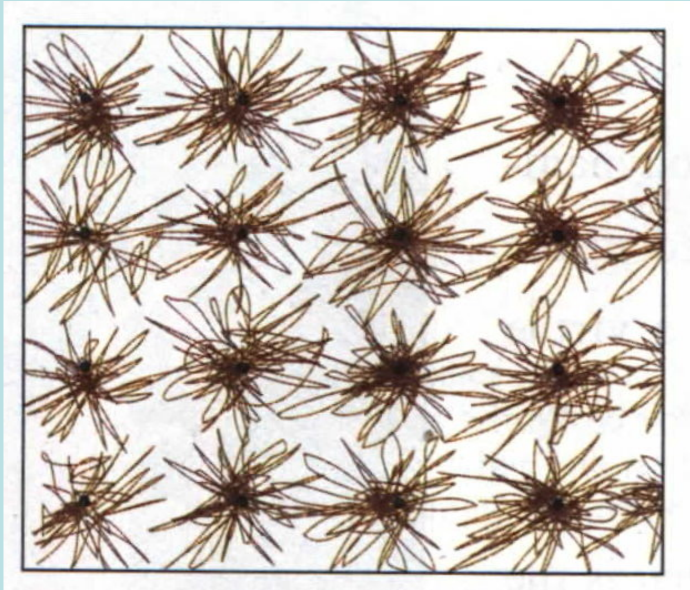
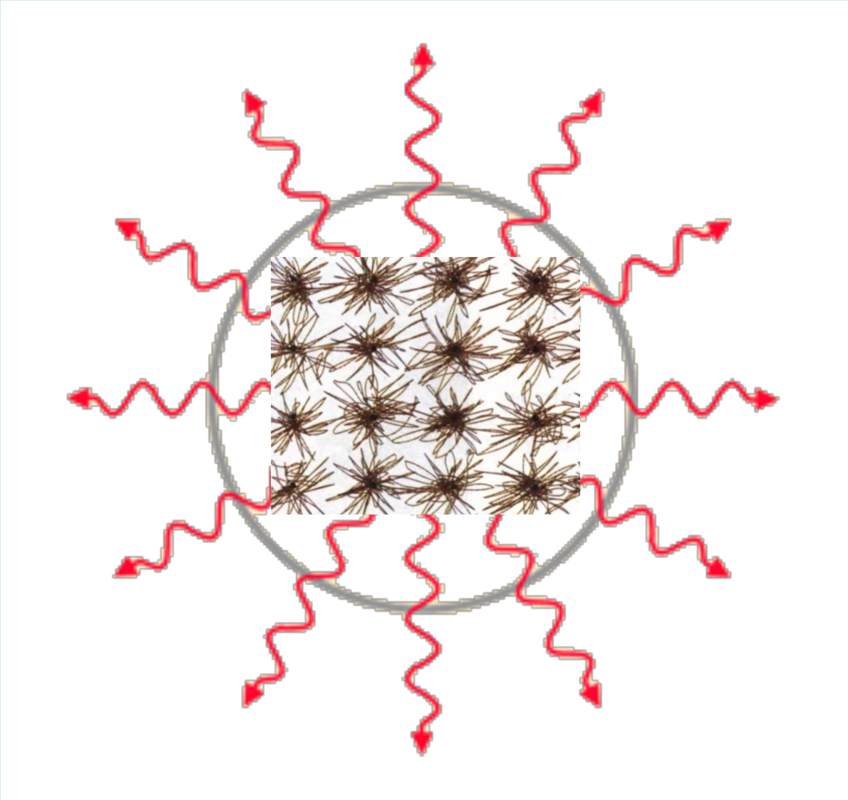
熱輻射（三種導熱方式中唯一不需要接觸者！）



物體內原子混亂的熱擾動會放出電磁波！

注意：電磁波無需介質即可傳播！



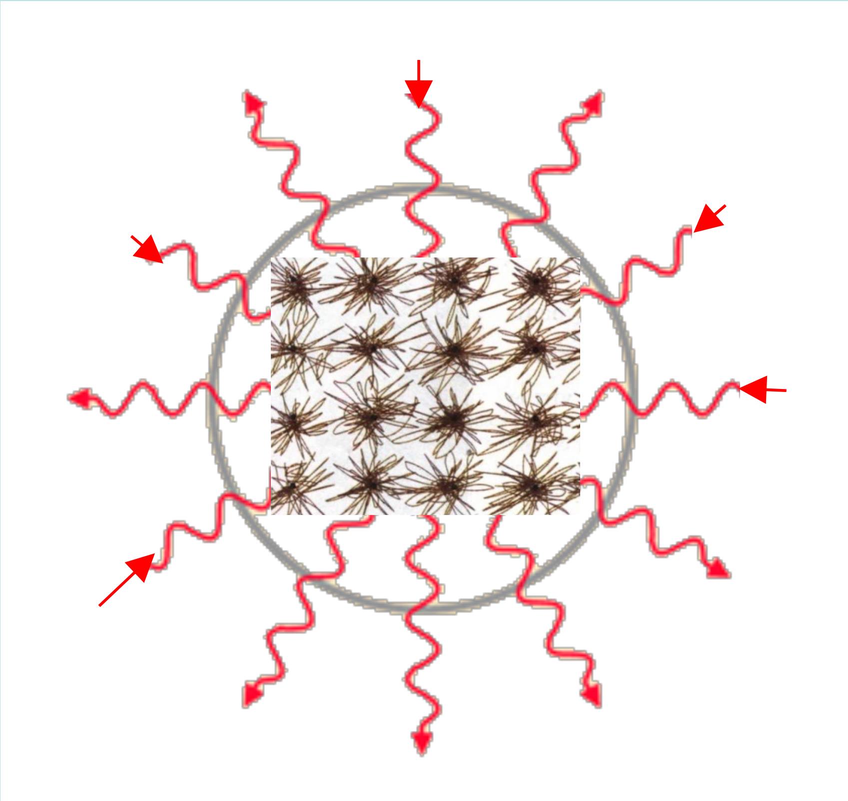
物體內的原子的熱擾動會放出電磁波，電磁波會帶走能量。

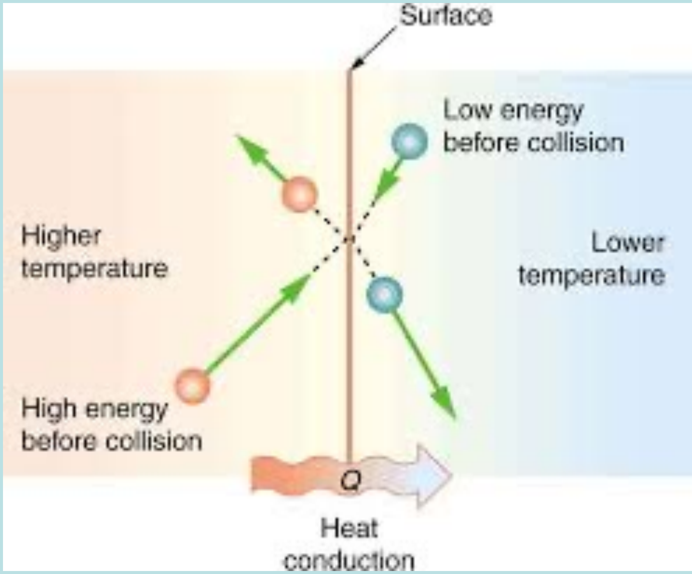
物體內能減少，溫度下降，這是放出熱量，因此熵會下降，亂度降低。

根據熱力學第二定律，熵不能減少，因此放出的電磁波，也要帶走熵。

換句話說，物體熱擾動放出的電磁波，必須夠亂，稱為熱輻射。

亂竟然也是有規則的！





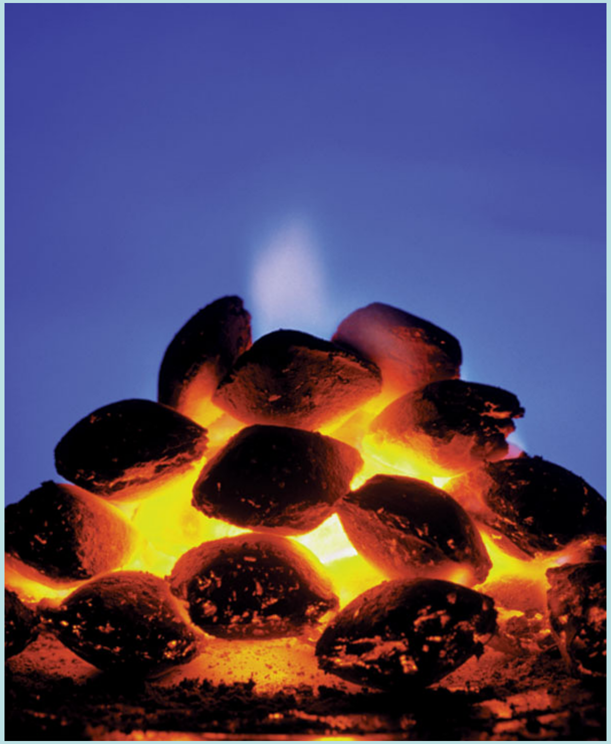
物體透過熱輻射，同時吸收環境的熱輻射，可以與環境交換熱。

效果與熱傳導相同，一段時間後，可以預期物體將與環境達到熱平衡。

這時物體的溫度T就與環境的溫度相等。

因此可以預期，根據第零定律，物體的熱輻射應該與物體的溫度T有關。

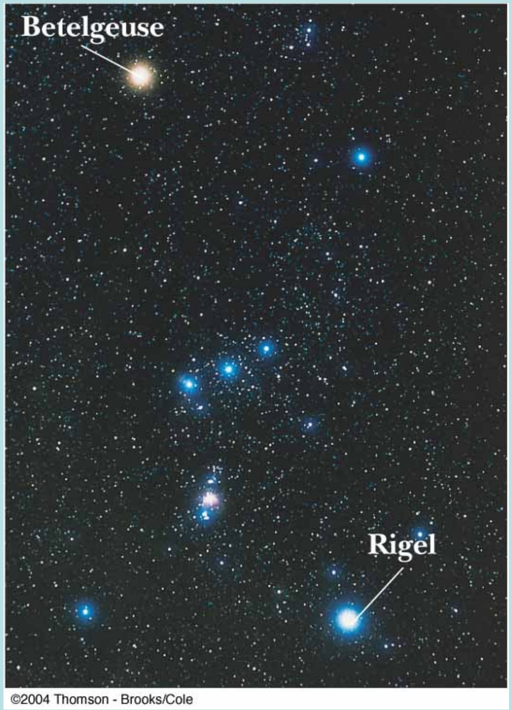
黑體輻射 Blackbody Radiation

表面不反射，而完全吸收電磁波的物體，稱為黑體。

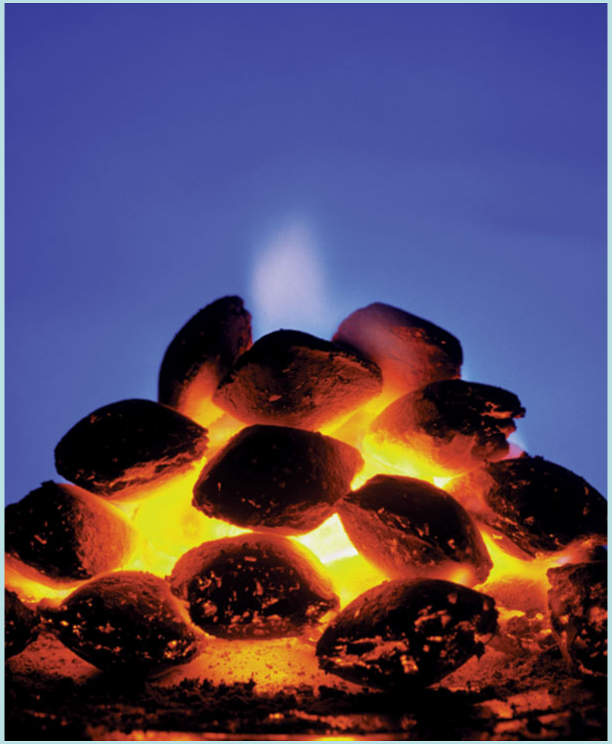
在紅外線的區域，大部分（非金屬）物體都是黑體！

黑體是完美的吸收者，因此也是好的輻射者，黑體發出的熱輻射稱為黑體輻射！



有點難想像，恆星、太陽、燈泡都近似是黑體！

黑體輻射 Blackbody Radiation



黑體輻射是涵蓋整個電磁波的範圍！在室溫以紅外線為主。

其實，在紅外線的區域，大部分（非金屬）物體都是黑體！

黑體輻射與黑體的其他性質無關，完全由黑體的溫度*T*決定。

如此，所有同溫度的黑體輻射都是一樣的！一定溫度下黑體輻射只有一種！

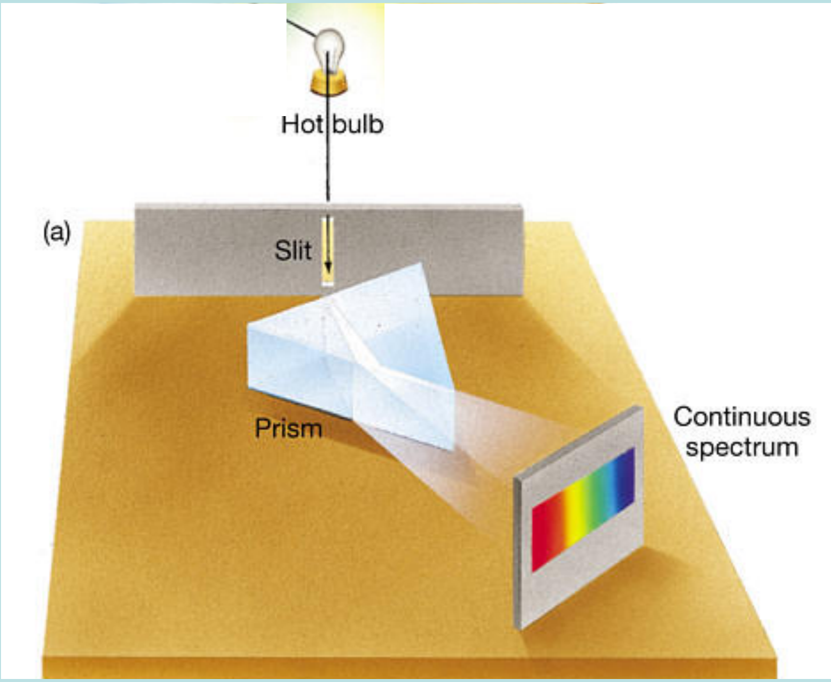
為什麼？

何謂一樣？熱輻射可測量的特性：

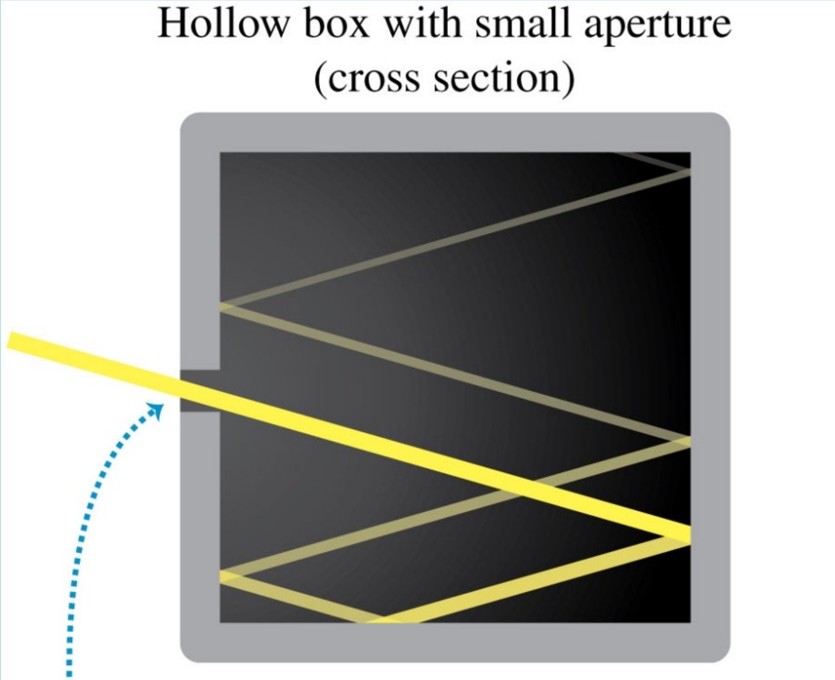
輻射總功率 P：單位時間的總輻射能量。

$$P = \frac{Q}{\Delta t}$$

輻射功率的波長分布： $P(\lambda)$



就以上兩個特性而言，所有同溫度的黑體輻射都是一模一樣！



一個小洞應該是非常好的吸收者，應該是黑體才對！

光透過小洞進入盒子，會被拘限在裡面。這稱為空腔 Cavity。

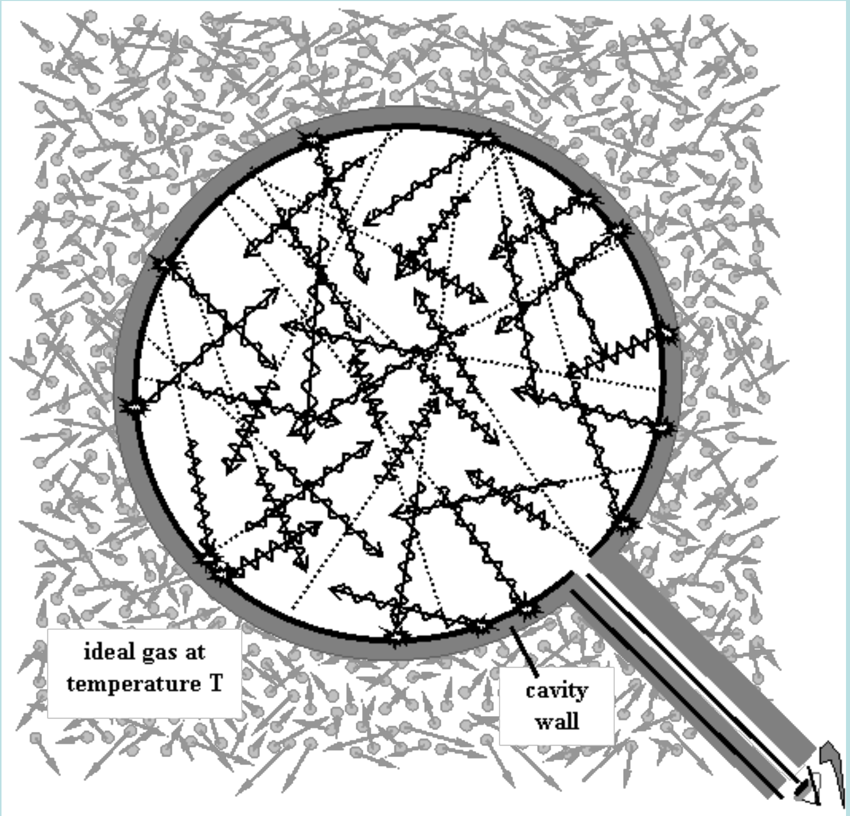
若洞很小，光會在盒子裡面待夠久，而與器壁達成熱平衡。

形成一個夠亂卻又穩定的狀態！

空腔內的輻射會與周圍溫度為T的空腔壁形成熱平衡。

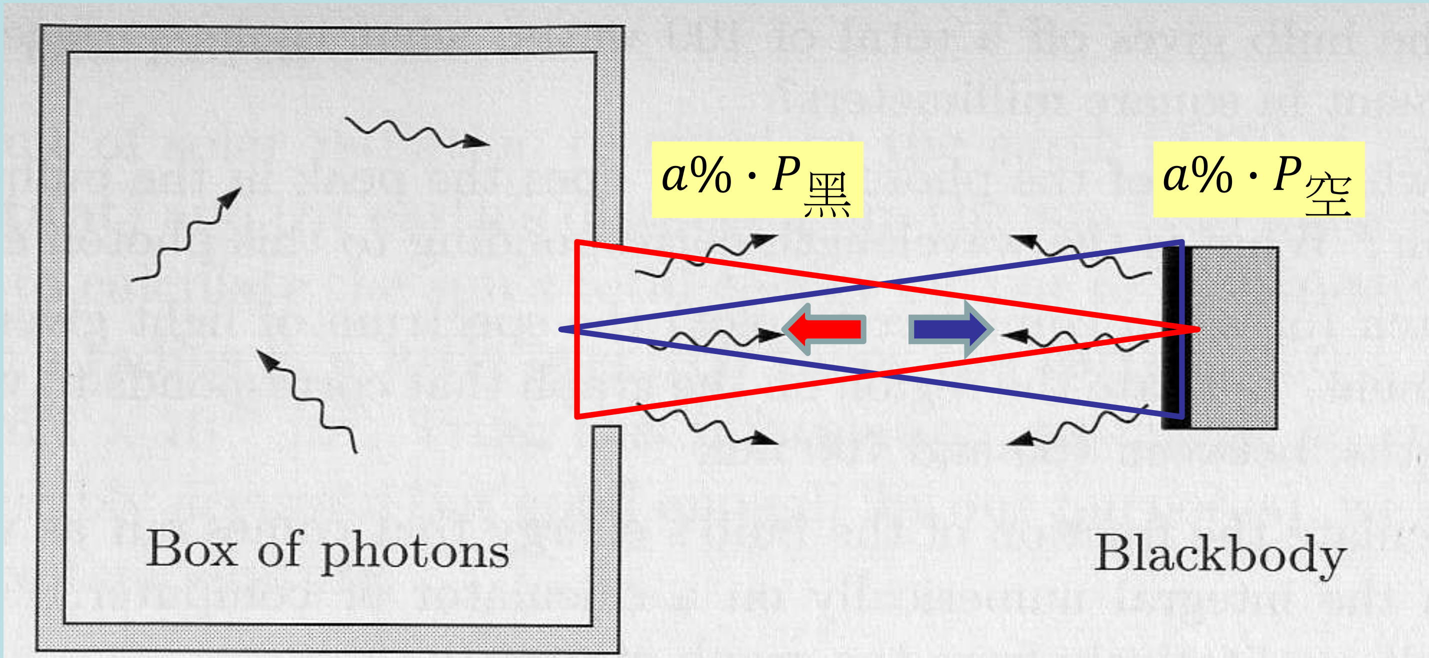
在如此狀態下由小洞發出的輻射就稱為空腔輻射 Cavity Radiation。

T就成為空腔輻射的溫度。



我要證明：溫度為T的空腔輻射，及溫度為T的黑體之輻射完全相同！

將一空腔輻射，及一個與其開口同面積的黑體輻射放在一起！



開口的空腔輻射被黑體吸收的比例，等於此黑體輻射被空腔吸收的比例。

當兩者達到熱平衡時，溫度相等。能量交換彼此抵消。 $a\% \cdot P_{黑} = a\% \cdot P_{空}$

因此空腔輻射與黑體輻射總量相等。 $P_{黑} = P_{空}$

溫度為T的空腔輻射，及溫度為T的黑體之輻射完全相同！

而同一溫度時的空腔輻射只有一種（空腔是空的！）

所以，所有黑體不論性質，同一溫度時發出的熱輻射都一樣！

所以所有黑體不論性質，同一溫度時發出的熱輻射都一樣！

我們就可以用黑體的溫度T來稱呼它的熱輻射，稱為溫度為T的黑體輻射！

經過測量，黑體輻射總功率，與黑體的面積及溫度的四次方成正比：

而與所有其他黑體性質都無關。

$$P = \frac{Q}{\Delta t} = \sigma A T^4$$

$$P_{黑} = P_{空}$$

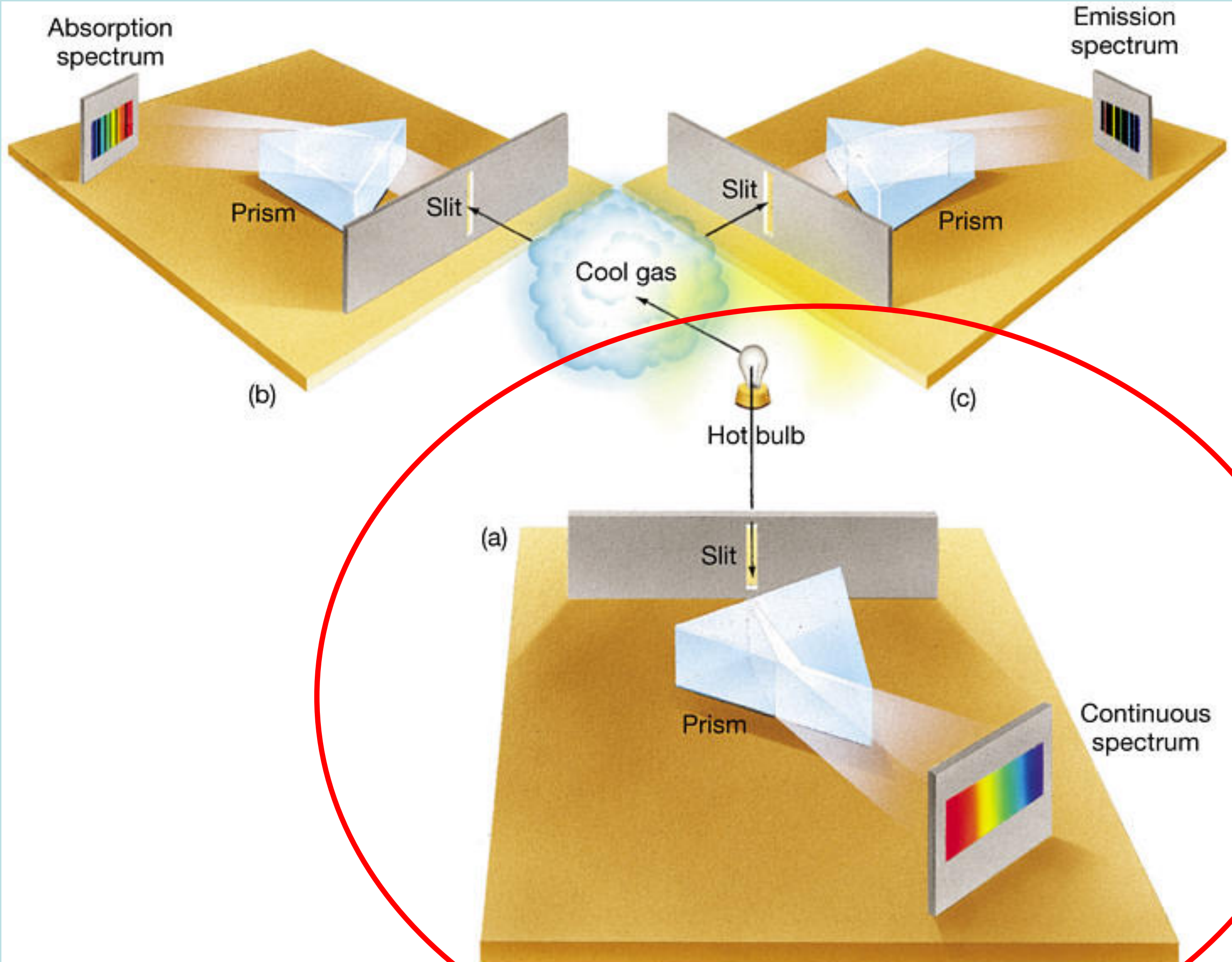
$$\sigma = 5.67\times10^{-8}\mathrm{W/m^2K^4}$$

