近代物理期中考

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1. Consider a quantum simple harmonic oscillator:$H=\frac{p^{2}}{2m}+\frac{1}{2}kx^{2}$. The angular frequency is $ω≡\sqrt{k/m}$. The ground state $\left.\left|0\right.\right⟩$ satisfies:$a\left.\left|0\right.\right⟩=0$and its energy eigenvalue is $\frac{1}{2}ℏω$*.*
2. The ket $\left.\left|n\right.\right⟩$ denotes energy eigenstates: $\left.\left|n\right.\right⟩=\frac{1}{\sqrt{n!}}\left(a^{†}\right)^{n}\left.\left|0\right.\right⟩$. They are orthonormal: $\left⟨m\right⟩=δ\_{nm}$*.* What is the energy eigenvalue of $\left.\left|n\right.\right⟩$? (5)
3. Calculate the expectation values: $\left⟨n\right⟩ $and $\left⟨n\right⟩$. (10)

You can use the following formula:

$$x=\sqrt{\frac{ℏ}{2mω}}\left(a+a^{†}\right), p=-i\sqrt{\frac{mωℏ}{2}}\left(a-a^{†}\right)$$

$$\left[a,a^{†}\right]=1$$

1. Calculate the matrix elements: $\left⟨n\right⟩$ and $\left⟨n\right⟩$. (10)

You can use $a^{†}\left.\left|n\right.\right⟩=\sqrt{n+1}∙\left.\left|n+1\right.\right⟩, a\left.\left|n\right.\right⟩=\sqrt{n}∙\left.\left|n-1\right.\right⟩$.

1. Do your results in B,C agree with the quantization condition: $\left[x,p\right]=iℏ$?

Solution:

1. Eigenvalue: $\left(n+\frac{1}{2}\right)ℏω$
2. Plug in the formulae of $x $and $p$:

$$\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩$$

Because $\left⟨n\right⟩\~$ $\left⟨n-2\right⟩=0$ and $\left⟨n\right⟩\~$ $\left⟨n+2\right⟩=0$, only the crossing terms remain:

$$\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}$$

And

$$\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩$$

$$=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}$$

This agrees with the quantization condition: $\left[x,p\right]=iℏ$

$$\left⟨n\right⟩-\left⟨n\right⟩=iℏ$$

1. $\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩$

$=-i\frac{ℏ}{2}\left⟨n\right⟩-i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}\left⟨n\right⟩$

 $=i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}\sqrt{n+1}\sqrt{n+2}\left⟨n+2\right⟩=i\frac{ℏ}{2}\sqrt{n+1}\sqrt{n+2}$

On the other hand:

$$\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩=-i\frac{ℏ}{2}\left⟨n\right⟩$$

$=i\frac{ℏ}{2}\left⟨n\right⟩=i\frac{ℏ}{2}\sqrt{n+1}\sqrt{n+2}$

This also agree with the quantization condition: $\left[x,p\right]=iℏ$

$$\left⟨n\right⟩-\left⟨n\right⟩=iℏ\left⟨n\right⟩=iℏ\left⟨n\right⟩=0$$

1. We denote the normalized eigenstates of $L^{2} and L\_{z} $as $\left.\left|l,m\right.\right⟩$.

For a quantum axially symmetric rotator, assume that its Hamiltonian can be written as

$$H=\frac{L\_{x}^{2}+L\_{y}^{2}}{2I}+\frac{L\_{z}^{2}}{I}$$

1. Consider the case $l=2$. How many $\left.\left|2,m\right.\right⟩$ states are possible? What are the possible quantum number $m$? (10)
2. Assume that the rotator is in the state of $\left.\left|2,2\right.\right⟩$. It decays into the state $\left.\left|1,1\right.\right⟩$ and emits one photon (there is a selection rule for this kind of process: $∆l=\pm 1,∆m=\pm 1). $ What is the energy of the photon it emits? (10)

Solution:

1. There are five $\left.\left|2,m\right.\right⟩$ states, with $m=-2,-1,0,1,,2$.
2. $L\_{x}^{2}+L\_{y}^{2}=L^{2}-L\_{z}^{2}$. Plug this into the formula:

$$H=\frac{L^{2}}{2I}+\left(\frac{1}{I}-\frac{1}{2I}\right)L\_{z}^{2}=\frac{1}{2I}\left(L^{2}+L\_{z}^{2}\right)$$

For eigenstates $\left.\left|l,m\right.\right⟩ $of $L^{2} and L\_{z}$, the energy eigenvalues are

$$E\_{lm}=\frac{1}{2I}ℏ^{2}\left[l\left(l+1\right)+m^{2}\right]$$

The emitted photon energy equals $E\_{22}-E\_{11}=\frac{7}{2I}ℏ^{2}$.

