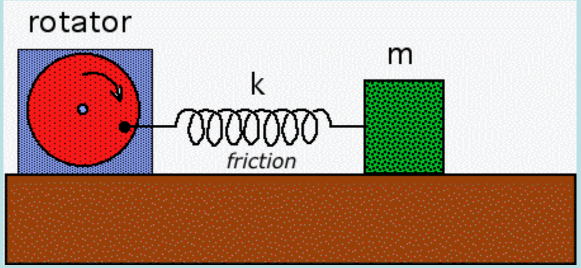
$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

General forced oscillation



$$f(t) \rightarrow F(\omega) \equiv \mathcal{F}\{f(t)\}$$

$$x(t) \rightarrow A(\omega) \equiv \mathcal{F}\{x(t)\}$$

Taking a Fourier Transformation on the whole differential equation:

$$(-i\omega)^2 A(\omega) + 2\alpha(-i\omega)^1 A(\omega) + \omega_0^2 A(\omega) = F(\omega)$$

Again, the differential equation is transformed into an algebraic equation for $A(\omega)$.

$$A(\omega) = \frac{F(\omega)}{\omega_0^2 - \omega^2 - 2\alpha i\omega}$$

Taking the inverse Fourier Transformation of $A(\omega)$, we solved the ODE.

$$\mathcal{F}^{-1}\{A(\omega)\} = x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(\omega)e^{-i\omega t}}{\omega_0^2 - \omega^2 - 2\alpha i\omega}\cdot d\omega$$

How to do the integral? Complex analysis.

Complex Analysis

Mathematical miracles do occur.

By making things more complicated, we could make them much more simple.

Elevate the real variable x of a continuous function $f(x)$ to a complex variable z.

$$f(x) \rightarrow f(z) \equiv u(x,y) + iv(x,y)$$

$$z = x + iy$$

For example:

$$z^2 = x^2 - y^2 + 2xyi$$

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

If $f(x)$ is continuous and good behaved like polynomials, $f(z)$ will be quite restricted.

Such complex functions are called analytic, meaning they have continuous derivatives.

Derivatives of complex functions can define as usual!

$$\frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

For example:

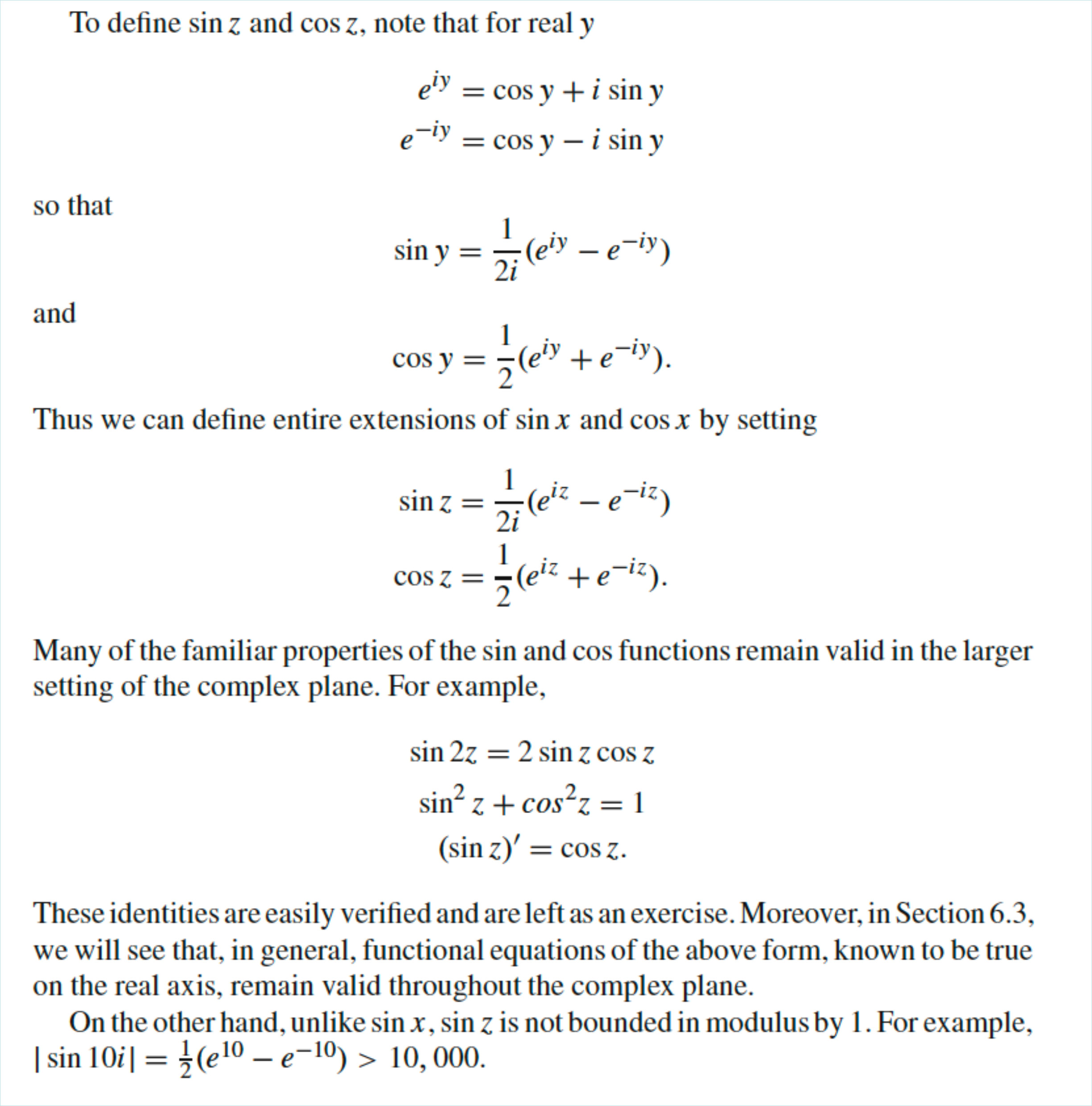
$$\frac{d}{dz} z^2 \equiv \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = 2z$$

