近代物理期中考

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The Following two problems are about a quantum simple harmonic oscillator. The mass of the particle is $m$. The angular frequency is $ω≡\sqrt{\frac{k}{m}}$。

1. The ground state $\left.\left|0\right.\right⟩$ of this system can be described by a wavefunction as

$$u\_{0}\left(x\right)=\left(\frac{mω}{πℏ}\right)^{1/4}e^{-\frac{mωx^{2}}{2ℏ}}≡Ce^{-\frac{mωx^{2}}{2ℏ}}$$

You can use the constant $C$ in the following to simplify your calculation. We use the raising operator $a^{†}$ to generate the first excited state from the ground state:

$$\left.\left|1\right.\right⟩=a^{†}\left.\left|0\right.\right⟩$$

We also know the raising operator is the linear combination of $\hat{x}$ and $\hat{p}$:

$$a^{†}=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{mω}{ℏ}}x-\sqrt{\frac{ℏ}{mω}}\frac{d}{dx}\right)$$

Find the wavefunction $u\_{1}\left(x\right)$ for $\left.\left|1\right.\right⟩$.

Draw a rough graph of the wavefunction $u\_{1}\left(x\right)$ versus $x$.

Hint:

$$u\_{1}\left(x\right)=a^{†}u\_{0}\left(x\right)$$

Solution:

$$u\_{1}\left(x\right)=a^{†}u\_{0}\left(x\right)=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{mω}{ℏ}}x-\sqrt{\frac{ℏ}{mω}}\frac{d}{dx}\right)\left(\frac{mω}{πℏ}\right)^{\frac{1}{4}}e^{-\frac{mωx^{2}}{2ℏ}}$$

$$=\frac{1}{\sqrt{2}}\left(\frac{mω}{πℏ}\right)^{\frac{1}{4}}\left(\sqrt{\frac{mω}{ℏ}}xe^{-\frac{mωx^{2}}{2ℏ}}+\sqrt{\frac{ℏ}{mω}}\frac{mω}{ℏ}xe^{-\frac{mωx^{2}}{2ℏ}}\right)=\frac{1}{\sqrt{2}}\left(\frac{mω}{πℏ}\right)^{\frac{1}{4}}2\sqrt{\frac{mω}{ℏ}}xe^{-\frac{mωx^{2}}{2ℏ}}$$

It is a gaussian form at infinity and cross the x axis once at the origin as an odd function.



1. Calculate $\left⟨0\right⟩ $for the quantum simple harmonic oscillator. You can use the following formula:

$$x=\sqrt{\frac{ℏ}{2mω}}\left(a+a^{†}\right)$$

And $a^{†}\left.\left|n\right.\right⟩=\sqrt{n+1}∙\left.\left|n+1\right.\right⟩, a\left.\left|n\right.\right⟩=\sqrt{n}∙\left.\left|n-1\right.\right⟩$.

$\left⟨\left.n\right|\right.a=\sqrt{n+1}∙\left⟨\left.n+1\right|\right., \left⟨\left.n\right|\right.a^{†}=\sqrt{n}∙\left⟨\left.n-1\right|\right.$

$$a\left.\left|0\right.\right⟩=0,\left⟨\left.0\right|\right.a^{†}=0$$

Hint: My first step is to show:

$$\left⟨0\right⟩\~\left(\frac{ℏ}{2mω}\right)\left⟨1\right⟩$$

using $a^{†}\left.\left|0\right.\right⟩=\left.\left|1\right.\right⟩,\left⟨\left.0\right|\right.a=\left⟨\left.1\right|\right.$. And we have calculated the right-hand side in class.

Solution:

$$\left⟨0\right⟩=\left(\frac{ℏ}{2mω}\right)^{2}\left⟨0\right⟩$$

Because $a\left.\left|0\right.\right⟩=0,\left⟨\left.0\right|\right.a^{†}=0$, the above can be simplified:

$$\left(\frac{ℏ}{2mω}\right)^{2}\left⟨0\right⟩$$

$$=\left(\frac{ℏ}{2mω}\right)^{2}\left⟨1\right⟩$$

Here we have used $a^{†}\left.\left|0\right.\right⟩=\left.\left|1\right.\right⟩,\left⟨\left.0\right|\right.a=\left⟨\left.1\right|\right.$.

We also know that

$$\left⟨1\right⟩=\left⟨1\right⟩=0$$

The above is further reduced to

$$=\left(\frac{ℏ}{2mω}\right)^{2}\left⟨1\right⟩$$

Using$ a^{†}\left.\left|1\right.\right⟩=\left.\sqrt{2}\left|2\right.\right⟩, a\left.\left|2\right.\right⟩=\sqrt{2}∙\left.\left|1\right.\right⟩,$ we can reduce the first term:

$$\left(\frac{ℏ}{2mω}\right)^{2}\left⟨1\right⟩=\left(\frac{ℏ}{2mω}\right)^{2}\left(\sqrt{2}\right)^{2}\left⟨1\right⟩$$

Using$ a\left.\left|1\right.\right⟩=\sqrt{1}∙\left.\left|0\right.\right⟩, a^{†}\left.\left|0\right.\right⟩=\left.\sqrt{1}\left|1\right.\right⟩,,$ we can reduce the second term:

$$\left(\frac{ℏ}{2mω}\right)^{2}\left⟨1\right⟩=\left(\frac{ℏ}{2mω}\right)^{2}\left(1\right)^{2}\left⟨1\right⟩$$

Hence

$$\left⟨0\right⟩=\left(\frac{ℏ}{2mω}\right)^{2}3\left⟨1\right⟩=\left(\frac{ℏ}{2mω}\right)^{2}3$$

1. We denote the eigenstates of $L^{2}, L\_{z} $as $\left.\left|l,m\right.\right⟩$ (normalized), with

$$\left.L^{2}\left|l,m\right.\right⟩=l\left(l+1\right)ℏ^{2}∙\left.\left|l,m\right.\right⟩,\left.L\_{z}\left|l,m\right.\right⟩=mℏ\left.∙\left|l,m\right.\right⟩$$

Consider the case $l=\frac{1}{2}$.

