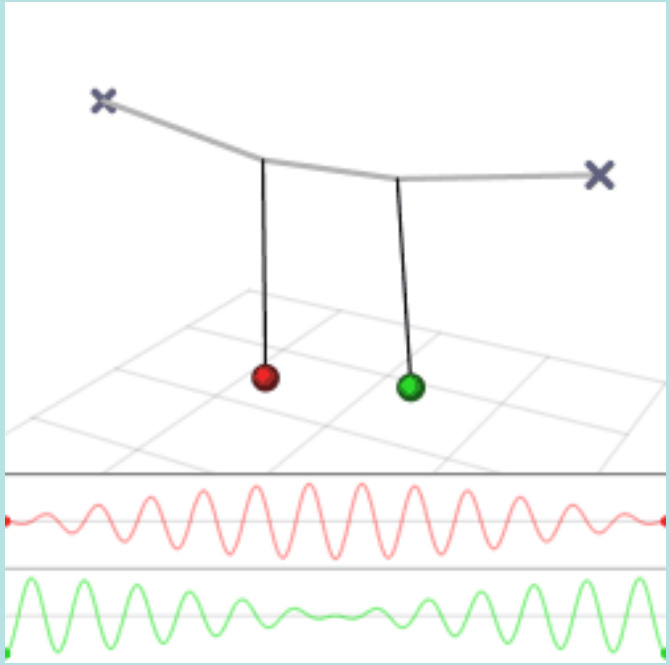
A 和 B 的運動軌跡如圖十所示，此為調幅振動（Amplitude-modulated Oscillation），其振動頻率為 $\frac{1}{2}(\omega' + \omega_0)$；振幅的調制頻率為 $\frac{1}{2}(\omega' - \omega_0)$。



圖十　實際拍攝的耦合擺振動情形。橫軸為時間，縱軸為振幅。當 $t = 0$ 時，B 為靜止，A 則給予一起始位移。由於摩擦損耗，振幅隨時間漸減。

Coupled Oscillators

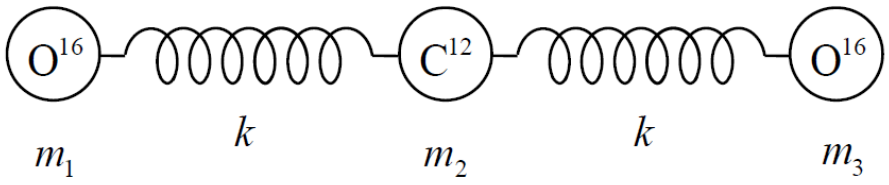




$\nu_e \rightleftarrows \nu_\mu$

【例 1】

CO_2 分子的振動情形可用圖十一所示的耦合系統來模擬。圖中的耦合彈簧相同，其力常數為 k。



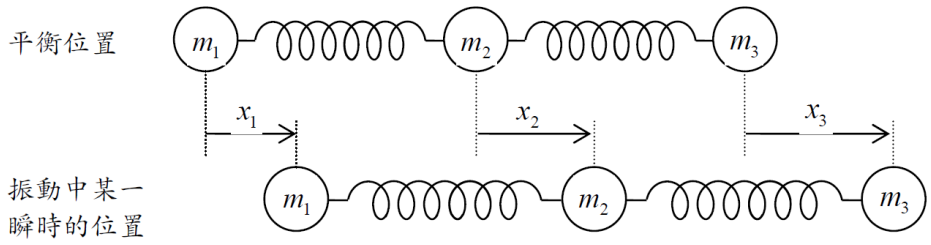
圖十一

(1)若僅考慮各原子的縱向振動（即沿原子中心連線的方向），試寫出其運動方程式。

(2)氧和碳的原子量分別為 16 和 12，試求此分子縱向振動的簡正模頻率的比值。

解：

(1)參考圖十二，各原子偏離其平衡位置的位移分別為 x_1，x_2 和 x_3。



圖十二

各原子的運動方程式如下：

$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = -k(x_1 - x_2) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \\ m_3 \dfrac{d^2 x_3}{dt^2} = -k(x_3 - x_2) \end{cases} \quad \text{.....(2-7)}$$

$x_n = a_n \cos(\omega_n t + \phi_n)$ 簡正模頻率，則 $x_n = a_n \cdot e^{i(\omega_n t + \phi_n)}$

$$x_n = a_n \cos \omega t \text{，} n = 1,2,3 \quad \text{.....(2-8)}$$

將(2-8)代入(2-7)可得

$$\begin{cases} \left(\omega^2 - \dfrac{k}{m_1}\right) a_1 + \left(\dfrac{k}{m_1}\right) a_2 = 0 \\ \left(\dfrac{k}{m_2}\right) a_1 + \left(\omega^2 - \dfrac{2k}{m_2}\right) a_2 + \left(\dfrac{k}{m_2}\right) a_3 = 0 \\ \left(\dfrac{k}{m_3}\right) a_2 + \left(\omega^2 - \dfrac{k}{m_3}\right) a_3 = 0 \end{cases} \quad \text{.....(2-9)}$$