Mononomials z^n have continuous derivatives and hence are analytic everywhere.

Finite degree Polynomials are analytic everywhere.

Some infinite degree Polynomials or power series are analytic, too.

For example, exponential function and Sine Cosine functions are analytic.



To define $\sin z$ and $\cos z$, note that for real y

$$\begin{aligned} e^{iy} &= \cos y + i \sin y \\ e^{-iy} &= \cos y - i \sin y \end{aligned}$$

so that

$$\sin y = \frac{1}{2i}(e^{iy} - e^{-iy})$$

and

$$\cos y = \frac{1}{2}(e^{iy} + e^{-iy}).$$

Thus we can define entire extensions of $\sin x$ and $\cos x$ by setting

$$\begin{aligned} \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\ \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}). \end{aligned}$$

Many of the familiar properties of the sin and cos functions remain valid in the larger setting of the complex plane. For example,

$$\begin{gathered} \sin 2z = 2 \sin z \cos z \\ \sin^2 z + cos^2 z = 1 \\ (\sin z)' = \cos z. \end{gathered}$$

These identities are easily verified and are left as an exercise. Moreover, in Section 6.3, we will see that, in general, functional equations of the above form, known to be true on the real axis, remain valid throughout the complex plane.

On the other hand, unlike $\sin x$, $\sin z$ is not bounded in modulus by 1. For example, $|\sin 10i| = \frac{1}{2}(e^{10} - e^{-10}) > 10,000$.

Analyticity poses extremely strong restrictions on the function.

The key is complex variables is essentially a two-dimensional plane.

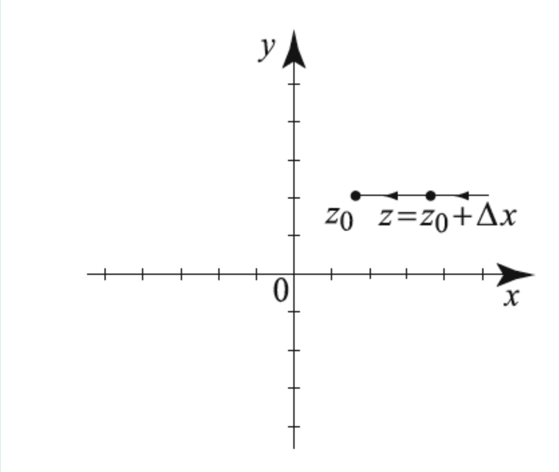
Its calculus is in essence vectorial calculus on plane.

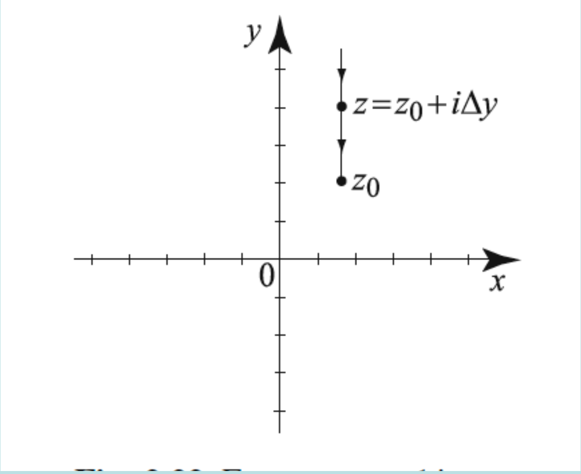
For derivatives, on a plane we can approach a point $\Delta z \to 0$ in multiple ways.

$$\frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$\Delta z = \Delta x \to 0$ $\quad$ $\Delta y = 0$ always