1. Consider the electron in a hydrogen atom. We learned in class that the radial part of its stationary state wave function can be written as:

$$R\_{nl}\left(r\right)\~e^{-\frac{r}{na\_{0}}}∙r^{l}∙H\left(\frac{2}{na\_{0}}r\right)$$

with $H\left(ρ\right)$ a polynomial of degree $n-l-1$($n-l-1$次的多項式). We also learn the angular part $Y\_{lm}\left(θ,ϕ\right) $of the wave function, called spherical harmonics, has the form

$$sin^{\left|m\right|}θ∙\left(a polynomial in\cos(θ)of degree l-m\right)∙e^{imϕ}$$

Now for a fixed principal quantum number $n$, consider the case $l = n - 1$, and $m = n - 1$.

1. Write down the radial function $R\_{nn-1}\left(r\right)$. You can keep an undetermined constant $c$ in your answer. ($c$ can be determined by normalization condition). (10)
2. Radial probability density $P\left(r\right)$ is defined so that $P\left(r\right)∙dr$ is the probability of finding the electron distance from origin to be between $r$ and $r+dr$. We showed in class that $P\left(r\right)=\left[R(r)\right]^{2}r^{2}$. Find the radius $r\_{0}$ where $P\left(r\right)$ has a maximum. (10)
3. Write down $Y\_{n-1n-1}\left(θ,ϕ\right)$. You can also keep an undetermined constant $c'$ in your answer. The angular probability density $\left|Y\_{n-1n-1}\left(θ,ϕ\right)\right|^{2} $peaks at a certain angle $θ\_{0}$. Find $θ\_{0}$. (5)



$$r\_{0}$$

Solution:

1. $R\_{nn-1}\left(r\right)\~e^{-\frac{r}{na\_{0}}}∙r^{n-1}$ since $H$ is a polynomials of $n-l-1=0$ order, a constant.
2. $P\left(r\right)=\left[R(r)\right]^{2}r^{2}=e^{-\frac{2r}{na\_{0}}}∙r^{2n}$The maximum occurs at

$$0=P^{'}\left(r\right)=-\frac{2}{na\_{0}}e^{-\frac{2r}{na\_{0}}}∙r^{2n}+2ne^{-\frac{2r}{na\_{0}}}∙r^{2n-1}$$

$$0=-\frac{1}{na\_{0}}+nr\_{0}$$

Therefore $r\_{0}=n^{2}a\_{0}$. This is equation (8-50) at p139 in our textbook.

1. $Y\_{n-1n-1}\left(θ,ϕ\right)\~sin^{n-1}θ$, since $the multiplying polynomial in\cos(θ)is of l-m=0 order$, again a constant. It peaks at the peak of $\sin(θ)$, ie $θ=\frac{π}{2}.$ That is on the $x-y$ plane.
2. As in class, use the two eigenstates $\left|z,\left.\uparrow \right⟩\right., \left|z,\left.\downright \right⟩\right.$ of the $z$-component spin $S\_{z} $to form a basis of the spin state space.
3. In this basis, the $y$-component of spin $S\_{y}$ can be written as the $2×2 matrix:S\_{y}=\frac{ℏ}{2}\left(\begin{matrix}0&-i\\i&0\end{matrix}\right)$. Find the eigenvalues and eigenvectors $\left|y,\left.\uparrow \right⟩,\right.\left|y,\left.\downright \right⟩\right. $of $S\_{y}$. Please normalize your answer so that $\left⟨y,\uparrow \right⟩=\left⟨y,\downright \right⟩=1$. (10)
4. What is the $2×2 matrix S\_{z}$ representing the $z$-component of spin?

Calculate the expectation value: $\left⟨y,\uparrow \right⟩$. (10)

1. In the state of $\left|y,\left.\uparrow \right⟩\right.$, measure $S\_{z}$. What are the possible values you can get and what are the corresponding probabilities? (10)
2. In the state of $\left|\left.α\right⟩\right.=\frac{1}{\sqrt{65}}\left(\begin{matrix}4\\7\end{matrix}\right)$, what is the probability that the measurement of $S\_{y} $yields the value of +$\frac{ℏ}{2}$?