各原子的運動方程式如下：

$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = -k(x_1 - x_2) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \\ m_3 \dfrac{d^2 x_3}{dt^2} = -k(x_3 - x_2) \end{cases} \quad \text{.....(2-7)}$$

(2)設 ω 為可能的簡正模頻率，則

$$x_n = a_n \cos \omega t \text{，} n = 1,2,3 \quad \text{.....(2-8)}$$

將(2-8)代入(2-7)可得

$$\begin{cases} \left(\omega^2 - \dfrac{k}{m_1}\right) a_1 + \left(\dfrac{k}{m_1}\right) a_2 = 0 \\ \left(\dfrac{k}{m_2}\right) a_1 + \left(\omega^2 - \dfrac{2k}{m_2}\right) a_2 + \left(\dfrac{k}{m_2}\right) a_3 = 0 \\ \left(\dfrac{k}{m_3}\right) a_2 + \left(\omega^2 - \dfrac{k}{m_3}\right) a_3 = 0 \end{cases} \quad \text{.....(2-9)}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\frac{d^2 X}{dt^2} = \begin{pmatrix} -\dfrac{k}{m_1} & \dfrac{k}{m_1} & 0 \\ \dfrac{k}{m_2} & -\dfrac{2k}{m_2} & \dfrac{k}{m_2} \\ 0 & \dfrac{k}{m_3} & -\dfrac{k}{m_3} \end{pmatrix} \cdot X \equiv AX$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i\omega t} \equiv a e^{i\omega t}$$

代入

$$\frac{d^2 X}{dt^2} = AX$$

$$Aa = -\omega^2 a$$

ω^2 及 a 分別為矩陣 A 的Eigenvalues本徵值及Eigenvectors本徵向量。

$$\begin{pmatrix} -\frac{k}{m_1} + \omega^2 & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -\frac{2k}{m_2} + \omega^2 & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} + \omega^2 \end{pmatrix} \cdot a = 0$$

a 只有當矩陣的行列式為零時有非零解：，

$$\begin{vmatrix} -\frac{k}{m_1} + \omega^2 & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -\frac{2k}{m_2} + \omega^2 & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} + \omega^2 \end{vmatrix} = 0$$

ω^2 有三個解Eigenvalues。每一個解對應一個向量 a：eigenvectors

取 $m_1 = m_3 = 16m_0$，$m_2 = 12m_0$，且令 $\omega_0^2 \equiv \dfrac{k}{m_0}$，則(2-9)可改寫為

$$\begin{cases} \left(\omega^2 - \dfrac{\omega_0^2}{16}\right)a_1 + \left(\dfrac{\omega_0^2}{16}\right)a_2 = 0 \\ \left(\dfrac{\omega_0^2}{12}\right)a_1 + \left(\omega^2 - \dfrac{\omega_0^2}{6}\right)a_2 + \left(\dfrac{\omega_0^2}{12}\right)a_3 = 0 \\ \left(\dfrac{\omega_0^2}{16}\right)a_2 + \left(\omega^2 - \dfrac{\omega_0^2}{16}\right)a_3 = 0 \end{cases} \quad \text{.....(2-10)}$$

(2-10)有非零解之條件為係數行列式等於零，即

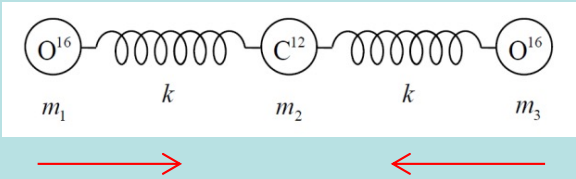
$$\begin{vmatrix} \omega^2 - \dfrac{\omega_0^2}{16} & \dfrac{\omega_0^2}{16} & 0 \\ \dfrac{\omega_0^2}{12} & \omega^2 - \dfrac{\omega_0^2}{6} & \dfrac{\omega_0^2}{12} \\ 0 & \dfrac{\omega_0^2}{16} & \omega^2 - \dfrac{\omega_0^2}{16} \end{vmatrix} = 0$$

解得 $\omega = \dfrac{\omega_0}{4}$，$\sqrt{\dfrac{11}{48}}\omega_0$，0（即為處於平衡狀態，不振動）

