動量 Momentum

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(m\frac{d\vec{r}}{dt}\right)$$

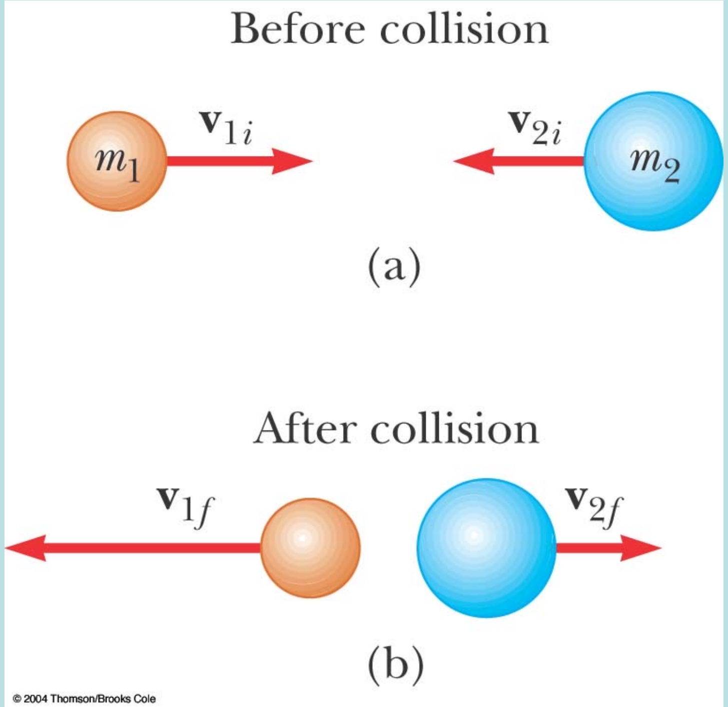
$$\vec{p} = m\frac{d\vec{r}}{dt} = m\vec{v}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$ 力是動量的變化率

若外力為零，粒子的動量是守恆量！

$$\vec{F}_{\text{ext}} = 0 = \frac{d\vec{p}}{dt}$$

動量守恆定律 Momentum Conservation Law



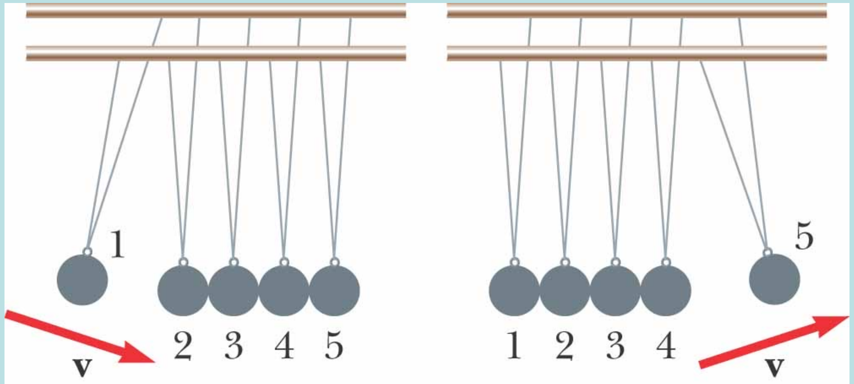
$$\vec{F}_{1\to 2} = -\vec{F}_{2\to 1}$$

$$\frac{d\vec{p}_2}{dt} = -\frac{d\vec{p}_1}{dt}$$

$$\frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = 0$$

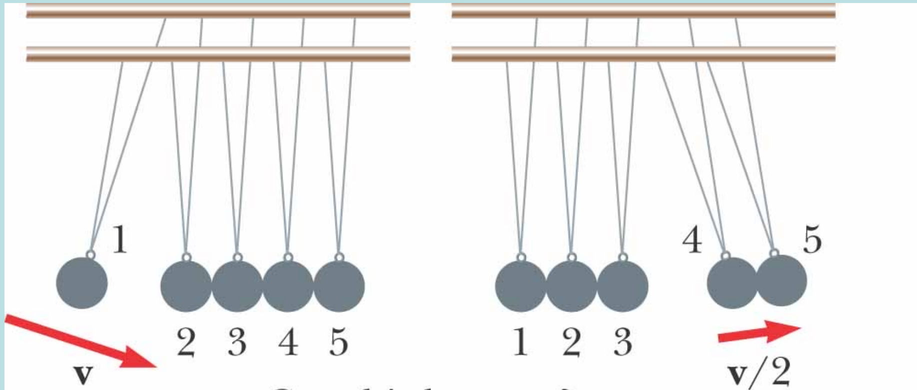
在碰撞過程中，兩粒子的動量和，也就是總動量，是守恆量！

$$m_1 v_{1i} - m_2 v_{2i} = -m_1 v_{1f} + m_2 v_{2f}$$



This can happen.

(b)

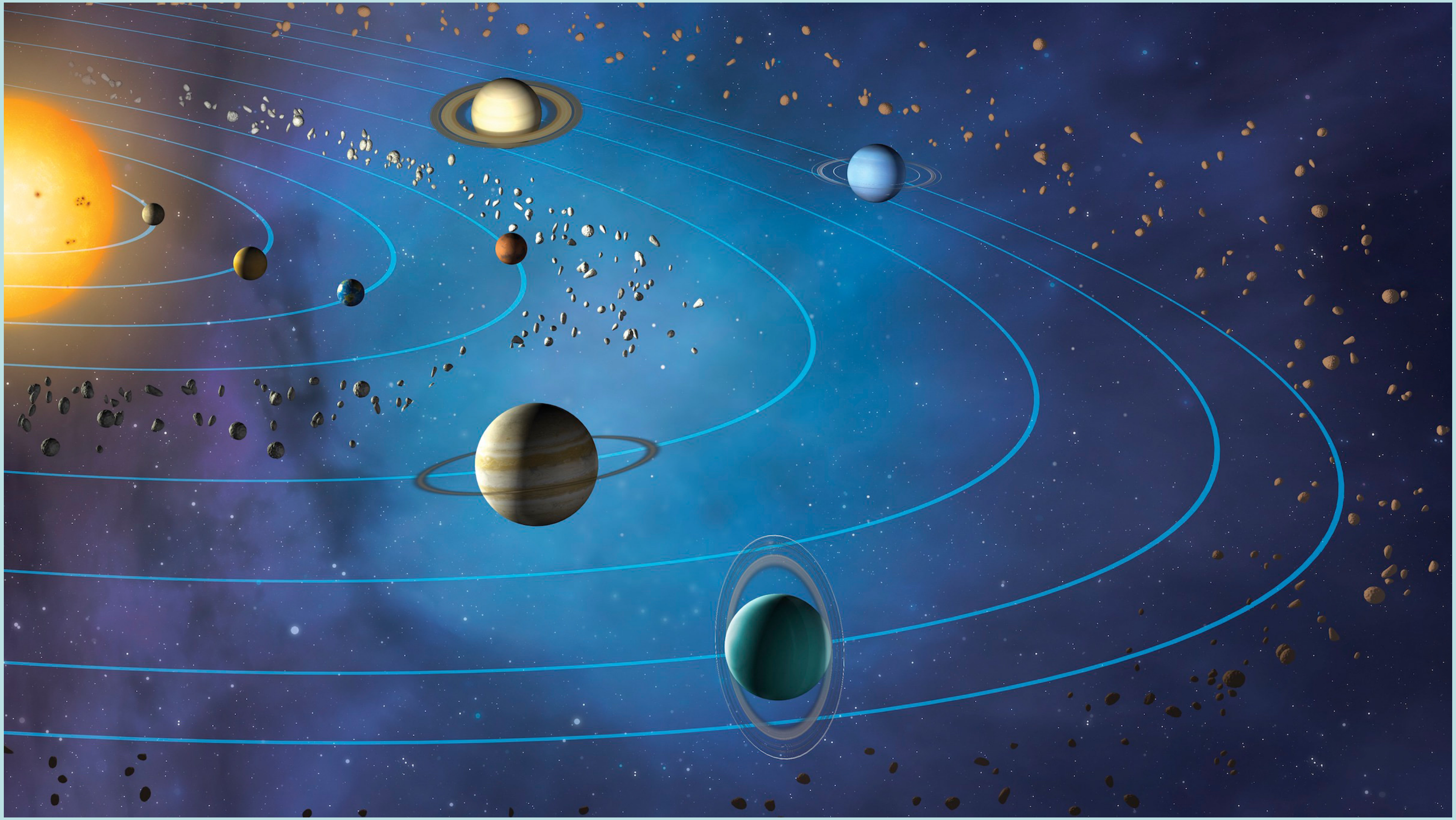


Can this happen?

