習題三

1. Solution:
2. $R\_{nn-1}\left(r\right)\~e^{-\frac{r}{na\_{0}}}∙r^{n-1}$ since $H$ is a polynomials of $n-l-1=0$ order, a constant.
3. The maximum occurs at $0=R\_{nn-1}^{'}\left(r\_{0}\right)\~\left(n-1\right)r\_{0}^{n-2}e^{-\frac{r\_{0}}{na\_{0}}}-r\_{0}^{n-1}\frac{1}{na\_{0}}e^{-\frac{r\_{0}}{na\_{0}}}$. Therefore $r\_{0}=n\left(n-1\right)a\_{0}$.
4. $Y\_{n-1n-1}\left(θ,ϕ\right)\~sin^{n-1}θ$, since $the multiplying polynomial in\cos(θ)is of l-m=0 order$, again a constant. It peaks at the peak of $\sin(θ)$, ie $θ=\frac{π}{2}.$ That is on the $x-y$ plane.
5. Ch. 10 10-1. For the eigenstate with the higher eigenvalue, measure $S\_{z}$.what are (or is) the possible values? What are (or is) the corresponding probabilities?

Solution:



The possible values: $\pm \frac{1}{2}$. The corresponding probabilities are respectively: $\left|\frac{1}{\sqrt{2}}\right|^{2}=\frac{1}{2}$ and $\left|\frac{i}{\sqrt{2}}\right|^{2}=\frac{1}{2}$

1. Ch. 10 $\left|\left.α\right⟩\right.=\frac{1}{\sqrt{65}}\left(\begin{matrix}4\\7\end{matrix}\right)$，calculate the expectation value of $S\_{x}^{4}:\left⟨α\right⟩$.

Measure $S\_{x}$, what are (or is) the possible values? What are (or is) the corresponding probabilities?

Hint: The expansion components 分量 of $\left|\left.α\right⟩\right.$ along $\left|\left.x,\uparrow \right⟩\right.$ equals $\left⟨α\right⟩$.

Solutions:

$$\left⟨α\right⟩=\left(\frac{1}{\sqrt{65}}\right)^{2}\left(\frac{ℏ}{2}\right)^{4}\left(\begin{matrix}4&7\end{matrix}\right)\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\left(\begin{matrix}4\\7\end{matrix}\right)=\left(\frac{ℏ}{2}\right)^{4}$$

The possible values are again: $\pm \frac{1}{2}$. The corresponding probabilities are respectively: $\left|\left⟨α\right⟩\right|^{2}=\left|\frac{1}{\sqrt{65}}\frac{1}{\sqrt{2}}\left(\begin{matrix}1&1\end{matrix}\right)\left(\begin{matrix}4\\7\end{matrix}\right)\right|^{2}=\frac{121}{130}$

 and $\left|\left⟨α\right⟩\right|^{2}=\left|\frac{1}{\sqrt{65}}\frac{1}{\sqrt{2}}\left(\begin{matrix}1&-1\end{matrix}\right)\left(\begin{matrix}4\\7\end{matrix}\right)\right|^{2}=\frac{9}{130}$.

1. As in class, use the eigenstates $\left|z,\left.\uparrow \right⟩\right., \left|z,\left.\downright \right⟩\right.$ of the spin operator in $z $direction $S\_{z} $as the basis of the electron spin States. Consider the spin operator $S\_{n}$ pointing in the direction of $\hat{n}=\left(\sin(θ)\cos(ϕ),\sin(θ)\sin(ϕ),\cos(θ)\right)$ with $ϕ=0$. This spin operator can be studied in the following setup. We assign the direction of electron beam as $y$ axis. Rotate the magnet in a SG experiment around $y$ axis by an angle $θ$. The magnetic field of this second SG is along $\hat{n}$.