兩振動中之簡正模頻率之比值為 $\sqrt{\dfrac{11}{3}}$。

三個頻率分別對應三個不同的(a_1, a_2, a_3)模式mode

例如$\omega = \dfrac{\omega_0}{4}$時，$a = (1,0,-1)$

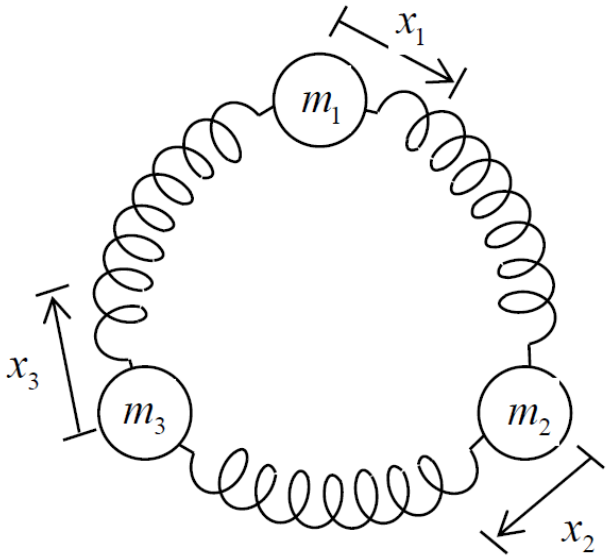


【例 2】

如圖十三所示，三個質量各為 m 的質點由三條相同的輕彈簧聯接在一起，彈簧的力常數為 k。當此系統處於平衡狀態時，各彈簧的長度為其自然長度。當處於運動狀態時，各質點被限制在同一圓周上運動。

(1)若此三質點偏離其平衡位置的小位移，分別為 x_1，x_2 和 x_3，試寫出每一質點的運動方程式。

(2)試求出此系統可能的簡正模頻率。



圖十三

解：

(1)

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} = -k(x_1 - x_2) - k(x_1 - x_3) \\ m\dfrac{d^2x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \\ m\dfrac{d^2x_3}{dt^2} = -k(x_3 - x_1) - k(x_3 - x_2) \end{cases} \quad \text{.....(2-11)}$$

(2)設 ω 為可能的簡正模頻率，則

$$x_n = a_n \cos\omega t \text{ ， } n = 1,2,3 \quad \text{.....(2-12)}$$

將(2-12)代入(2-11)可得

$$\begin{cases} \left(\dfrac{\omega^2}{\omega_0^2} - 2\right)a_1 + a_2 + a_3 = 0 \\ a_1 + \left(\dfrac{\omega^2}{\omega_0^2} - 2\right)a_2 + a_3 = 0 \\ a_1 + a_2 + \left(\dfrac{\omega^2}{\omega_0^2} - 2\right)a_3 = 0 \end{cases} \quad \text{.....(2-13)}$$

其中 $\omega_0 \equiv \sqrt{\dfrac{k}{m}}$

(2-13)式有非零解之條件為係數行列式等於零，即

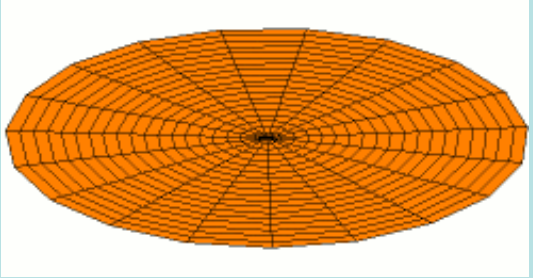
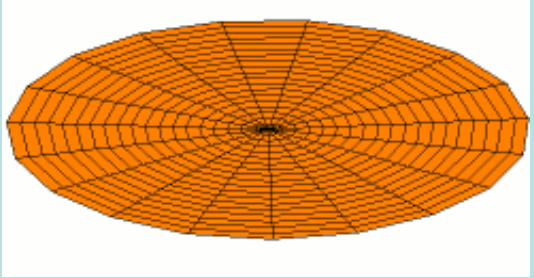
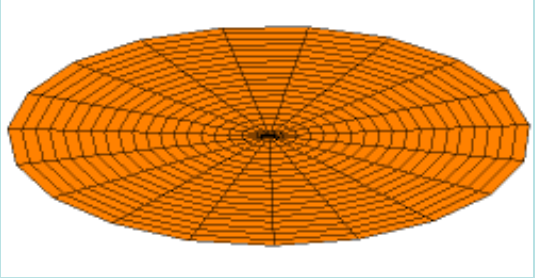
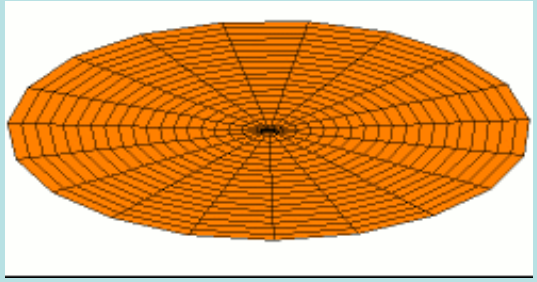
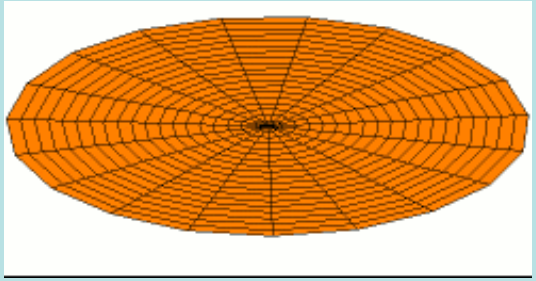
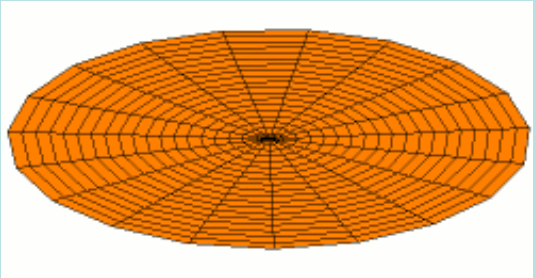
$$\begin{vmatrix} \dfrac{\omega^2}{\omega_0^2} - 2 & 1 & 1 \\ 1 & \dfrac{\omega^2}{\omega_0^2} - 2 & 1 \\ 1 & 1 & \dfrac{\omega^2}{\omega_0^2} - 2 \end{vmatrix} = 0$$