(c)

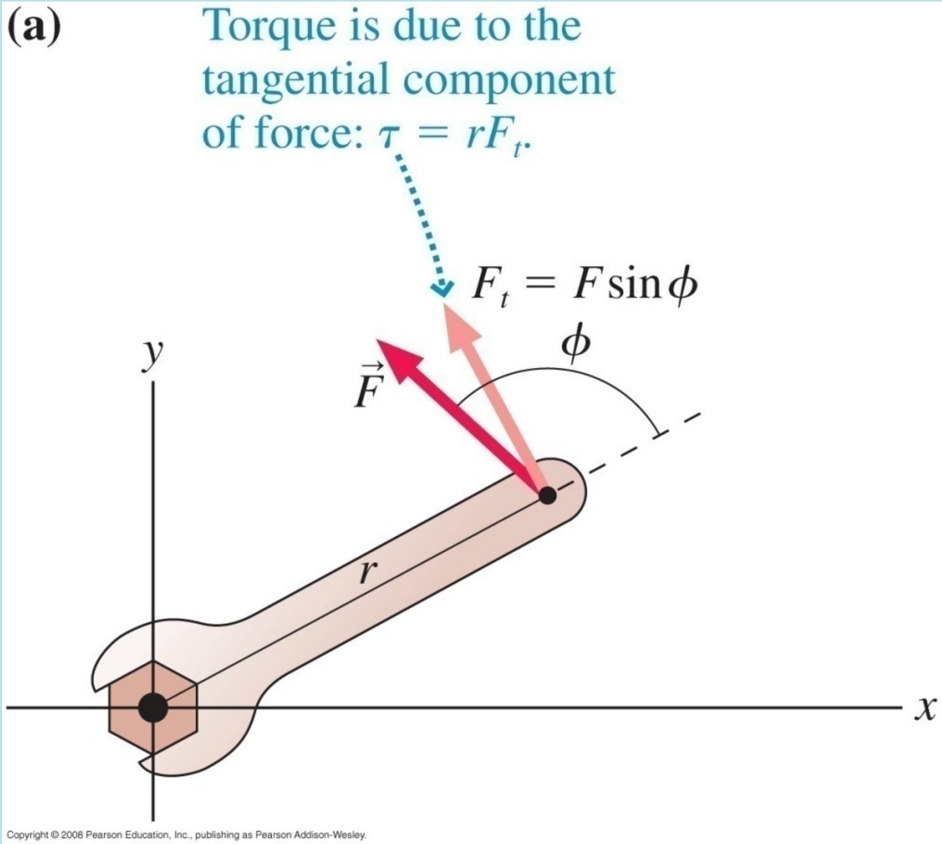
粒子在外力影響下作旋轉運動 Rotation

旋轉運動依舊由牛頓運動定律控制，但可以作一些設計使討論更容易！



物體也可受力旋轉！

造成旋轉的物理量，應該與力、力的方向、及施力的位置同時有關！



平行於力臂的力對旋轉沒有幫助！因此將垂直力臂的力乘上力臂。

$$F_t \cdot r = F\sin\phi \cdot r \equiv \tau$$

力矩 Torque 扭力

大膽的猜想：力在旋轉運動的對應是力矩！

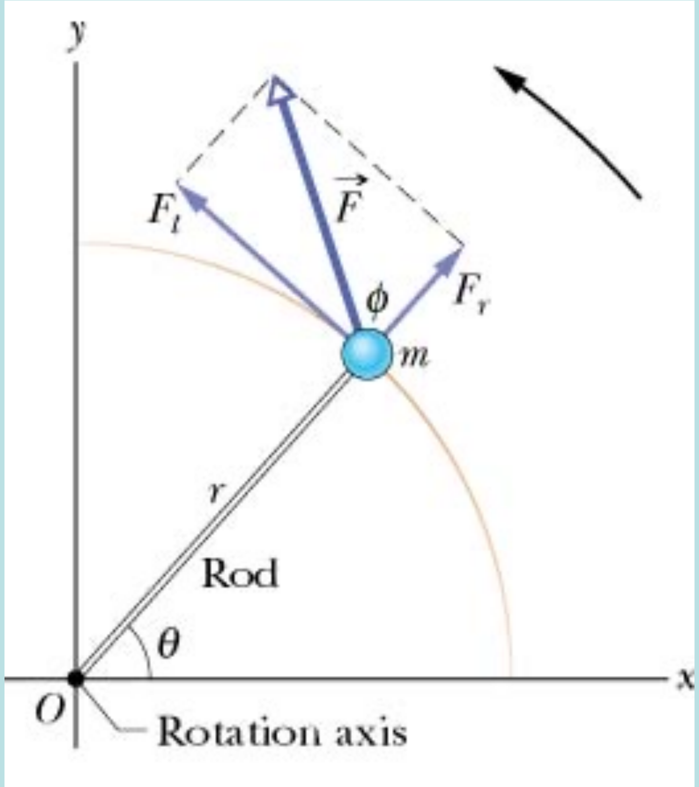
力矩 Torque 扭力

$\tau \longleftrightarrow F$

$$\tau = r \cdot F_t = r \cdot F \sin\phi = \left|\vec{r}\times\vec{F}\right|$$

如此可以定義力矩為一向量$\vec{\tau}$：

$$\vec{\tau} \equiv \vec{r}\times\vec{F}$$



$\vec{\tau} \equiv \vec{r} \times \vec{F}$　這樣的定義有用嗎？

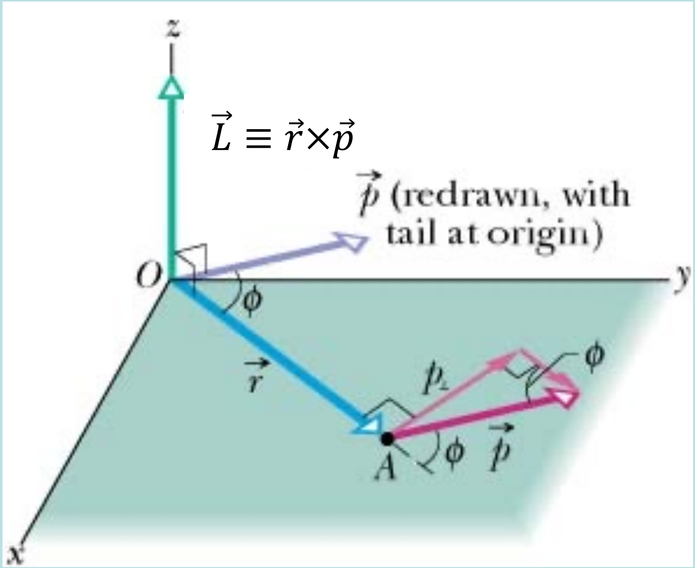
考慮力矩作用於一個粒子之上所產生的效應。

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) - \cancel{\frac{d\vec{r}}{dt} \times \vec{p}} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\vec{v} \times m\vec{v} = 0$$