$\Delta z = i\Delta y \to 0$ $\quad$ $\Delta x = 0$ always

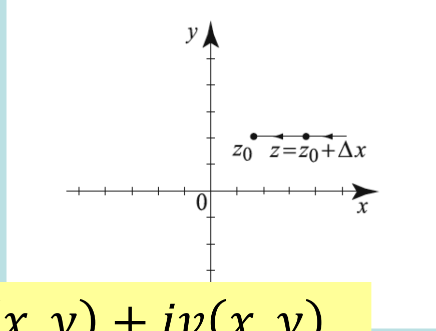




If $\Delta z = \Delta x \to 0$

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) + iv(x,y)}{\Delta x}$$

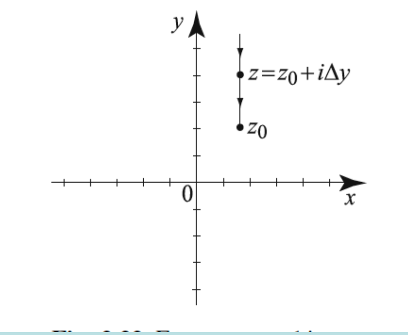
$$= \lim_{\Delta x \to 0} \frac{\frac{\partial u}{\partial x}\Delta x + i\frac{\partial v}{\partial x}\Delta x}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$



If $\Delta z = i\Delta y \to 0$

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y\Delta y) - u(x,y) + iv(x,y)}{i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\partial u}{\partial y}\Delta y + i\frac{\partial v}{\partial y}\Delta y}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$



The two expressions must equal to each other.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Cauchy-Riemann Condition for the real and imaginary parts of an analytic function.

$$z^2 = x^2 - y^2 + 2xyi \qquad u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$$

$$\int_{\Gamma} f(z)dz$$

Complex integration is along a path on the complex plane and path dependent!

It is basically a line integral on 2D plane.

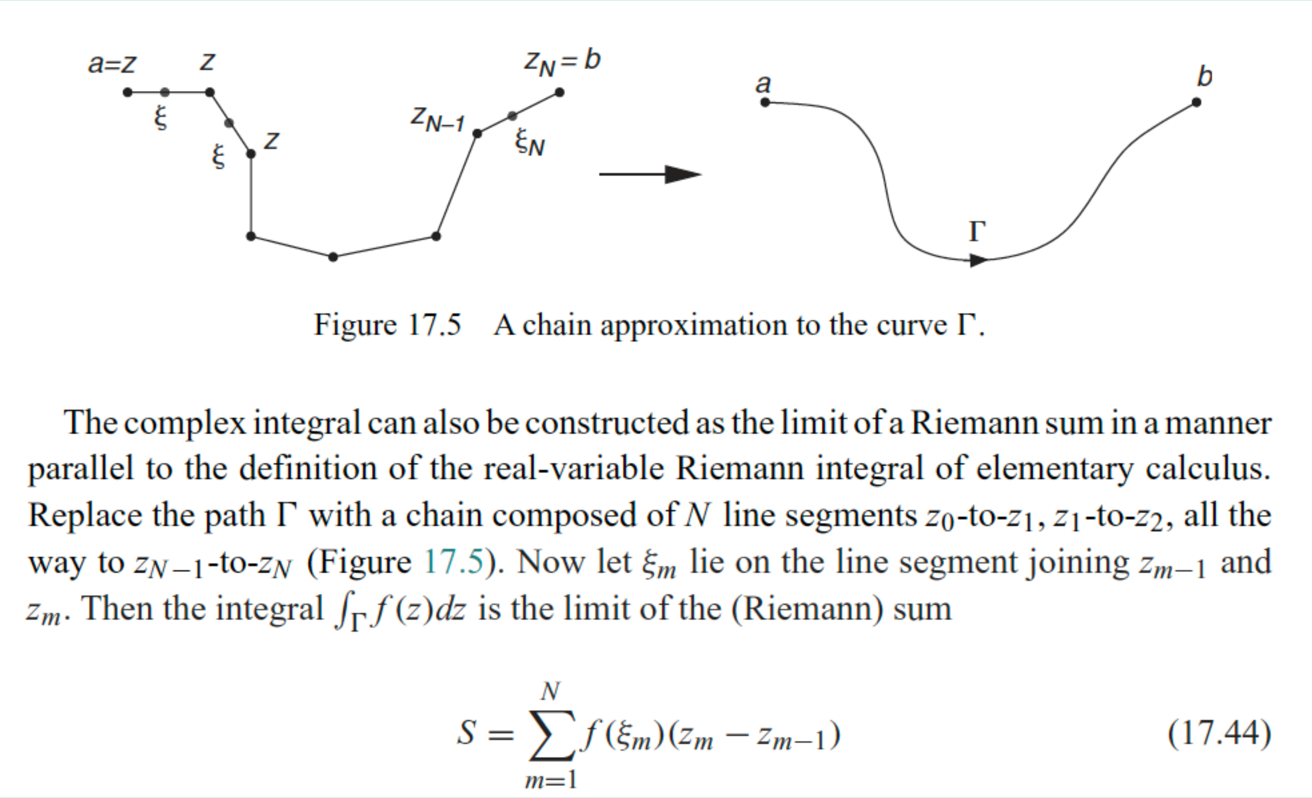


Figure 17.5 A chain approximation to the curve Γ.