Stefan-Boltzmann Constant

© 2004 Thomson - Brooks/Cole

測量輻射功率的波長分布： $E(\lambda, T)$。

$$E_{黑} = E_{空}$$

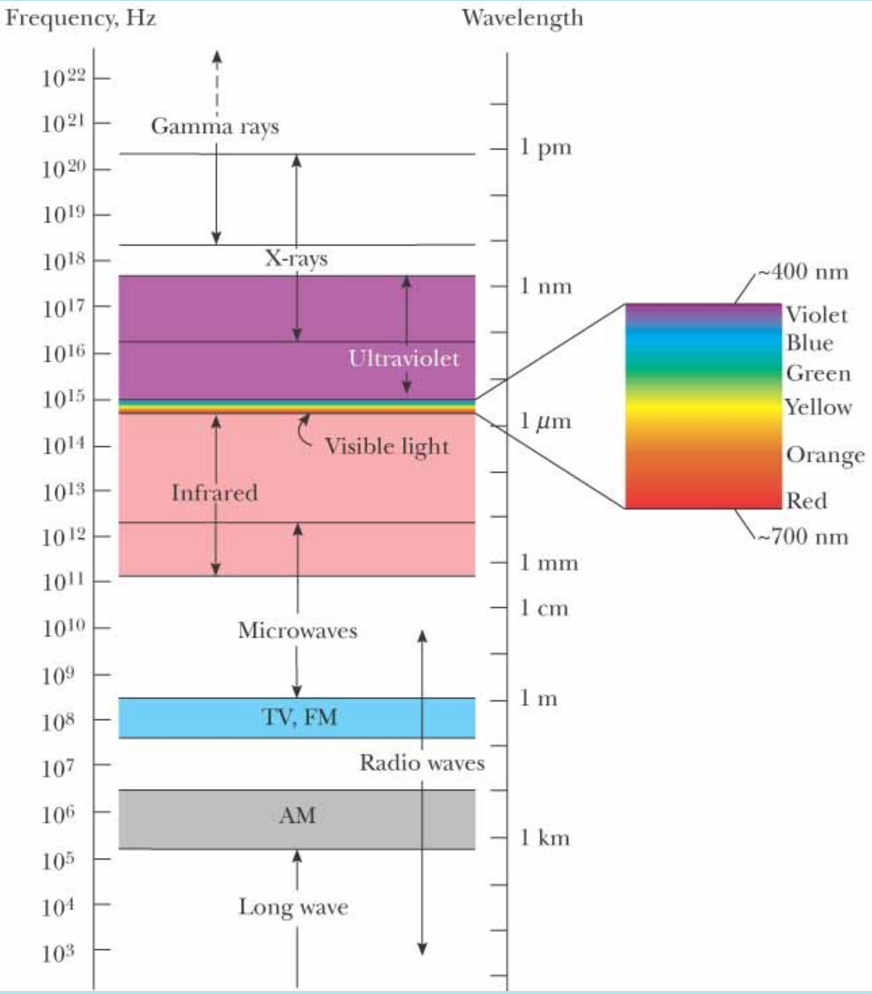


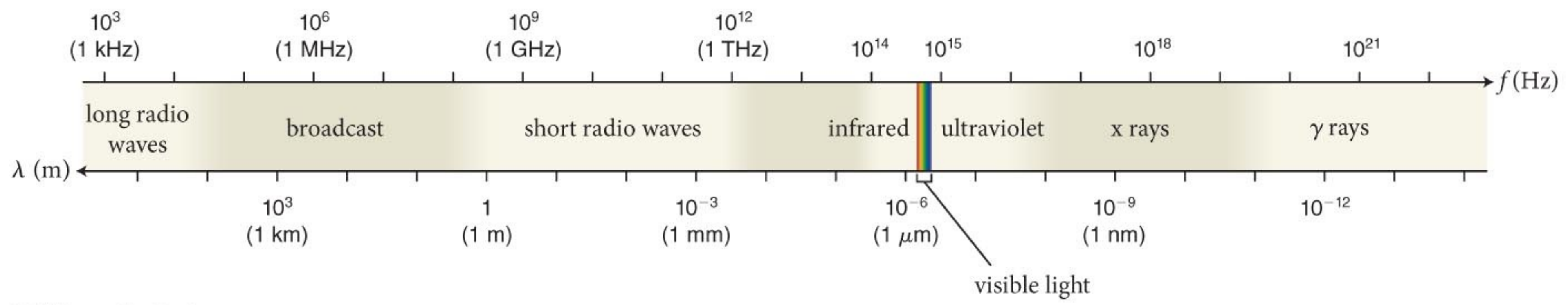
$$P = \int_0^{\infty} E(\lambda, T) \cdot d\lambda$$

注意：黑體輻射的波長分布是連續的！

電磁波以頻率或波長為特徵

$\lambda \cdot f = c$

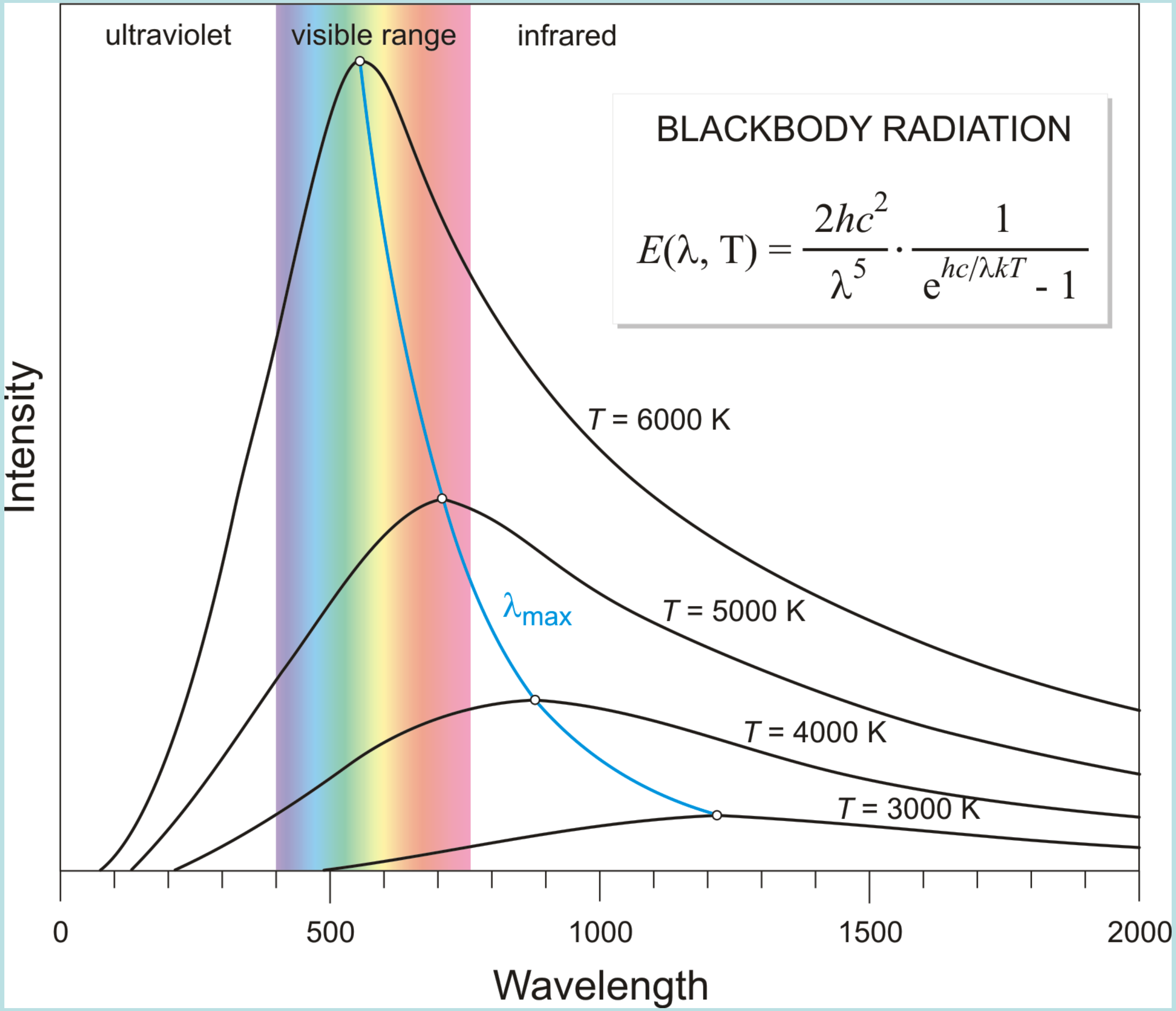


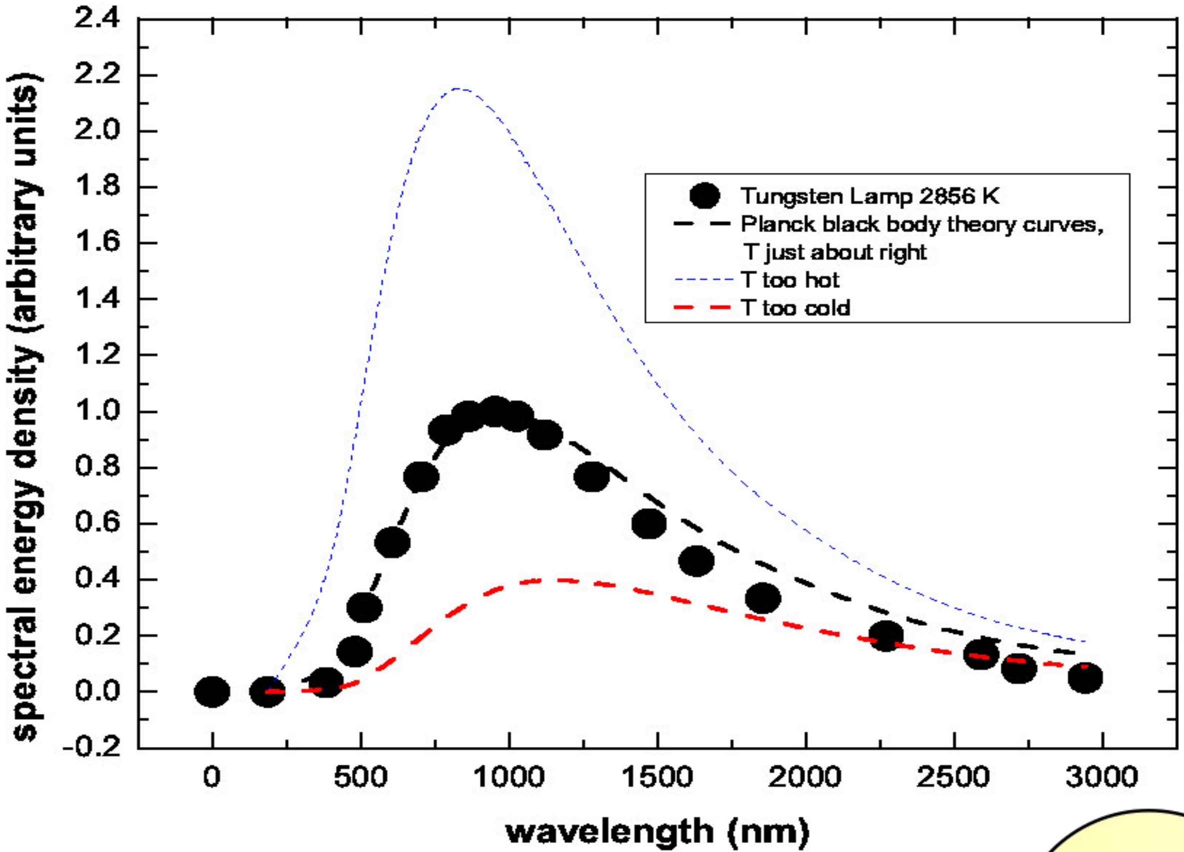


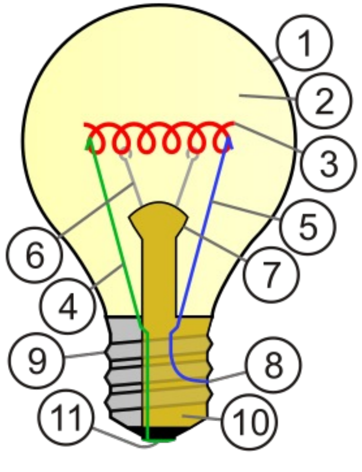
峰值的波長λ_{max}與溫度成反比關係！

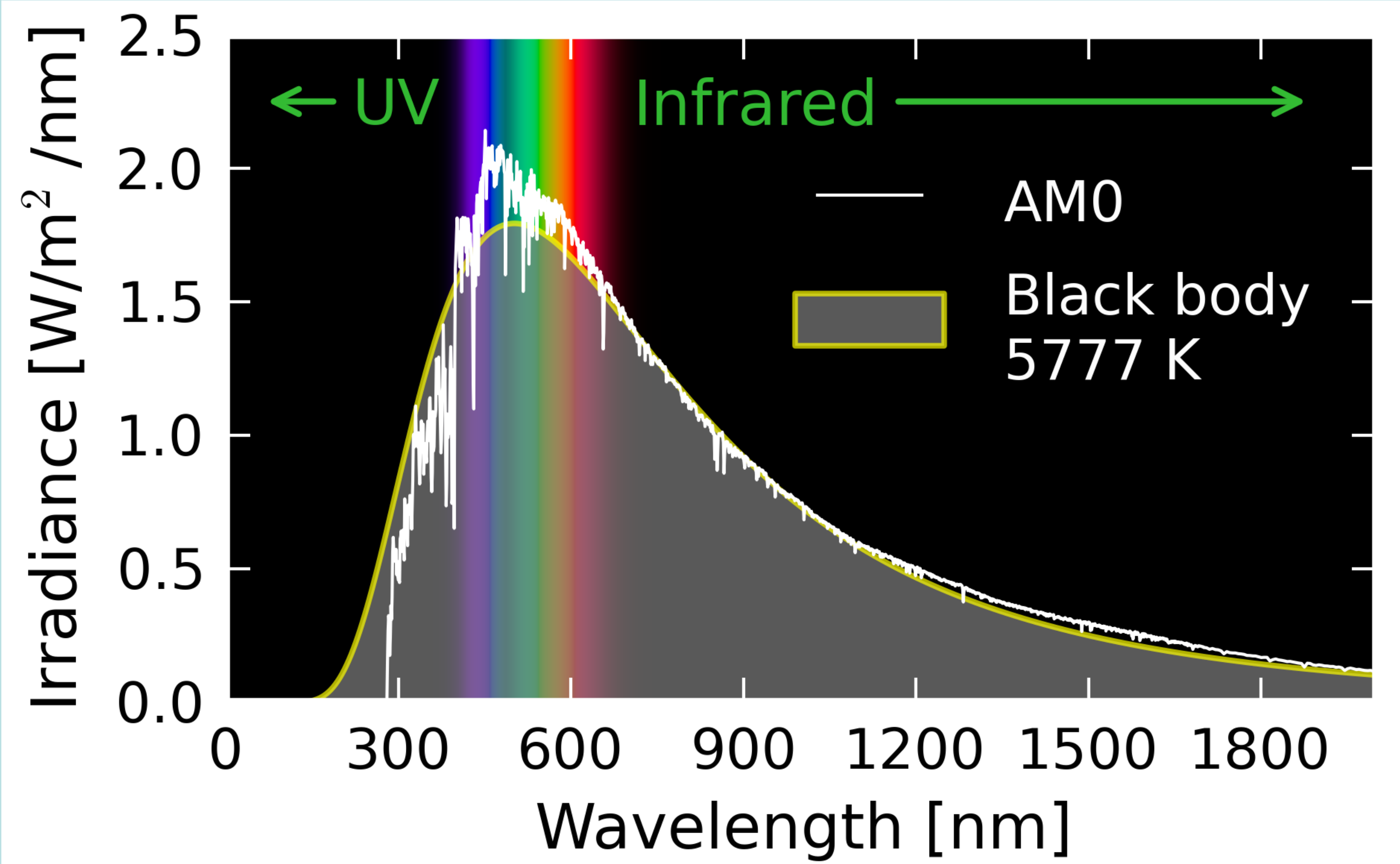
$$\lambda_{max} \cdot T = 2.898 \times 10^{-3} \mathrm{m} \cdot \mathrm{K}$$

Wien's displacement constant




spectral energy density (arbitrary units)
wavelength (nm)
Tungsten Lamp 2856 K
Planck black body theory curves,
T just about right
T too hot
T too cold
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
500
1000
1500
2000
2500
3000


1
2
3
5
6
7
4
9
8
11
10



Emission spectrum of the sun as measured above the Earth's atmosphere (AM0) compared to the black body spectrum of an object at 5777 K. Image Credit: Solar AM0 spectrum with visible spectrum background (en) by Danmichaelo [Public domain], from Wikimedia Commons

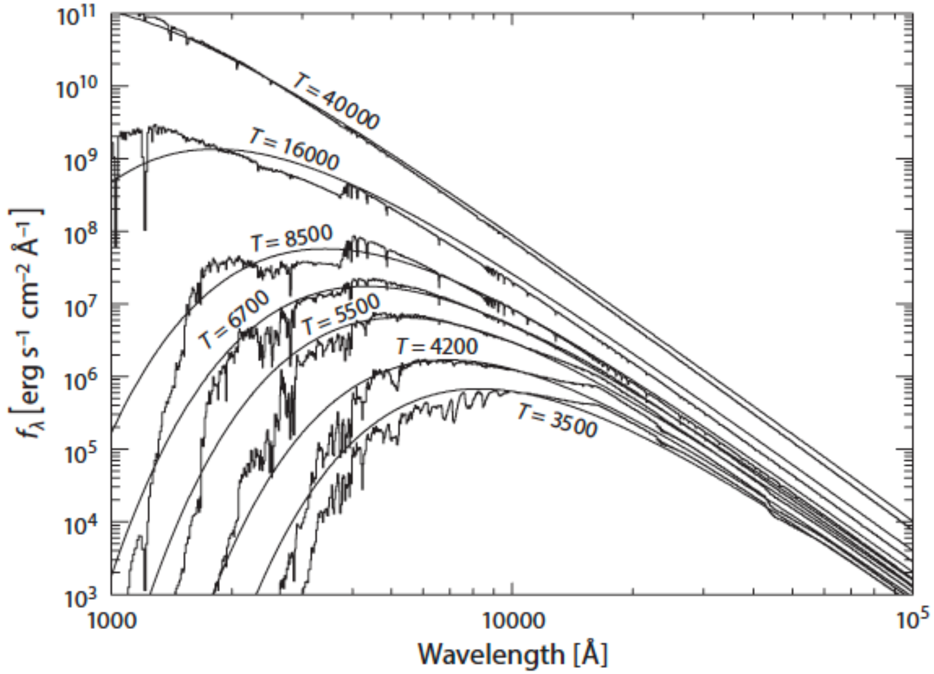
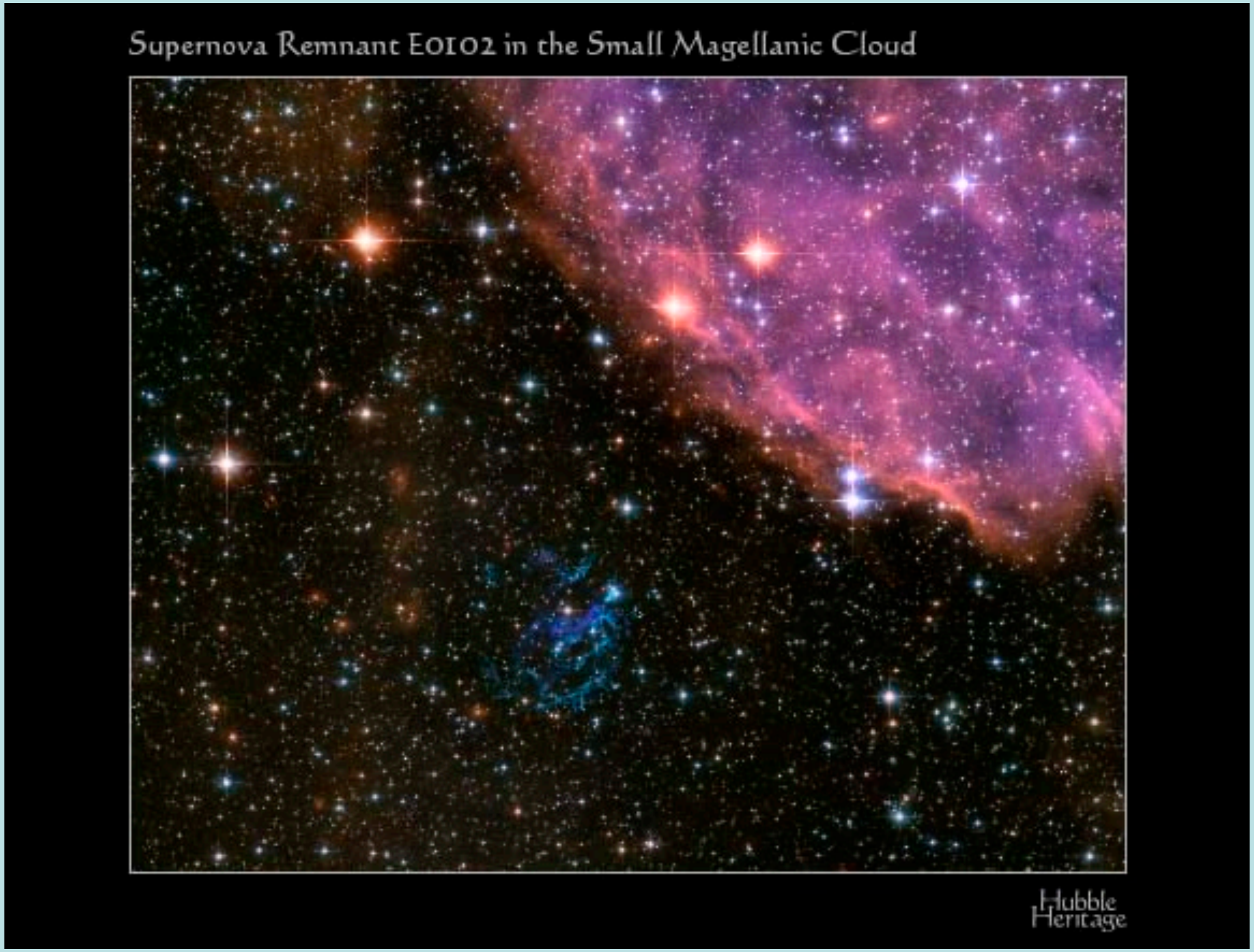


FIGURE 2. Stars are almost black bodies! Flux per wavelength interval emitted by different types of stars. This is fitted to the black body spectrum at various temperatures ranging from $T = 3{,}500$ K to $T = 40{,}000$ K. Note that the Planck distribution is plotted here as a function of wavelength, not frequency. Maoz, D. *Astrophysics in a Nutshell*. Princeton University Press, 2016.

Exercise

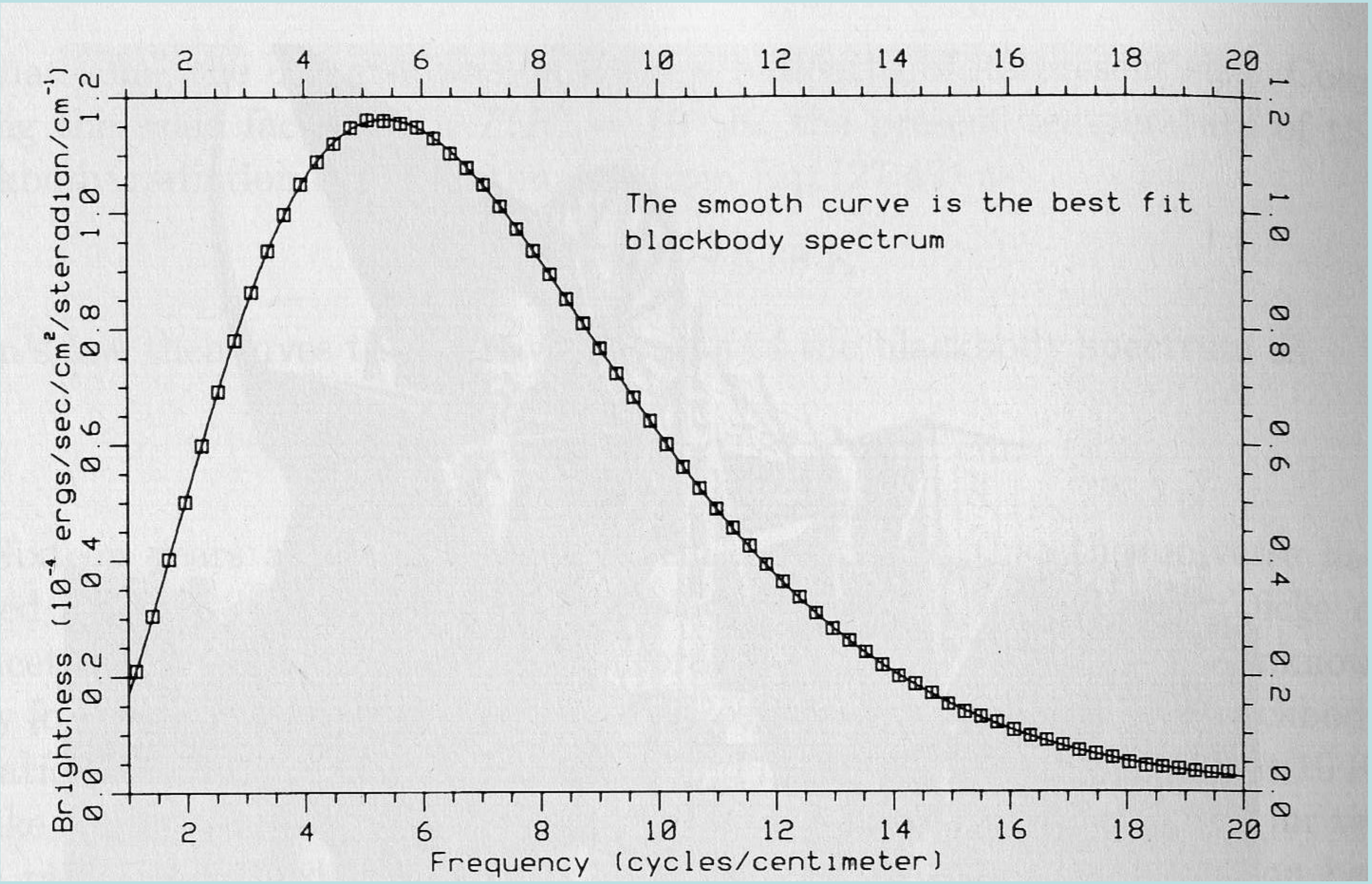
(1) Wien's displacement law, which played such a crucial role in the eventual discovery of quantum mechanics, is actually quite general. Show that if the distribution of a physical quantity ζ has the form $\zeta^a F(\zeta/T)$, with a an arbitrary constant and $F(x)$ an arbitrary function, the peak value ζ_{max} of the distribution increases linearly with T. Indeed, see also the caption to figure 1.