$\vec{\tau} = \frac{d\vec{L}}{dt}$　力矩可以寫成一個向量物理量的變化率：

$\vec{L} \equiv \vec{r} \times \vec{p}$　一個粒子的角動量

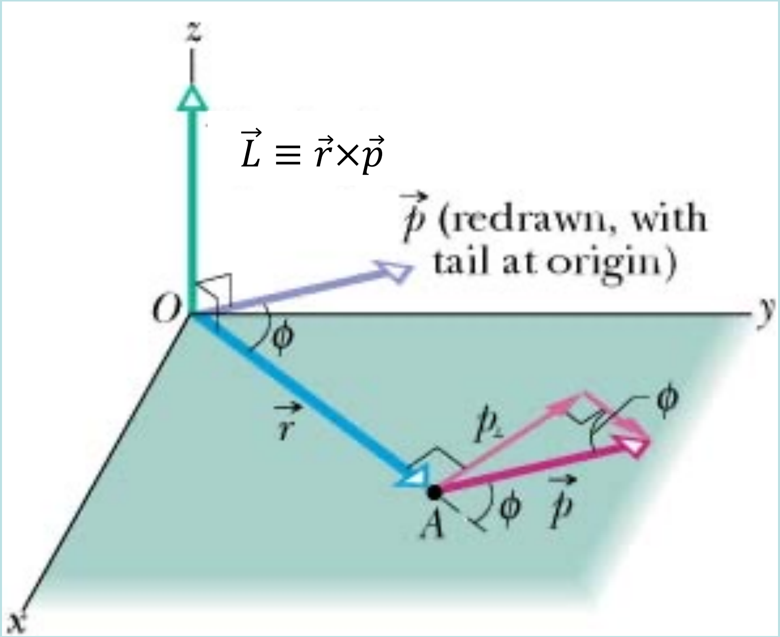


此公式決定了粒子的旋轉運動。

$\vec{F} = \frac{d\vec{p}}{dt}$　如同牛頓第二定律。

力在旋轉運動的對應真的是是力矩！

角動量的物理意義最重要就是它的守恆律，現在以軌道運動來說明。



$\vec{\tau} = \dfrac{d\vec{L}}{dt}$ 力矩真的可以寫成一個向量物理量的變化率：

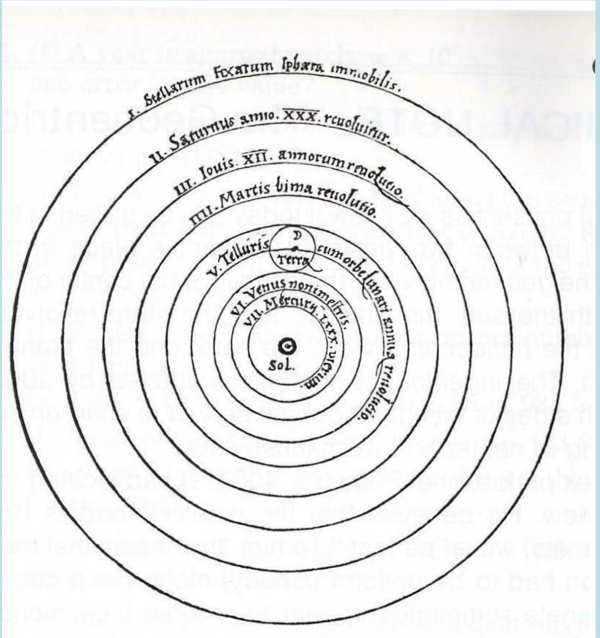
$\vec{L} \equiv \vec{r}\times\vec{p}$ 一個粒子的角動量

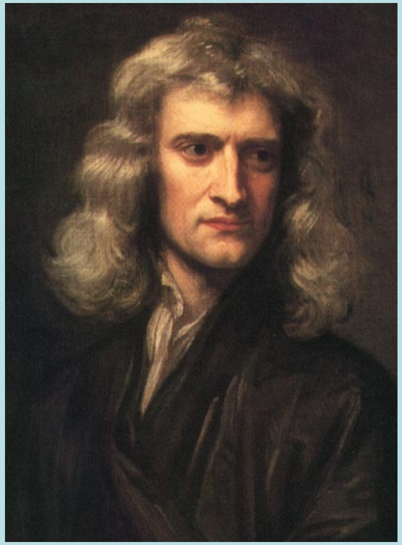
注意：$\vec{L}$的方向垂直於$\vec{r},\vec{p}$兩個向量所展開的平面，如上圖。

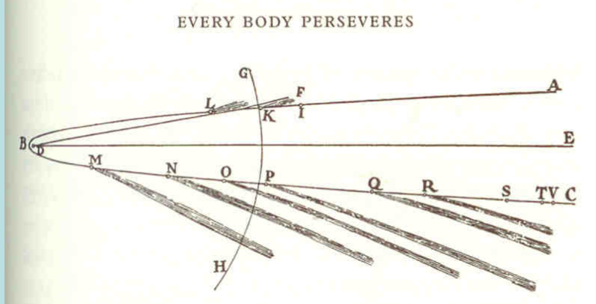
若$\vec{\tau} = 0$，則角動量$\vec{L}$守恆。

星體的軌道運動就是角度量守恆的運動。

行星、彗星繞太陽是由於太陽的萬有引力：

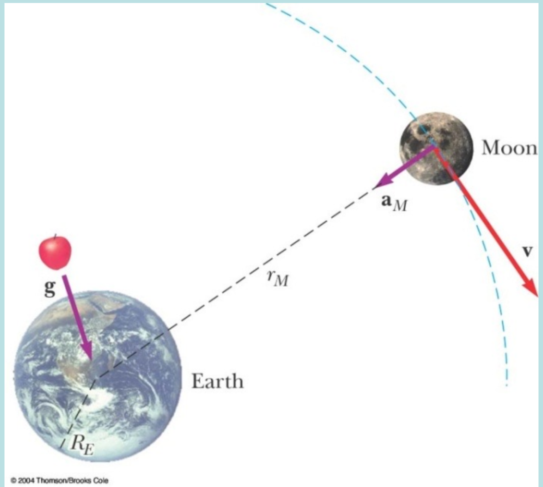






The comet of 1680—"as observed by Flamsteed" and "corrected by Dr. Halley." Newton also collated sightings by Ponthio in Rome, Gallet in Avignon, Ango at La Fleche, "a young man" at Cambridge, and Mr. Arthur Storer near Hunting Creek, in Maryland, in the confines of Virginia. "Thinking it would not be improper, I have given . . . a true representation of the orbit which this comet described, and of the tail which it emitted in several places." He concludes that the tails of comets always rise away from the sun and "must be derived from some reflecting matter"—smoke, or vapor.

萬有引力是指向原點的施力者：$\vec{F} = -G\dfrac{mM}{r^2}\hat{r}$



沿力臂方向$\hat{r}$的力產生的力矩為零。

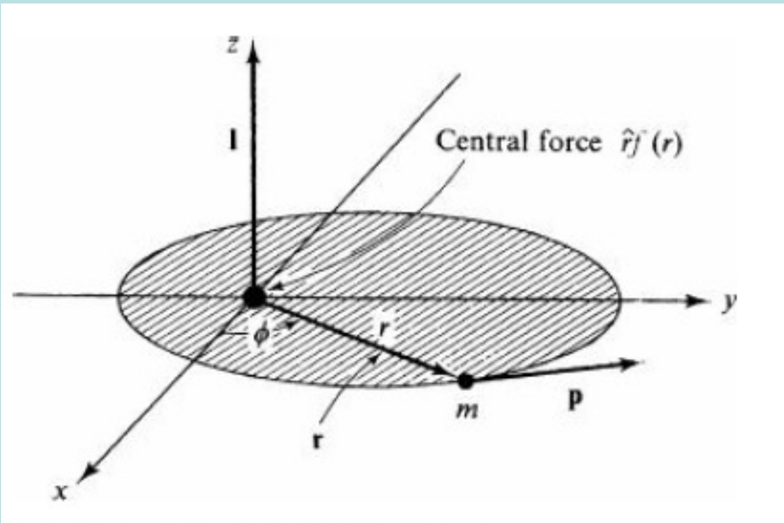
$$\vec{\tau} \equiv \vec{r}\times\vec{F} \sim \vec{r}\times\hat{r} = 0$$