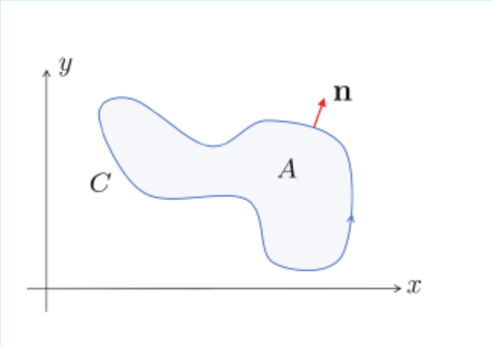
The complex integral can also be constructed as the limit of a Riemann sum in a manner parallel to the definition of the real-variable Riemann integral of elementary calculus. Replace the path Γ with a chain composed of N line segments z_0-to-z_1, z_1-to-z_2, all the way to z_{N-1}-to-z_N (Figure 17.5). Now let ξ_m lie on the line segment joining z_{m-1} and z_m. Then the integral $\int_\Gamma f(z)dz$ is the limit of the (Riemann) sum

$$S = \sum_{m=1}^{N} f(\xi_m)(z_m - z_{m-1}) \tag{17.44}$$

$$\int_\Gamma f(z)dz = \int_\Gamma (u+iv)\cdot d(x+iy) = \int_\Gamma (udx - vdy) + i\int_\Gamma (vdx + udy)$$

When Γ is a closed curve, we call it a contour C!

$$\oint f(z)dz = \oint (udx - vdy) + i\oint (vdx + udy)$$



Apply Stokes' Law to the contour on a plane.

$$\oint \vec{E}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$$

$$\oint (E_x dx + E_y dy) = \int \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\cdot \Delta x\Delta y$$

Assign u as E_x and $-v$ as E_y and apply the Cauchy-Riemann Condition:

$$\oint (udx - vdy) = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\cdot \Delta x\Delta y = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Assign v as E_x and u as E_y:

$$\oint (vdx + udy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\cdot \Delta x\Delta y = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\oint f(z)dz = 0$$

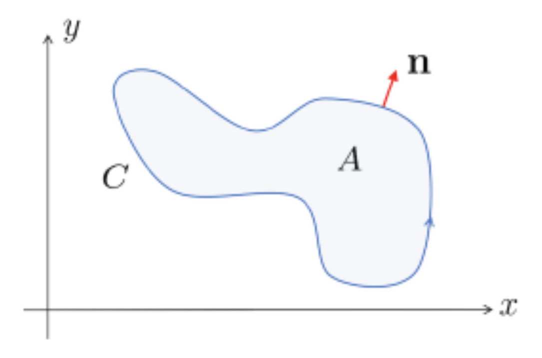
Contour integral is zero for analytic functions.

Cauchy Theorem

$$\oint f(z)dz = 0$$

Contour integral is zero for analytic functions.

Cauchy Theorem

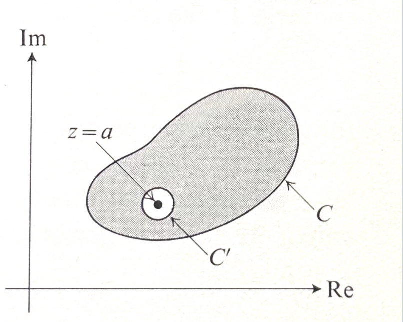


Contour integral is zero only for analytic functions.

Nonanalytic functions could be:

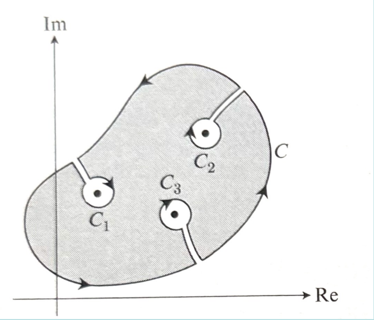
$$\frac{1}{z-a}$$

This function diverges at $z = a$. We call it a pole of order one!

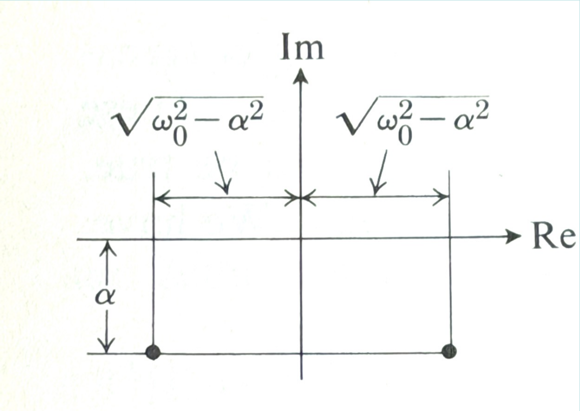


$$\frac{1}{(z-a_1)(z-a_2)(z-a_3)}$$

This function has 3 poles at $z = a_1, a_2, a_3$!