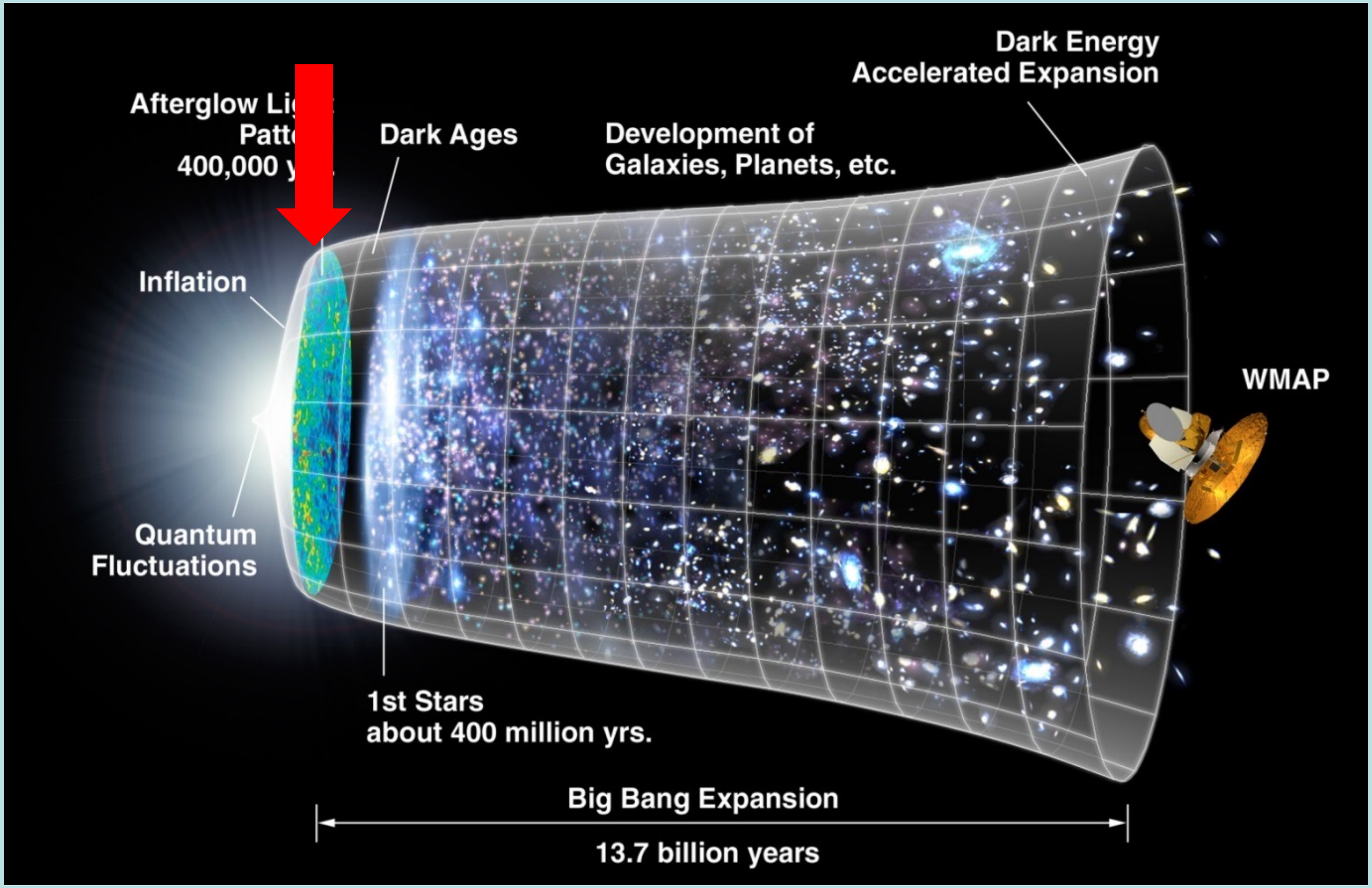
宇宙也是有溫度的！

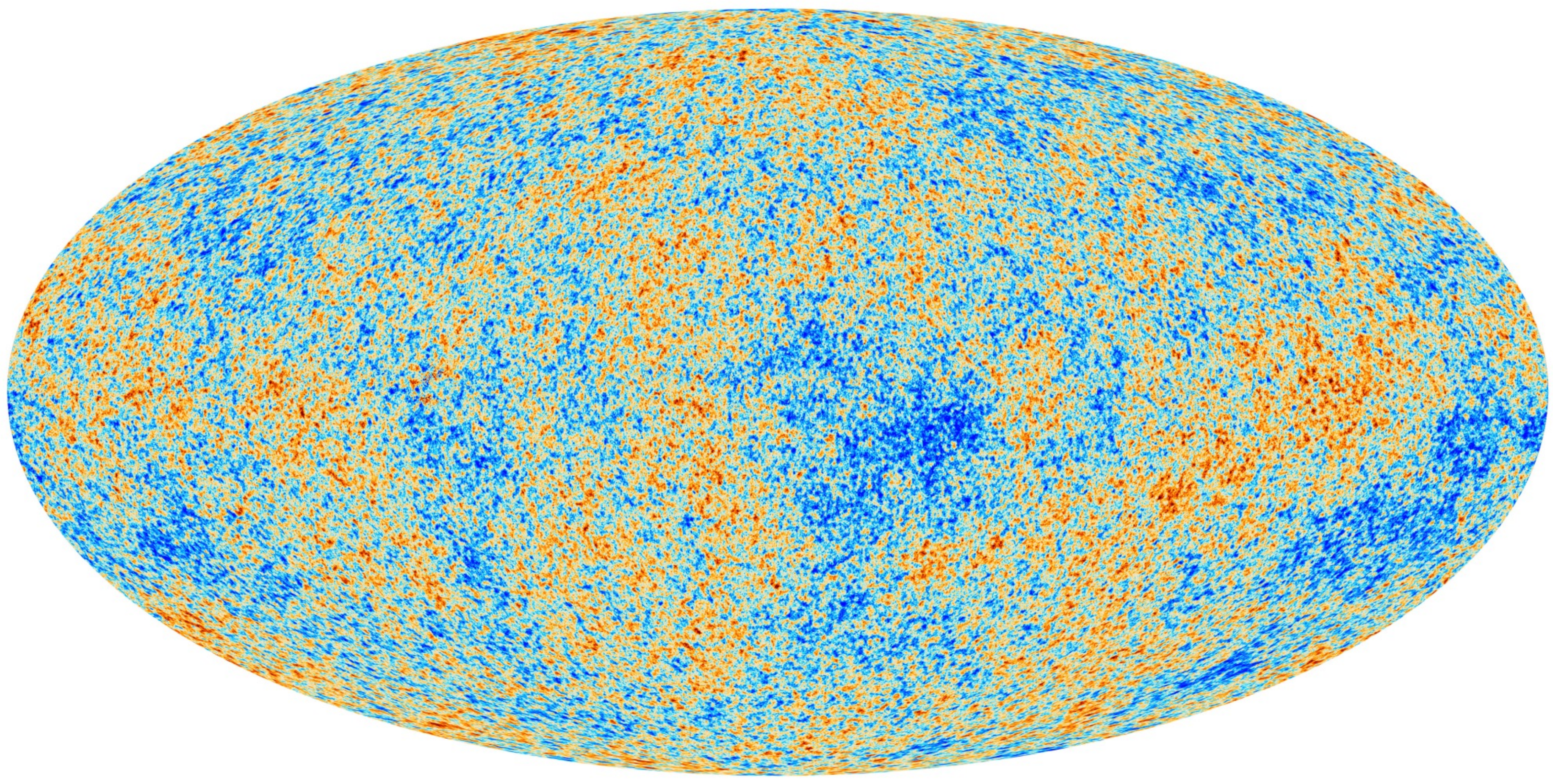

Supernova Remnant E0102 in the Small Magellanic Cloud
Hubble
Heritage

Mather and George F. Smoot *"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"*

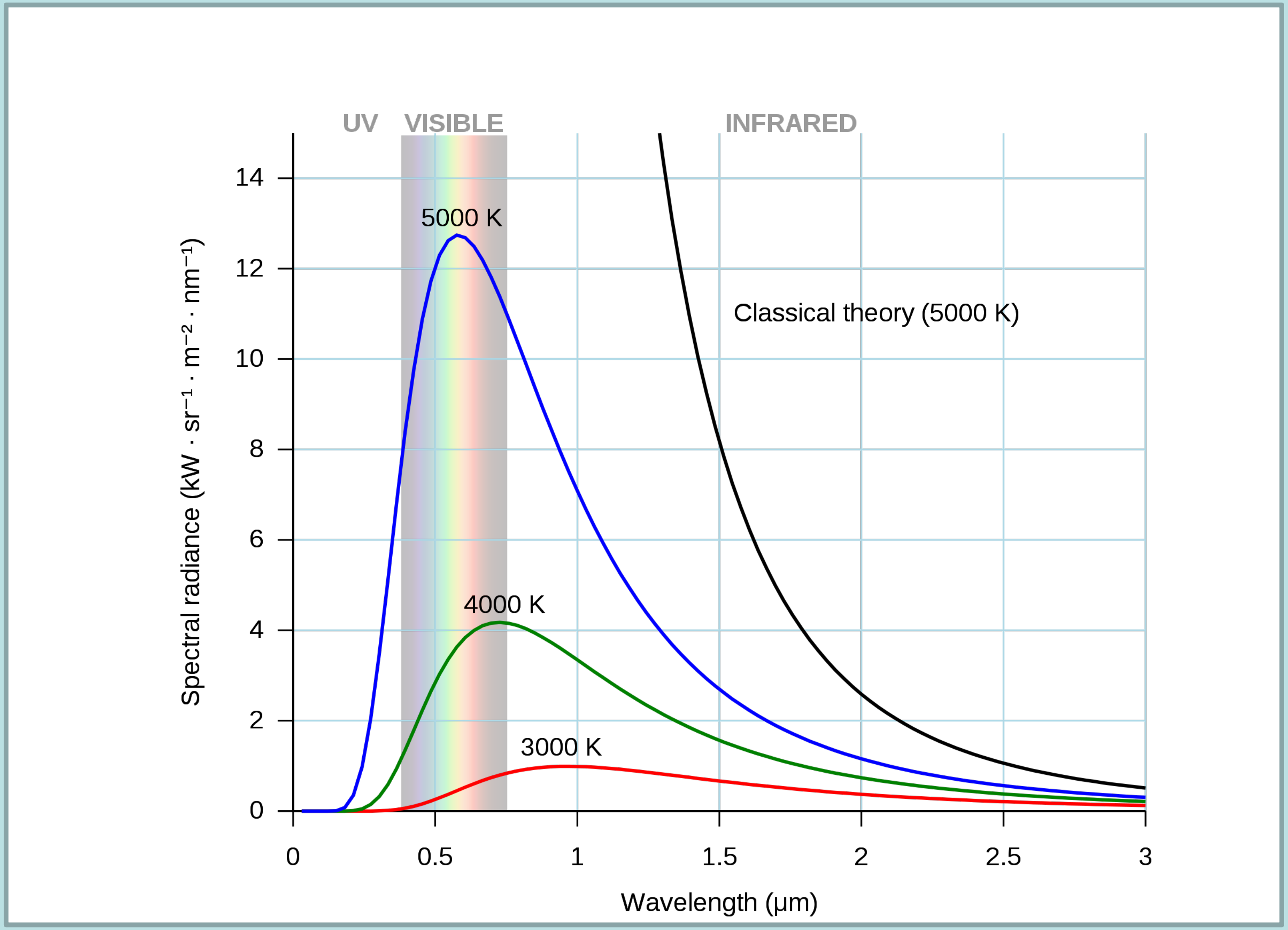
2.72548±0.00057 K

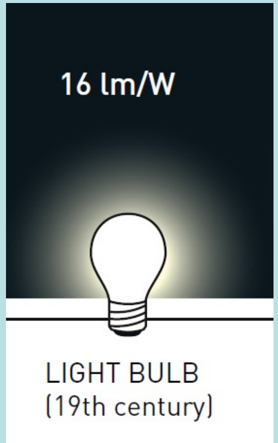



Dark Energy
Accelerated Expansion
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years



The anisotropies of the Cosmic Microwave Background (CMB) as observed by Planck 2013.





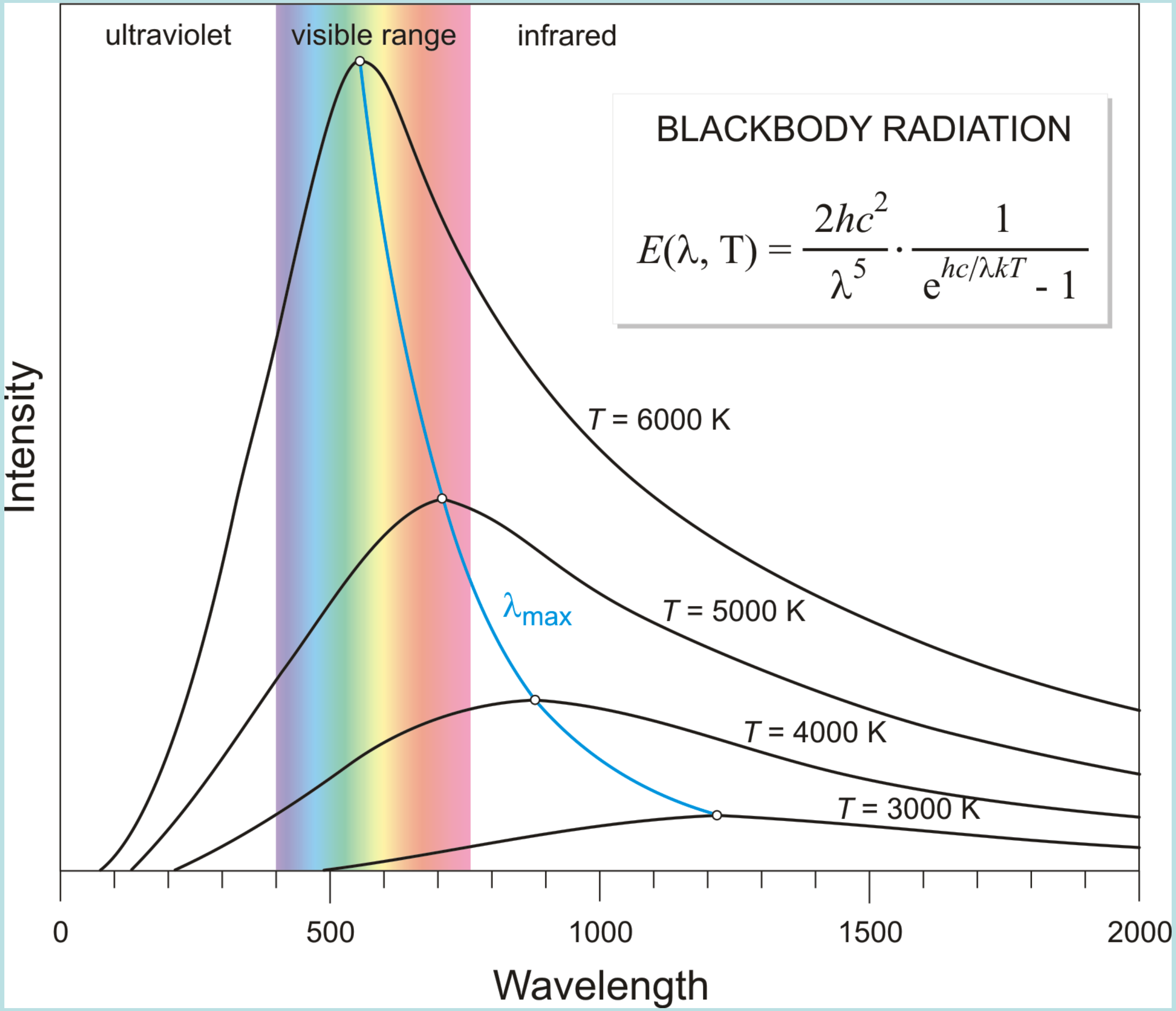
室溫下黑體輻射幾乎沒有可見光。

放出熱輻射幾乎都是紅外線！

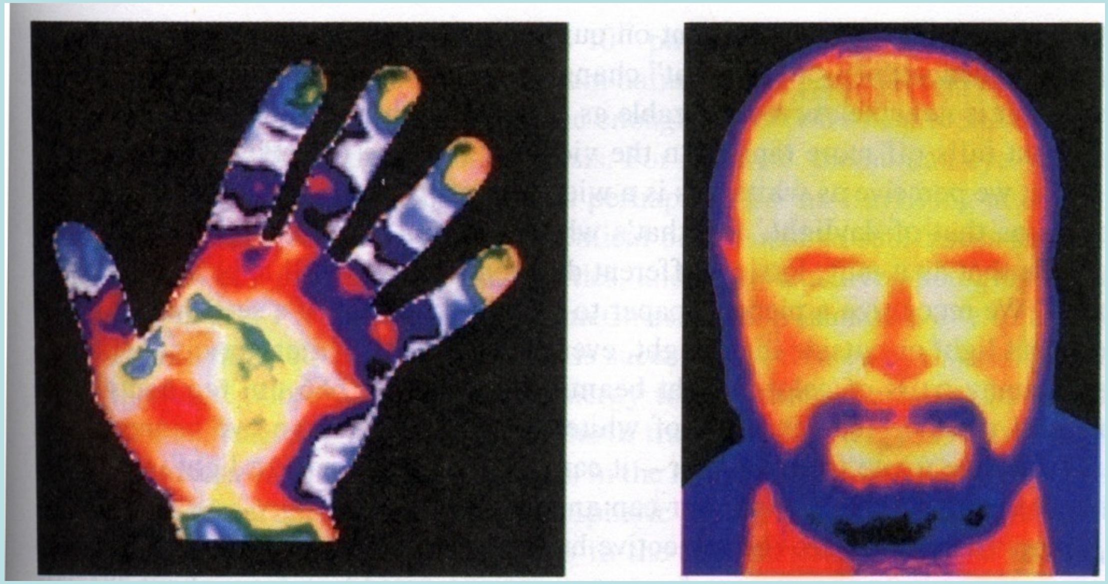
即使是約三千度的燈泡，大部分的能量都轉變為紅外線帶走的熱！

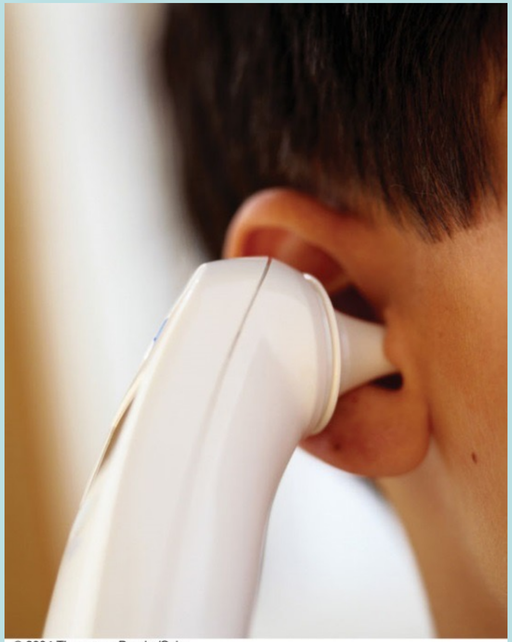
隨著溫度增加，峰值的波長λ_{max}會減小，

可見光的黑體輻射會慢慢出現。



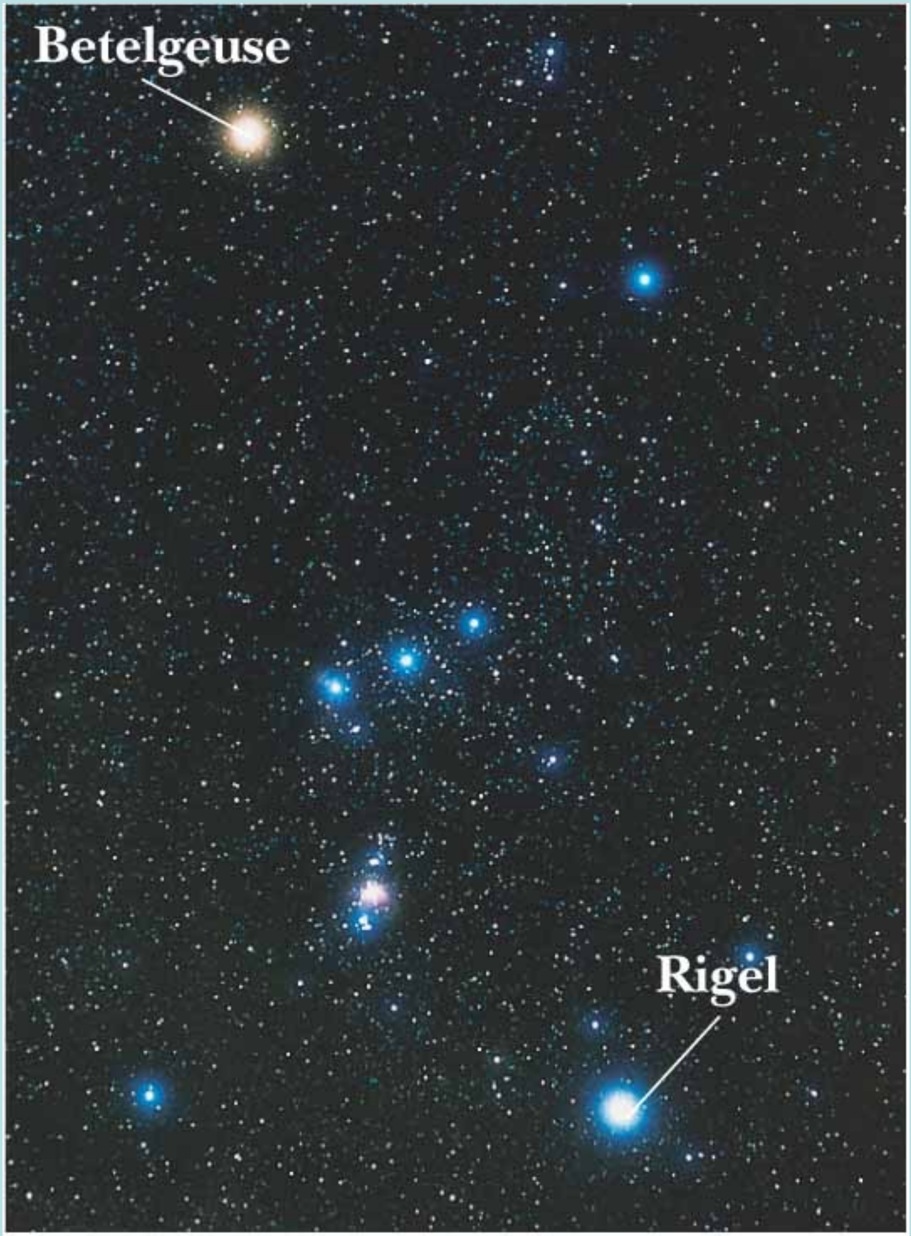
室溫下大部分輻射為紅外線





紅外線黑體輻射可以用特殊儀器觀察！

紅外線甚至可以穿透塑膠！



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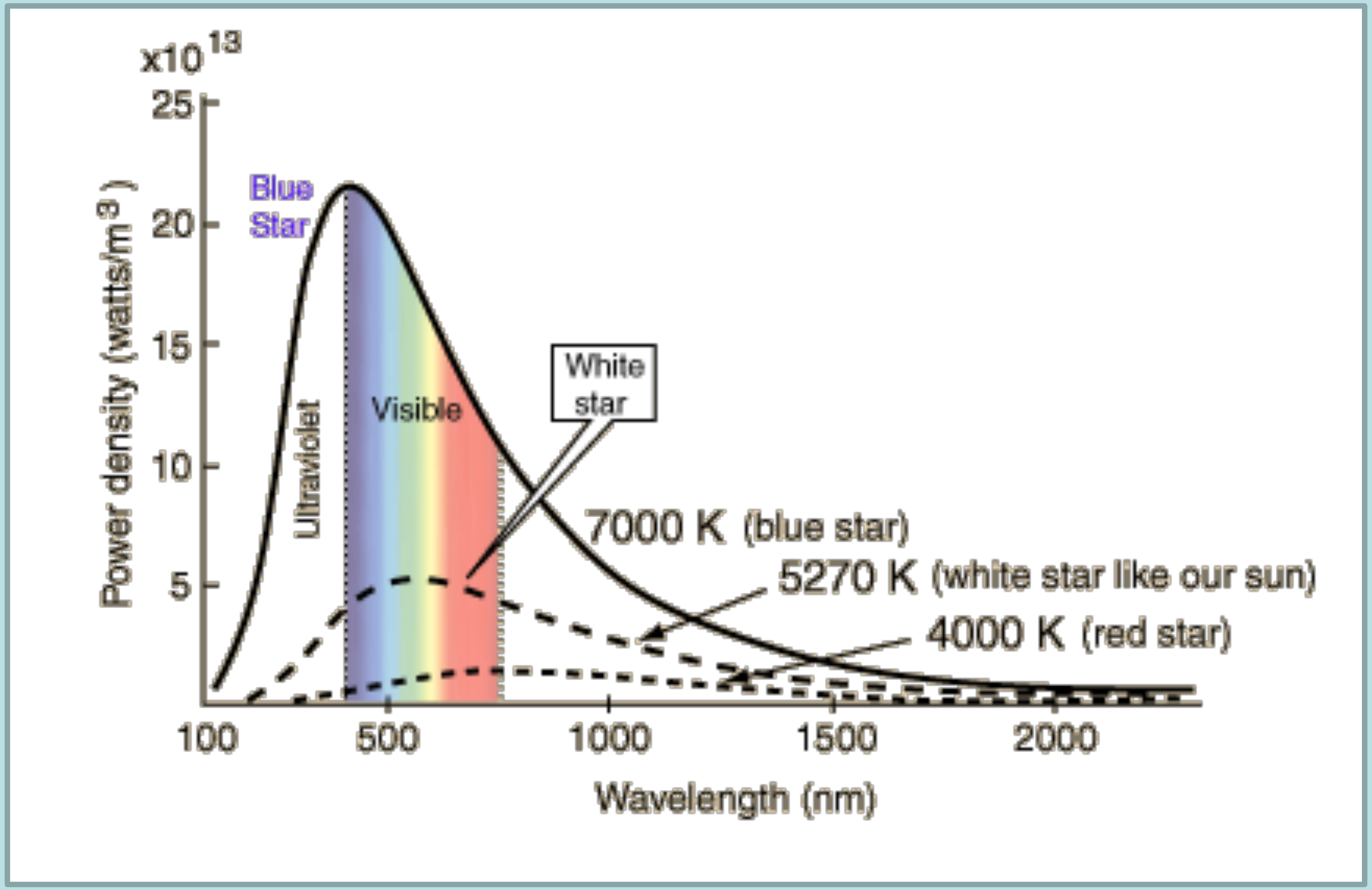
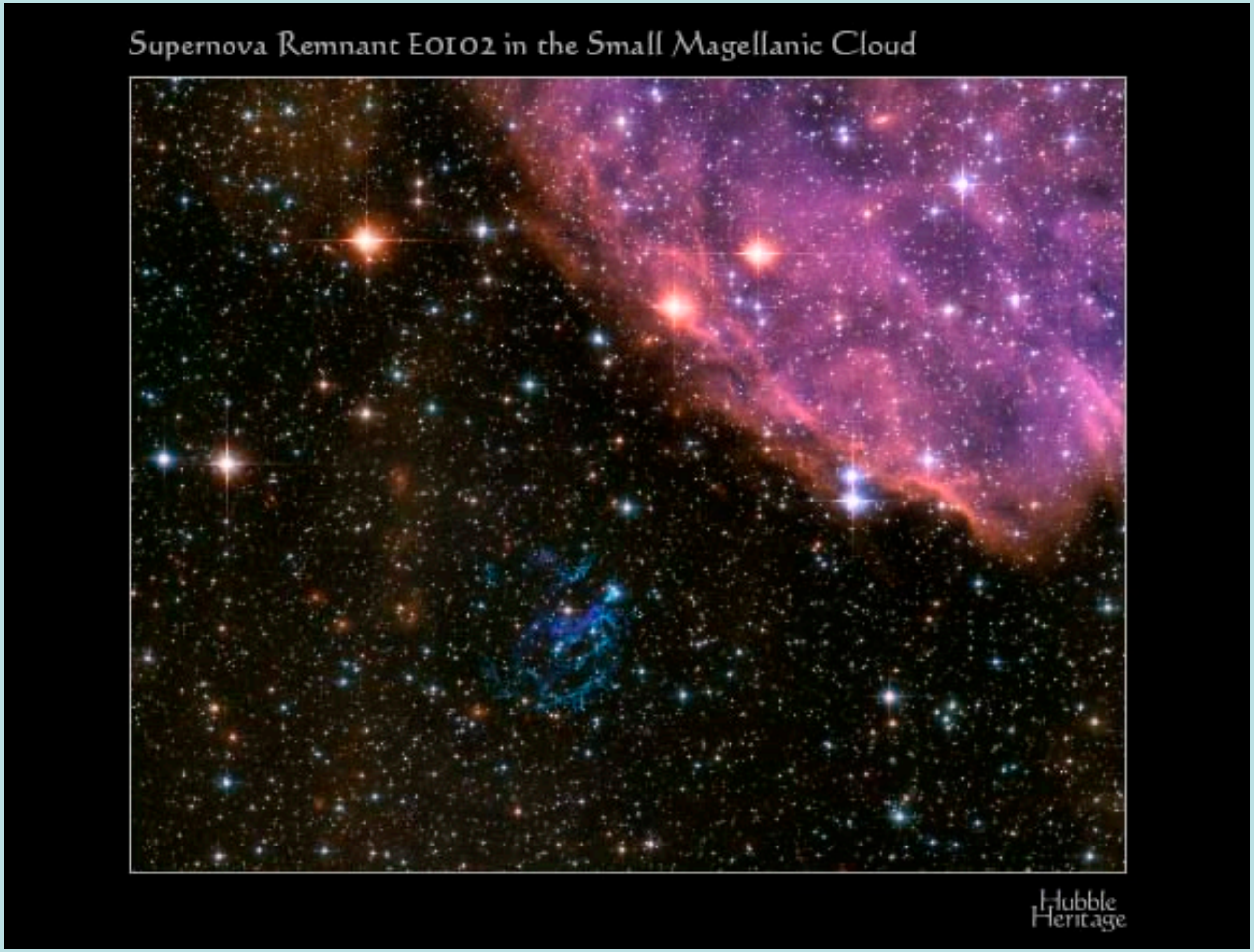
哪一個恆星溫度較高？



x10^13
25
20
15
10
5
Power density (watts/m^3)
Blue Star
Ultraviolet
Visible
White star
7000 K (blue star)
5270 K (white star like our sun)
4000 K (red star)
100
500
1000
1500
2000
Wavelength (nm)