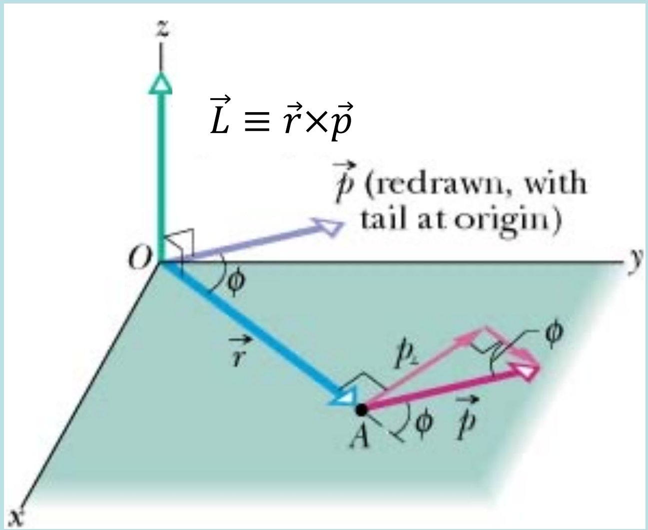
$$\frac{d\vec{L}}{dt} = 0$$

$\vec{\tau} = 0$，角動量$\vec{L}$守恆。對萬有引力作用下的運動，角動量守恆！

$\vec{L}$的方向垂直於$\vec{r},\vec{p}$兩個向量所展開的平面，因此此平面亦不變。

行星運動中$\vec{r},\vec{p}$都會維持在一個固定平面上。因此軌道運動是一如下圖的平面運動



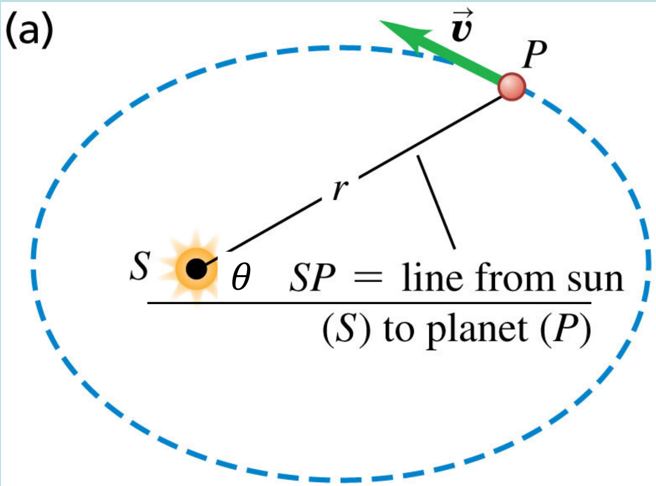


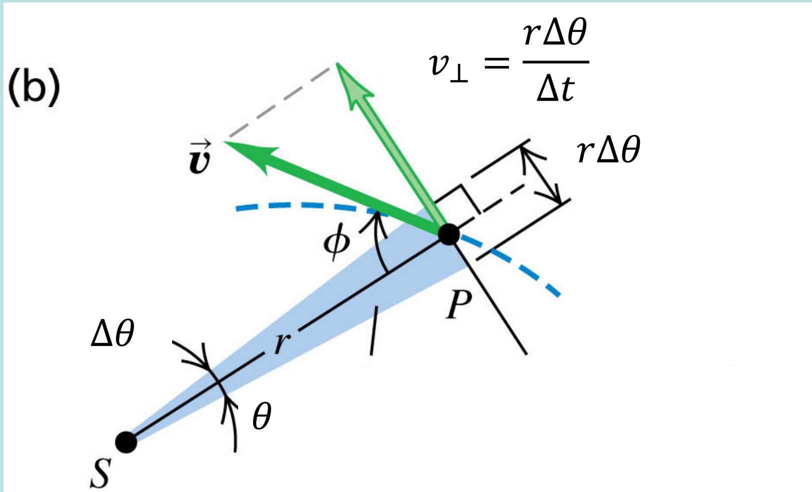
將平面軌道位置的垂直座標以距離$r(t)$及角度$\theta(t)$取代，稱為極座標！

$$L \equiv |\vec{r} \times \vec{p}| = r \cdot p_{\perp} = r \cdot m \frac{弧長}{\Delta t} = r \cdot m \frac{r\Delta\theta}{\Delta t} \rightarrow mr^2 \frac{d\theta}{dt} \equiv mr^2\omega$$

$$\frac{d\theta}{dt} \equiv \omega$$

角度的變化率ω稱為角速度，在軌道運動中角速度會改變，是時間的函數。



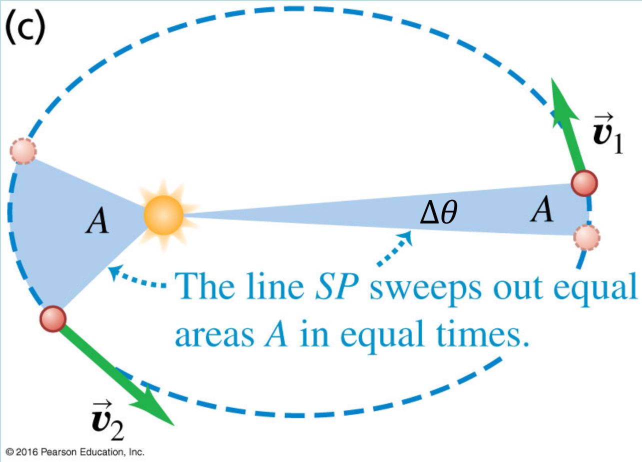


$L = mr^2\omega$ 當粒子的角動量守恆，大小記為l。

$$\omega = \frac{l}{mr^2}$$

角速度ω由距離r決定！距離r小的時候，角速度ω大，轉得快。

而且$L = mr^2\omega$有一個很簡單的幾何意義。



將角動量乘上單位時間差Δt

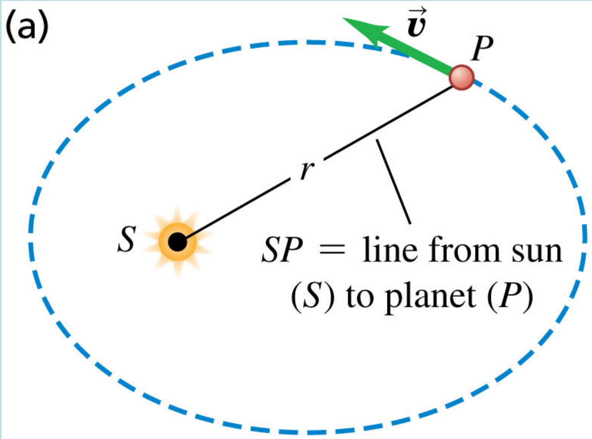
$$l \cdot \Delta t = mr^2 \frac{\Delta\theta}{\Delta t} \cdot \Delta t \propto r \cdot r\Delta\theta = r \cdot \text{弧長} \propto 2A$$