$$\frac{F(\omega)e^{i\omega t}}{\omega_0^2 - \omega^2 - 2\alpha i\omega}$$

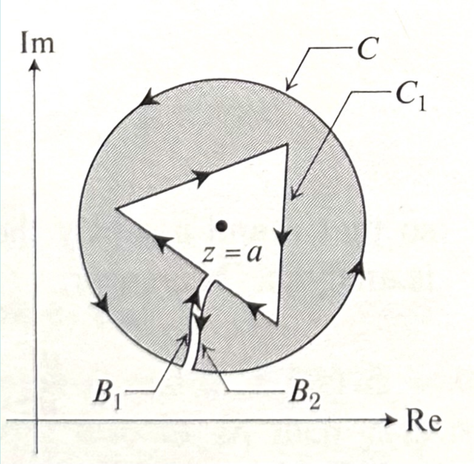


These functions have isolated singularities.

With Cauchy Theorem, we can devise contours that circumvent poles.

Along these contours, integrals equal zero.

This zero integral equals the difference between the integrals along contours C, C_1.

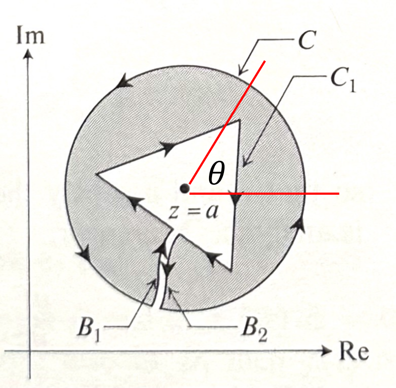


$$\oint_{C_1} \frac{1}{z-a} dz = \oint_{C} \frac{1}{z-a} dz = 2\pi i$$

Contour integrals of the function with poles are equal, independent of contours.

We can compute contour integral of functions with a pole choosing the best contour: a circle with center at $z = a$.

$$\oint_C \frac{1}{z-a}dz = \int_0^{2\pi} \frac{1}{Re^{i\theta}} Re^{i\theta} i d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$



$$\oint_C \frac{1}{z-a}dz = 2\pi i$$

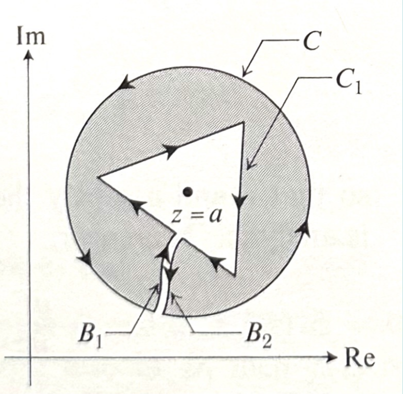
The result is independent of R.

Contour integral of functions with poles of order two or higher is zero.

$$\oint_C \frac{1}{(z-a)^2}dz = \int_0^{2\pi} \frac{1}{R^2 e^{i2\theta}} Re^{i\theta} i d\theta = \frac{2}{R}\int_0^{2\pi} e^{-i\theta} d\theta = 0$$

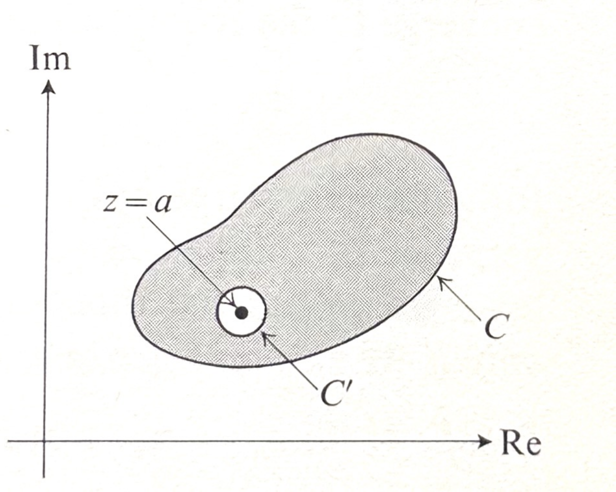
$$\oint_C \frac{1}{(z-a)^n}dz = 0$$

Contour integrals of a function around its pole are equal, independent of contours.



$$\oint_{C_1} \frac{1}{z-a}dz = \oint_C \frac{1}{z-a}dz = 2\pi i$$

Cauchy Integral Formula



$$\oint_C \frac{g(z)}{z-a}dz = \oint_{C'} \frac{g(z)}{z-a}dz$$

$$\oint_{C'} \frac{g(z)}{z-a}dz \xrightarrow[R\to 0]{} \int_0^{2\pi} \frac{g(a)}{Re^{i\theta}} Re^{i\theta} i d\theta = 2\pi i g(a)$$

$$\oint_C \frac{g(z)}{z-a}dz = 2\pi i g(a)$$

We can calculate the function value in the middle from values on the edge.