宇宙也是有溫度的！


Supernova Remnant E0102 in the Small Magellanic Cloud
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Mather and George F. Smoot *"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"*

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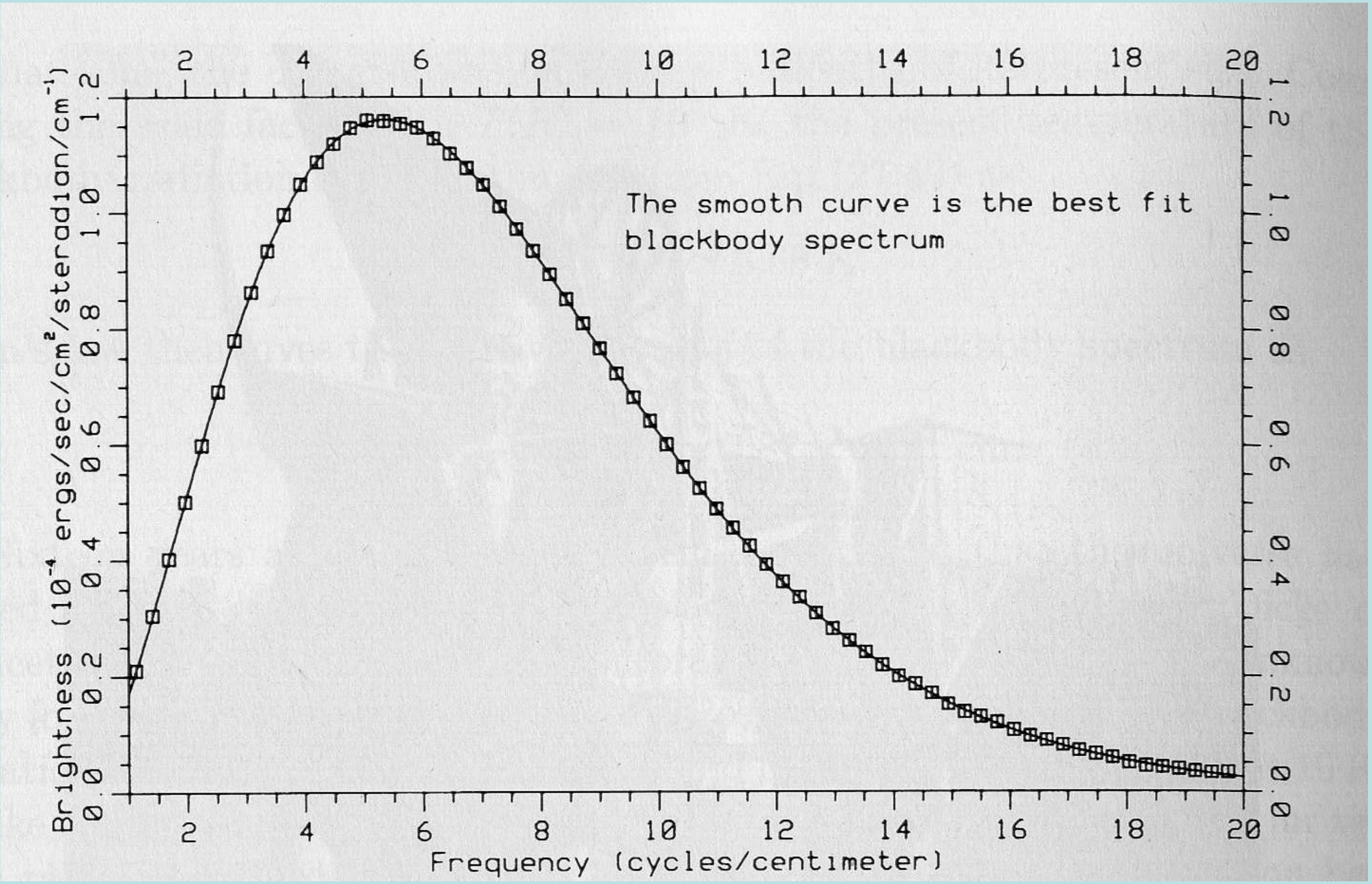
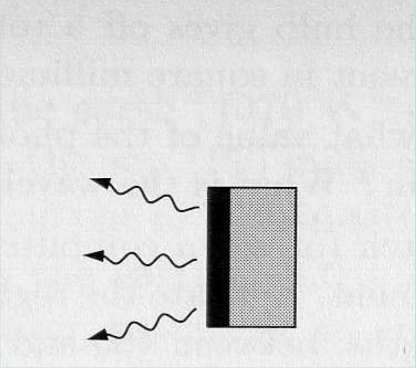


一般物體不完全是黑體，所照到的輻射一部分會反射回去：

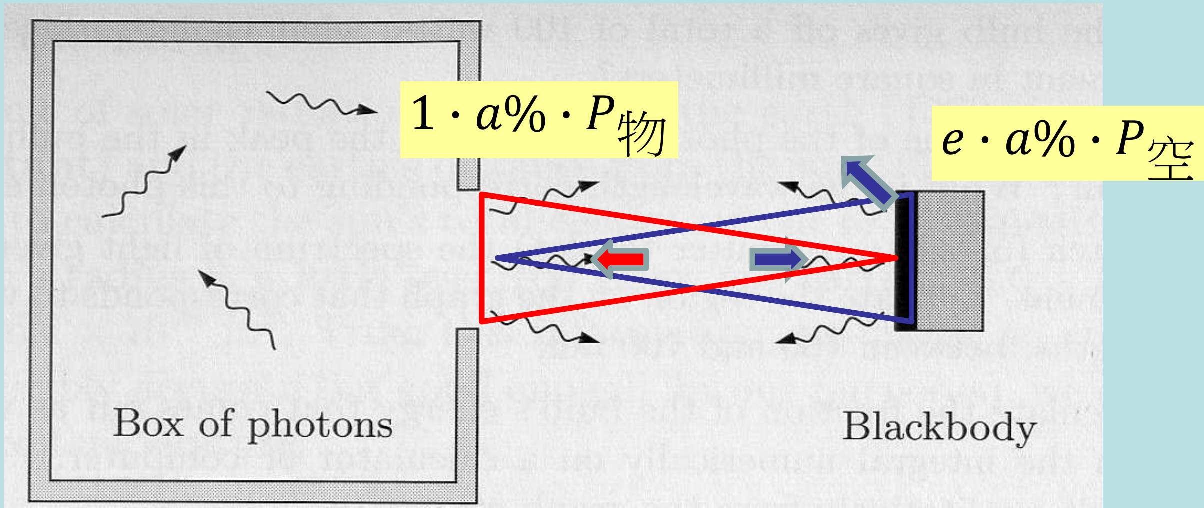
吸收的比例一般記為吸收率 e。



吸收率e也會是放射率emissivity，因此它的熱輻射功率為黑體的e倍。

$$P = eP_{黑體} = e\sigma AT^4$$

e 一般來說與溫度及波長都有關 $e_\lambda(T)$ ，但變化不大，可視為常數 e 。



該物體只吸收空腔射到該物體表面的輻射的 e 倍，

然而它對空腔的輻射，空腔會完全吸收。

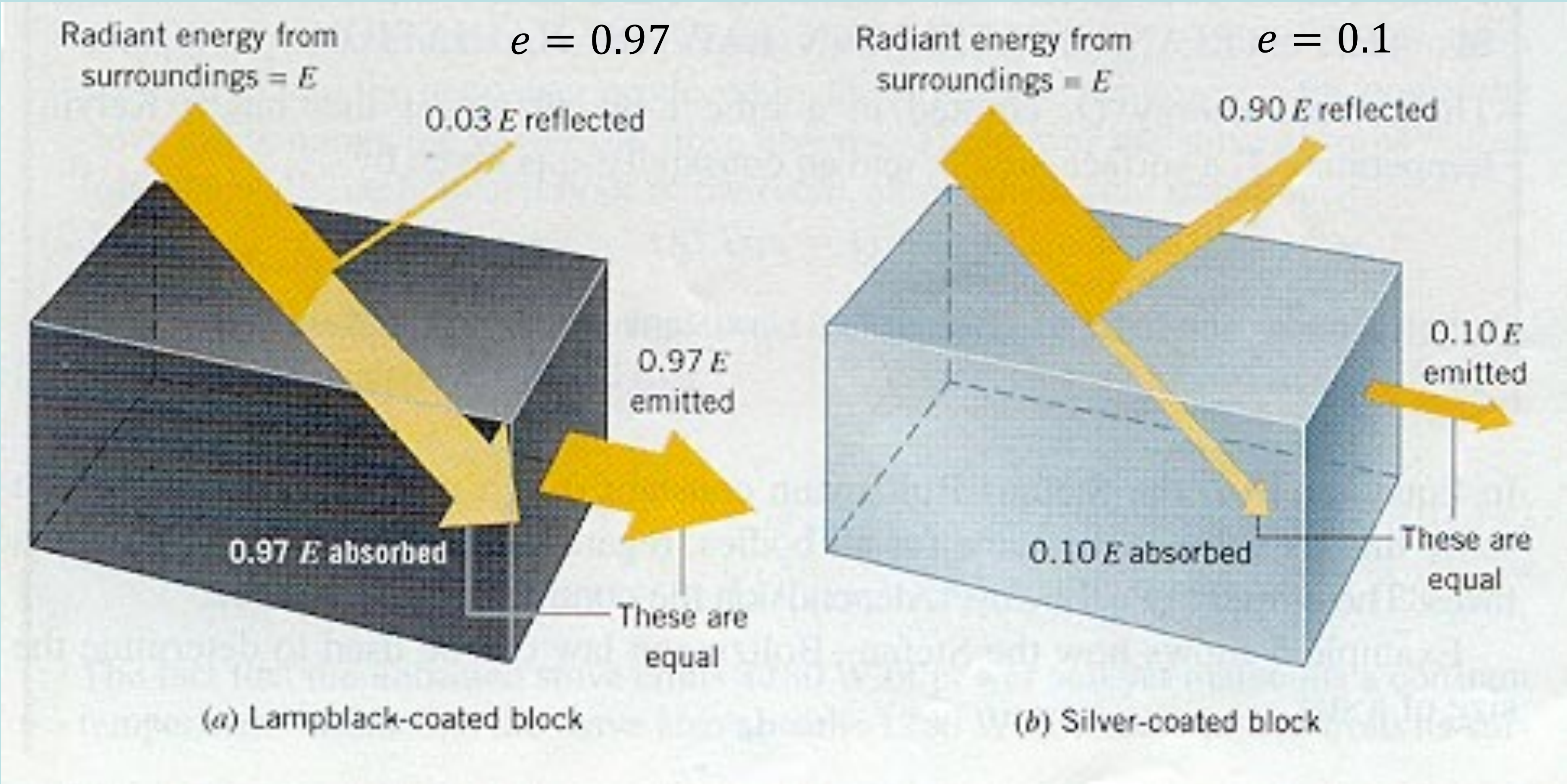
等溫時，兩者熱量吸收相等，該物體的輻射就是空腔輻射的 e 倍，

$$P_{物} = e \cdot P_{空} = e\sigma AT^4$$

所以 e 也是放射率emissivity

好的吸收者，也會是好的放射者！

Material	Emissivity
Aluminum foil	0.03
Aluminum, anodized	0.9[11]
Asphalt	0.88
Brick	0.90
Concrete, rough	0.91
Copper, polished	0.04
Copper, oxidized	0.87
Glass, smooth	0.95
Ice	0.97
Limestone	0.92
Marble (polished)	0.89 to 0.92
Paint (including white)	0.9
Paper, roofing or white	0.88 to 0.86
Plaster, rough	0.89
Silver, polished	0.02
Silver, oxidized	0.04
Snow	0.8 to 0.9
Water, pure	0.96



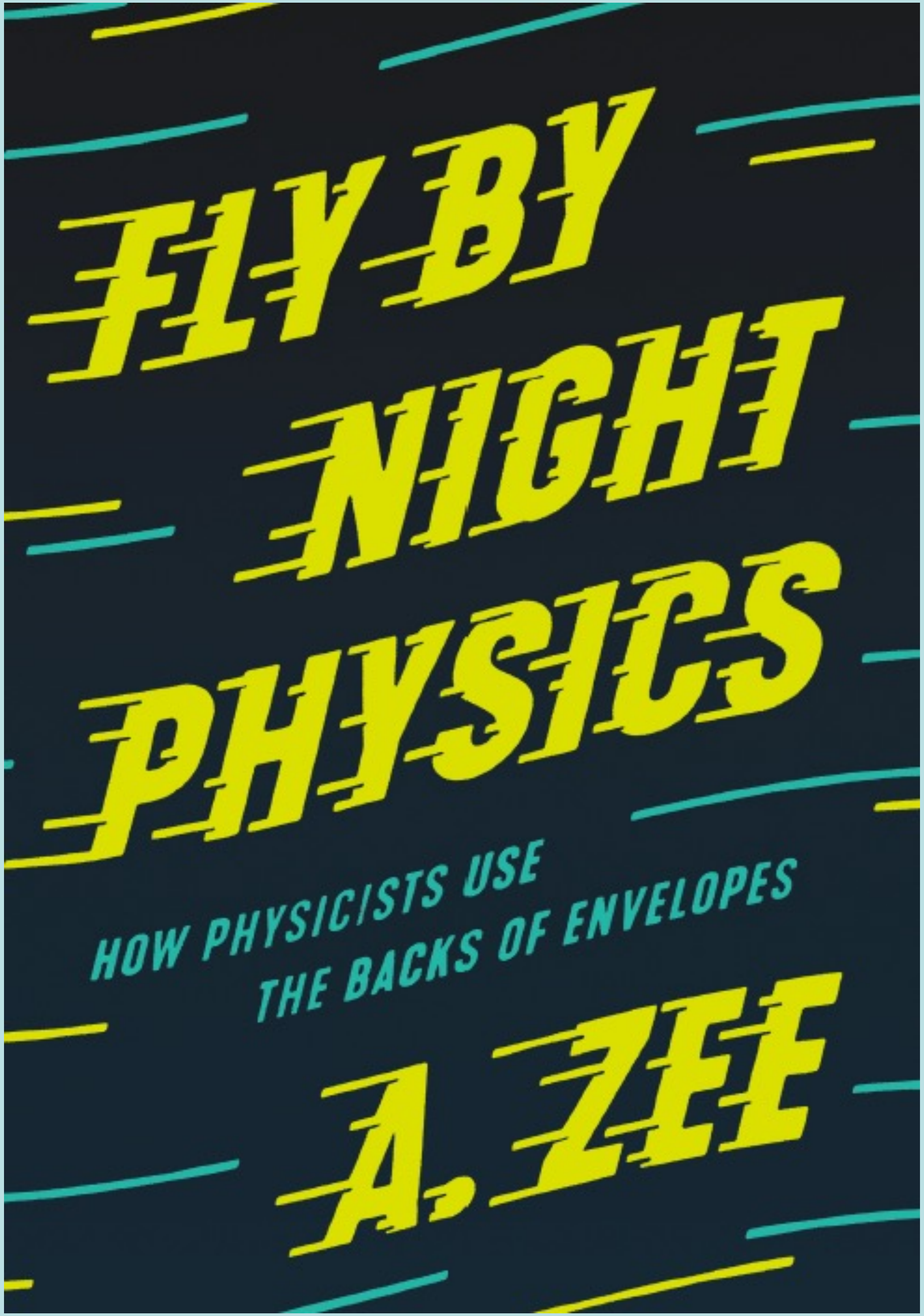
Radiant energy from surroundings = E
$e = 0.97$
0.03 E reflected
0.97 E emitted
0.97 E absorbed
These are equal
(a) Lampblack-coated block
Radiant energy from surroundings = E
$e = 0.1$
0.90 E reflected
0.10 E emitted
0.10 E absorbed
These are equal
(b) Silver-coated block

物體除了放出熱幅射，也接受來自環境的熱幅射：

如果環境充滿了T_0的黑體輻射，那物體的吸收如同空腔輻射的開口！

那表面積為A物體的吸收功率恰為 $P = e\sigma A T_0^4$

單位時間淨輻射熱量： $P = e\sigma A(T^4 - T_0^4)$


FLY BY
NIGHT
PHYSICS
HOW PHYSICISTS USE
THE BACKS OF ENVELOPES
A. ZEE

III.5
CHAPTER

Black body radiation

Black is not dark

When I was a student, I was puzzled by two aspects of black body radiation. I could follow the math, but what business does a black body have radiating light? And why were the Germans heating up everything in sight and measuring the frequency of the radiation?

Years later, I learned that the words "black" and "dark" have distinct meanings in physics. An ideal black body is a perfect absorber (hence, black) and by time reversal,[1] is also a perfect emitter of electromagnetic waves. In contrast, dark matter does not even interact with the electromagnetic field. Dark will never be the new black.* As for my historical question, I learned[2] decades later that the industrial nations were racing to find the most efficient electric light bulb.

I mentioned in the preface that I would have to cut corners once in a while. In this chapter, I will have to do a bit of that.

So, let us review the story of how Planck, by trying to understand cavities filled with electromagnetic waves, found the quantum!

Maxwell and Boltzmann

To begin, we have to go back to the Maxwell-Boltzmann distribution. Once physicists suspected that gases consist of atoms or molecules moving around and that temperature T corresponds to the typical energy of the atom, they wanted to know the probability distribution of the momenta $\vec{p}$ of the atoms. Evidently, at any given instant, some atoms are moving fast, others more slowly.

*English is particularly obscure in this connection. The distinction is clearer in other languages.

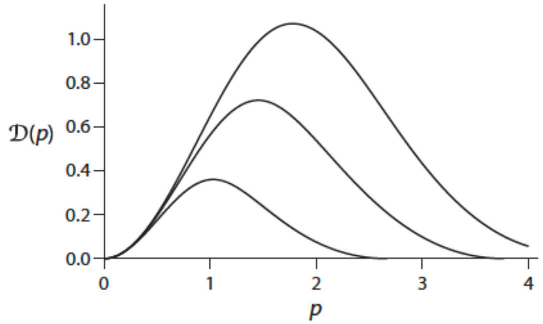


FIGURE 1. Maxwell-Boltzmann distribution plotted for $mT = 0.5, 1, 1.5$ (in arbitrary units); note that the location of the peak of the distribution as a function of p increases as $\sqrt{2mT}$.

A central result of statistical mechanics[3] states that the probability of an atom having energy E is proportional to $e^{-E/T}$. Applied to our gas, this implies that the probability for an atom to have momentum $\vec{p}$ is proportional to $e^{-\varepsilon(p)/T}$. Here $\varepsilon(p) = p^2/2m$, and p denotes the magnitude of $\vec{p}$.

To find the statistical average of various physical quantities $X(\vec{p})$, we need to integrate over all momenta with the measure d^3p. For quantities independent of the direction of $\vec{p}$, the measure or geometric factor d^3p may be replaced effectively by $4\pi dpp^2$, and thus we encounter integrals of the form $\int_0^\infty dpp^2 e^{-p^2/2mT} X(p)$.

Figure 1 shows the Maxwell-Boltzmann distribution $\mathcal{D}(p) \equiv p^2 e^{-p^2/2mT}$ for various values of mT.

An act of desperation

Now let us join Planck puzzling over the black body data. Even better, since this is a totally ahistorical account,[4] imagine yourself as a young Planck in a civilization far far away in another galaxy.[5] You understand that the cavity is full of electromagnetic waves zipping this way and that,* with wave vector $\vec{k}$ and frequency $\omega = c|\vec{k}| = ck$. Again, the geometric factor d^3k can (often) be replaced effectively by $4\pi dkk^2 \propto d\omega\omega^2$. In analogy with the Maxwell-Boltzmann distribution, denote the frequency distribution by $\mathcal{D}(w)$.

With $\mathcal{D}(\omega)$ plotted as a function of ω, the data look very similar to that shown in figure 1. So you try to mimic Maxwell and Boltzmann, and write something like $\mathcal{D}(\omega) \equiv \omega^2 e^{-\omega/T}$ for the frequency distribution.

*Note that even in Maxwell's electromagnetism, an electromagnetic wave of frequency ω carries momentum, according to the Poynting vector, proportional to $\omega/c = k$ in the direction of $\vec{k}$.

But this fails resoundingly! Dimensions do not match: The frequency ω has dimension of an inverse time so that $[\omega] = 1/T$, while the temperature* T has dimension of energy $[T] = [E] = ML^2/T^2$.

In physics, revolutions are marked by the introduction of a hitherto unknown fundamental constant: Newton with his G, and Einstein with his[6] c. Only the brave, or at least the reckless, can drive physics through perceived impasses.

You (or Planck) boldly invent a brand new fundamental constant $\hbar$ just so that $\hbar\omega$ has the same dimension as temperature T, namely, that of energy: $[\hbar\omega] = E$. Thus, the new constant must have dimension of energy times[7] time:

$$[\hbar] = ET \tag{1}$$

But what is the energy $\hbar\omega$? As you know, Planck made an inspired leap of faith,† guessing that an electromagnetic wave of frequency ω actually consists of tiny packets[8] of energy $\hbar\omega$, later named photons. Surely one of the most amazing guesses in the history[9] of physics!

Distribution of frequencies in a cavity filled with electromagnetic waves

With the new constant $\hbar$, you are tempted to propose the distribution $\mathcal{D}(\omega) \equiv \omega^2 e^{-\hbar\omega/T}$, but this only fits the data at high ω. So, let us retreat and write $\mathcal{D}(\omega) \equiv \omega^2 f(\hbar\omega/T)$ without specifying what $f(\hbar\omega/T)$ is.

Well, we already "know" that $f(x) \to e^{-x}$ for x large.

How about the behavior of $f(x)$ as $x \to 0$?

First, note that the energy density in the cavity is given by $\int_0^\infty d\omega\, \omega^2 f(\hbar\omega/T)\hbar\omega$. Thus, the integral converges exponentially at the high-frequency end. What we are asking for is the behavior at the low-frequency end.

Remarkably, without knowing what f is, we can still do the integral by scaling. Write $\hbar\omega = xT$. The integral scales like the 4th power of ω, and so instantly we obtain

$$\frac{E}{V} \propto T^4 \tag{2}$$

Historically, this corresponds to the Stefan-Boltzmann law. Later, at our leisure, we will fill in $\hbar$ and c by dimensional analysis.

Let us now see whether we can extract more information about the function $f(\hbar\omega/T)$, which determines the spectrum (namely, the energy $d\omega\, \omega^2 f(\hbar\omega/T)\hbar\omega$ contained in waves with frequency between ω and $\omega + d\omega$).

One clue is that at the low-frequency limit the energy spectrum should go over to the classical limit, known as the Rayleigh-Jeans law[10] $d\omega\, \omega^2 T$.

---

*Again I plead the poverty of the alphabet: The two Ts mean different things, but surely you get it.

†Planck later described his move as "an act of desperation."

(My understanding is that this law was "derived" invoking the equipartition theorem, assigning T to each degree of freedom, and mixing it with some hocus-pocus about oscillators, but it is all history now, given the Planck law to be presented presently.) In other words, we want $f(\hbar\omega/T)\hbar\omega \to T$ for $\omega \to 0$ or equivalently, for $T \to \infty$.

Note that the mysterious Mr. H-bar goes away in this limit, as he should. High temperature should correspond to the classical limit.

Thus, $f(x) \to 1/x$ as $x \to 0$.

The Planck distribution

I will pause and let you think of a function that has this small-x behavior, as well as $f(x) \to e^{-x}$ as $x \to \infty$.

The brain of the fly by night physicist finds $1/x$ harder to grasp than x, so the first step[11] is to define $g(x) = 1/f(x)$. We want $g(x) \to e^x$ for $x \to \infty$ and $g(x) \to x$ for $x \to 0$. The reader with a good memory might recall that I already posed this question back in chapter I.4.

A plausible interpolation would be[12] $g(x) = e^x - 1$. So, did you guess $f(x) = \frac{1}{e^x - 1}$?

Thus, the Planck distribution is given by, in the notation used here,

$$\mathcal{D}(\omega) = \omega^2 \frac{1}{e^{\frac{\hbar\omega}{T}} - 1} \tag{3}$$

For use below, let us define $n(\omega) = 1/(e^{\frac{\hbar\omega}{T}} - 1)$, namely, $\mathcal{D}(\omega)$ shorn of the kinematic ω^2.

Of course, all is clear in hindsight, but still, that shouldn't have been too hard after the act of faith of introducing the new constant $\hbar$. Historically, Planck had more to guide him. In particular, Wien's displacement law[13] states how the peak frequency $\omega_{\max}$ of the distribution increases linearly with T.

An important point and the moral of the story. Derive? I don't see Planck's result being derived here.[14] Only an inspired guess backed by deep insight.[15]

Phase space

We can now put $\hbar$ and c into (2), as promised. Since $[\hbar c] = ET(L/T) = EL$, we note that $[T/\hbar c] = E/EL = 1/L$ is an inverse length, and so $(T/\hbar c)^3$ is an inverse volume. Hence, the energy per unit volume $E/V \sim T(T/\hbar c)^3 \sim T^4/(\hbar c)^3$.

Indeed, you could almost write down the exact energy spectrum, if you know[16] that phase space in statistical mechanics is given by $Vd^3p/(2\pi\hbar)^3 = Vd^3p/h^3$.

Trading the angular integration for 4π as usual, writing $p=\hbar k=\hbar\omega/c$ so that $d^3p\to 4\pi(\hbar/c)^3 d\omega\omega^2$, we obtain[17] the energy density

$$\varepsilon\equiv\frac{E}{V}=\frac{1}{\pi^2c^3}\int_0^\infty d\omega\omega^2\frac{\hbar\omega}{e^{\frac{\hbar\omega}{T}}-1}=\frac{T^4}{\pi^2(\hbar c)^3}\int_0^\infty dx\frac{x^3}{e^x-1} \tag{4}$$

In the last step, we scaled by writing $\hbar\omega=xT$.[18]

I will leave it to you to fly by day[19] and show off your prowess in doing the integral in (4), if you wish.

That amazing Einstein

Any thinking theoretical physicist should become more and more in awe of that amazing Einstein as the years go by. Looking at (3), Einstein discovered yet another interesting phenomenon: stimulated emission.

Caution: Our treatment of black body radiation is necessarily sketchy and schematic, and that of Einstein's stimulated emission, even more so. The interested reader is referred to more specialized textbooks for more thorough discussions.[20]

Consider a bunch of two-state atoms in equilibrium with a gas of photons. Let the difference in energy between the excited and the ground states be $E_1-E_0=\hbar\omega$. Boltzmann tells us that the ratio of excited atoms to grounded atoms is equal to* $\frac{N_1}{N_0}=e^{-\hbar\omega/T}$. On absorbing a photon of frequency ω, an atom in the ground state jumps to the excited state, while an atom in the excited state relaxes to the ground state by emitting a photon of frequency ω.