1. Derive from $S\_{n}=\vec{S}∙\hat{n}$ that $S\_{n}=\frac{ℏ}{2}\left(\begin{matrix}\cos(θ)&\sin(θ)\\\sin(θ)&-\cos(θ)\end{matrix}\right)$ .
2. Find the eigenvectors of $S\_{n}$, expressed as column vectors in the basis of $\left|z,\left.\uparrow \right⟩\right., \left|z,\left.\downright \right⟩\right.$: $\left|n,\left.\uparrow \right⟩\right.=\left(\begin{matrix}c\_{1}\\c\_{2}\end{matrix}\right), \left|n,\left.\downright \right⟩\right.=\left(\begin{matrix}d\_{1}\\d\_{2}\end{matrix}\right)$. You can choose the coefficient $c\_{1,2},d\_{1,2} $to be all real. Calculate the coefficients $c\_{1,2}$. Please normalize the coefficients so that $c\_{1}^{2}+c\_{2}^{2}=1$.
3. As in the setup above, place a typical SG with $z$ magnetic field and then a second SG with $\hat{n}$ magnetic field. In the first SG only spin-up electron is allowed to pass, ie: $\left|\left.z,\uparrow \right⟩=\right.\left(\begin{matrix}1\\0\end{matrix}\right)$. Calculate the probability for it to pass the second SG as $\left|n,\left.\uparrow \right⟩\right., $ie with $σ\_{n}=\frac{ℏ}{2}$ measured? It is a function of $θ$. Think a little bit when the probability would be the largest and the smallest and whether it makes sense. (25)

Solution:

1. $S\_{n}=\vec{S}∙\hat{n}=\frac{ℏ}{2}\left[\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\sin(θ)\cos(ϕ)+\left(\begin{matrix}0&-i\\i&0\end{matrix}\right)\sin(θ)\sin(ϕ)+\left(\begin{matrix}1&0\\0&-1\end{matrix}\right)\cos(θ)\right]=\frac{ℏ}{2}\left[\left(\begin{matrix}0&1\\1&0\end{matrix}\right)\sin(θ)+\left(\begin{matrix}1&0\\0&-1\end{matrix}\right)\cos(θ)\right]=\frac{ℏ}{2}\left(\begin{matrix}\cos(θ)&\sin(θ)\\\sin(θ)&-\cos(θ)\end{matrix}\right)$*.*
2. $\left(\begin{matrix}\cos(θ)&\sin(θ)\\\sin(θ)&-\cos(θ)\end{matrix}\right)\left(\begin{matrix}u\\v\end{matrix}\right)=λ\left(\begin{matrix}u\\v\end{matrix}\right)$*,* 此式只有在行列式為零時有非零解：$\left|\begin{matrix}\cos(θ)-λ&\sin(θ)\\\sin(θ)&-\cos(θ)-λ\end{matrix}\right|=0$, $λ^{2}-\left(\cos(θ)\right)^{2}-\left(\sin(θ)\right)^{2}=0$,

如預期$λ=\pm 1$。

若$λ=1$，$\left(\cos(θ)-1\right)u+\sin(θ)v=0$

$$\left(\begin{matrix}c\_{1}\\c\_{2}\end{matrix}\right)=\left(\begin{matrix}\sin(θ)\\1-\cos(θ)\end{matrix}\right)\frac{1}{\sqrt{2-2\cos(θ)}}=\left(\begin{matrix}2\sin(\cos(\frac{θ}{2}))\frac{θ}{2}\\2sin\frac{θ}{2}sin\frac{θ}{2}\end{matrix}\right)\frac{1}{2sin\frac{θ}{2}}=\left(\begin{matrix}\cos(\frac{θ}{2})\\sin\frac{θ}{2}\end{matrix}\right)$$

若$λ=-1$，$\left(\cos(θ)+1\right)u+\sin(θ)v=0$

$$\left(\begin{matrix}d\_{1}\\d\_{2}\end{matrix}\right)=\left(\begin{matrix}-\sin(θ)\\1+\cos(θ)\end{matrix}\right)\frac{1}{\sqrt{2+2\cos(θ)}}=\left(\begin{matrix}-2\sin(\cos(\frac{θ}{2}))\frac{θ}{2}\\2\cos(\frac{θ}{2})\cos(\frac{θ}{2})\end{matrix}\right)\frac{1}{2\cos(\frac{θ}{2})}=\left(\begin{matrix}-sin\frac{θ}{2}\\\cos(\frac{θ}{2})\end{matrix}\right)$$

1. The probability for $\left|\left.z,\uparrow \right⟩=\right.\left(\begin{matrix}1\\0\end{matrix}\right)$ to pass as $\left|n,\left.\uparrow \right⟩\right.,$ equals:

$$\left|\left⟨z,\uparrow \right⟩\right|^{2}=\left|\left(\cos(\frac{θ}{2}), sin\frac{θ}{2} \right)\left(\begin{matrix}1\\0\end{matrix}\right)\right|^{2}=\left(\cos(\frac{θ}{2})\right)^{2}=\frac{1+\cos(θ)}{2}$$

check$ θ=0$，$\hat{n}$ is the same as $\hat{z}$, the probability is 1, correct.