For a function $f(z)$ with a pole at $z = a$:

(you could add analytic function $A(z)$ or higher poles $\frac{1}{(z-a)^n}$, only pole order 1 matters)

$$f(z) \equiv \frac{g(z)}{z-a} + \cdots$$

$$\oint A(z)dz = 0$$

$$\frac{1}{2\pi i}\oint_{C'} f(z)dz = g\,(a) \equiv \text{Res}\, f(a)$$

$$\oint_C \frac{1}{(z-a)^n}dz = 0$$

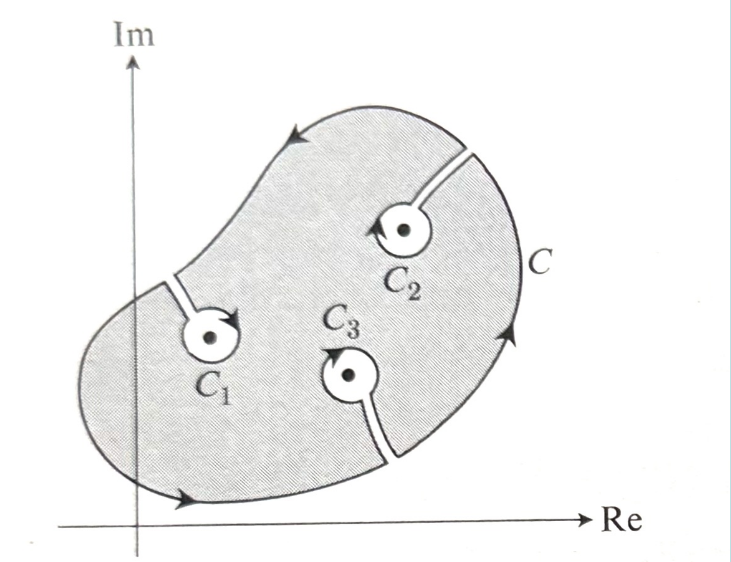
This only non-zero contour integral of $f(z)$ is called its residue 留數.

If the function has more than one poles, contour integral would be the sum of its residue at all the poles, defined as the contour integral along small circles round the poles.

$$f(z) \equiv \frac{g(z)}{(z-a_1)(z-a_2)(z-a_3)}$$

$$\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_{C_1} f(z)dz + \frac{1}{2\pi i}\oint_{C_2} f(z)dz + \frac{1}{2\pi i}\oint_{C_3} f(z)dz =$$

$$= \text{Res}\, f(a_1) + \text{Res}\, f(a_2) + \text{Res}\, f(a_3)$$



$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(a_k)$$

Contour integrals of a function equal the sum of residues of poles inside the contour.

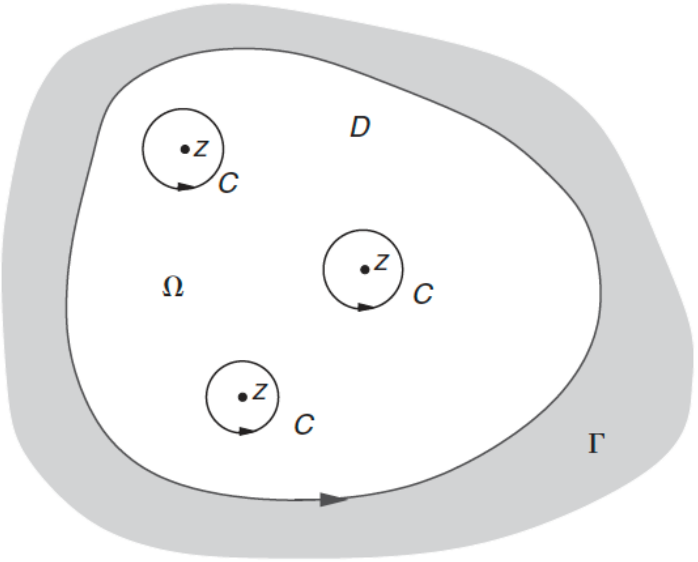
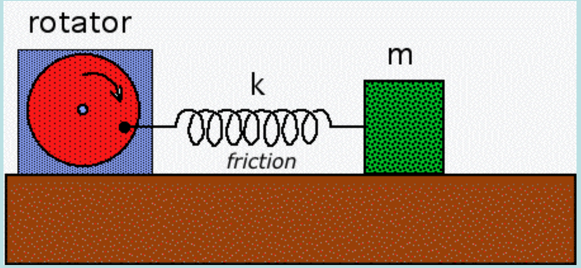


Figure 17.6 Circles for the residue theorem.

$$f(z) \equiv \frac{g(z)}{z-a}$$

$$\operatorname{Res} f(a) = \frac{1}{2\pi i}\oint_C f(z)dz = g(a)$$

The residue at a pole is defined as the contour integral along a small circle around the pole.



$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$ General forced oscillation

$$A(\omega) = \frac{F(\omega)}{\omega_0^2 - \omega^2 - 2\alpha i\omega}$$

Taking the inverse Fourier Transformation of $A(\omega)$, we solved the ODE.