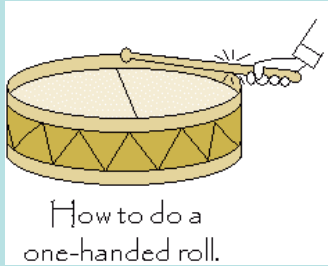
(2-13)式有非零解之條件為係數行列式等於零，即

$$\begin{vmatrix} \frac{\omega^2}{\omega_0^2}-2 & 1 & 1 \\ 1 & \frac{\omega^2}{\omega_0^2}-2 & 1 \\ 1 & 1 & \frac{\omega^2}{\omega_0^2}-2 \end{vmatrix} = 0$$

解得 $\left(\frac{\omega^2}{\omega_0^2}\right)\left(\frac{\omega^2}{\omega_0^2}-3\right)^2 = 0$

因此振動中之簡正模頻率為 $\sqrt{3}\omega_0 = \sqrt{\frac{3k}{m}}$



原則上物體的變形模式有無限多個，但並不是連續分布。

因此可以分離地一個一個編號。

每一個模式，如同簡諧運動，對應一個內在的特定的振動頻率！

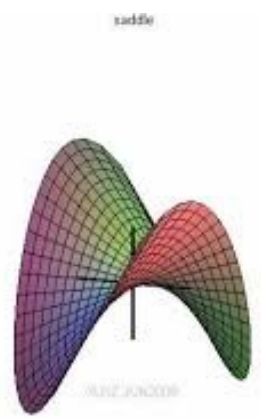
不同模式頻率不同。一般來說，頻率越高的模式，越難激發。

物體的所有變形就是以這些模式或它們的疊加來進行！

一個物體有那些振盪模式 Norm以及對應的頻率，就是該物體的一個特徵。

3. 考慮一個剛體的馬鞍面，垂直放置，使此表面滿足方程式：$z = \frac{1}{R}(x^2 - y^2)$，選$z$軸為垂直軸（考慮重力）。在此表面上放一個小球，質量為m。假設在球運動的過程，一直保持在馬鞍面上。此球極小，它的旋轉效應可以忽略。使馬鞍面繞z軸以固定角速度ω旋轉。選在相對於馬鞍面靜止的旋轉座標系中觀察，此球的位置記為$\vec{r}' = (x', y', z')$。假設時間為零時，靜止座標與旋轉座標的軸重合，因此$x' = x\cos\omega t + y\sin\omega t$，$y' = y\cos\omega t - x\sin\omega t$。因為旋轉坐標系為非慣性座標系，在此座標系中觀察時，所有物體都會受到假力：$m\omega^2\vec{r}' - 2m\vec{\omega} \times \dot{\vec{r}}'$。在此$\vec{\omega} \equiv \omega\hat{e}_z$。第一項即是離心力，而第二項與運動速度有關，稱為科氏力。

(a) 寫下此球的x'座標以及y'座標所必須滿足的方程式。



球的運動限制在面上，其三個座標滿足一個條件：

$$z = \frac{1}{R}(x^2 - y^2)$$

$$E = \frac{m}{2}\left[v_x^2 + v_y^2 + v_z^2\right] + U$$

$$v_z = \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \propto x\cdot v_x + y\cdot v_y \sim 0$$ 當球離原點不遠

$$E = \frac{m}{2}\left[v_x^2 + v_y^2 + v_z^2\right] + U \rightarrow \frac{m}{2}\left[v_x^2 + v_y^2\right] + mgz$$