軌道運動的角動量l洽正比於單位時間內$\vec{r}$掃過的面積A！

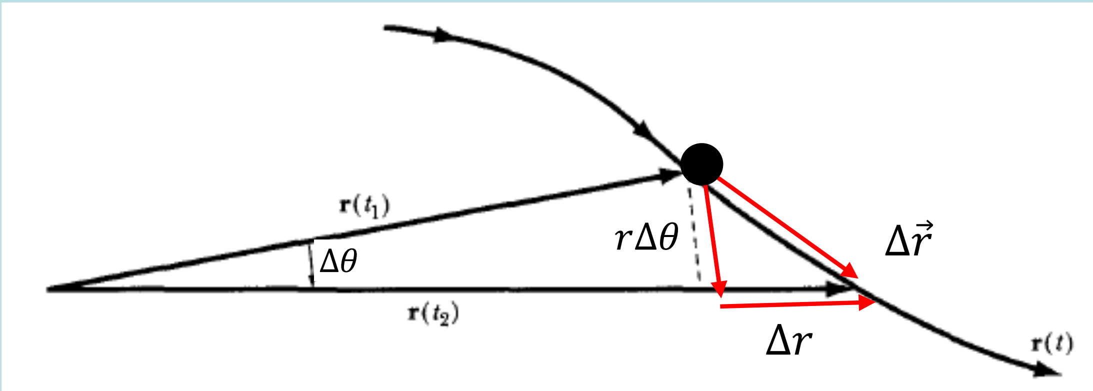
角動量守恆表示在軌道上各處，此面積是固定的。

這就是克卜勒第三定律！

面積若是固定的，距離r小的時候，轉得快，距離r遠的時候，轉得慢。。



考慮行星軌道運動的如下一小段。

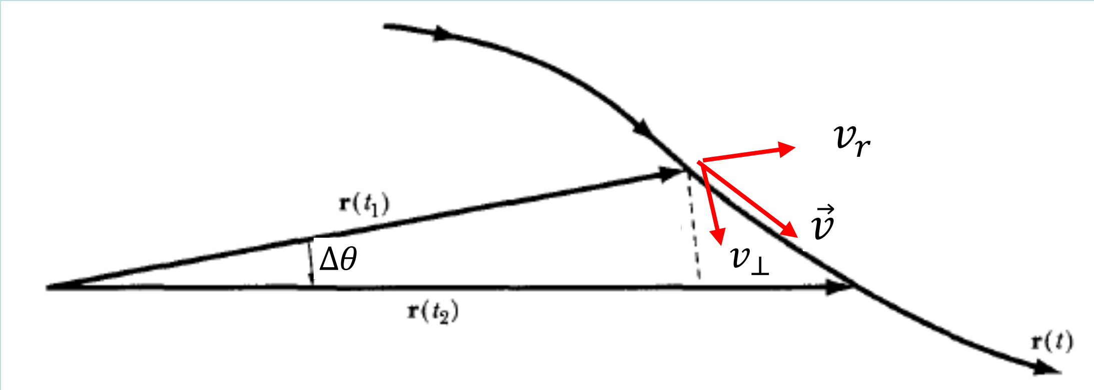


位移$\Delta\vec{r}$時，行星一邊改變與原點的距離r，位移是沿r的徑向： Δr

一邊繞原點旋轉，位移近似是一個弧長，與弧角$\Delta\theta$對應： $r\Delta\theta(t)$

關鍵是：這兩種運動近似是彼此垂直的！

所以沿這兩個方向的速度分量，就是沿這兩個方向位移的時間微分。



速度$\vec{v}$沿$\hat{r}$的方向的分量，就是距離的變化率！

$$v_r = \frac{\Delta r}{\Delta t} \to \frac{dr}{dt}$$

沿垂直於$\hat{r}$的方向的分量，就是旋轉的速率。

$$v_\perp = \frac{r\Delta\theta}{\Delta t} \to r\frac{d\theta}{dt} = r\omega$$

這兩個分量彼此垂直，速度大小就可以寫成：

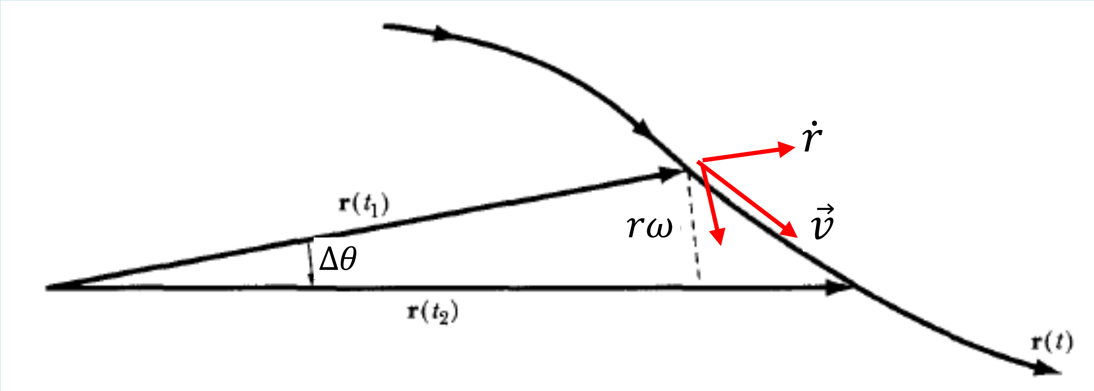
$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2$$

動能也有兩項：

轉動動能

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\omega^2\right]$$

似乎轉動與徑向運動可以分開看！



但旋轉的快慢，因為角動量守恆，完全由距離決定了。

$$\omega = \frac{l}{mr^2}$$

角速度ω由距離r決定！因此轉動動能完全由距離決定。

轉動動能$\frac{1}{2}mr^2\omega^2$如同位能一樣！將轉動動能併入位能之中！

$$\frac{1}{2}mr^2\omega^2 + V(r) \equiv V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)$$

動能就剩下：$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2$，如同一維$r$軸上的運動一般。

二維運動等價於一個在有效位能$V_{\text{eff}}(r)$作用下的一維運動！

機械能的公式可以很明顯看出來：

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\omega^2 + V(r) = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r)$$

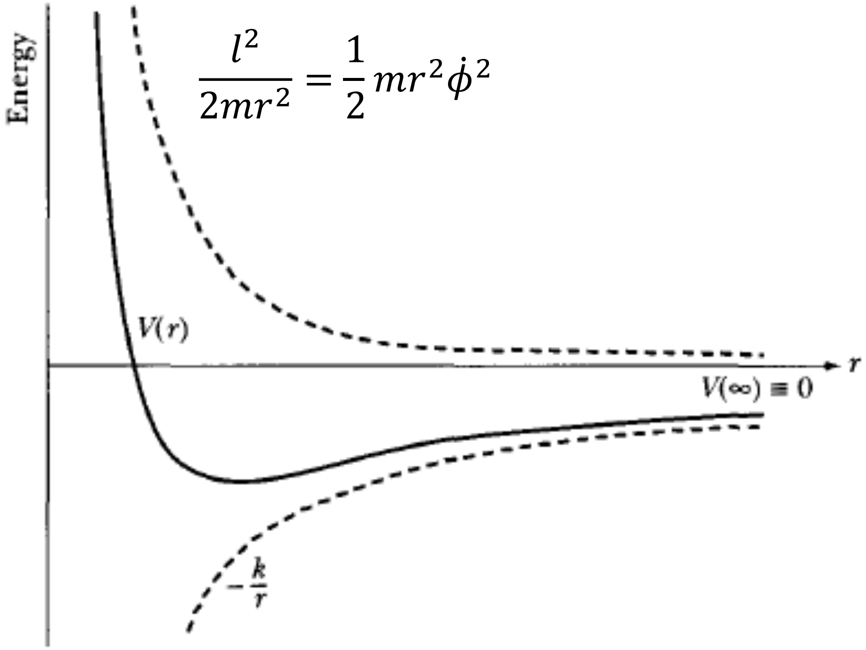


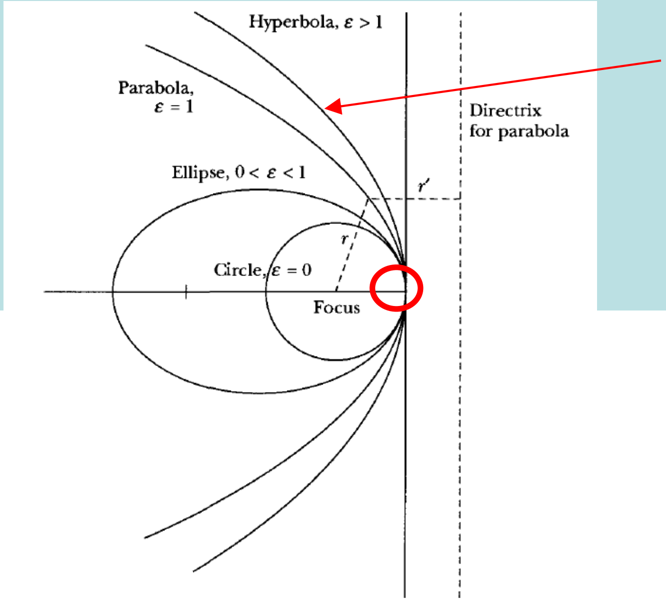
FIGURE 8-5 The effective potential for gravitational attraction $V(r)$ is composed of the real potential $-k/r$ term and the centrifugal potential energy $l^2/2\mu r^2$.

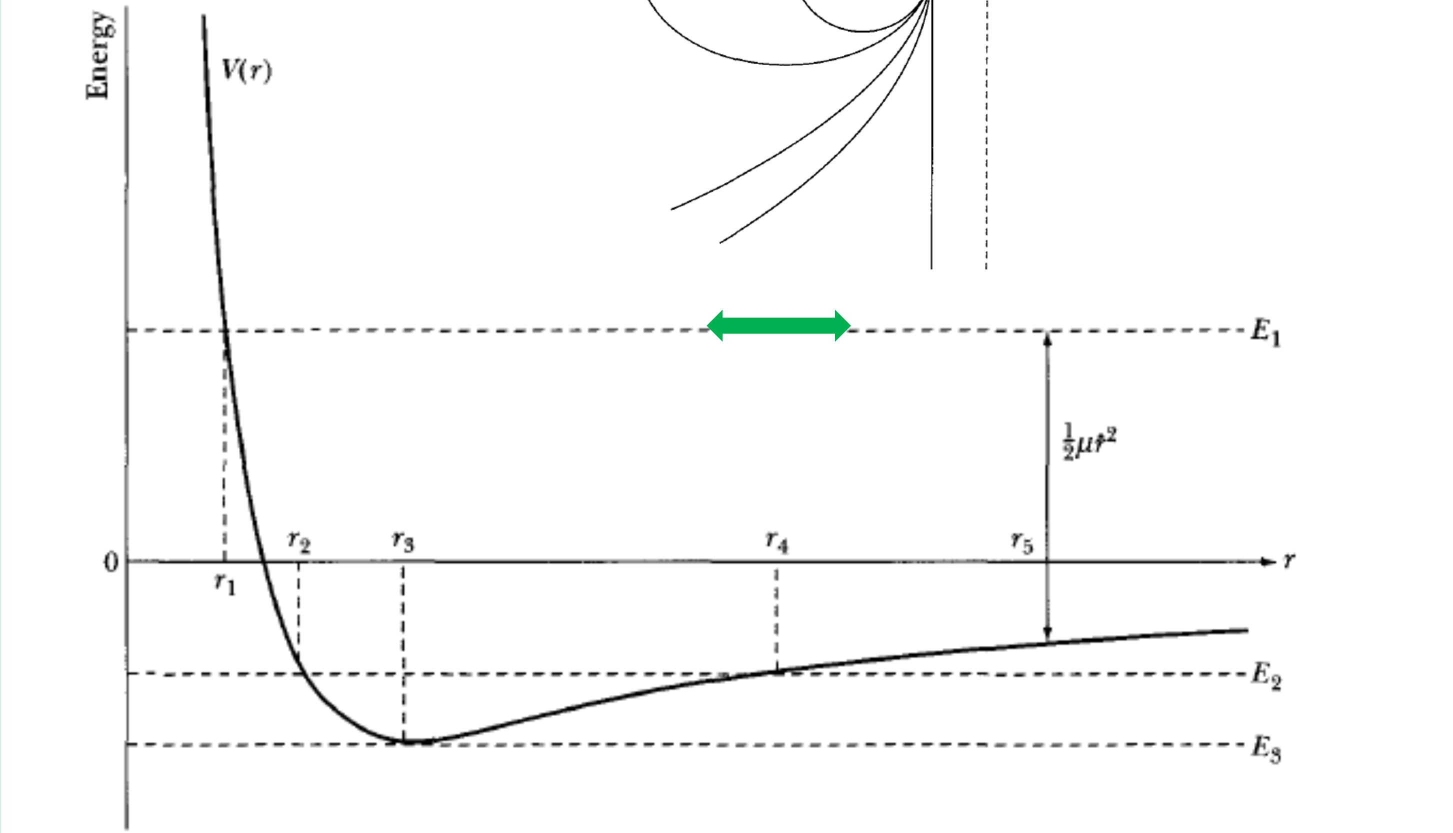
$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r)$$