Einstein now exploits the fact that, in equilibrium, the two rates must be equal.

From (3) and (4), we know that the number of photons with frequency in the interval $(\omega,\omega+d\omega)$ is equal to $\frac{V}{\pi^2c^3}d\omega\omega^2 n(\omega)$, with $n(\omega)=1/(e^{\frac{\hbar\omega}{T}}-1)$. But to keep things as clear as possible, let us strip the inessential factor $\frac{V}{\pi^2c^3}d\omega\omega^2$.

The rate at which photons of frequency ω are being absorbed is evidently proportional to N_0 and $n(\omega)$: hence, $n(\omega)N_0$. The corresponding rate at which photons of frequency ω are being emitted by the excited atoms is similarly proportional to $e(\omega)N_1$ (which you can regard as the definition of $e(\omega)$ if you like).

If I have ever seen an elegant back of the envelope calculation, then this is it. Equating the two rates, we obtain, following Einstein of course,

$$e(\omega)=n(\omega)\frac{N_0}{N_1}=n(\omega)e^{\frac{\hbar\omega}{T}}=\frac{e^{\frac{\hbar\omega}{T}}}{e^{\frac{\hbar\omega}{T}}-1}=\left(1+\frac{1}{e^{\frac{\hbar\omega}{T}}-1}\right)=1+n(\omega) \tag{5}$$

I trust that you agree with me that the way $n(\omega)$ is defined, with ω^2, π, and whatnot suppressed, enables us to see more clearly the forest for the trees.

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*This and the relations we obtain below are to be understood probabilistically.

So amazingly, $e(\omega) = 1 + n(\omega)$.

The "1" represents the fact that the excited atoms would still come down to ground even when a photon gas is not present, that is, when $n(\omega) = 0$. What is amazing about the result $e(\omega) = 1 + n(\omega)$ is the $n(\omega)$ term: The presence of a photon gas enhances the tendency of the excited atoms to emit.

The inclination of photons to hang out with each other[21] also leads them to encourage the atoms to create more photons for them to hang out with. We all know that the laser and much else in our so-called civilization depend on stimulated emission.

Note that the derivation in (5) depends explicitly on the actual form of $n(\omega)$.

Creation and annihilation operators

For readers who know about raising and lowering operators for the harmonic oscillator and how they morph into creation ($a^\dagger$) and annihilation (a) operators in quantum field theory, I sketch how contemporary textbooks derive the Planck distribution starting with the commutation relation $[a, a^\dagger] = 1$.

First, a few words about quantum field theory. After being Fourier transformed, the electromagnetic potential $A_\mu(\vec{x}, t)$ can be seen[22] to behave like harmonic oscillators,[23] one at each point $\vec{x}$ in space.

Let us now quantize A_μ, but stay inside a large box of volume V, so that the modes of the electromagnetic field, labeled by $\vec{k}$ (with $k = \omega/c$), are discrete and countable. Focus on a specific mode. Then it makes sense to talk about the number n of photons in that mode. (Again, apologies for the finite number of letters in the alphabet. In this section, n will denote an integer, not to be confused with $n(\omega)$.)

In standard textbooks on quantum mechanics, it is explained that the states of the harmonic oscillator are labeled by an integer n (we derived in chapter III.2 the equal spacing law of the energy levels, which fact is crucial for the formulation of quantum field theory, as was also mentioned), written as $|n\rangle$ in Dirac's notation. Then the commutation relation $[a, a^\dagger] = 1$ is solved* by

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{6}$$

and[24]

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{7}$$

Denote by $|n, 0\rangle$ the quantum state consisting of an atom in the ground state $|0\rangle$ in the presence of n photons of the appropriate frequency. Similarly, $|n, 1\rangle$ denotes the state with the atom in the excited state $|1\rangle$ in the presence of n photons. The amplitude for the absorption process discussed in the preceding section is then given by $\langle n-1, 1|a\mathcal{O}|n, 0\rangle = \langle n-1|a|n\rangle\langle 1|\mathcal{O}|0\rangle = \sqrt{n}\langle 1|\mathcal{O}|0\rangle$, where $\mathcal{O}$ represents an operator acting on the atom that turns

*$a^\dagger a|n\rangle = \sqrt{n}a^\dagger|n-1\rangle = n|n\rangle$, and $aa^\dagger|n\rangle = \sqrt{n+1}a|n+1\rangle = (n+1)|n\rangle$, so that $[a, a^\dagger]|n\rangle = 1|n\rangle$.

$|0\rangle$ into $|1\rangle$. Similarly, the amplitude for the emission process is given by $\langle n+1, 0|a^\dagger \mathcal{O}^\dagger|n, 1\rangle = \langle n+1|a^\dagger|n\rangle\langle 0|\mathcal{O}^\dagger|1\rangle = \sqrt{n+1}\langle 0|\mathcal{O}^\dagger|1\rangle$. Squaring these amplitudes to obtain the transition probabilities, we see that they are in the ratio n to $n+1$, in accordance with Einstein.

Indeed, including the relative probability $e^{-\frac{\hbar\omega}{T}}$ of finding an atom in the excited state versus the ground state and playing a bit fast and loose, we could derive the Planck distribution: $(n+1)e^{-\frac{\hbar\omega}{T}} = n$ implies $n = e^{-\frac{\hbar\omega}{T}}/(1 - e^{-\frac{\hbar\omega}{T}}) = 1/(e^{\frac{\hbar\omega}{T}} - 1)$.

Time reversing the invention of integral calculus

If the probability of a system (here we have in mind the electromagnetic field, but the discussion to follow is quite general) at temperature T having energy E is given by $e^{-E/T}$, and if E can take on a continuum of values, then the expected value of E equals $\int_0^\infty dEEe^{-E/T}/\int_0^\infty dEe^{-E/T} = T$. We all learned how Newton and Leibniz invented integral calculus.

Suppose we time reverse this amazing creation, and replace the integrals just written down by sums. Allow E to take on only the discrete values $E_n = n\varepsilon$ with $n = 0, 1, \ldots, \infty$. The expected value of E is now given by

$$\sum_{n=0}^{\infty} n\varepsilon e^{-n\varepsilon/T} \Big/ \sum_{n=0}^{\infty} e^{-n\varepsilon/T} = \frac{\varepsilon}{(e^{\varepsilon/T} - 1)} \tag{8}$$

We obtain[25] the Planck distribution, with the benefit of hindsight (and of Planck's and Einstein's profound insights). In the limit $\varepsilon \to 0$ (or equivalently, $T \to \infty$), this expression $\to T$, and we recover the continuum result, as Newton and Leibniz discovered.

This sum is of course done in any statistical mechanics textbook, but I feel that, for many students, the physics is often lost amid all the πs, k_B, and other distracting stuff zinging around.

The universe and stars as black bodies

Going back to the confusion that I suffered from when I first learned about black bodies (which the professor was unable to explain), I show you in figure 2 the actual spectrum[26] of different types of stars, with temperature ranging from 3,500 K to 40,000 K, being fitted to Planck's distribution.

Admit it, you don't normally think of stars as black bodies!

We will also see (in chapter VII.3) that the early universe, amazingly enough, can be considered as a box filled with black body radiation. Furthermore, the cosmic microwave background of the present universe fits Planck's distribution almost perfectly. Indeed, it provides a "history book" of the universe!

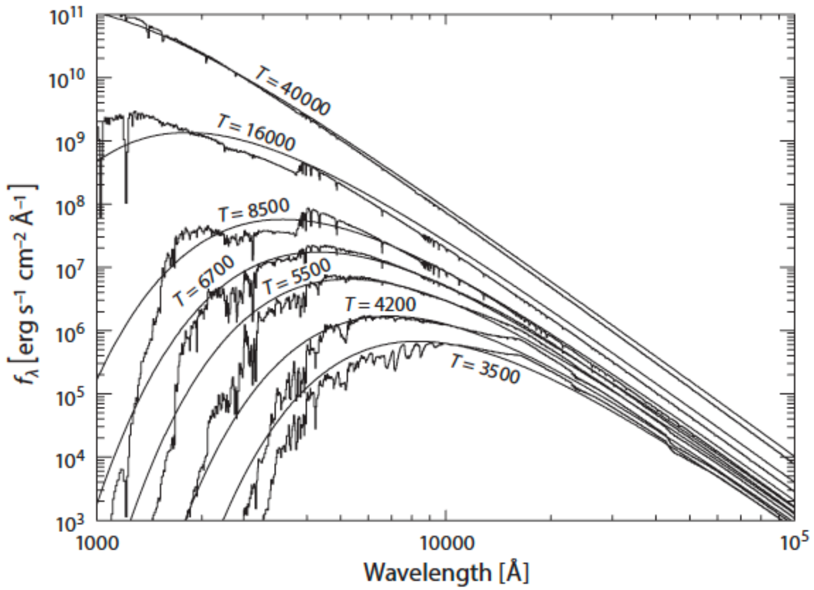

FIGURE 2. Stars are almost black bodies! Flux per wavelength interval emitted by different types of stars. This is fitted to the black body spectrum at various temperatures ranging from $T = 3{,}500$ K to $T = 40{,}000$ K. Note that the Planck distribution is plotted here as a function of wavelength, not frequency. Maoz, D. *Astrophysics in a Nutshell*. Princeton University Press, 2016.

Exercise

(1) Wien's displacement law, which played such a crucial role in the eventual discovery of quantum mechanics, is actually quite general. Show that if the distribution of a physical quantity ζ has the form $\zeta^a F(\zeta/T)$, with a an arbitrary constant and $F(x)$ an arbitrary function, the peak value $\zeta_{\max}$ of the distribution increases linearly with T. Indeed, see also the caption to figure 1.

IV.1
CHAPTER

Planck gave us God-given units

To understand the universe at a fundamental level

Once upon a time, boys and girls, we used some English king's feet to measure lengths with. You laugh, but a metal bar preserved in Paris and decreed by a bunch of French revolutionaries is not much more intrinsic. To understand the universe at a fundamental level, we ought not to have to use some absurdly human invention, such as the imperial or metric system.*

Einstein recognized that with the universal speed of light c, we no longer need separate units for length and time. Even the proverbial guy and gal in the street understand that henceforth we could measure length in light years.

We and another civilization, be they in some other galaxy, would now be able to agree on a unit of distance, if we could only communicate to them what we mean by one year or one day. Therein lies the rub: our unit for measuring time derives from how fast our home planet spins and revolves around its star. Only homeboys would know. How could we possibly communicate to a distant civilization this period of rotation we call a "day," which is merely an accident of how some interstellar debris came together to form the rock we call home?

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*Notions we take for granted today still had to be thought up by someone. Maxwell, in his magnum opus on electromagnetism, proposed that the meter be tied to the wavelength of light emitted by some particular substance, adding that such a standard "would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body." The various eminences of our subject could be quite sarcastic.

Two out of three taken care of

Newton's discovery of the law of gravity brought the first universal constant G into physics. The emphasis here is on the first, and the one and only one at that time.

Next, Maxwell and Einstein brought the second fundamental constant, c, into physics. Henceforth, physicists proudly possessed two universal constants, G and c.

Comparing the kinetic energy $\frac{1}{2}mv^2$ of a particle of mass m in a gravitational potential with its potential energy $-GMm/r$ and canceling off m, we see that the combination[1] GM/c^2 has dimension of length.

Now we are empowered to measure masses in terms of our unit for length (or equivalently, time), or lengths in terms of our unit for mass.

Some readers might argue that we could also use something like the mass of a particle such as the proton, or the electron, or perhaps even the quark. That would indeed suffice for communicating with another civilization, but particle theorists generally believe that these masses were generated.[2] Specifically, in the early universe, these particles are thought to have been massless, or nonexistent (the proton would have fallen apart into three quarks).

We would prefer to base our units on the fundamental laws of physics.

You realize that to do physics at the fundamental level, without recourse to somebody's foot, and not even to a possibly ephemeral particle (such as the proton), we need another constant on the same level as G and c. What could that be? Can you guess?

Planck's great contribution to physics: for all civilizations, extraterrestrial and nonhuman

> The two[3] constants . . . which occur in the equation for radiative entropy offer the possibility of establishing a system of units for length, mass, time and temperature which are independent of specific bodies or materials and which necessarily maintain their meaning for all time and for all civilizations, even those which are extraterrestrial and non-human. [Max Planck]

Surely you guessed! Max Planck[4] is properly revered for his introduction of the fundamental constant $\hbar$ into physics. With this far-reaching and magnanimous gesture, he gave us a natural system of units, sometimes known as God-given units.

In a tremendously insightful paper, Planck pointed out that with the three fundamental constants[5] G, c, and $\hbar$, in order of their entrance into the grand drama of physics, we finally have a universal set of units for mass M, length L, and time T, the three basic concepts we need to do physics with.

Three big names, three basic principles, three natural units

To see how these units are defined, note that Heisenberg's uncertainty principle tells us that $\hbar$ over the momentum Mc is a length. Equating the two lengths GM/c^2 and $\hbar/Mc$, we see that the combination $\hbar c/G$ has dimension of mass squared. In other words, the three fundamental constants G, c, and $\hbar$ allow us to define a mass,[6] known rightfully as the Planck mass:

$$M_P = \sqrt{\frac{\hbar c}{G}} \tag{1}$$

We can then immediately define, with Heisenberg's help, a Planck length:

$$l_P = \frac{\hbar}{M_P c} = \sqrt{\frac{\hbar G}{c^3}} \tag{2}$$

and, with Einstein's help, a Planck time:

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \tag{3}$$

Newton, Einstein, Heisenberg, three big* names, three basic principles, three natural units to measure space, time, and energy. We have reduced the MLT system to "nothing"! We no longer have to invent or find some unit, such as the transition frequency of some agreed-on atom,[7] to measure the universe with. We measure mass in units of M_P, length in units of l_P, and time in units of t_P.

Another way of saying this is that in these natural units, $c = 1$, $G = 1$, and $\hbar = 1$. The natural system of units would be understood no matter where your travels might take you, within this galaxy or far beyond.

Newton small, so Planck huge, and the Mother of All Headaches

The Planck mass works out to be about 10^{19} times the proton mass m_p. That humongous number 10^{19} is responsible for the Mother of All Headaches plaguing fundamental physics today.[8] That M_P is so gigantic compared to the known particles can be traced back to the extreme feebleness (as was mentioned in chapter I.2) of gravity: G tiny, so M_P enormous.

As the Planck mass is huge, the Planck length and time are tiny. If you insist on contaminating the purity of natural units by human-made units, t_P comes out to be $\simeq 5.4 \times 10^{-44}$ second, the Planck length $l_P \simeq 1.6 \times 10^{-33}$ centimeter, and the Planck mass[9] $M_P \simeq 2.2 \times 10^{-5}$ gram!

*As big as they come!

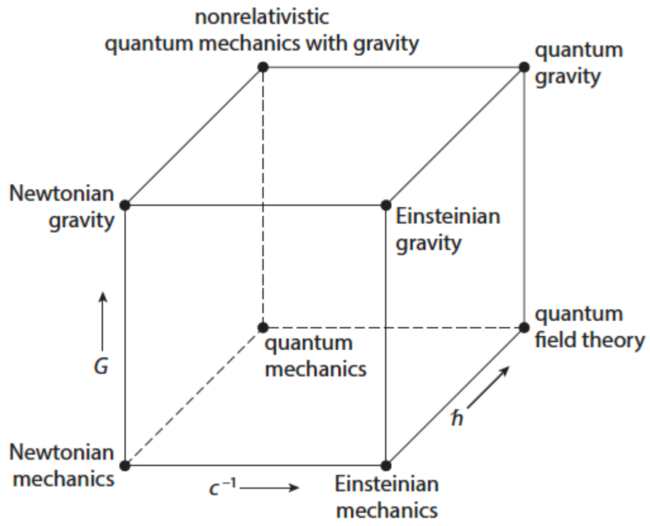


FIGURE 1. The cube of physics. From Zee, A. *Einstein Gravity in a Nutshell*. Princeton University Press, 2013.

It is important to realize[10] how significant Planck's insight was. Nature Herself, far transcending any silly English king or some self-important French revolutionary committee, gives us a set of units to measure by. We have managed to get rid of all human-made units. We needed three fundamental constants, each associated with a fundamental principle, and we have precisely three!

This suggests that we have discovered all[11] the fundamental principles that there are. Had we not known about the quantum, then we would have to use one human-made unit to describe the universe, which would be weird. From that fact alone, in my opinion, we would have to go looking for quantum physics.

The cube of physics

That we need three fundamental constants associated with three fundamental principles suggests that we could neatly summarize all of physics as a cube. See figure 1.

Physics started with Newtonian mechanics at one corner of the cube, and is now desperately trying to get to the opposite corner, where sits the alleged "Holy Grail." The three fundamental constants, c^{-1}, $\hbar$, and G, characterizing Einstein, Planck[12] or Heisenberg, and Newton, label the three axes. As we "turned on" one or the other of three constants, in other words, as each of these constants came into physics, we took off from the home base of Newtonian mechanics.[13]

Much of 20th-century physics consisted of getting from one corner of the cube to another. Consider the lower face[14] of the cube. When we turned on c^{-1}, we went from Newtonian mechanics to special relativity. When we turned on $\hbar$, we went from Newtonian mechanics to quantum mechanics. When we turned on both c^{-1} and $\hbar$, we arrived at quantum field theory, in my opinion the greatest monument of 20th-century physics.

Newton himself already moved up the vertical axis from Newtonian mechanics to Newtonian gravity by turning on G. Turning on c^{-1}, Einstein took us from that corner to general relativity or Einsteinian gravity.

All the Stürm und Drang of the past few decades is the attempt to cross from that corner to the Holy Grail of quantum gravity, when (glory glory hallelujah) all three fundamental constants are turned on.[15]

You might be wondering about the corner with $c^{-1} = 0$ but $\hbar \neq 0$ and $G \neq 0$. That corner, relatively unpublicized and generally neglected, covers phenomena described adequately by nonrelativistic quantum mechanics in the presence of a gravitational field.[16]

In our everyday existence, we are aware of only two corners of this cube, because these three fundamental constants are either absurdly small or absurdly large compared to what humans experience.[17]

Setting $\hbar$, c, and G to unity, or not

The Planckian system of units amounts to setting $\hbar$, c, and G to unity, but often, depending on which area of physics you work in (or which corner of the cube of physics you are on), it would be inconvenient, or even inappropriate, to do so. For instance, in electromagnetism, $\hbar$ and G do not even enter. Setting $c = 1$ would be quite appropriate, but occasionally it would be useful to keep it around, for example, to show the weakness of the magnetic force compared to the electric force.

Particle physicists deal with relativistic quantum phenomena, and so routinely set $\hbar$ and c to 1, but not G, which does not enter until one starts discussing quantum gravity. In theories of quantum gravity (for example, string theory), G is routinely also set to 1.

Did we need to include k?

And now I come to my pet peeve. No doubt that Boltzmann's constant[18] k played a pivotal role in physicists' struggle to understand the discreteness of matter. But now that the reality of the atom has long been established, k should be retired. Temperature is an energy, period. Boltzmann's constant k is just a conversion factor between energy units and some quaint markings on a tube of mercury.

Of course I do not object to the fact that temperature is often measured in degrees, but then degree should be considered a unit for energy, just like the erg or the British Thermal Unit, albeit a rather peculiar unit. The constant k could then be suppressed. Otherwise, why not introduce a fundamental constant called $\kappa = 2.54$ cm/in, measure in inches, and pepper formulas in physics with expressions like $Gm_1m_2/(\kappa r)^2$? The appearance of k similarly stings my eyes.

We can imagine a world with $\hbar = 0$. Indeed, physicists lived in that world until Planck came along. Similarly, we can imagine worlds with $G = 0$, or $c^{-1} = 0$. But what would it mean to have a world with $k = 0$? Instead of filling glass tubes with mercury, we fill them with a liquid whose coefficient of thermal expansion is infinite?

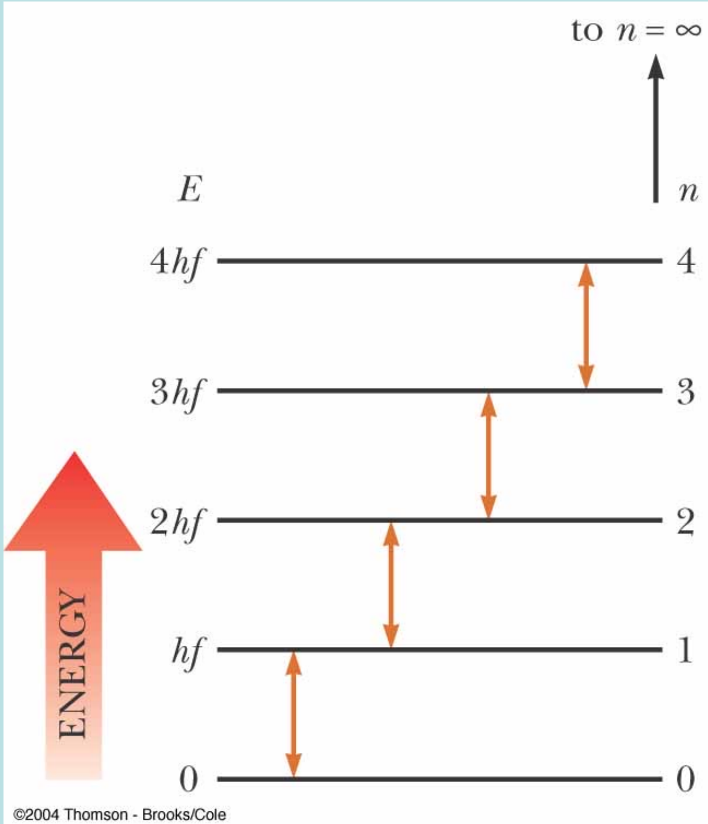
I have been surprised that distinguished physicists continue to write kT, where T would have sufficed. Perhaps they are so used to it that they think of kT as a new letter[19] in some exotic alphabet.

Postscript: Some colleagues who read this chapter in manuscript urge me to strengthen my rant and rave[20] against k even more. Why keep on writing k after an entire century has passed and after our long sojourn in the quantum world? Is it merely to confuse some of the weaker students into thinking that k has the same status as the three fundamental constants G, c, and $\hbar$?

黑體輻射可不可以計算預測？

如果要解釋黑體輻射，量子彈簧的能量變化必需是能階狀的！

Max Planck 在1900 開啟了量子革命的第一槍！



量子彈簧能吸收的能量不是連續的，而是固定量子的整數倍（離散型式）

量子(Quantum)的大小與頻率成正比！

$E_n = n \cdot hf$ h: Planck Constant $h = 6.625 \times 10^{-34} \mathrm{J} \cdot s$

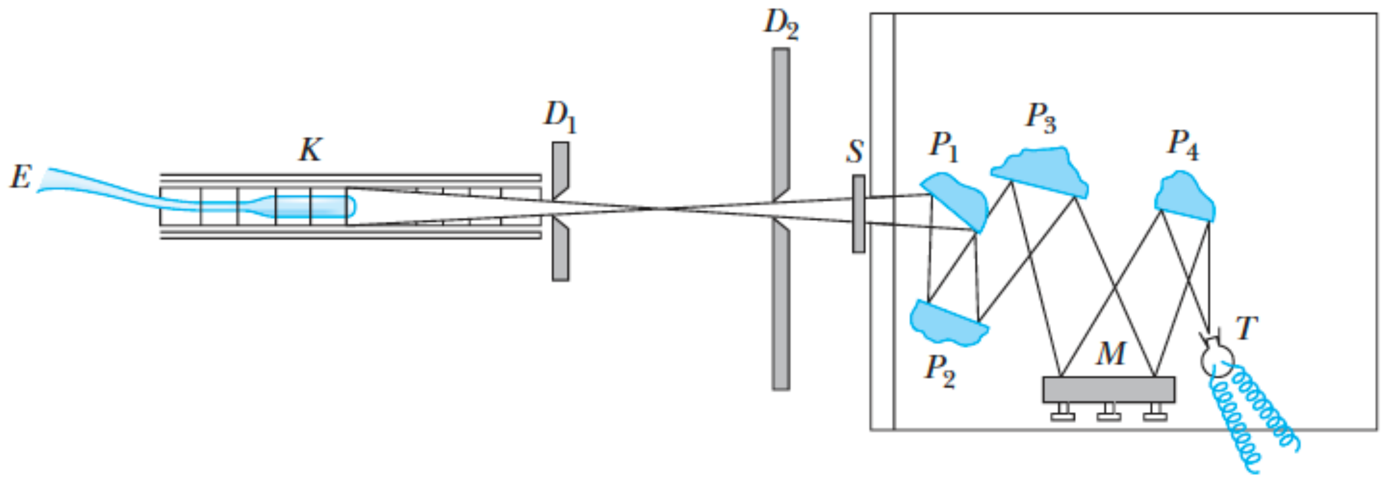
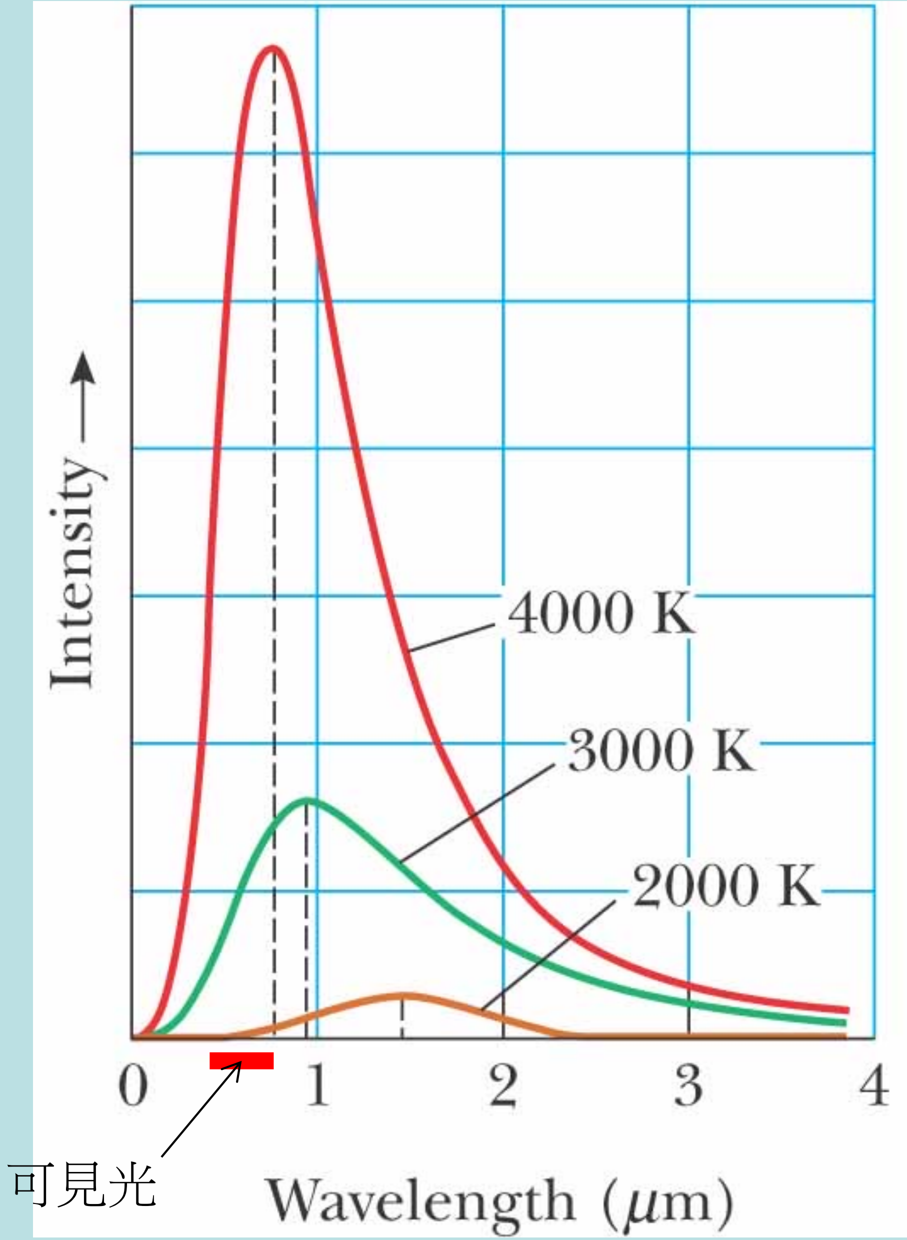


Figure 3.6 Apparatus for measuring blackbody radiation at a single wavelength in the far infrared region. The experimental technique that disproved Wien's law and was so crucial to the discovery of the quantum theory was the method of residual rays (*Restrahlen*). In this technique, one isolates a narrow band of far infrared radiation by causing white light to undergo multiple reflections from alkalide halide crystals (P_1–P_4). Because each alkali halide has a maximum reflection at a characteristic wavelength, quite pure bands of far infrared radiation may be obtained with repeated reflections. These pure bands can then be directed onto a thermopile (T) to measure intensity. E is a thermocouple used to measure the temperature of the blackbody oven, K.