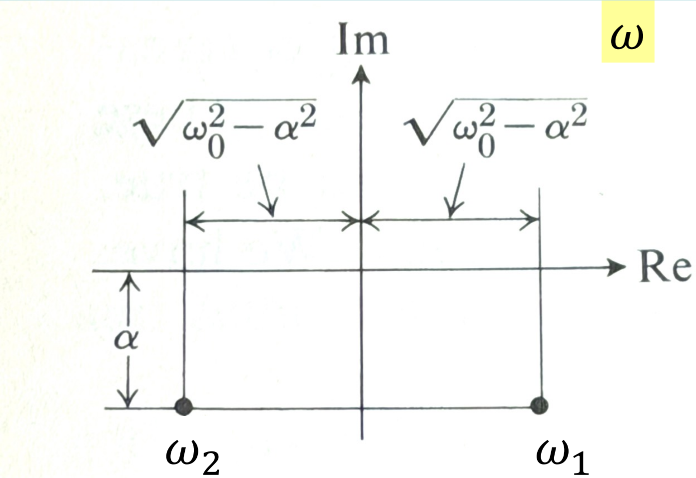
$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(\omega)e^{-i\omega t}}{\omega_0^2 - \omega^2 - 2\alpha i\omega}\cdot d\omega$$

How to do the integral? Complex analysis.

$$\frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}$$

Elevate ω to complex number.

$$\frac{F(\omega)e^{-i\omega t}}{(\omega-\omega_2)(\omega_1-\omega)}$$



There are Poles at: $\omega=-\alpha i\pm\sqrt{\omega_0^2-\alpha^2}\equiv\omega_{1,2}$

$$f(z)\equiv\frac{g(z)}{z-a}$$

$$\text{Res}\,f(a)=g(a)$$

For the pole at ω_1: $\frac{F(\omega)e^{i\omega t}}{(\omega-\omega_2)}\cdot\frac{1}{(\omega_1-\omega)}$

Res at ω_1 $=\frac{F(\omega_1)e^{i\omega_1 t}}{(\omega_1-\omega_2)}$

For the pole at ω_2: $\frac{F(\omega)e^{i\omega t}}{(\omega_1-\omega)}\cdot\frac{1}{(\omega-\omega_2)}$

Res at ω_2 $=-\frac{F(\omega_2)e^{i\omega_2 t}}{(\omega_1-\omega_2)}$

Consider a contour of lower half circle:

$$\oint \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega$$

The contour integral equals sum of the residues at $\omega_{1,2}$.

The contour integral is the same regardless of R.

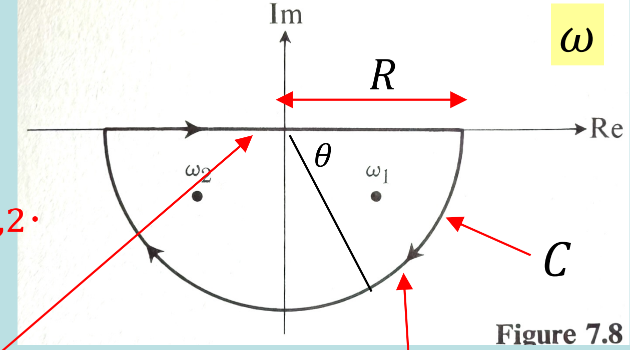


Figure 7.8

$$\oint \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega = \int_{-R}^{R} \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega + \int_{C} \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega$$

The integral on real axis is part of this contour integral.

On the lower circle C, the imaginary part of ω in the exponential is proportional to R.

$e^{-i\omega t}$ becomes small as the circle becomes large if $t > 0$

$$\mathrm{Im}\,\omega = -R\sin\theta$$

$$e^{-i\omega t} \propto e^{-R\sin\theta t} \xrightarrow[\substack{R\to\infty\\ t>0}]{} 0$$

Take the limit $R \to \infty$, the integral along C vanishes.

$$\int_{C} \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega \xrightarrow[R\to\infty]{} 0$$

Contour integrals of a function equal the sum of residues of poles inside.

The contour integral equals the integral on the real axis:

$$\oint \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega \xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} \frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega = 2\pi i(\text{Res at }\omega_1+\text{ Res at }\omega_2)$$

$$x(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(\omega)e^{-i\omega t}}{\omega_0^2-\omega^2-2\alpha i\omega}\cdot d\omega = i\sqrt{2\pi}\left[\frac{F(\omega_1)e^{-i\omega_1 t}}{(\omega_1-\omega_2)}-\frac{F(\omega_2)e^{-i\omega_2 t}}{(\omega_1-\omega_2)}\right]$$

$$=i\sqrt{2\pi}\frac{1}{(\omega_1-\omega_2)}\left[F(\omega_1)e^{-i\omega_1 t}-F(\omega_2)e^{-i\omega_2 t}\right]$$