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} - \frac{GMm}{r}$$

這等價於一個沿一維r軸，位能為$V_{\text{eff}}(r)$的運動。

一維運動能量圖的討論就能適用於這個情況！

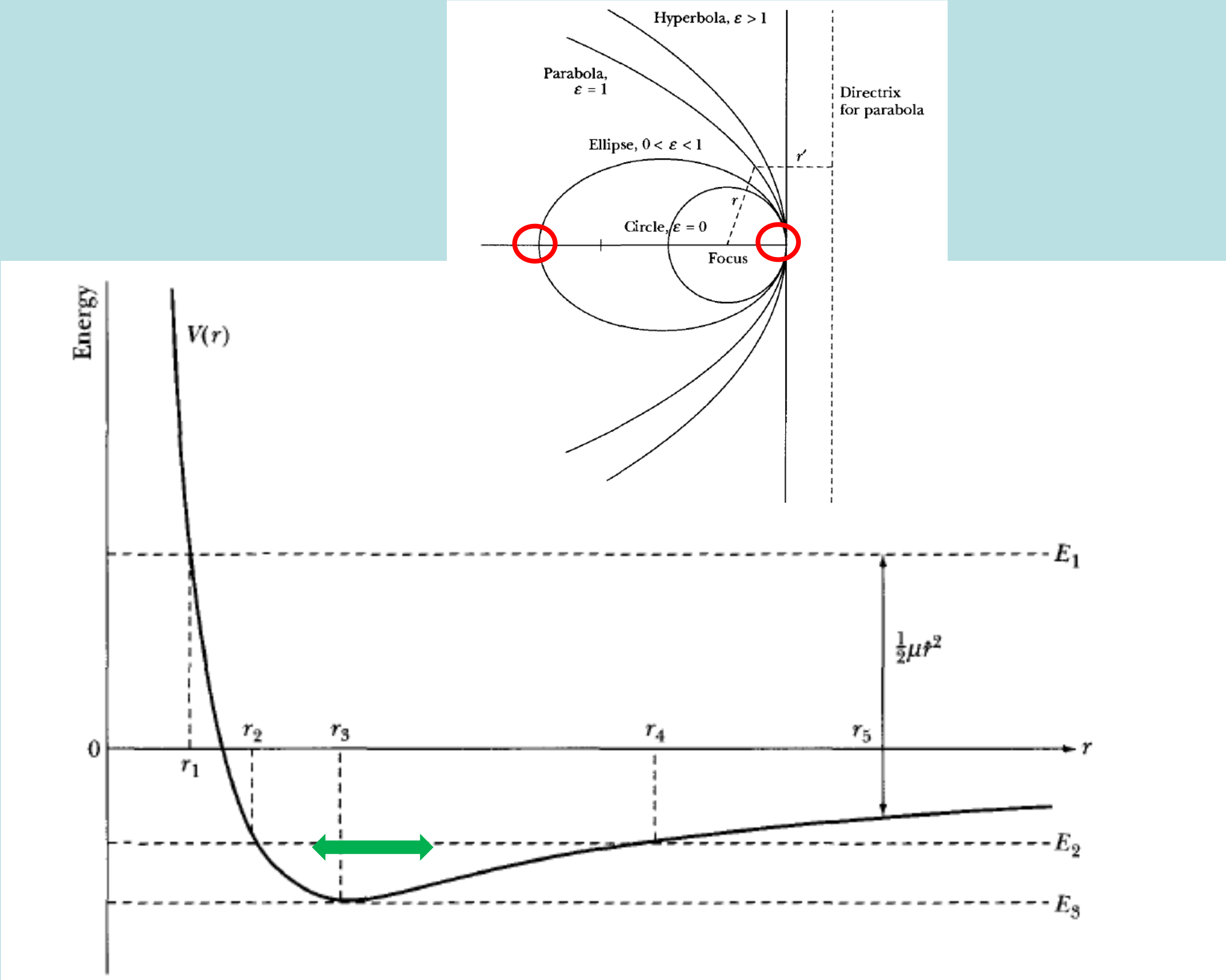




$E = E_1 > 0$ 則粒子成自由態。

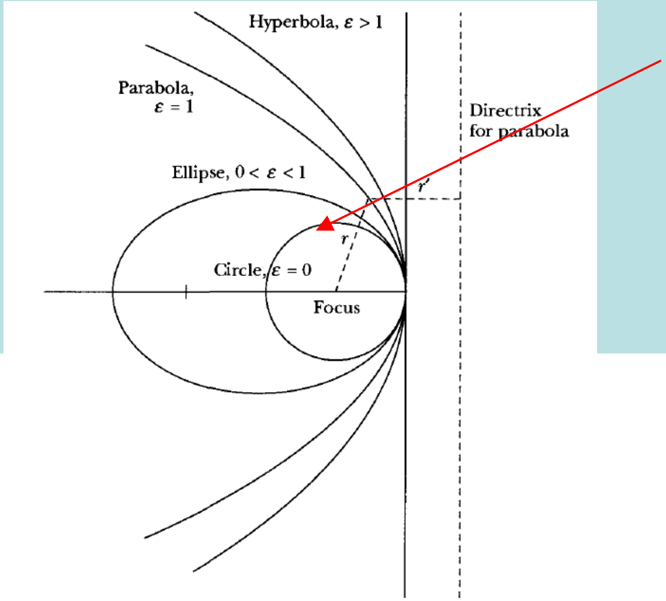
距離由∞減小到折返點的距離，再遠離回到∞，

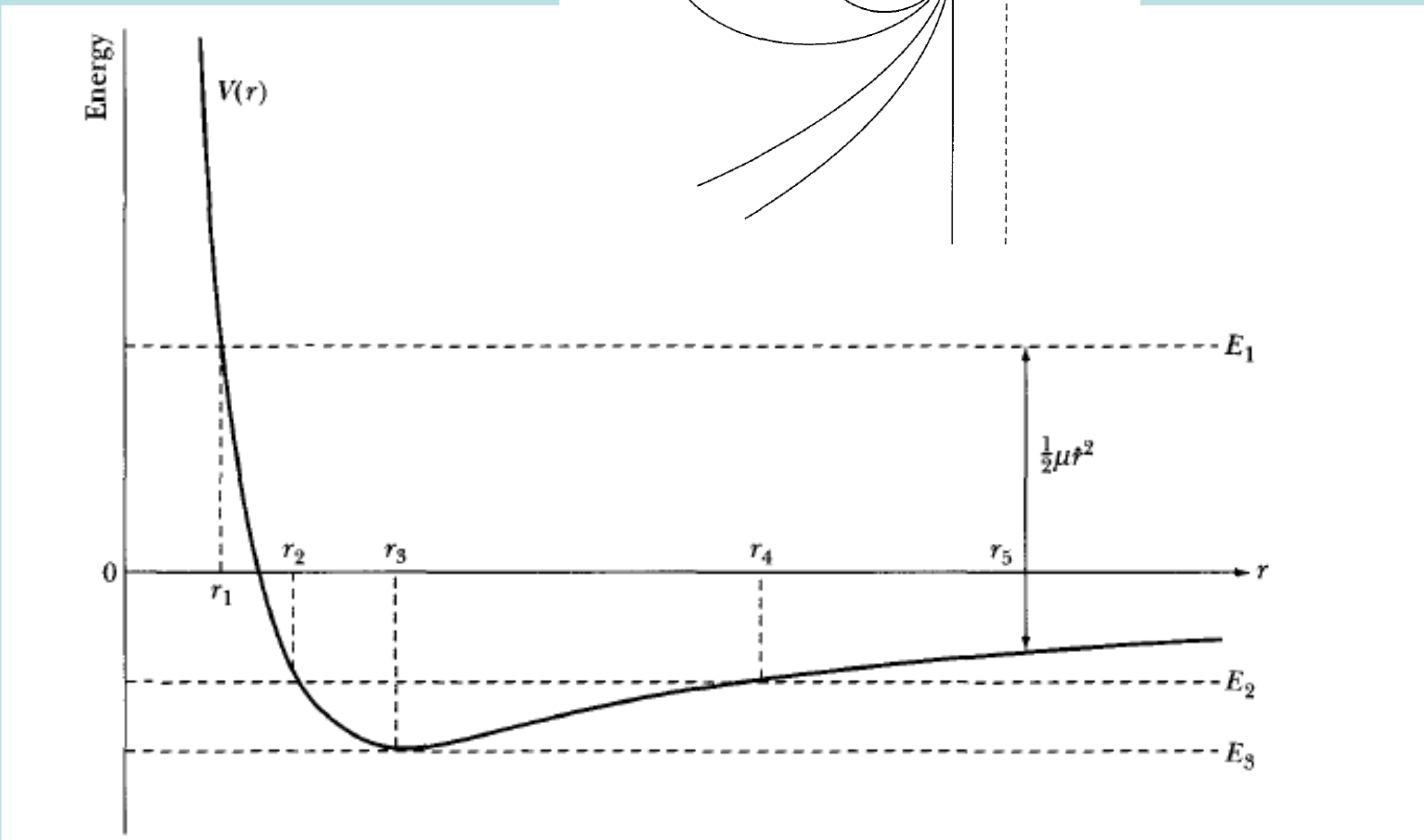
這是如彗星軌道的距離變化過程。



$E = E_2 < 0$ 則粒子成束縛態。距離在兩折返點間$r_2 < r < r_4$。。

