習題三

4. We need to calculate

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a dx x^2 \sin^2 \frac{n\pi x}{a}$$

With $\pi x/a = \mathrm{u}$ we have

$$\langle x^2 \rangle = \frac{2}{a}\frac{a^3}{\pi^3}\int_0^\pi du u^2 \sin^2 nu = \frac{a^2}{\pi^3}\int_0^\pi du u^2 (1-\cos 2nu)$$

The first integral is simple. For the second integral we use the fact that

$$\int_0^\pi du u^2 \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \int_0^\pi du \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \frac{\sin\alpha\pi}{\alpha}$$

At the end we set $\alpha = n\pi$. A little algebra leads to

$$\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2}$$

For large n we therefore get $\Delta x = \frac{a}{\sqrt{3}}$. Since $\langle p^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}$, it follows that $\Delta p = \frac{\hbar \pi n}{a}$, so that

$$\Delta p \Delta x \approx \frac{n\pi\hbar}{\sqrt{3}}$$

The product of the uncertainties thus grows as n increases.

5. With $E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2$ we can calculate

$$E_2 - E_1 = 3\frac{(1.05\times10^{-34} J.s)^2}{2(0.9\times10^{-30} kg)(10^{-9} m)^2}\frac{1}{(1.6\times10^{-19} J/eV)} = 0.115 eV$$

We have $\Delta E = \frac{hc}{\lambda}$ so that $\lambda = \frac{2\pi\hbar c}{\Delta E} = \frac{2\pi(2.6\times10^{-7} ev.m)}{0.115 eV} = 1.42\times10^{-5} m$

where we have converted $\hbar c$ from J.m units to eV.m units.

以上解答的 a，答案數值上忘了乘上π^2。

9. The general solution is

$$\psi(x,t)=\sum_{n=1}^{\infty}C_n u_n(x)e^{-iE_n t/\hbar}$$

with the C_n defined by

$$C_n=\int_{-a/2}^{a/2}dx u_n^*(x)\psi(x,0)$$

(a) It is clear that the wave function does not remain localized on the l.h.s. of the box at later times, since the special phase relationship that allows for a total interference for $x>0$ no longer persists for $t\neq 0$.

(b) With our wave function we have $C_n=\sqrt{\frac{2}{a}}\int_{-a/2}^{0}dx u_n(x)$. We may work this out by using the solution of the box extending from $x=0$ to $x=a$, since the shift has no physical consequences. We therefore have

$$C_n=\sqrt{\frac{2}{a}}\int_0^{a/2}dx\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}=\frac{2}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^{a/2}=\frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right]$$

Therefore $P_1=|C_1|^2=\frac{4}{\pi^2}$ and $P_2=|C_2|^2=\frac{1}{\pi^2}|(1-(-1))|^2=\frac{4}{\pi^2}$

這個題目有點問題，原因是原來題目給的波函數，在原點並不滿足無限大位能井的邊界條件。比較合理的題目應該是：

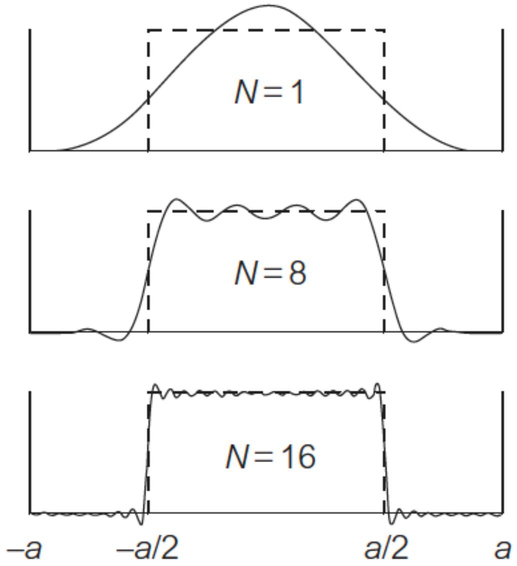
$$\psi(x)=\sqrt{\frac{2}{a}}\quad \frac{a}{4}<x<\frac{3a}{4},\quad \psi(x)=0\quad x<\frac{a}{4},x>\frac{3a}{4}$$