球的運動等價於二維運動：

$$= \frac{m}{2}\left[v_x^2 + v_y^2\right] + \frac{mg}{R}(x^2 - y^2)$$

對二維表面來說，此位能會對球施予一個力：

$$F_x = -\frac{\partial U}{\partial x'} = -\frac{2mg}{R}x', F_y = -\frac{\partial U}{\partial y'} = \frac{2mg}{R}y'$$

再加上科氏力：$-2m\vec{\omega}\times\dot{\vec{r}}' = (+2\omega\dot{y}' \quad -2\omega\dot{x}')$及離心力，可以得到$x', y'$

座標滿足的方程式：

$m\omega^2\vec{r}'$

$$\ddot{x}' = +2\omega\dot{y}' + \left(\omega^2 - \frac{2g}{R}\right)x'$$

$$\ddot{y}' = -2\omega\dot{x}' + \left(\omega^2 + \frac{2g}{R}\right)y'$$

(b) 球有可能滑下馬鞍面，也有可能留在原點附近來回移動。以普遍的方法，來求解這兩個方程式最方便的辦法，就是推廣兩個座標為複數，而將其可能的解寫成一個複數αt的指數函數：$x' = a \cdot e^{\alpha t}$，$y' = b \cdot e^{\alpha t}$，常數$a, b, \alpha$為複數。得到這樣的解後，作線性組合，將適當的實數起始條件代入後，上述的解就自然維持是實數。

若$x' = a \cdot e^{\alpha t}$，$y' = b \cdot e^{\alpha t}$是球的運動方程式的解，寫下常數$\alpha$必須滿足的條件。

當角速度ω大於某值ω_{cr}，所有的解，無論時間多長，都會留在有限的範圍內，不會運動到無限遠處（也就是球不會滑下馬鞍面）。找出ω_{cr}。

當$\omega > \omega_{\mathrm{cr}}$，針對所有可能的常數$\alpha$，寫下對應的常數$a, b$彼此的關係。

(c) 設$\omega > \omega_{\mathrm{cr}}$。在時間為零時，將球放置於原點，相對於靜止實驗室朝y軸方向給予一速度v_0。寫下在旋轉的座標系中觀察時，此球的y'座標及x'座標與時間的關係。

(b) 將$x' = (a \cdot e^{\alpha t})$，$y' = (b \cdot e^{\alpha t})$代入以上方程式：

$$\ddot{x}' = +2\omega\dot{y}' + \left(\omega^2 - \frac{2g}{R}\right)x'$$

$$\ddot{y}' = -2\omega\dot{x}' + \left(\omega^2 + \frac{2g}{R}\right)y'$$

$$\left[\alpha^2 - \left(\omega^2 - \frac{2g}{R}\right)\right]a - 2\omega\alpha b = 0$$

$$2\omega\alpha a + \left[\alpha^2 - \left(\omega^2 + \frac{2g}{R}\right)\right]b = 0$$

若a, b有不全為零的解，α必須滿足：

$$\left[\alpha^2 - \left(\omega^2 - \frac{2g}{R}\right)\right] \cdot \left[\alpha^2 - \left(\omega^2 + \frac{2g}{R}\right)\right] + 4\omega^2\alpha^2 = 0$$

化簡後得：

$$\alpha^4 + 2\omega^2\alpha^2 + \left(\omega^4 - \frac{4g^2}{R^2}\right) = 0$$

$$\alpha^2 = -\omega^2 \pm \frac{2g}{R}$$

如果α不是虛數，在時間無限大時，$e^{\alpha t}$也將趨近無限大。因此如果要求球不會運動到無限遠處，角速度ω必須滿足：

$$\omega^2 > \frac{2g}{R}, \omega > \sqrt{\frac{2g}{R}}$$

若$\omega > \sqrt{\frac{2g}{R}}$，$\alpha$有四個虛數解：

$$\alpha = \pm i\left(\sqrt{\omega^2 + \frac{2g}{R}}\right) \equiv \pm i\omega_1$$，對應的常數滿足：$\omega a_{1\pm} = \pm i\omega_1 b_{1\pm}$。

$$\alpha = \pm i\left(\sqrt{\omega^2 - \frac{2g}{R}}\right) \equiv \pm i\omega_2$$，對應的常數滿足：$\omega b_{2\pm} = \mp i\omega_2 a_{2\pm}$