In the state of $\left|\left.β\right⟩\right.=\frac{1}{\sqrt{65}}\left(\begin{matrix}4\\7i\end{matrix}\right)$, what is the probability that the measurement of $S\_{y} $yields the value of +$\frac{ℏ}{2}$? (10)

Hint: Expand the state $\left|\left.α\right⟩\right.$ using $S\_{y}$ eigenstates $\left|y,\left.\uparrow \right⟩\right.$ and $\left|y,\left.\downright \right⟩\right.$: $\left|\left.α\right⟩\right.=d\_{1}\left|\left.y,\uparrow \right⟩\right.+d\_{2}\left|\left.y,\downright \right⟩\right.$. Like components of 3D vectors, $d\_{1}=\left⟨α\right⟩, d\_{2}=\left⟨α\right⟩$. The expansion components $d\_{1,2}$ are the amplitudes of $S\_{y}$ measurement $ yielding the values of \pm \frac{ℏ}{2}$. The absolute value squared of the amplitudes are the probabilities.

Solution:

1. $\left(\begin{matrix}0&-i\\i&0\end{matrix}\right)\left(\begin{matrix}u\\v\end{matrix}\right)=λ\left(\begin{matrix}u\\v\end{matrix}\right)$*,* 此式只有在行列式為零時有非零解：

$\left|\begin{matrix}-λ&-i\\i&-λ\end{matrix}\right|=0$, $λ^{2}-1=0$,

如預期$λ=\pm 1$。

若$λ=1$，eigevalue is $\frac{ℏ}{2}$. $-u-iv=0, eigenstate\left(\begin{matrix}u\\v\end{matrix}\right)∝\left(\begin{matrix}1\\i\end{matrix}\right)\rightarrow \frac{1}{\sqrt{2}}\left(\begin{matrix}1\\i\end{matrix}\right)$

若$λ=-1$，eigevalue is $-\frac{ℏ}{2}, u-iv=0$,$ eigenstate \left(\begin{matrix}u\\v\end{matrix}\right)∝\left(\begin{matrix}1\\-i\end{matrix}\right)\rightarrow \frac{1}{\sqrt{2}}\left(\begin{matrix}1\\-i\end{matrix}\right)$

1. $\left⟨y,\uparrow \right⟩=\frac{1}{2}\left(1,i\right)^{\*}\frac{ℏ}{2}\left(\begin{matrix}1&0\\0&-1\end{matrix}\right)\left(\begin{matrix}1\\i\end{matrix}\right)=\frac{ℏ}{2}\left(1,i\right)\left(\begin{matrix}1\\i\end{matrix}\right)=0$
2. In $\left|\left.y,\uparrow \right⟩=\right.\frac{1}{\sqrt{2}}\left(\begin{matrix}1\\i\end{matrix}\right)$ measure $S\_{z}$. The probability of yielding $\frac{ℏ}{2}$ equal $\left|\frac{1}{\sqrt{2}}\right|^{2}=\frac{1}{2}.$

The probability of yielding $-\frac{ℏ}{2}$ equal $\left|\frac{i}{\sqrt{2}}\right|^{2}=\frac{1}{2}.$.

1. In the state of $\left|\left.α\right⟩\right.=\frac{1}{\sqrt{65}}\left(\begin{matrix}4\\7\end{matrix}\right)$, the probability that the measurement of $S\_{y} $yields the value of +$\frac{ℏ}{2}$ equals $\left|\left⟨α\right⟩\right|^{2}=\left|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{65}}\left(1,i\right)^{\*}\left(\begin{matrix}4\\7\end{matrix}\right)\right|^{2}=\left|\frac{4+7i}{\sqrt{130}}\right|^{2}=\frac{16+49}{130}=\frac{65}{130}=\frac{1}{2}$

In the state of $\left|\left.β\right⟩\right.=\frac{1}{\sqrt{65}}\left(\begin{matrix}4\\7i\end{matrix}\right)$, the probability that the measurement of $S\_{y} $yields the value of +$\frac{ℏ}{2}$ equals $\left|\left⟨α\right⟩\right|^{2}=\left|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{65}}\left(1,i\right)^{\*}\left(\begin{matrix}4\\7i\end{matrix}\right)\right|^{2}=\left|\frac{4+7}{\sqrt{130}}\right|^{2}=\frac{121}{260}$