這樣就滿足邊界條件。我們就能預期$\psi(x)$可以以u_n展開：$\psi(x)=\sum C_n u_n$。

$$C_n=\sqrt{\frac{2}{a}}\int_0^a dx\sin\frac{n\pi x}{a}\psi(x)=\sqrt{\frac{2}{a}}\int_{a/4}^{3a/4}dx\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}=\frac{2}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_{a/4}^{3a/4}$$

$$C_1=-\frac{2}{a}\frac{a}{\pi}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)=\frac{2\sqrt{2}}{\pi}$$

$$C_2=-\frac{2}{a}\frac{a}{2\pi}(0-0)=0$$



10. (a) We use the solution of the above problem to get

$$P_n = |C_n|^2 = \frac{4}{n^2\pi^2} f_n$$

where $f_n = 1$ for n = odd integer; $f_n = 0$ for $n = 4,8,12,\ldots$ and $f_n = 4$ for $n = 2,6,10,\ldots$

(b) We have

$$\sum_{n=1}^{\infty} P_n = \frac{4}{\pi^2}\sum_{odd}\frac{1}{n^2} + \frac{4}{\pi^2}\sum_{n=2,6,10,..}\frac{4}{n^2} = \frac{8}{\pi^2}\sum_{odd}\frac{1}{n^2} = 1$$

Note. There is a typo in the statement of the problem. The sum should be restricted to *odd* integers.

11. We work this out by making use of an identity. The hint tells us that

$$(\sin x)^5 = \left(\frac{1}{2i}\right)^5 (e^{ix} - e^{-ix})^5 = \frac{1}{16}\frac{1}{2i}(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix})$$

$$= \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$$

Thus

$$\psi(x,0) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x) - 5u_3(x) + 10u_1(x)\right)$$

(a) It follows that

$$\psi(x,t) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x)e^{-iE_5t/\hbar} - 5u_3(x)e^{-iE_3t/\hbar} + 10u_1(x)e^{-iE_1t/\hbar}\right)$$

(b) We can calculate A by noting that $\int_0^a dx\,|\psi(x,0)|^2 = 1$. This however is equivalent to the statement that the sum of the probabilities of finding *any* energy eigenvalue adds up to 1. Now we have

$$P_5 = \frac{a}{2}A^2\frac{1}{256}; P_3 = \frac{a}{2}A^2\frac{25}{256}; P_1 = \frac{a}{2}A^2\frac{100}{256}$$

so that

$$A^2 = \frac{256}{63a}$$

The probability of finding the state with energy E_3 is 25/126.

12. The initial wave function vanishes for $x \le -a$ and for $x \ge a$. In the region in between it is proportional to $\cos\frac{\pi x}{2a}$, since this is the first nodeless trigonometric function that vanishes at $x = \pm a$. The normalization constant is obtained by requiring that

$$1 = N^2\int_{-a}^{a} dx\cos^2\frac{\pi x}{2a} = N^2\left(\frac{2a}{\pi}\right)\int_{-\pi/2}^{\pi/2} du\cos^2 u = N^2 a$$

so that $N = \sqrt{\frac{1}{a}}$. We next expand this in eigenstates of the infinite box potential with boundaries at $x = \pm b$. We write

$$\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a} = \sum_{n=1}^{\infty} C_n u_n(x;b)$$

so that

$$C_n = \int_{-b}^{b} dx u_n(x;b)\psi(x) = \int_{-a}^{a} dx u_n(x;b)\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a}$$

In particular, after a little algebra, using cosu cosv=(1/2)[cos(u-v)+cos(u+v)], we get

$$C_1 = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \cos\frac{\pi x}{2b}\cos\frac{\pi x}{2a} = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \frac{1}{2}\left[\cos\frac{\pi x(b-a)}{2ab} + \cos\frac{\pi x(b+a)}{2ab}\right]$$
$$= \frac{4b\sqrt{ab}}{\pi(b^2 - a^2)}\cos\frac{\pi a}{2b}$$

so that

$$P_1 = |C_1|^2 = \frac{16ab^3}{\pi^2(b^2 - a^2)^2}\cos^2\frac{\pi a}{2b}$$

The calculation of C_2 is trivial. The reason is that while $\psi(x)$ is an *even* function of x, $u_2(x)$ is an *odd* function of x, and the integral over an interval symmetric about $x = 0$ is zero. Hence P_2 will be zero.