鹼金屬鹵化物

黑體輻射的波長分布是固定的，與材質無關！



波長介於λ及$\lambda + d\lambda$的黑體輻射功率等於：

$$E(\lambda, T) \cdot d\lambda$$

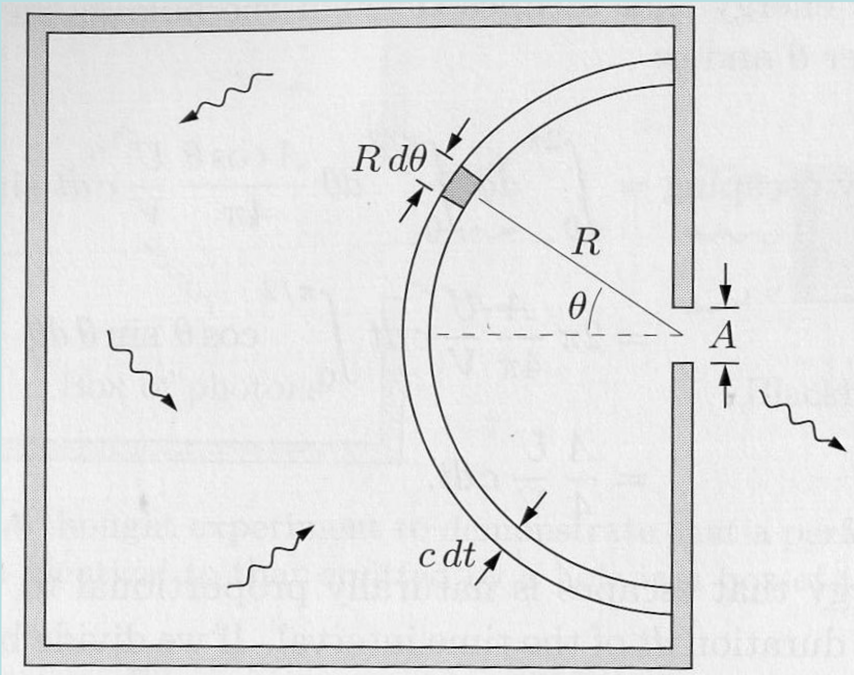
波長介於λ_1及λ_2的黑體輻射功率等於：

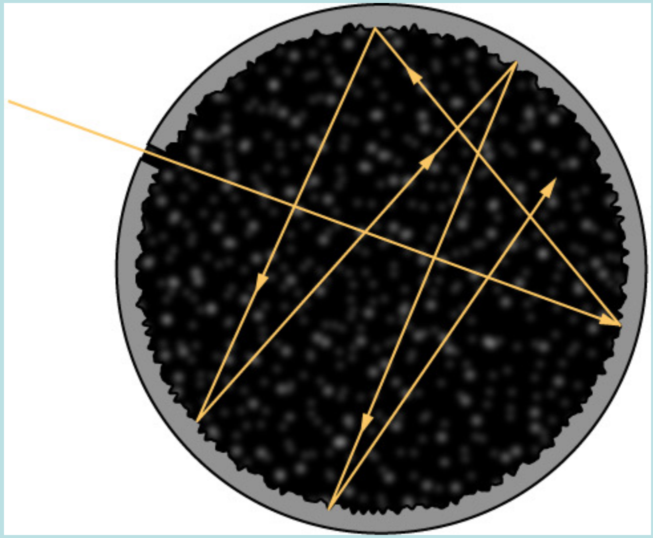
$$\int_{\lambda_1}^{\lambda_2} E(\lambda) \cdot d\lambda$$

$E(\lambda, T)$可以稱為功率對波長的密度。

黑體輻射完全等於空腔輻射：因此$E(\lambda,T)$也是空腔輻射功率對波長的密度。

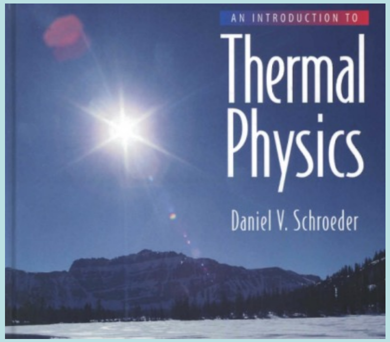
而空腔輻射是來自空腔中的來回反射的電磁波，透過小洞出口放射出來。





由小洞放出的空腔輻射應是空腔內的電磁波的一個樣本，

$$E(\lambda,T) = \frac{c}{4} \cdot w(\lambda,T) \qquad 1\text{-}1$$



空腔內單位體積的輻射能量對波長密度：$w(\lambda,T)$。

空腔輻射功率與空腔內的輻射能量對波長密度$w(\lambda,T)$成正比！

The photons that escape now, during a time interval dt, were once pointed at the hole from somewhere within a hemispherical shell, as shown in Figure 7.23. The radius R of the shell depends on how long ago we're looking, while the thickness of the shell is $c\,dt$. I'll use spherical coordinates to label various points on the shell, as shown. The angle θ ranges from 0, at the left end of the shell, to $\pi/2$, at the extreme edges on the right. There's also an azimuthal angle ϕ, not shown, which ranges from 0 to 2π as you go from the top edge of the shell into the page, down

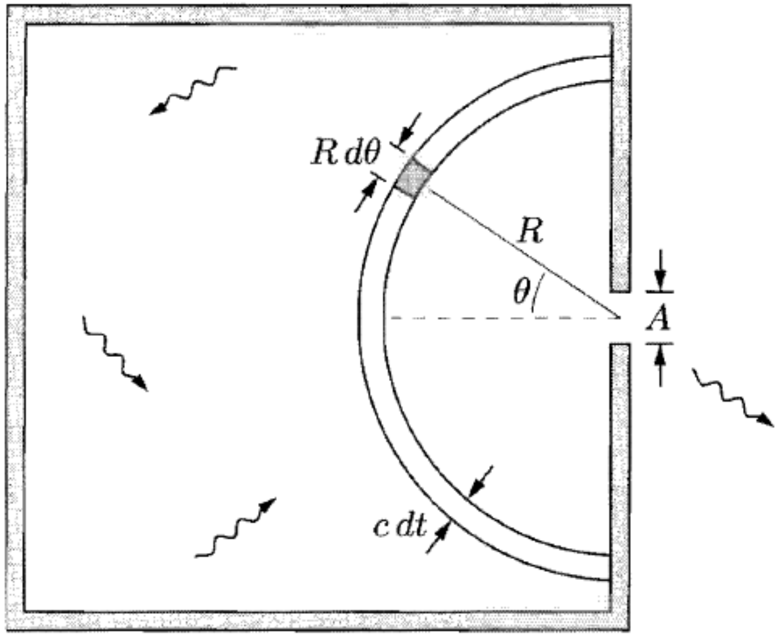


Figure 7.23. The photons that escape now were once somewhere within a hemispherical shell inside the box. From a given point in this shell, the probability of escape depends on the distance from the hole and the angle θ.

to the bottom, out of the page, and back to the top.

Now consider the shaded chunk of the shell shown Figure 7.23. Its volume is

$$\text{volume of chunk} = (R\,d\theta) \times (R \sin\theta\, d\phi) \times (c\,dt). \qquad (7.90)$$

(The depth of the chunk, perpendicular to the page, is $R \sin\theta \, d\phi$, since $R \sin\theta$ is the radius of a ring of constant θ swept out as ϕ ranges from 0 to 2π.) The energy density of the photons within this chunk is given by equation 7.86:

$$\frac{U}{V} = \frac{8\pi^5}{15} \frac{(kT)^4}{(hc)^3}. \tag{7.91}$$

In what follows I'll simply call this quantity U/V; the total energy in the chunk is thus

$$\text{energy in chunk} = \frac{U}{V} \, c \, dt \, R^2 \sin\theta \, d\theta \, d\phi. \tag{7.92}$$

But not all the energy in this chunk of space will escape through the hole, because most of the photons are pointed in the wrong direction. The probability of a photon being pointed in the *right* direction is equal to the apparent area of the hole, as viewed from the chunk, divided by the total area of an imaginary sphere of radius R centered on the chunk:

$$\text{probability of escape} = \frac{A \cos\theta}{4\pi R^2}. \tag{7.93}$$

Here A is the area of the hole, and $A \cos\theta$ is its foreshortened area, as seen from the chunk. The amount of energy that escapes from this chunk is therefore

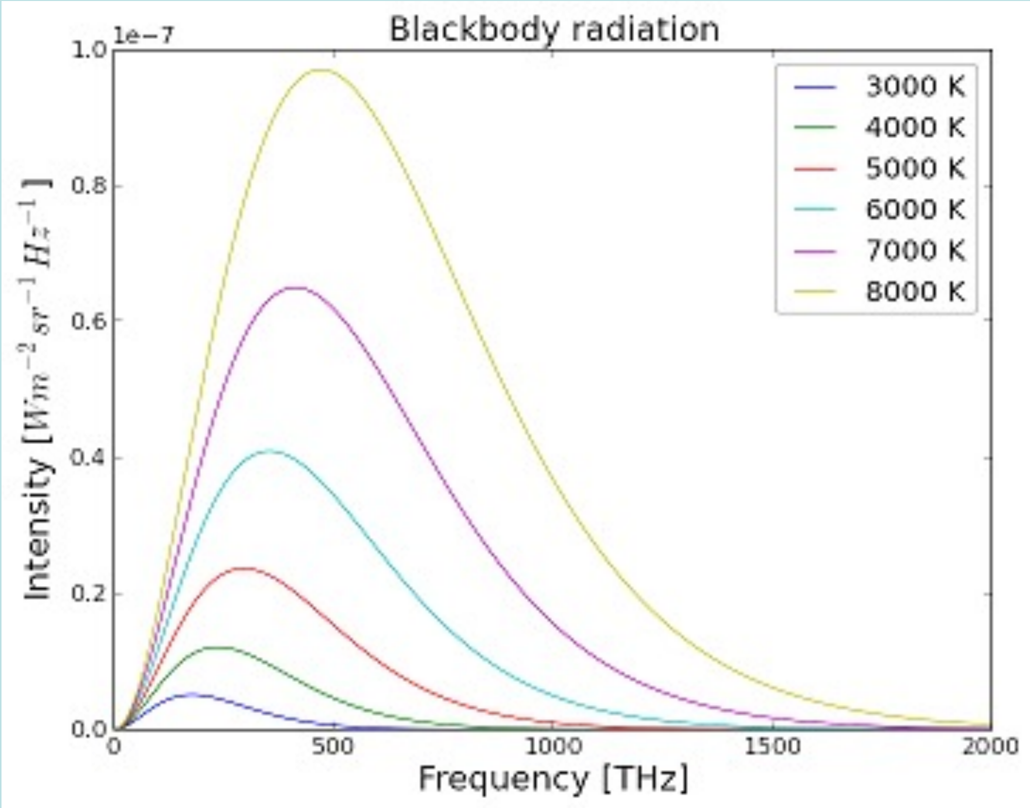
$$\text{energy escaping from chunk} = \frac{A \cos\theta}{4\pi} \frac{U}{V} \, c \, dt \, \sin\theta \, d\theta \, d\phi. \tag{7.94}$$

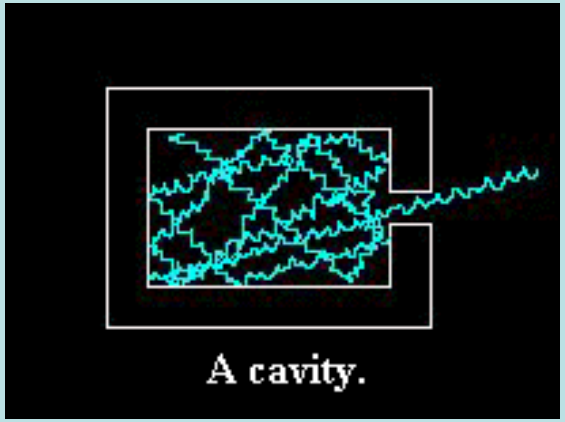
To find the *total* energy that escapes through the hole in the time interval dt, we just integrate over θ and ϕ:

$$\begin{aligned}
\text{total energy escaping} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \frac{A\cos\theta}{4\pi} \frac{U}{V} c\,dt \, \sin\theta \\
&= 2\pi \frac{A}{4\pi} \frac{U}{V} c\,dt \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta \qquad (7.95) \\
&= \frac{A}{4} \frac{U}{V} c\,dt.
\end{aligned}$$

The amount of energy that escapes is naturally proportional to the area A of the hole, and also to the duration dt of the time interval. If we divide by these quantities we get the *power* emitted per unit area:

$$\text{power per unit area} = \frac{c}{4}\frac{U}{V}. \qquad (7.96)$$





單位體積空腔輻射能量對波長密度： $w(\lambda, T)$。

波長介於λ及$\lambda + d\lambda$的cavity energy density(per unit volumn)等於：

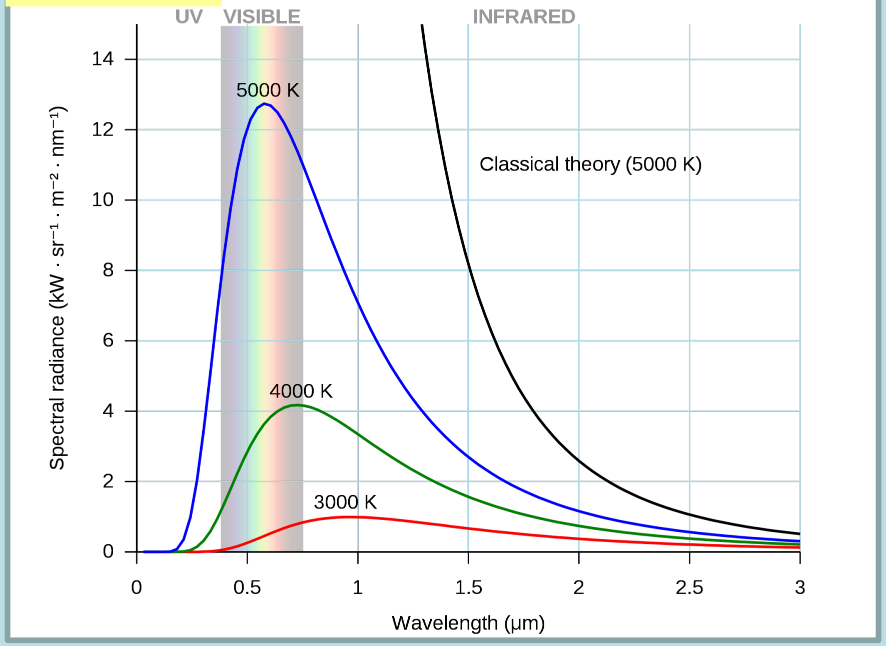
$$w(\lambda, T) \cdot d\lambda$$

單位體積空腔輻射能量對頻率密度：$u(\nu, T)$。

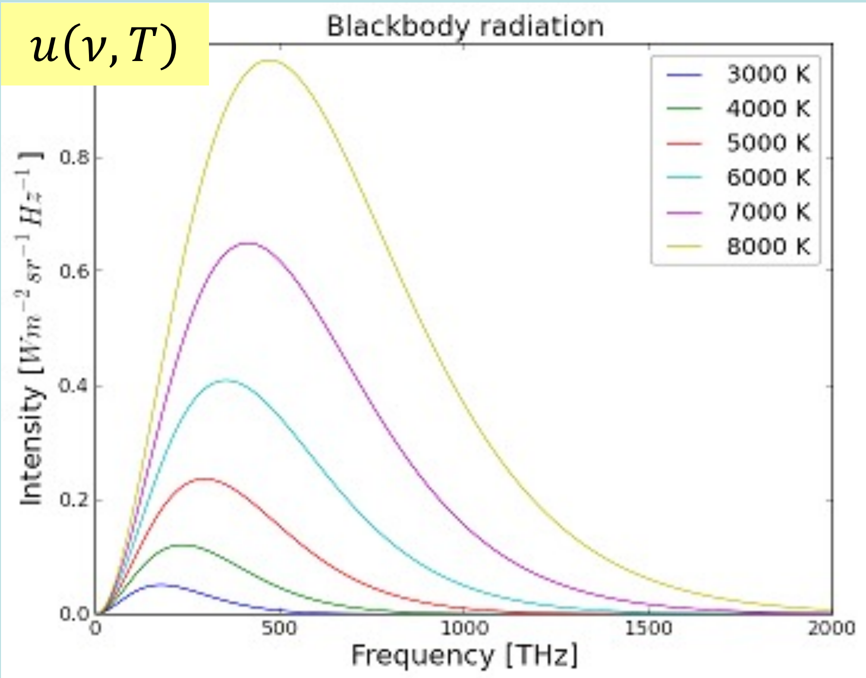
頻率介於ν及$\nu + d\nu$的cavity energy density (per unit volumn)等於：

$$u(\nu, T) \cdot d\nu \qquad E(\nu, T) = \frac{c}{4} \cdot u(\nu, T)$$

$w(\lambda, T)$

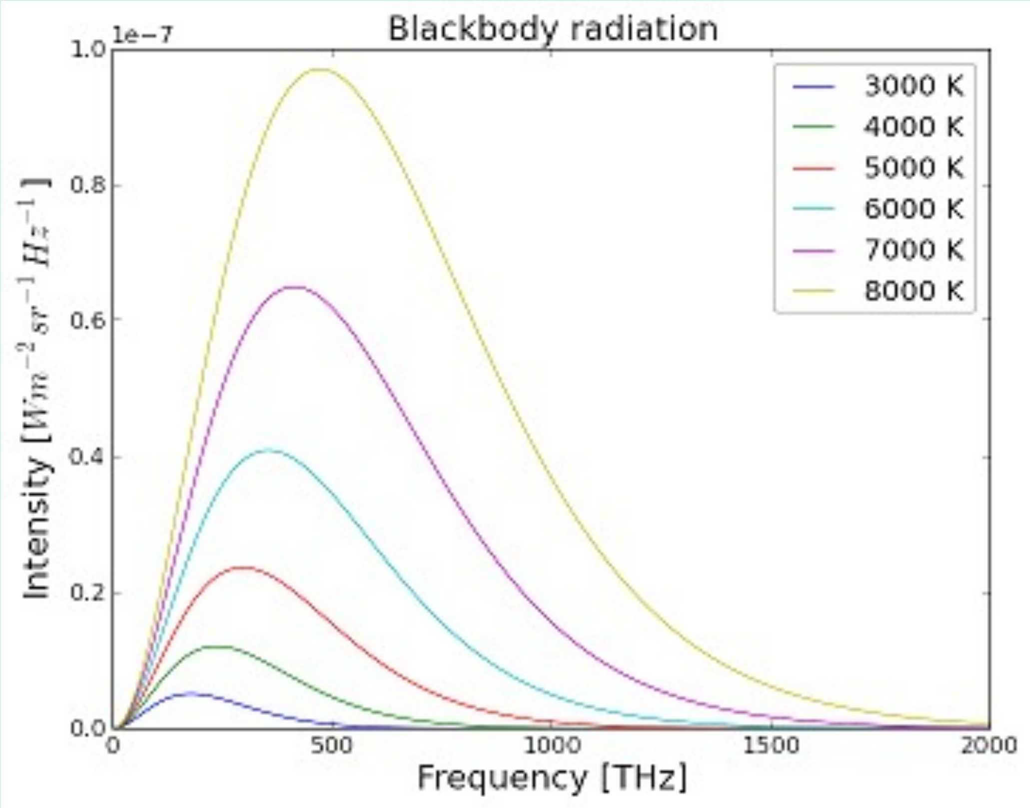


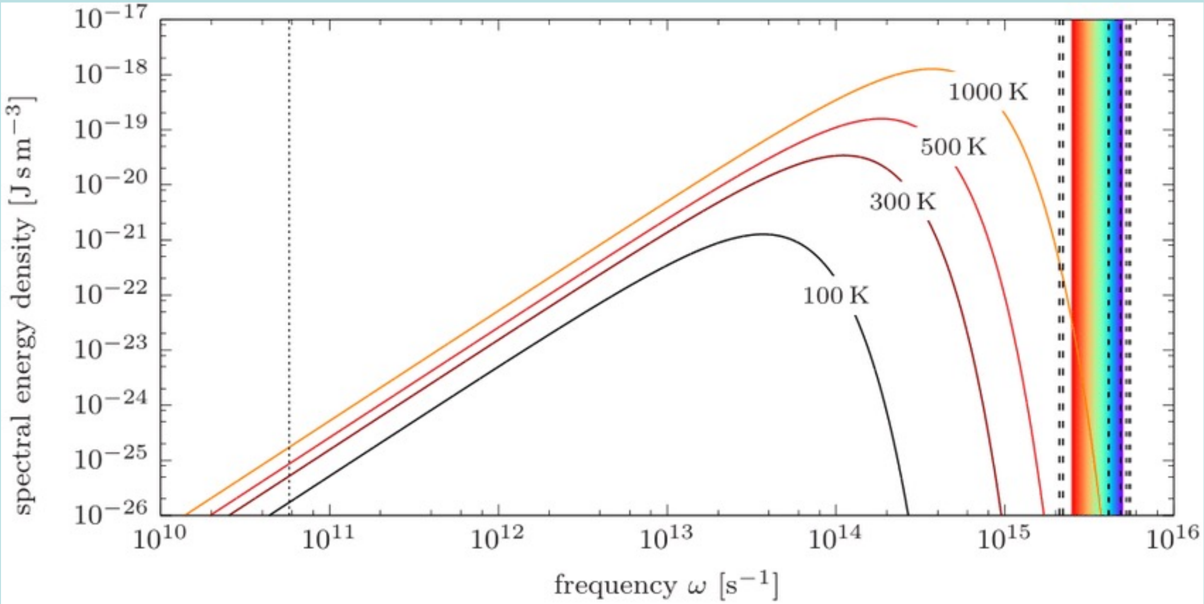
$u(\nu, T)$



$$u(\nu, T) = w(\lambda, T) \cdot \left|\frac{d\lambda}{d\nu}\right| = w\left(\frac{c}{\nu}, T\right) \cdot \frac{c}{\nu^2}$$

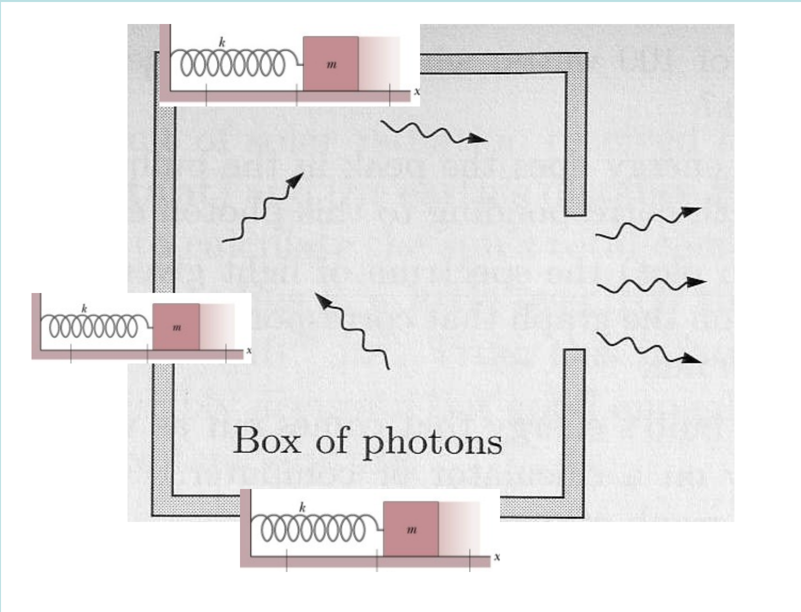
1-4

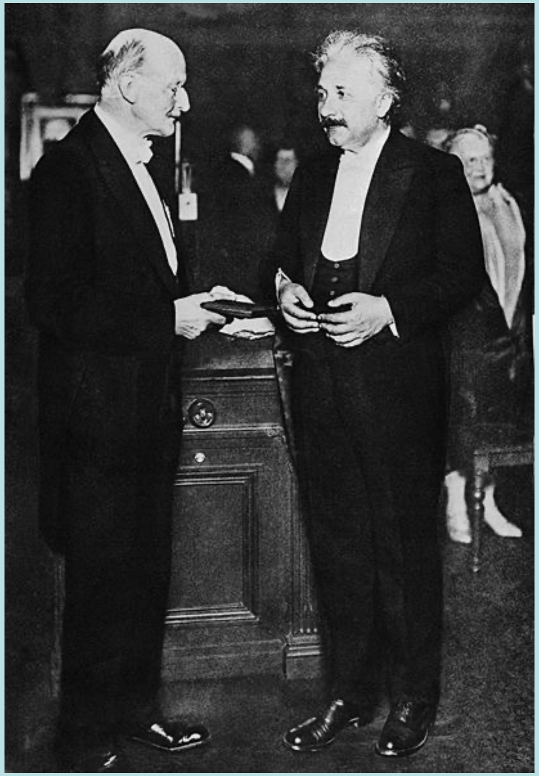


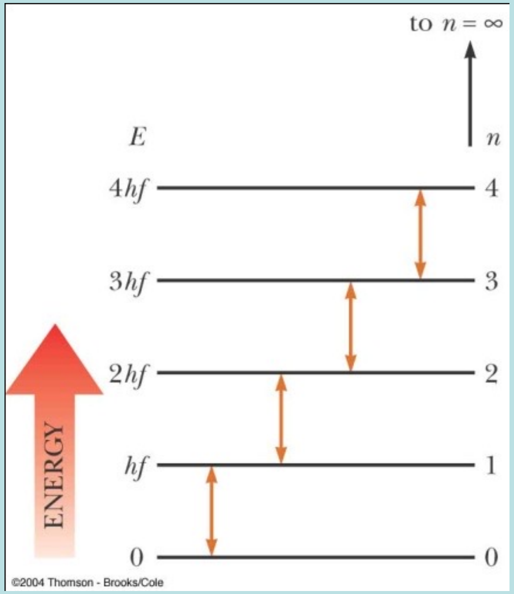


這是我們的目標！

$$u(\nu) = \frac{8\pi\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1}$$







Planck 的分析是針對空腔器壁上的量子彈簧（簡諧振盪器），

由黑體輻射觀察到的光譜結果倒推，得到量子彈簧的能量必須是hf的整數倍，如此與它們達到熱平衡的輻射，才會呈現所觀察的光譜。

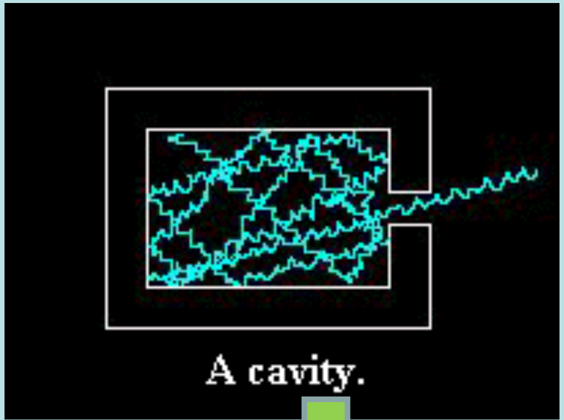
Einstein 在5年後提出另一個推導！而且他的想法更進了一步！

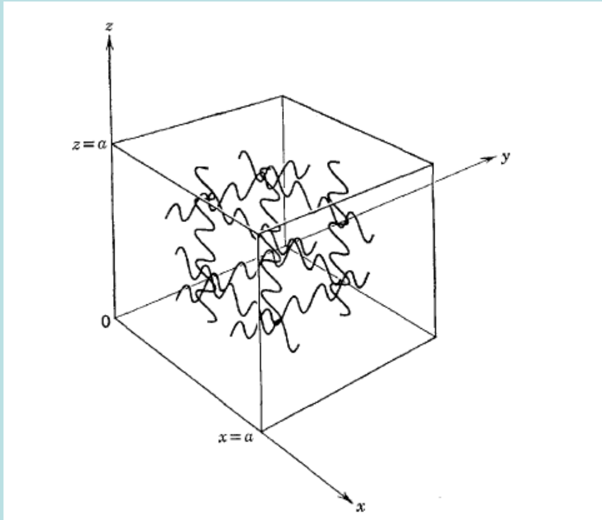
黑體輻射可不可以計算預測？Einstein的版本。

研究空腔內平衡時的輻射電磁波能量密度$u(\nu, T)$！

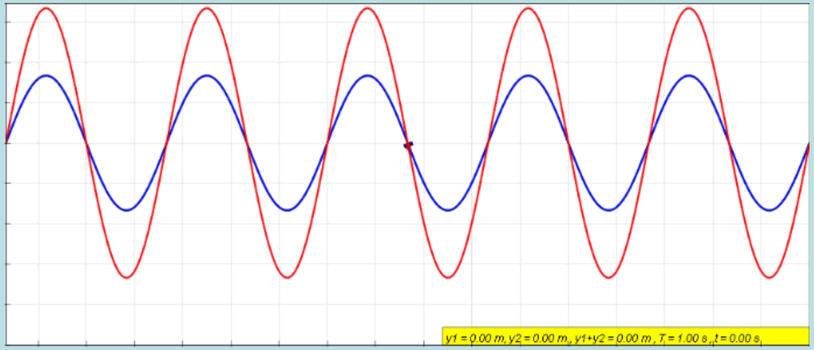
空腔內的輻射電磁波會來回反射，不斷疊加，

這些輻射穩定後會形成駐波。



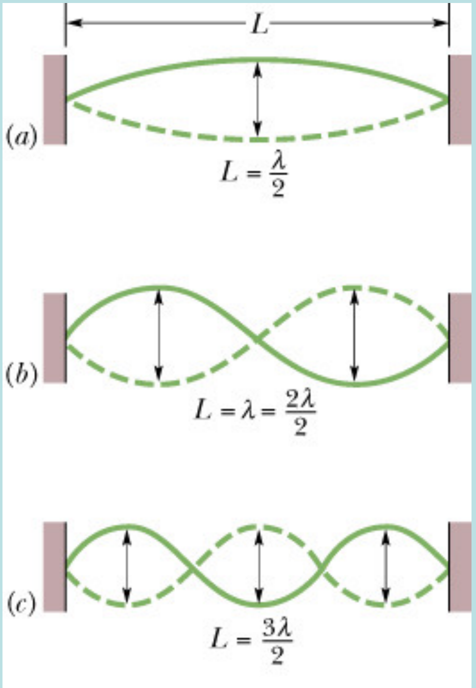


為簡單起見，先考慮一維的空腔，來回反射的電磁波形成駐波。



空腔中的電磁波是駐波，被腔壁邊界限制的波，這樣的波不傳播。

$E_0 \sin(kx - \omega t) - E_0 \sin(-kx - \omega t)$ → $E_0 \sin kx \cdot \cos \omega t$