1. A state $\left.\left|ψ\right.\right⟩ $is a combination of $\left.\left|\frac{1}{2},\frac{1}{2}\right.\right⟩$ and $\left.\left|\frac{1}{2},-\frac{1}{2}\right.\right⟩$：$\left.\left|ψ\right.\right⟩=\frac{1}{\sqrt{2}}\left.\left|\frac{1}{2},\frac{1}{2}\right.\right⟩+\frac{1}{\sqrt{2}}\left.\left|\frac{1}{2},-\frac{1}{2}\right.\right⟩$. When I measure $L\_{z}$ on $\left.\left|ψ\right.\right⟩$, what are (or is) the possible values? What are (or is) the corresponding probabilities?
2. Now switch to the state of $\left.\left|\frac{1}{2},\frac{1}{2}\right.\right⟩$. When I measure $L\_{x}^{2}+L\_{y}^{2}$ on this state, what are (or is) the possible values? What are (or is) the corresponding probabilities?

Solution:

1. The measured values of $L\_{z} $could only be its eigenvalues. For $l=\frac{1}{2}$, the eigenvalues are $\frac{1}{2}$ and $-\frac{1}{2}$. The probabilities are the square of the components of the expansion. In this case,$\left.\left|ψ\right.\right⟩=\frac{1}{\sqrt{2}}\left.\left|\frac{1}{2},\frac{1}{2}\right.\right⟩+\frac{1}{\sqrt{2}}\left.\left|\frac{1}{2},-\frac{1}{2}\right.\right⟩$*.* The components are$\frac{1}{\sqrt{2}}$*.* Hence the probabilities are $\frac{1}{2}$ for both $L\_{z}=\frac{1}{2}$ and $L\_{z}=-\frac{1}{2}$.
2. $L\_{x}^{2}+L\_{y}^{2}=L^{2}-L\_{z}^{2}$. Since $\left.\left|\frac{1}{2},\frac{1}{2}\right.\right⟩$ is the eigenstate of $L^{2}, L\_{z}$, it is also the eigenstate of $L\_{x}^{2}+L\_{y}^{2}$. Hence the measurement will give a certain values:

$$l\left(l+1\right)ℏ^{2}-m^{2}ℏ^{2}=\frac{3}{4}ℏ^{2}-\frac{1}{4}ℏ^{2}=\frac{1}{2}ℏ^{2}$$

1. In this problem, **the comments are points I hope you will think about later after exam and there is no need to answer them.**

Consider a hydrogen atom. We learned in class that the radial function of its stationary state wave function can be written as:

$$R\_{nl}\left(r\right)\~e^{-\frac{r}{na\_{0}}}∙r^{l}∙H\left(\frac{2}{na\_{0}}r\right)$$

with $H\left(ρ\right)$ a polynomial of $n-l-1$ order. We also learn the angle-depending part $Y\_{lm}\left(θ,ϕ\right) $of the wave function, called spherical harmonics, has the form

$$sin^{\left|m\right|}θ∙\left(a polynomial in\cos(θ)of l-m order\right)$$

Now for a fixed principal quantum number $n$, consider the case $l = n - 1$, and $m = n - 1$.

1. Write down the radial function $R\_{nn-1}\left(r\right)$, up to normalization.
2. $R\_{nn-1}\left(r\right)$ has a maximum. Find the radius $r\_{0}$ where $R\_{nn-1}\left(r\_{0}\right)$ has a maximum. Comments: Observe that as $n$ increase, the peak becomes sharper. Hence the probability density concentrates on a certain radius $r\_{0}$.
3. Write down $Y\_{n-1n-1}\left(θ,ϕ\right)$ up to normalization. It also peaks at a certain angle $θ\_{0}$. Find this angle.

Comments: Observe that as $n$ increase, the peak becomes sharper. Hence the probability density concentrates on a plane.

 

$$r\_{0}$$

$$r\_{0}$$

Solution:

1. $R\_{nn-1}\left(r\right)\~e^{-\frac{r}{na\_{0}}}∙r^{n-1}$ since $H$ is a polynomials of $n-l-1=0$ order, a constant.
2. The maximum occurs at $0=R\_{nn-1}^{'}\left(r\_{0}\right)\~\left(n-1\right)r\_{0}^{n-2}e^{-\frac{r\_{0}}{na\_{0}}}-r\_{0}^{n-1}\frac{1}{na\_{0}}e^{-\frac{r\_{0}}{na\_{0}}}$. Therefore $r\_{0}=n\left(n-1\right)a\_{0}$.
3. $Y\_{n-1n-1}\left(θ,ϕ\right)\~sin^{n-1}θ$, since $the multiplying polynomial in\cos(θ)is of l-m=0 order$, again a constant. It peaks at the peak of $\sin(θ)$, ie $θ=\frac{π}{2}.$ That is on the $x-y$ plane.