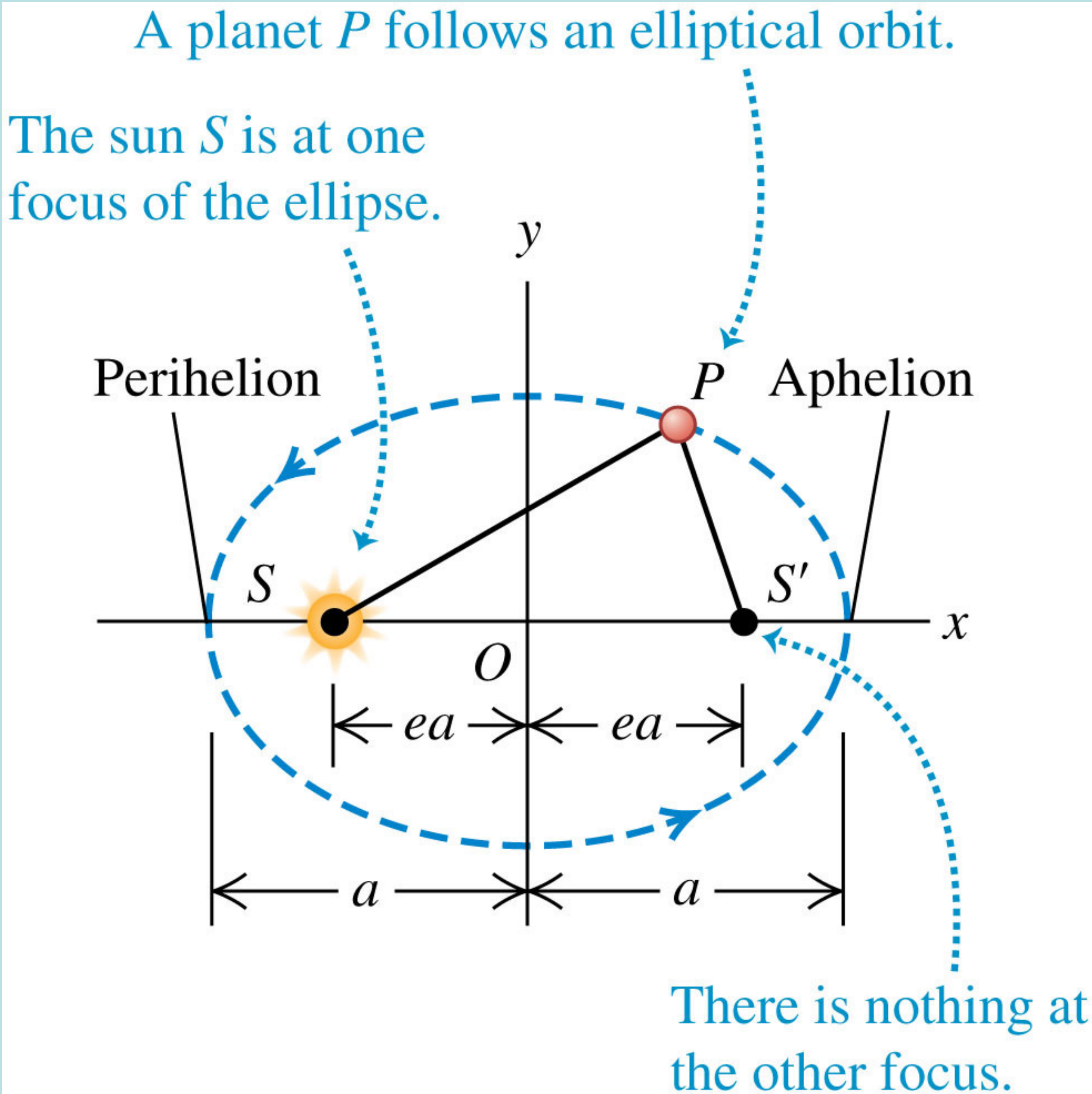
這正是橢圓形行星軌道的距離變化過程，兩折返點對應近日點與遠日點。



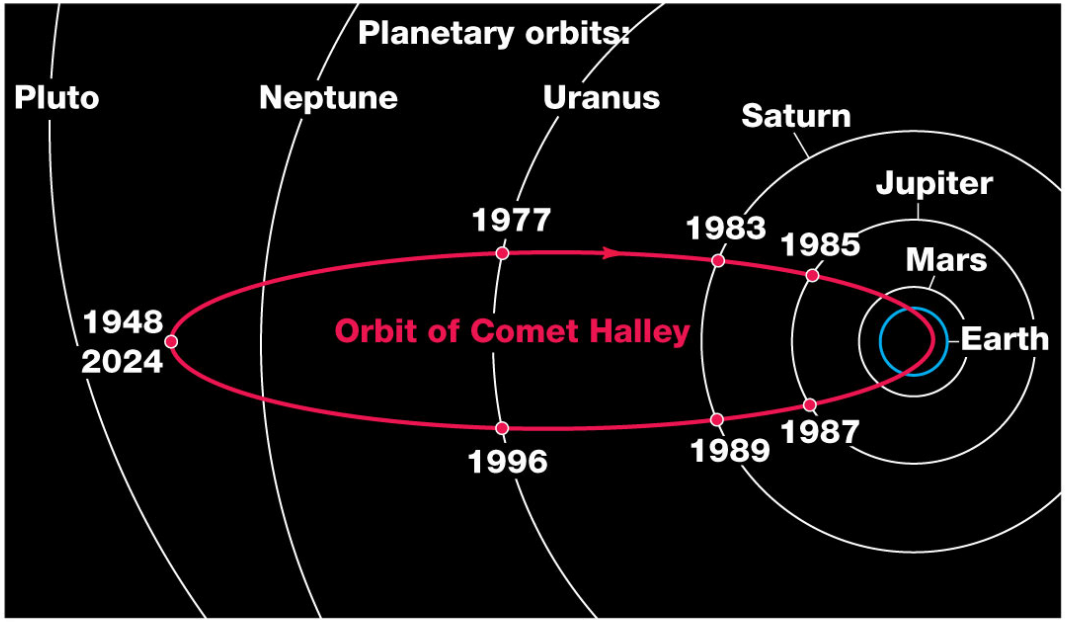


$E = E_3 < 0$，粒子距離會固定在有效位能的平衡點

軌道是圓形。距離在$r = r_3$。


A planet P follows an elliptical orbit.
The sun S is at one focus of the ellipse.
y
Perihelion
P
Aphelion
S
S'
x
O
ea
ea
a
a
There is nothing at the other focus.