穩定的駐波態還必須滿足邊界條件： $y=(2E_0\cdot\sin kx)\cdot\cos\omega t$

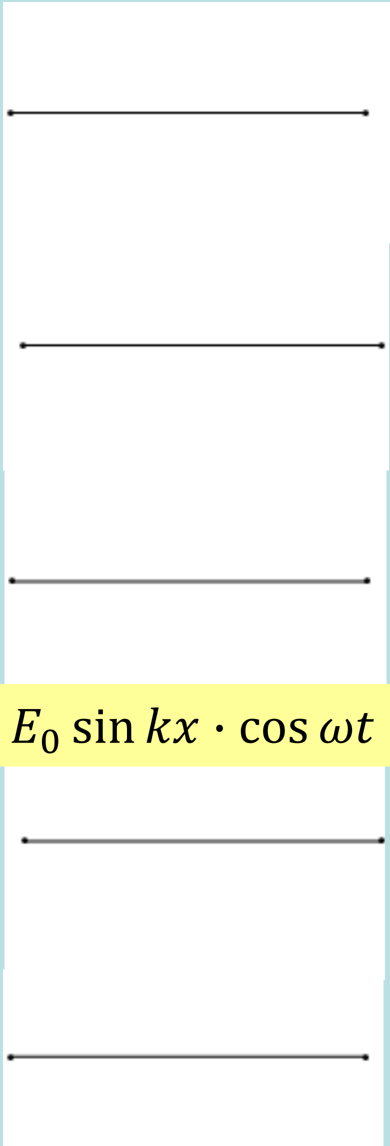
$y(0)=0$ 可以自動滿足！

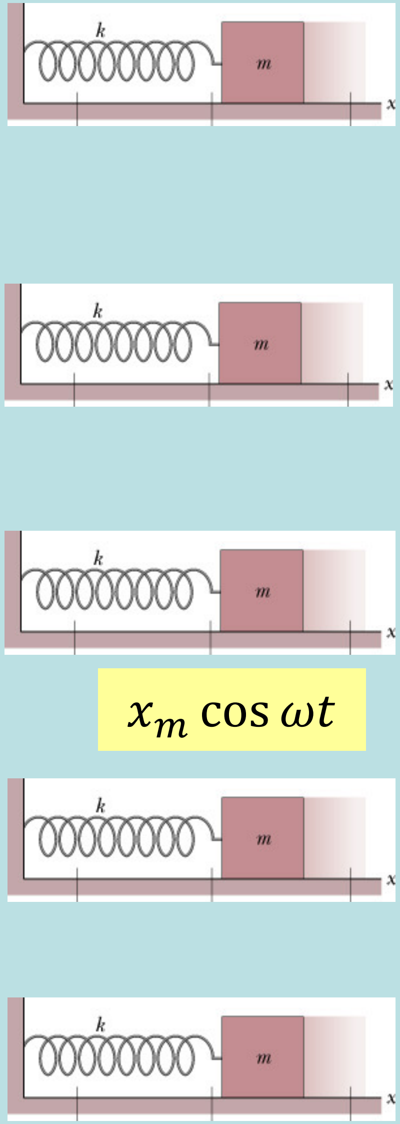
$y(L)=0$

波長λ不能任意： $kL=n\pi=\frac{2\pi}{\lambda}L$ $k=\frac{\pi}{L}n$ $\lambda=\frac{2L}{n}$

頻率ν不能任意 $\nu=\frac{v_{波}}{\lambda}=n\frac{v_{波}}{2L}$

因為邊界的限制，駐波的模式要求：頻率與一自然數n成正比。





$E_0 \sin kx \cdot \cos \omega t$

$x_m \cos \omega t$

因此一維的空腔內，電磁波是一系列駐波模式。

注意：每一個駐波模式如同一個振動的彈簧：

根據能量均分原理，無論頻率大小，每一個模式在熱平衡時，可以得到 kT 的能量！

所以要得到空腔輻射的總能量與能量分布，只要數一下對應的駐波數目即可！

策略：

計算單位體積內，頻率介於ν及$\nu + d\nu$之間的駐波狀態總數！

每一個駐波模式就對應一個彈簧，結果就是彈簧數$N(\nu) \cdot d\nu$。

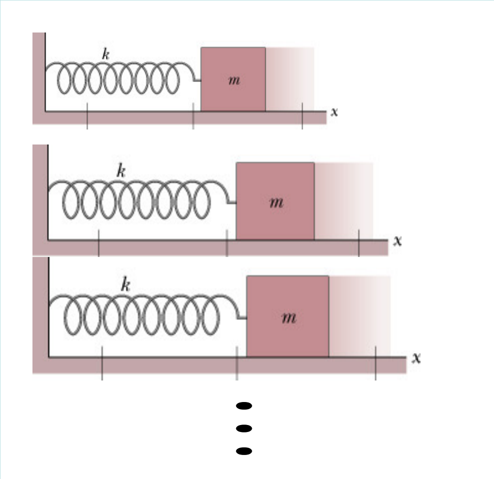
一個彈簧在溫度為T的環境中的能量平均值：

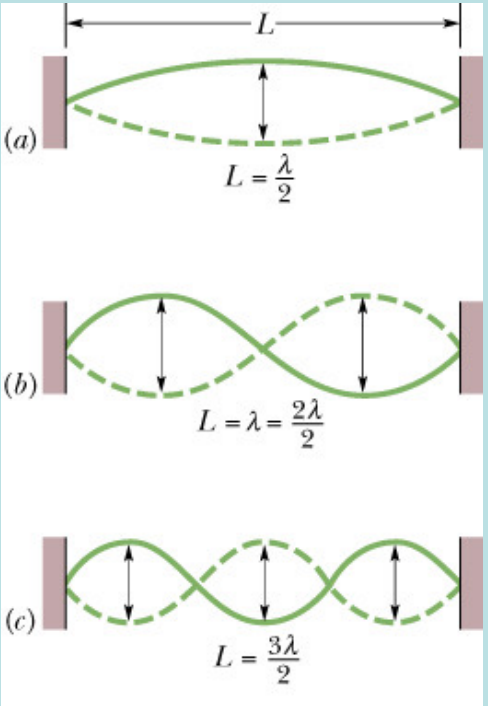
$$\langle E \rangle = kT$$

頻率介於ν及$\nu + d\nu$之間，彈簧總能量：

$$u(\nu) = N(\nu)kT \cdot d\nu$$

這就是頻率介於ν及$\nu + d\nu$之間的單位體積空腔輻射能量。



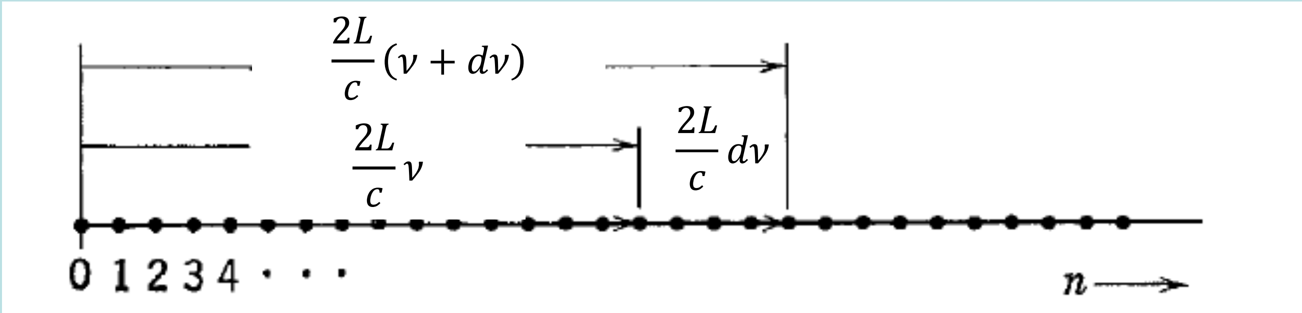


以一維駐波為例：n洽等於$\frac{2L}{c}$ 。

$$\nu = n\frac{v_{\text{波}}}{2L} \quad \rightarrow \quad n = \frac{2L}{c}\nu$$

在自然數 n 的軸上來討論最容易！

一個自然數 n ，對應一個點，對應一個駐波態。



頻率小於ν的駐波態，其n小於$\frac{2L}{c}\nu$ 。

頻率小於$\nu + d\nu$的駐波態，其n小於$\frac{2L}{c}(\nu + d\nu)$ 。

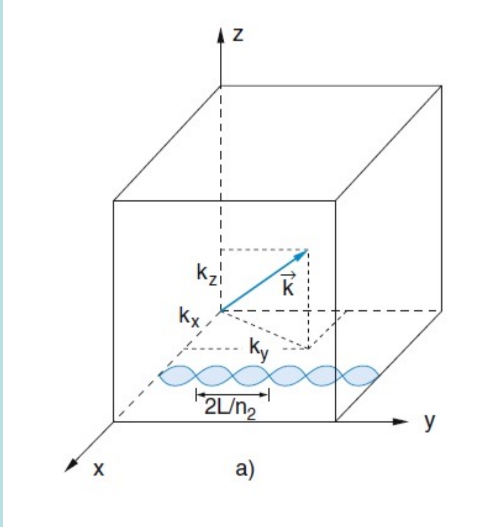
頻率介於ν及$\nu + d\nu$對應的駐波n的數目為兩者的差：

$$N(\nu) \cdot d\nu = \frac{2L}{c}d\nu$$

這就是頻率介於ν及$\nu + d\nu$之間的駐波的數目。

以上結果可以很容易推廣到三度空間：

考慮三度空間一個立方盒子，波與駐波分別可以寫成：



$$E_0 \sin(kx - \omega t) \rightarrow E_0 \sin(\vec{k} \cdot \vec{r} - \omega t)$$

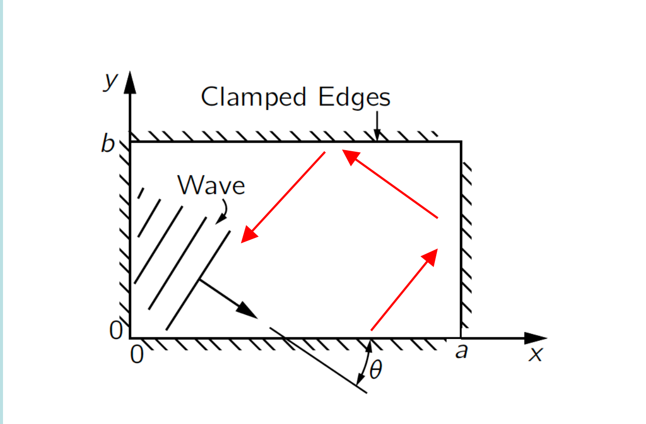
角波數 $\vec{k}$ 是一個向量，方向即傳播方向：

以二維為例： $E_0 \sin(k_x x + k_y y - \omega t)$

駐波可以寫成反射波疊加： $-E_0 \sin(k_x x - k_y y - \omega t)$

$E_0 \sin(-k_x x - k_y y - \omega t)$

$-E_0 \sin(-k_x x + k_y y - \omega t)$



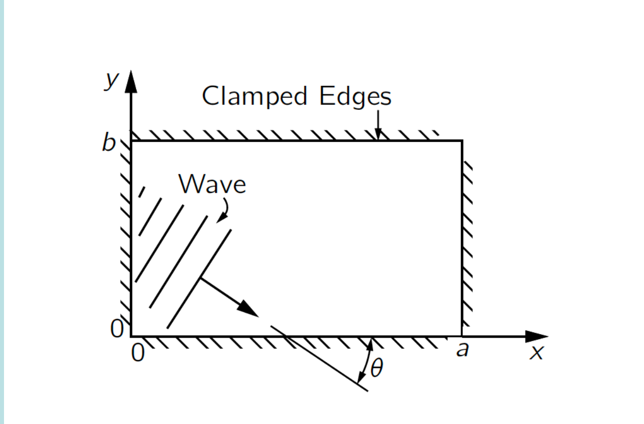
疊加等於：

$$\sum_{\pm} E_0 \sin(\pm k_x x \pm k_y y - \omega t)$$

$$\Downarrow$$

$$E_0 \sin kx \cdot \cos \omega t \rightarrow E_0 \sin k_x x \cdot \sin k_y y \cdot \sin \omega t$$

二維駐波： $E_0 \sin k_x x \cdot \sin k_y y \cdot \sin \omega t$

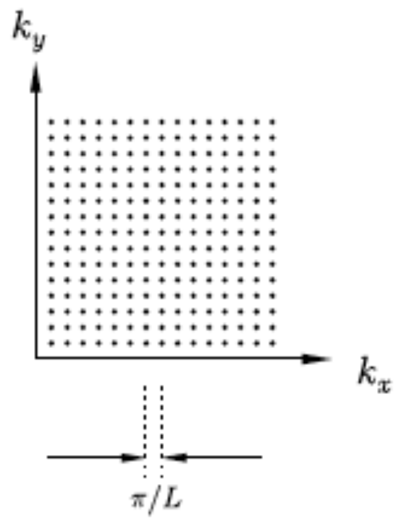


若要求在四個邊界上，波函數都是零：

$$k_x = \frac{\pi}{L} n_x \qquad k_y = \frac{\pi}{L} n_y$$

在三度空間中，駐波的條件會擴大到三個方向：

$$k = \frac{\pi}{L} n \quad \longrightarrow \quad k_x = \frac{\pi}{L} n_x \qquad k_y = \frac{\pi}{L} n_y \qquad k_z = \frac{\pi}{L} n_z$$



一組自然數 n_x, n_y, n_z的選擇，對應一個駐波態。

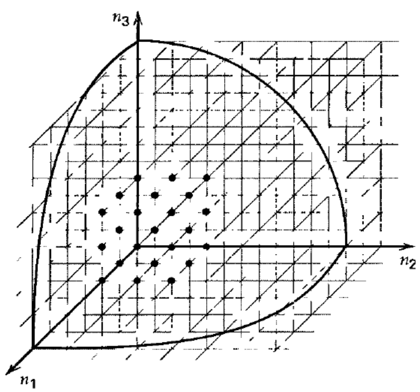
以(n_x, n_y, n_z)畫一個三維空間，一個駐波態即是一個點。

角波數向量 $\vec{k}$ 大小：

$$k = \sqrt{n_x^2 + n_y^2 + n_z^2} \cdot \frac{\pi}{L}$$

而駐波態對應的頻率為：

$$\nu = \frac{kc}{2\pi} = \sqrt{n_x^2 + n_y^2 + n_z^2} \cdot \frac{c}{2L}$$



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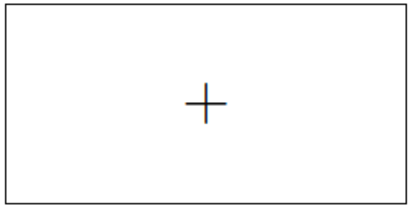
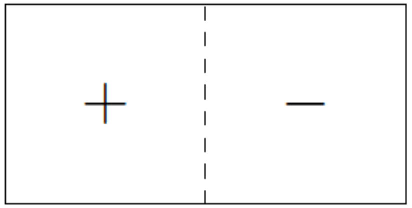
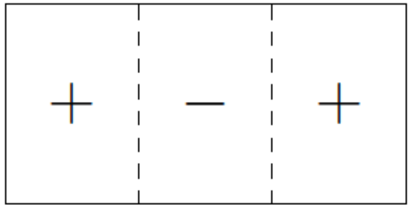
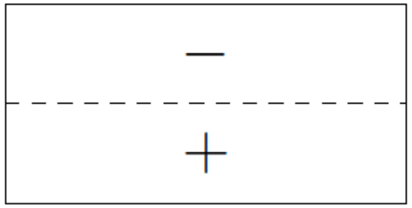
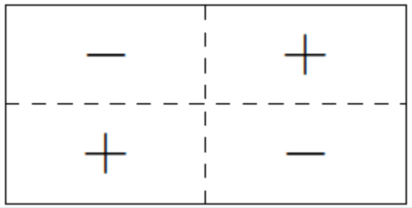
	1	1
	1	2
	1	3
	2	1
	2	2

$$\nu = \sqrt{n_x^2 + n_y^2 + n_z^2} \cdot \frac{c}{2L}$$

$$\nu = \sqrt{n_x^2 + n_y^2 + n_z^2} \cdot \frac{c}{2L}$$

$$\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{2L}{c} \cdot \nu$$

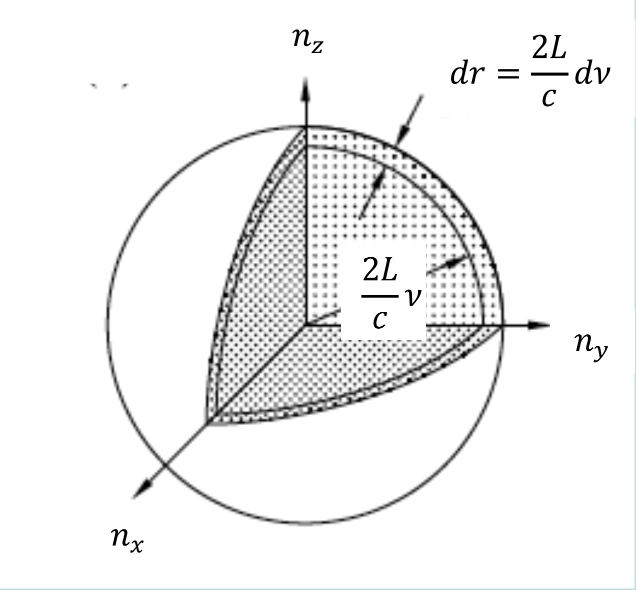
$\sqrt{n_x^2 + n_y^2 + n_z^2}$是在$(n_x, n_y, n_z)$空間中，與原點的距離。恰等於$\frac{2L}{c}\nu$。

此距離，如同一維空腔的n，是與頻率成正比。

頻率小於ν的駐波態， 距離$\sqrt{n_x^2 + n_y^2 + n_z^2}$自然小於$\frac{2L}{c}\nu$ ，點位在半徑為$\frac{2L}{c}\nu$的球內。

同理，頻率小於$\nu + d\nu$的駐波態，點就位在半徑為$\frac{2L}{c}(\nu + d\nu)$的球內。

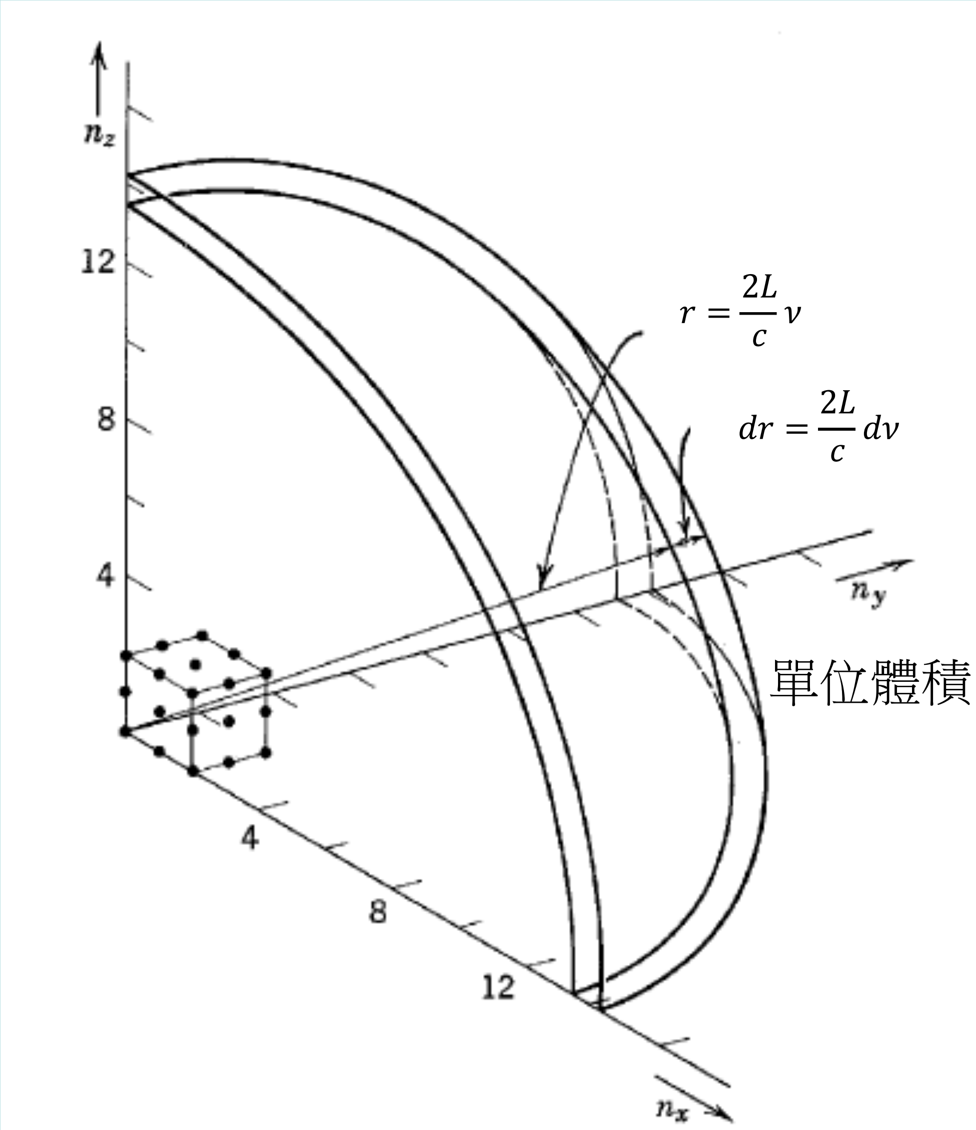
因此，頻率介於ν及$\nu + d\nu$之間的駐波態，點就位於半徑$\frac{2L}{c}\nu$、厚度$\frac{2L}{c}d\nu$的球殼內。



頻率介於ν及$\nu + d\nu$之間的(n_x, n_y, n_z)近似就位於半徑$\frac{2L}{c}\nu$、厚度$\frac{2L}{c}d\nu$的球殼內：

因自然數為正，應為$\frac{1}{8}$球殼，體積等於：

$$4\pi\left(\frac{2L}{c}\nu\right)^2 \cdot \left(\frac{2L}{c}d\nu\right) \cdot \frac{1}{8} = \frac{4\pi\nu^2}{c^3}d\nu \cdot V$$