不同的模式

(c) 以上得到四個解，兩個座標可以寫成四個解的和：

$$x' = a_{1+} \cdot e^{i\omega_1 t} + a_{1-} \cdot e^{-i\omega_1 t} + a_{2+} \cdot e^{i\omega_2 t} + a_{2-} \cdot e^{-i\omega_2 t}$$

$$y' = b_{1+} \cdot e^{i\omega_1 t} + b_{1-} \cdot e^{-i\omega_1 t} + b_{2+} \cdot e^{i\omega_2 t} + b_{2-} \cdot e^{-i\omega_2 t}$$

$$= -i\frac{\omega}{\omega_1} a_{1+} \cdot e^{i\omega_1 t} + i\frac{\omega}{\omega_1} a_{1-} \cdot e^{-i\omega_1 t} - i\frac{\omega_2}{\omega} a_{2+} \cdot e^{i\omega_2 t} + i\frac{\omega_2}{\omega} a_{2+} \cdot e^{-i\omega_2 t}$$

起始條件為 $x(0) = 0$，$\dot{x}(0) = 0$，$y(0) = 0$，$\dot{y}(0) = v_0$，

$$x' = a_{1+} \cdot e^{i\omega_1 t} + a_{1-} \cdot e^{-i\omega_1 t} + a_{2+} \cdot e^{i\omega_2 t} + a_{2-} \cdot e^{-i\omega_2 t}$$

$$y' = b_{1+} \cdot e^{i\omega_1 t} + b_{1-} \cdot e^{-i\omega_1 t} + b_{2+} \cdot e^{i\omega_2 t} + b_{2-} \cdot e^{-i\omega_2 t}$$

$$= -i\frac{\omega}{\omega_1} a_{1+} \cdot e^{i\omega_1 t} + i\frac{\omega}{\omega_1} a_{1-} \cdot e^{-i\omega_1 t} - i\frac{\omega_2}{\omega} a_{2+} \cdot e^{i\omega_2 t} + i\frac{\omega_2}{\omega} a_{2+} \cdot e^{-i\omega_2 t}$$

起始條件為 $x(0) = 0$，$\dot{x}(0) = 0$，$y(0) = 0$，$\dot{y}(0) = v_0$，

利用座標的轉換關係 $x' = x\cos\omega t + y\sin\omega t$，$y' = y\cos\omega t - x\sin\omega t$，

可以得到旋轉座標系中的起始條件：

$$x'(0) = 0，\dot{x}'(0) = 0，$$

$$y'(0) = 0，\dot{y}'(0) = v_0，$$

將座標的解x', y'代入此四個起始條件：

$$a_{1+} + a_{1-} + a_{2+} + a_{2-} = 0$$

$$\omega_1 a_{1+} - \omega_1 a_{1-} + \omega_2 a_{2+} - \omega_2 a_{2-} = 0$$

$$-\frac{\omega}{\omega_1} a_{1+} + \frac{\omega}{\omega_1} a_{1-} - \frac{\omega_2}{\omega} a_{2+} + \frac{\omega_2}{\omega} a_{2+} = 0$$

$$-\omega a_{1+} - \omega a_{1-} - \frac{\omega_2^2}{\omega} a_{2+} - \frac{\omega_2^2}{\omega} a_{2+} = v_0$$

$$a_{1+} + a_{1-} + a_{2+} + a_{2-} = 0$$

$$\omega_1 a_{1+} - \omega_1 a_{1-} + \omega_2 a_{2+} - \omega_2 a_{2-} = 0$$

$$-\frac{\omega}{\omega_1} a_{1+} + \frac{\omega}{\omega_1} a_{1-} - \frac{\omega_2}{\omega} a_{2+} + \frac{\omega_2}{\omega} a_{2+} = 0$$

$$-\omega a_{1+} - \omega a_{1-} - \frac{\omega_2^2}{\omega} a_{2+} - \frac{\omega_2^2}{\omega} a_{2+} = v_0$$

由第二式及第三式可以得到：$a_{1+} = a_{1-}$，$a_{2+} = a_{2-}$。

因此第一式及第四式可以化簡為：

$$a_{1+} + a_{2+} = 0$$

$$-2\omega a_{1+} - 2\frac{\omega_2^2}{\omega} a_{2+} = v_0$$

求解可以得到：

$$a_{1+} = -a_{2+} = \frac{\omega v_0}{\frac{4g}{R}} = \frac{\omega R v_0}{4g}$$

因此

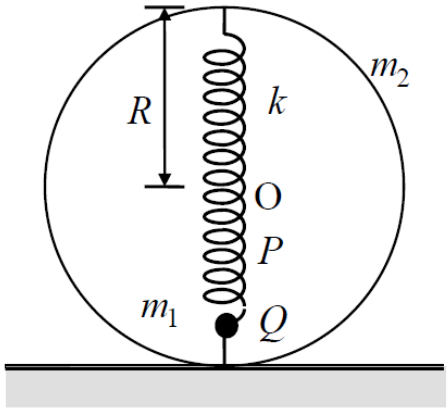
$$y' = i\frac{\omega}{\omega_1} a_{1+} \cdot \left(-e^{i\omega_1 t} + e^{-i\omega_1 t}\right) + i\frac{\omega_2}{\omega} a_{2+} \cdot \left(-e^{i\omega_2 t} + e^{-i\omega_2 t}\right)$$

$$= \frac{R\omega^2 v_0}{2\omega_1 g} \sin \omega_1 t - \frac{R\omega_2 v_0}{2g} \sin \omega_2 t$$

$$x' = \frac{\omega R v_0}{4g} \cdot \cos \omega_1 t - \frac{\omega R v_0}{4g} \cdot \cos \omega_2 t$$