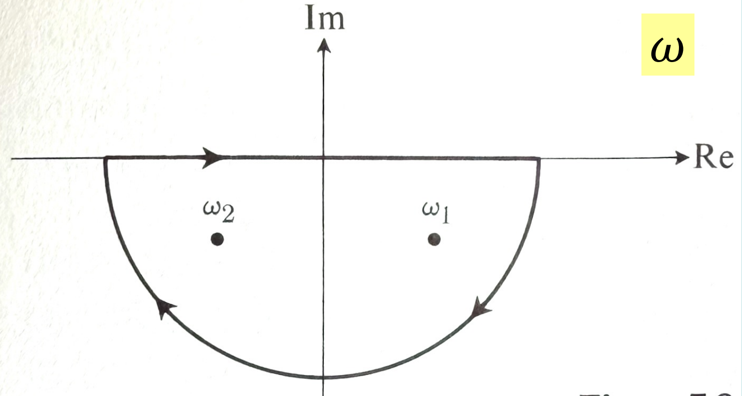
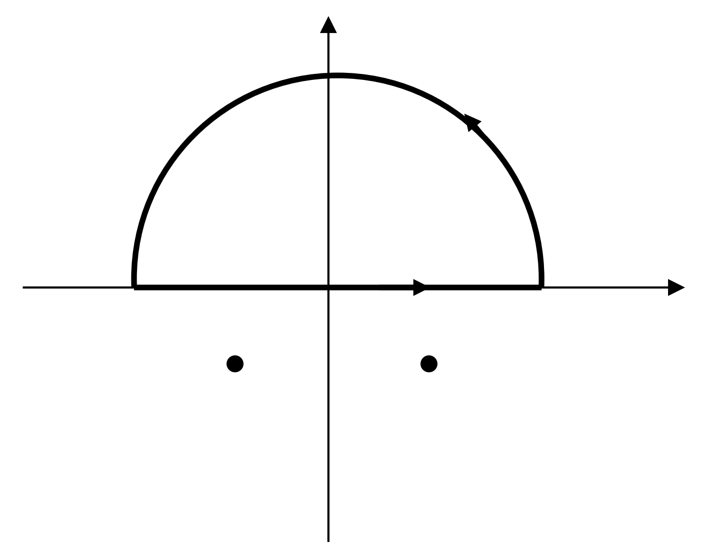


Figure 7.8

What if $t < 0$



Now we would choose an upper half circle.

On the upper circle C, the imaginary part of ω is positive. $\operatorname{Im}\omega = R\sin\theta$

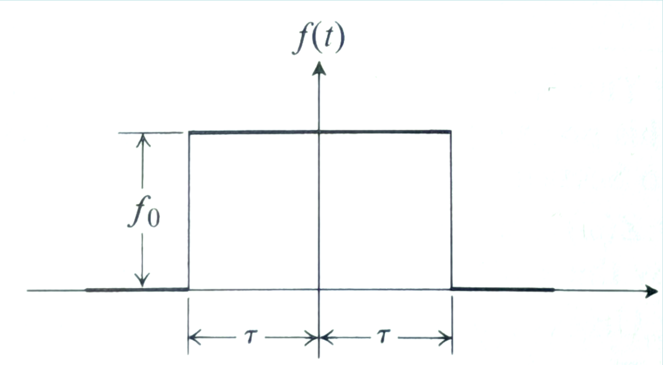
$e^{-i\omega t}$ again becomes small as the circle becomes large.

$$e^{-i\omega t} \propto e^{R\sin\theta t} \xrightarrow[\substack{R\to\infty \\ t<0}]{} 0$$

$$\oint \frac{F(\omega)e^{-i\omega t}}{\omega_0^2 - \omega^2 - 2\alpha i\omega} \cdot d\omega \xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} \frac{F(\omega)e^{-i\omega t}}{\omega_0^2 - \omega^2 - 2\alpha i\omega} \cdot d\omega = 0$$

But now the contour does not surround the poles, and the integral is zero.

As an example, consider:



$$F(\omega) \equiv \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega t} \cdot dt = \frac{1}{\sqrt{2\pi}} \int_{-\tau}^{\tau} f_0 e^{i\omega t} \cdot dt$$

$$F(\omega) = \sqrt{\frac{2}{\pi}} f_0 \frac{\sin \omega\tau}{\omega}$$

$$x(t) = 2if_0 \left[\frac{\sin \omega_1 \tau}{\omega_1(\omega_1 - \omega_2)} e^{-i\omega_1 t} - \frac{\sin \omega_2 \tau}{\omega_2(\omega_1 - \omega_2)} e^{-i\omega_2 t}\right]$$

$$\omega_{1,2} = \pm\sqrt{\omega_0^2 - \alpha^2} - \alpha i \equiv= \pm p - \alpha i$$

$x(t)$ can be simplified into a real form:

$$x(t) = \frac{f_0}{\omega_0^2}\left[\cos p(t-\tau) + \frac{\alpha}{p} \sin p(t-\tau)\right] e^{-\alpha(t-\tau)}$$

$$-\frac{f_0}{\omega_0^2}\left[\cos p(t+\tau) + \frac{\alpha}{p} \sin p(t+\tau)\right] e^{-\alpha(t+\tau)}$$