(a)


Planetary orbits:
Pluto
Neptune
Uranus
Saturn
Jupiter
Mars
Earth
1977
1983
1985
1948
2024
Orbit of Comet Halley
1996
1989
1987

(b)

$$E=\frac{1}{2}m\left(\frac{dr}{dt}\right)^2+\frac{l^2}{2mr^2}+V(r)=\frac{1}{2}m\left(\frac{dr}{dt}\right)^2+V_{\text{eff}}(r)$$

有了兩個守恆量l, E做標記，粒子的運動可以解出來：

$$\frac{dr}{dt}=\sqrt{2m^{-1}}\cdot\sqrt{E-V_{\text{eff}}(r)}$$

$$\sqrt{\frac{m}{2}}\,dr\cdot\frac{1}{\sqrt{E-V_{\text{eff}}(r)}}=dt$$

$$\sqrt{\frac{m}{2}}\int_{r_0}^{r}dr\cdot\frac{1}{\sqrt{E-V_{\text{eff}}(r)}}=t$$

兩邊積分

距離 r 與時間 t 的關係可以解出來！

$$\frac{dr}{d\phi}=\frac{\dot{r}}{\dot{\phi}}=\frac{mr^2}{l}\sqrt{2m^{-1}}\cdot\sqrt{E-V_{\text{eff}}(r)}$$

$$l\sqrt{\frac{1}{2m}}\int_{r_0}^{r}dr\cdot\frac{1}{r^2\sqrt{E-V_{\text{eff}}(r)}}=\phi-\phi_0$$

距離r與角度ϕ的關係，也就是整個軌跡，可以解出來！

The equation for the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance between the particle and the force center can be obtained (see Equation 8.17) from

$$\theta(r) = \int \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} + \text{constant} \tag{8.38}$$

The integral can be evaluated if the variable is changed to $u \equiv l/r$ (see Problem 8-2). If we define the origin of θ so that the minimum value of r is at $\theta = 0$, we find

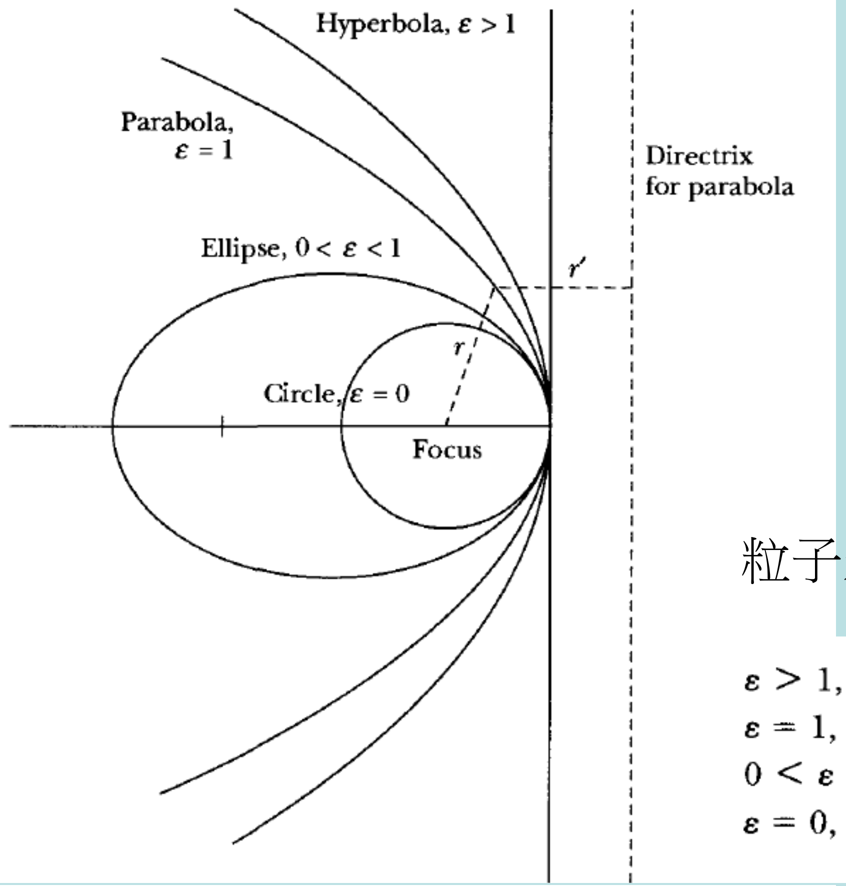
$$\cos\theta = \frac{\frac{l^2}{\mu k}\cdot\frac{1}{r} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}} \tag{8.39}$$

Let us now define the following constants:

$$\left.\begin{aligned} \alpha &\equiv \frac{l^2}{\mu k} \\ \varepsilon &\equiv \sqrt{1 + \frac{2El^2}{\mu k^2}} \end{aligned}\right\} \tag{8.40}$$

Equation 8.39 can thus be written as

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta} \tag{8.41}$$



粒子永遠不會墜入到原點。

$\varepsilon > 1,$	$E > 0$	Hyperbola
$\varepsilon = 1,$	$E = 0$	Parabola
$0 < \varepsilon < 1,$	$V_{\min} < E < 0$	Ellipse
$\varepsilon = 0,$	$E = V_{\min}$	Circle

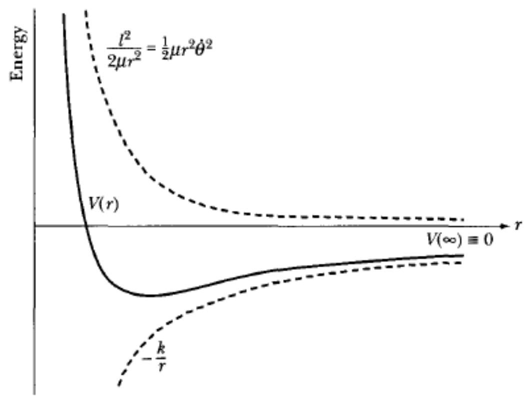


FIGURE 8-5 The effective potential for gravitational attraction $V(r)$ is composed of the real potential $-k/r$ term and the centrifugal potential energy $l^2/2\mu r^2$.

$$a = \frac{\alpha}{1 - \varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

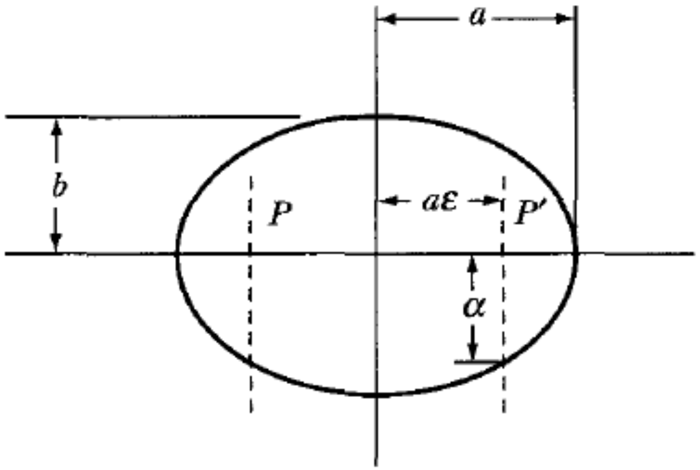


FIGURE 8-9 The geometry of elliptic orbits is shown in terms of parameters α, ε, a, and b. P and P' are the foci.

and P' are the foci. From this diagram, we see that the apsidal distances ($r_{\min}$ and $r_{\max}$ as measured from the foci to the orbit) are given by

$$\left.\begin{aligned} r_{\min} &= a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon} \\ r_{\max} &= a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon} \end{aligned}\right\} \tag{8.44}$$