耦合震盪

一、一力常數為 k 的輕彈簧下端懸掛一質量為 m_1 的小球，彈簧的上端則連在質量為 m_2 的均勻薄球殼內壁上，如右圖所示。彈簧的自然長度與球殼半徑均為 R。開始時球殼靜置於水平地面上，彈簧在球殼內處於鉛直狀態，將小球自其力平衡點 P（見附註一）下拉至 Q 點後，用一輕直的細繩使與球殼底端相連。設 P 和 Q 兩點間的距離為 H，細繩在某一時刻突然斷開，爾後發現球殼時而升離地面，時而無碰撞地落地（見附註二），且在離地與落地的兩個瞬間，小球的離地高度皆相同。



〔附註一〕: P 點是指小球自然垂直地懸掛在彈簧下端時的力平衡點（這時未用細繩連到球殼底端）。

〔附註二〕: 所謂「無碰撞的落地」，係指球殼落地的瞬間，其相對於地面的速度為零。

(1)在繩斷開之後，球殼離地之前，小球做上升的簡諧運動，求其週期及振幅。

(2)球殼升離地面的瞬間，令小球的位置稱為 S 點，求 S 點到球心 O 點的距離。

(3)從繩斷開到球殼離地，共經歷多少時間?

(4)試求小球上升到 S 點時的速度 v_s，以已知量表示之。

(5)以小球上升至 S 點的瞬間做為計時的起點 $t=0$，這時的球殼中心位置做為座標原點，令 y 座標向上為正。設小球上升至 S 點以後，小球及球殼中心的 y 座標分別為 y_1 及 y_2，求 $y_1(t)$ 及 $y_2(t)$。

(6)若要球殼無碰撞落地，寫出 v_s 必須滿足的方程式。

(7)若要球殼無碰撞落地，求 H 的最小值所必須滿足的算式。

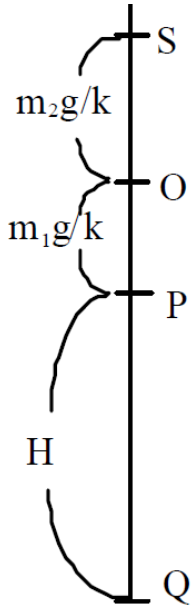
(1)在繩斷開之後，球殼離地之前，小球做上升的簡諧運動，求其週期及振幅。

(2)球殼升離地面的瞬間，令小球的位置稱為 S 點，求 S 點到球心 O 點的距離。

(3)從繩斷開到球殼離地，共經歷多少時間?

(1) 在繩斷開之後，球殼離地之前，彈簧處於伸長狀態，小球做上升的簡諧運動，其角頻率 $\omega_1=\sqrt{k/m_1}$ ，週期 $T_1=2\pi\sqrt{m_1/k}$ ，振幅 $A_1=H$ 。

(2) 如圖四所示，小球經 $\frac{T_1}{4}$ 時間，行經 H 路程後，到達其力平衡點 P，彈簧在此處時的伸長量為 $\frac{m_1g}{k}$ ，此時地面給予球殼的正向力 $N=m_1g+m_2g$。小球再經過 $\frac{m_1g}{k}$ 路程，到達球心 O，彈簧恢復自然長度，此時 N 減小到等於 m_2g。再後，小球繼續向上運動，彈簧被壓縮，N 繼續減小。當小球到達 S 點時，彈簧被壓縮 $\frac{m_2g}{k}$ 的長度，其作用於球殼的恢復力恰等於球殼的重力 m_2g。因此 N 降至零，球殼開始升離地面，故 S 點到 O 點的距離為 $\frac{m_2g}{k}$ 。



圖四

O是彈簧的平衡點

P是簡諧運動的平衡點

(3) 小球從 P 點到達 S 點所經歷的時間 t_{PS}，可從下式求出：

$$\frac{(m_1+m_2)g}{k}=A_1\sin\omega_1 t_{PS}$$

因此小球從 Q 點到 S 點所經歷的時間為

$$\Delta t_1=\frac{T_1}{4}+t_{ps}=\sqrt{\frac{m_1}{k}}\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{(m_1+m_2)g}{kH}\right)\right]$$

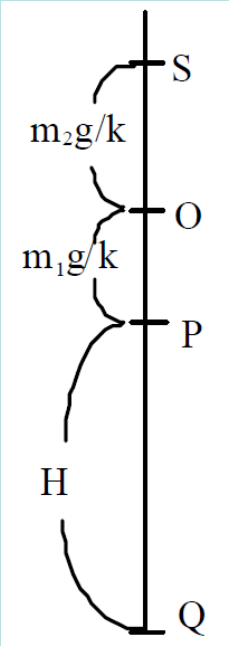
(4)試求小球上升到 S 點時的速度 v_s，以已知量表示之。

(4) 取 Q 點為重力位能的基準點，則由力學能守恆定律可得

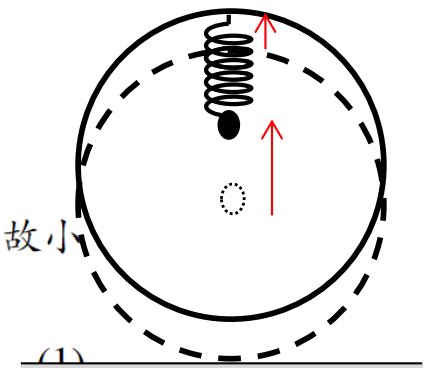
$$\frac{1}{2}mv_s^{\ 2}+m_1g(H+\overline{PS})+\frac{1}{2}k\overline{OS}^2=\frac{1}{2}k(H+\overline{PO})^2$$

式中 $\overline{PS}=\dfrac{(m_1+m_2)g}{k}$，$\overline{OS}=\dfrac{m_2g}{k}$，$\overline{PO}=\dfrac{m_1g}{k}$，解之得

$$v_s=\sqrt{\frac{k}{m_1}\left(H^2-\frac{(m_1+m_2)^2g^2}{k^2}\right)}$$



(5)以小球上升至 S 點的瞬間做為計時的起點 $t=0$，這時的球殼中心位置做為座標原點，令 y 座標向上為正。設小球上升至 S 點以後，小球及球殼中心的 y 座標分別為 y_1 及 y_2，求 $y_1(t)$ 及 $y_2(t)$。



(5) 設小球的位置為 y_1，球殼中心的位置為 y_2，則 y_1-y_2 等於彈簧被壓縮的長度，故小球和球殼中心的運動方程式分別為

$$m_1\ddot{y}_1=-k(y_1-y_2)-m_1g \quad (1)$$

$$m_2\ddot{y}_2=k(y_1-y_2)-m_2g \quad (2)$$

彈簧是經由球殼頂點而施力於球殼。(1)式和(2)式可簡化為

$$\ddot{y}_1=-\frac{k}{m_1}(y_1-y_2)-g \quad (3)$$

Coupled SHM

$$\ddot{y}_2=\frac{k}{m_2}(y_1-y_2)-g \quad (4)$$

兩式相減得

$$\ddot{y}_1-\ddot{y}_2=-\left(\frac{m_1+m_2}{m_1m_2}\right)k(y_1-y_2) \quad (5)$$

對 y_1-y_2 而言，這是標準的簡諧振動方程式，其解可寫為

$$y_1-y_2=A\cos(\omega t+\phi) \quad (6)$$

$$\omega=\sqrt{\frac{k(m_1+m_2)}{m_1m_2}}$$

利用小球運動的起始條件，即當 t=0 時，

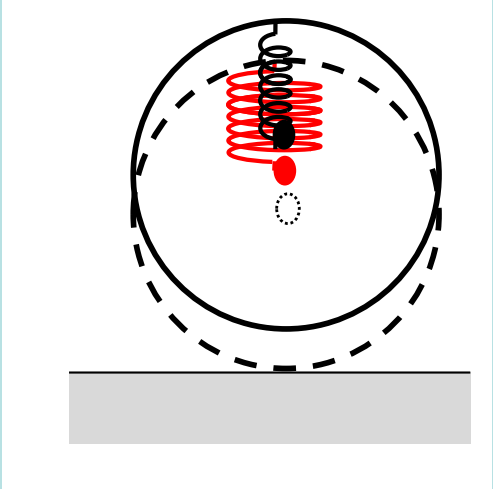
$$y_1 = \frac{m_2 g}{k} \equiv h\,, \quad \dot{y}_1 = v_s$$

$$y_2 = 0\,, \quad \dot{y}_2 = 0$$

代入(6)式，可解得

$$A\cos\phi = h\,,\ A\sin\phi = -\frac{v_s}{\omega} \quad \Rightarrow \quad A = \sqrt{h^2 + \frac{v_s^{\,2}}{\omega^2}}$$

$$\tan\phi = -\frac{v_s}{\omega h} \quad \Rightarrow \phi = -\tan^{-1}\left(\frac{v_s}{\omega h}\right)$$



$$y_1 - y_2 = A\cos(\omega t + \phi)$$

這是相對位置，要得到質點個別位置還需要質心的位置。