而(n_x, n_y, n_z)空間內狀態的密度是1！

頻率介於ν及$\nu + d\nu$之間的狀態總數為

$$\frac{4\pi\nu^2}{c^3}d\nu \cdot V$$

一個狀態的電磁波可以有兩個偏振：

單位體積內，頻率介於ν及$\nu + d\nu$之間的狀態總數為

$$\frac{8\pi\nu^2}{c^3}d\nu$$

單位體積內，頻率介於ν及$\nu + d\nu$之間的電磁駐波狀態總數為 $\frac{8\pi\nu^2}{c^3}d\nu$

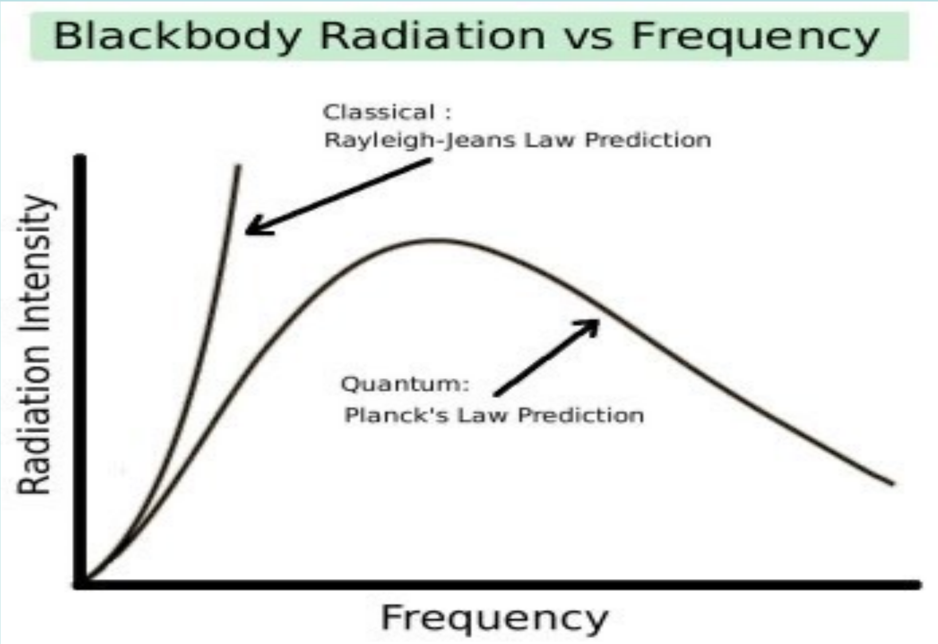
每一個駐波模式就對應一個量子彈簧。

根據能量均分原理，無論頻率大小，每一個彈簧在熱平衡時，可以得到kT能量！

因此頻率介於ν及$\nu + d\nu$之間的空腔輻射能量密度：$u(\nu)d\nu = kT \cdot \frac{8\pi\nu^2}{c^3}d\nu$

$$u(\nu) = kT \cdot \frac{8\pi\nu^2}{c^3} \qquad \text{1-5}$$

Rayleigh-Jeans Prediction

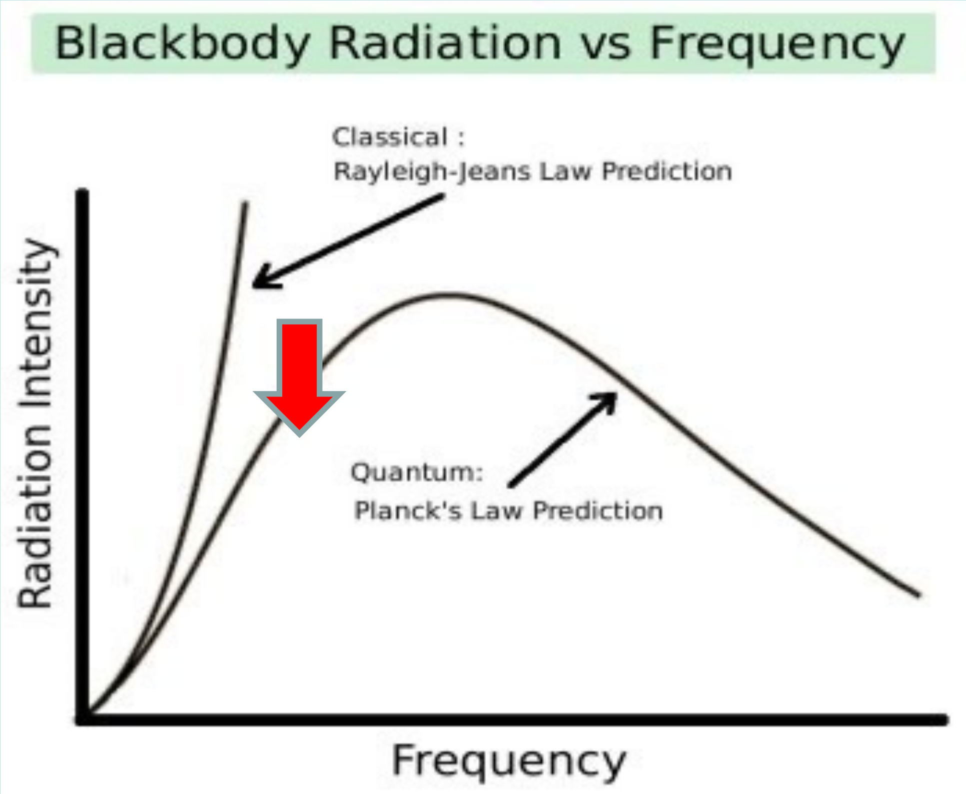


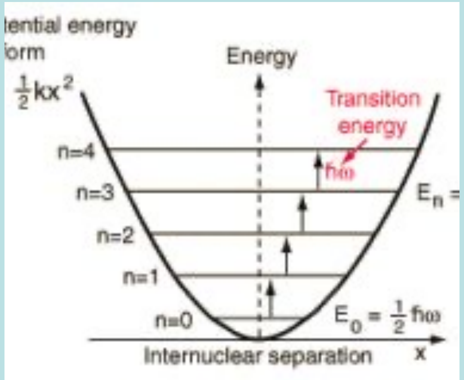
$$E(\nu) = \frac{c}{4} \cdot u \sim kT \cdot \frac{2\pi\nu^2}{c^2}$$

駐波模式數目隨頻率增加而一直增加，

輻射能量應該隨頻率增加而一直無限地增加！

這顯然與實驗結果不符！



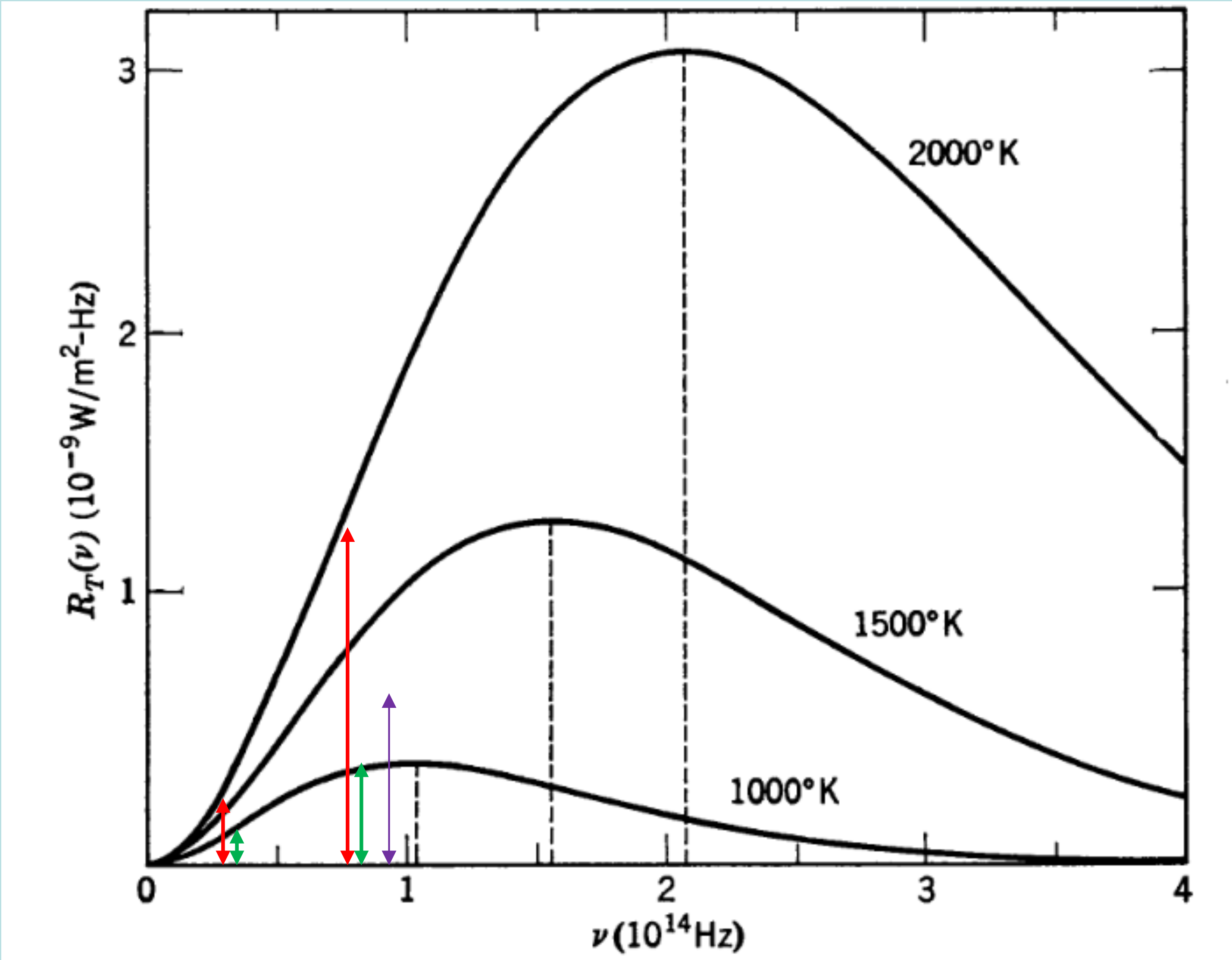


但注意：在低頻率區，以上的古典計算非常準確！

在高頻率區，能量則比預期低，似乎熱平衡的能量難以輸入到彈簧之中！

頻率越高，門檻越高，越難把能量輸入！

這感覺非常熟悉，這是量子彈簧的特徵！



在低頻率區，固定頻率時，溫度降低為一半時，功率也降低為一半。

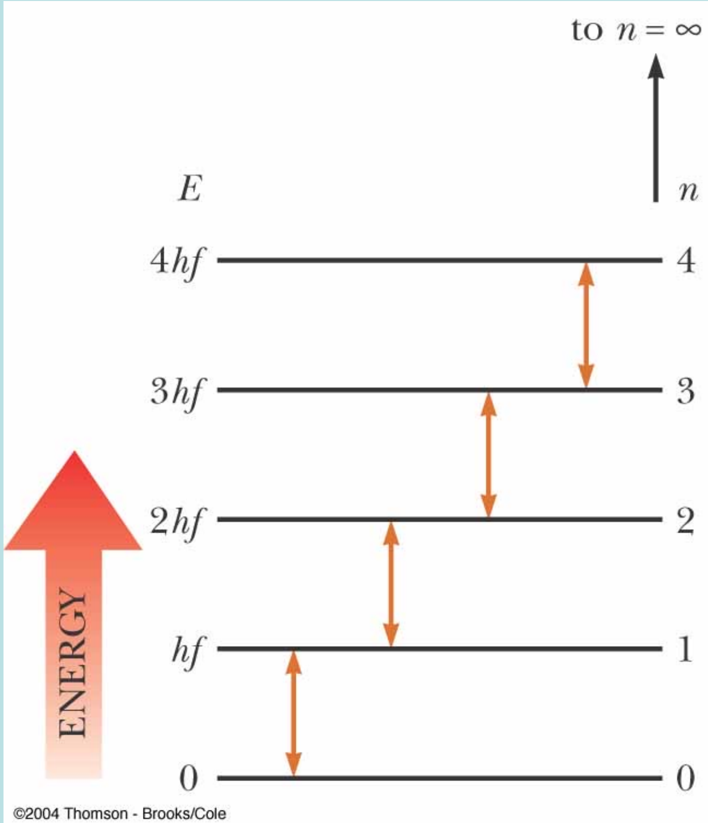
固定頻率時，狀態數固定，符合古典能量均分原理的計算，$P \propto kT$！

但在高頻率區，功率則降低遠超過一半。

可見熱能在低溫時，進不了彈簧，跨不過吸收的門檻。頻率越高，門檻越高！

量子彈簧吸收能量是有門檻的！而門檻與頻率成正比。

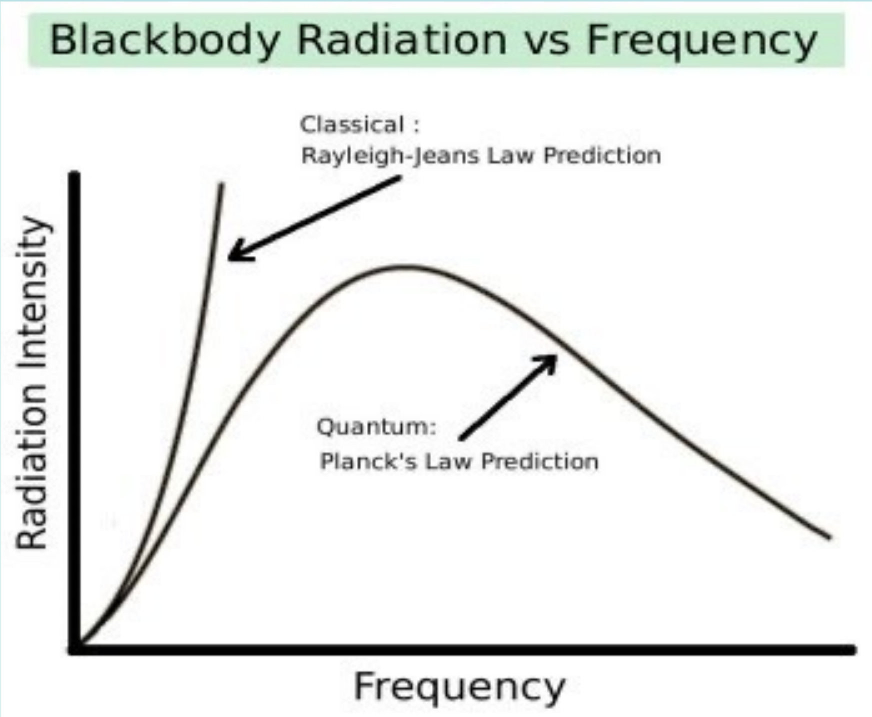
Max Planck 在1900 開啟了量子革命的第一槍！

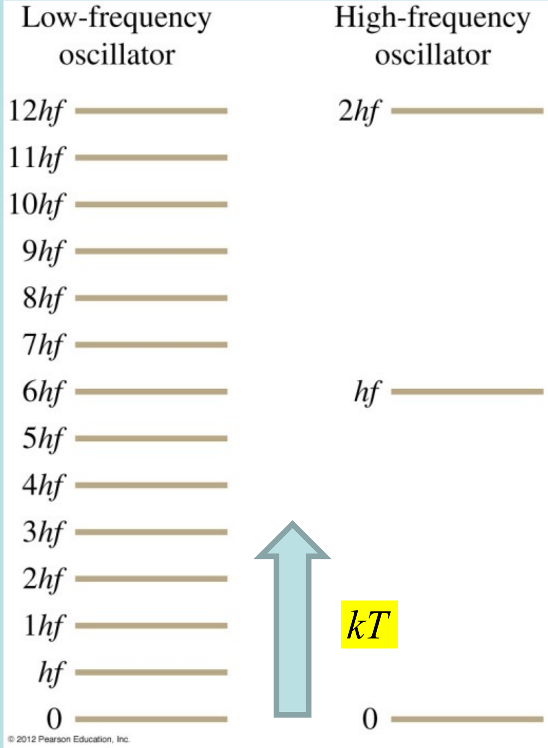


量子彈簧能吸收的能量不是連續的，而是固定量子的整數倍（離散型式）

量子(Quantum)的大小與頻率成正比！

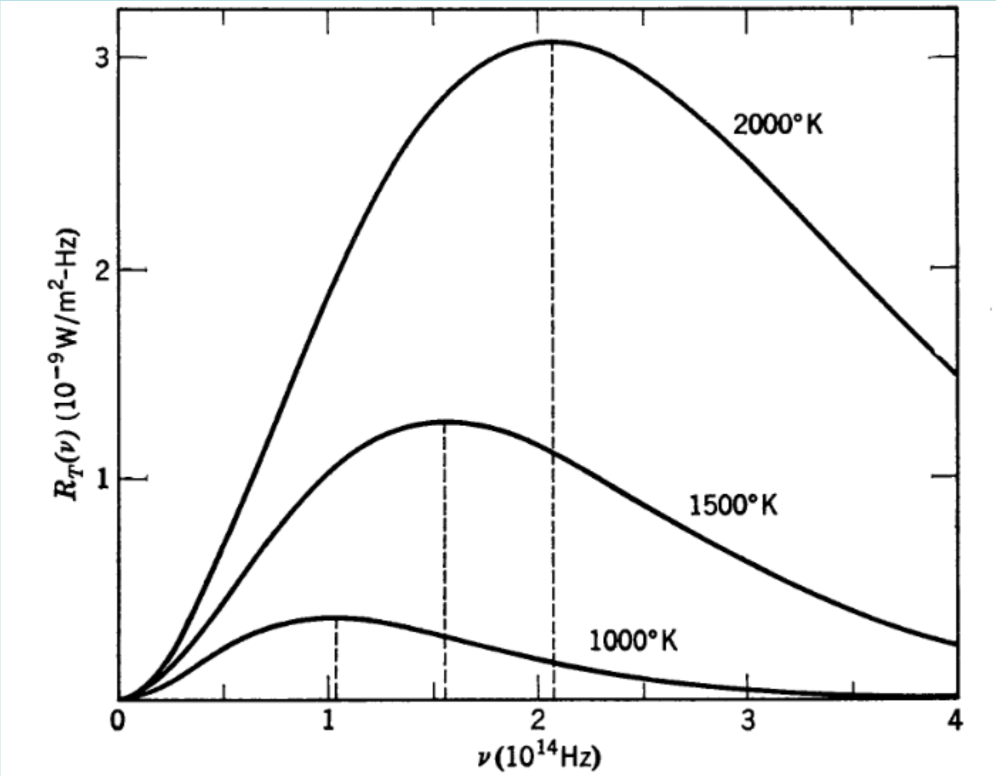
$E_n = n \cdot hf$ h: Planck Constant $h = 6.625 \times 10^{-34} \mathrm{J} \cdot s$

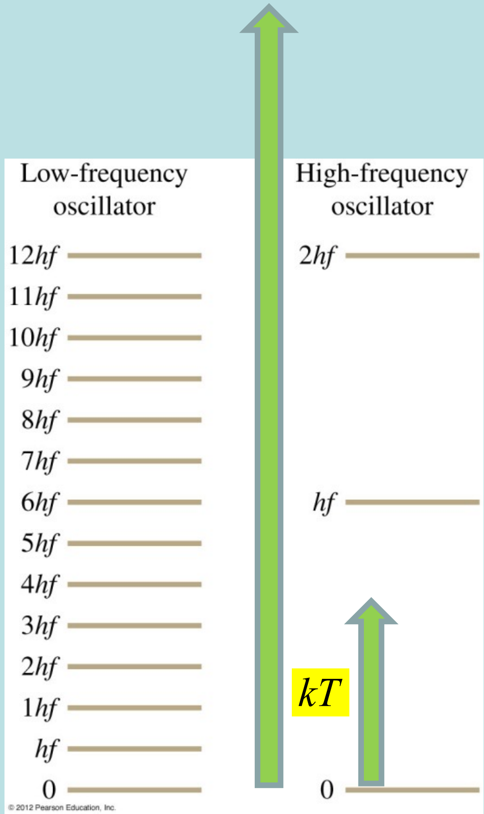




在高頻率區域，因能階差較大，平均為 kT 的熱能有較大機會會小於一個量子而無法被吸收，所以此區的駐波膜式能量吸收效率就遠小於能量均分原理所要求的 kT。

而在低頻率區域，能階差相對於平均為 kT 的熱能很小，所以能階差幾乎可以忽略，此區的駐波膜式能量吸收效率就大約與古典彈簧能量均分原理所要求的 kT 差距不大。古典的結果大致與實驗吻合。





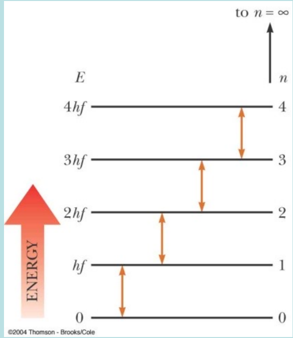
而溫度越高，能夠被激發的駐波頻率越高，

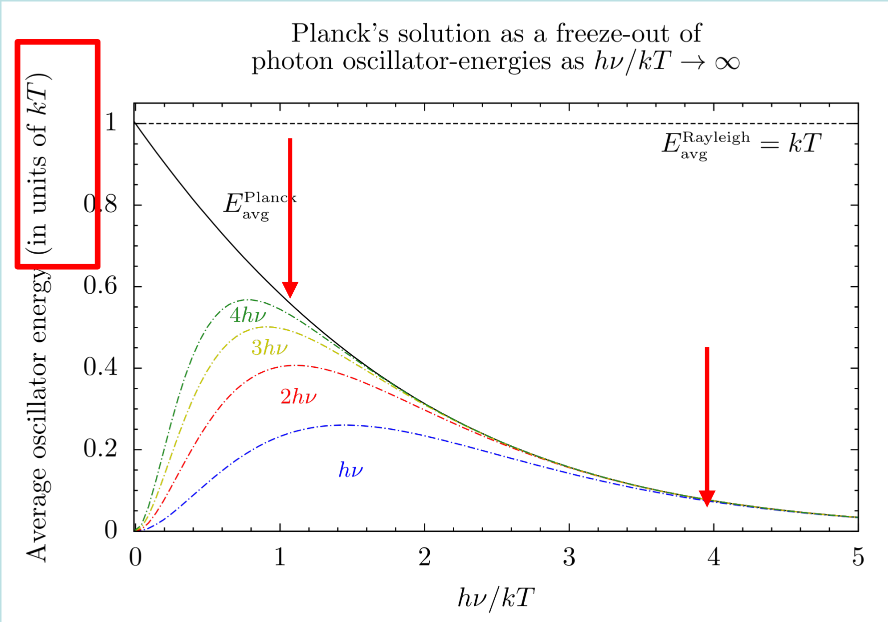
所以溫度越高，高頻率的輻射能量就會增加！

以上這些性質都表現在量子彈簧在溫度為T時的能量平均值之中！

$$P_n = \frac{e^{-E_n/kT}}{\sum_{n=0}^{\infty} e^{-E_n/kT}}$$

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$





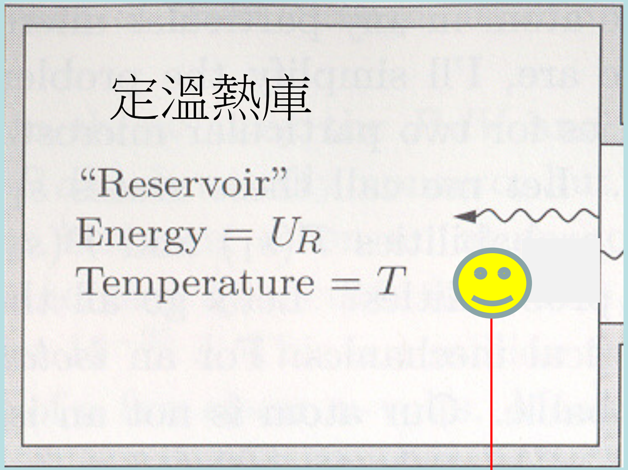
只有在頻率與溫度的比$h\nu/kT$為無限小時，彈簧吸收的能量才是 kT。

頻率增加，會溫度降低，都使門檻效果顯著，吸收比例變小。

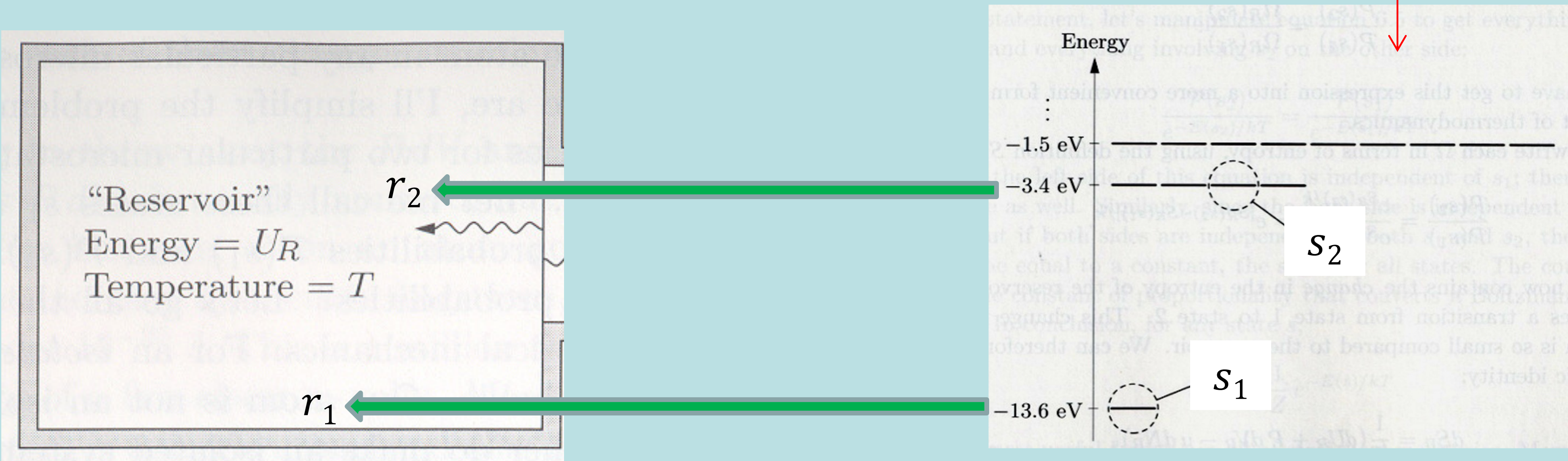
固定溫度但不定能量的熱系統的統計描述 Gibbs 稱為 Canonical Ensemble 正則系集。

所研究的系統可以小到為一原子，

而環境為相對極大的溫度不變的系統，



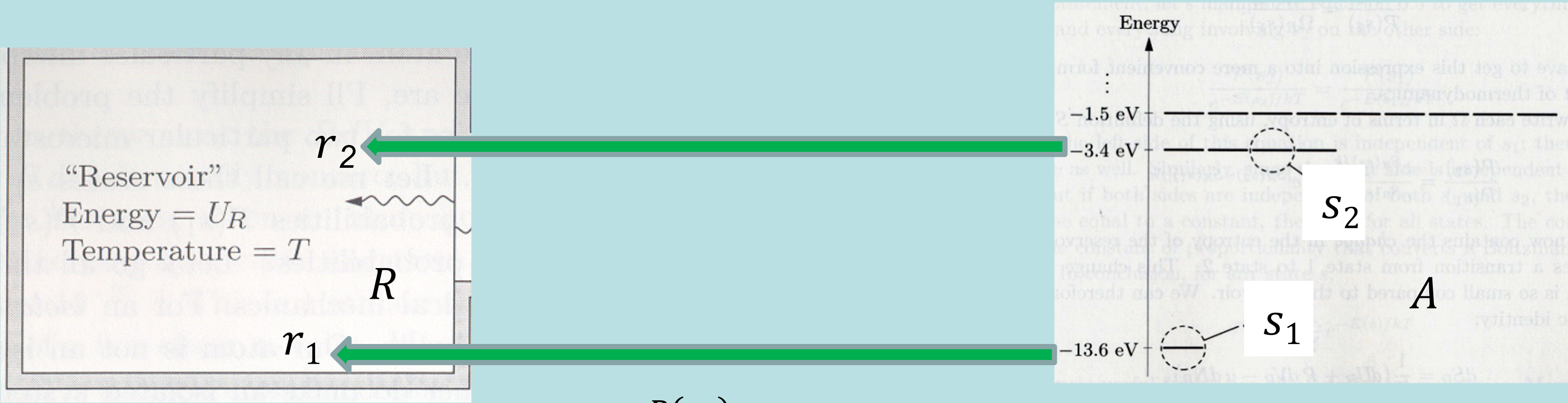
兩者不斷進行熱交互作用，因此原子的能量不固定。



s_1 及 s_2 分別對應熱庫的Macrostates r_1 及 r_2

假設原子的狀態是由分離的能階來描述，以 s_1 及 s_2 來標記兩個分立的能態。

總系統為封閉，故總能量 E 固定。



原子處於狀態 s_1 及 s_2 的機率比$\frac{P(s_1)}{P(s_2)}$是多少？

$$P(s) \propto W(s) = W_A(s) \cdot W_R(r)$$

注意：s_1 及 s_2的W_A都是1。原子處於狀態 s_1 及 s_2 代表熱庫狀態處於 r_1 及 r_2。

$$\frac{P(s_1)}{P(s_2)} = \frac{W(s_1)}{W(s_2)} = \frac{W_R(r_1) \cdot 1}{W_R(r_2) \cdot 1}$$

此機率比等於熱庫的狀態 r_1 及 r_2 的W_R的比。

而環境的W_R可以以環境的熵S_R表示：

$$S_R = k \ln W_R \qquad W_R = e^{S_R/k}$$

$$\frac{W_R(r_1)}{W_R(r_2)} = \frac{e^{S_R(r_1)/k}}{e^{S_R(r_2)/k}} = e^{[S_R(r_1)-S_R(r_2)]/k} = e^{\Delta S_R/k}$$

以一熱過程連接 r_1 及 r_2

$$\Delta S_R = \frac{Q}{T} = \frac{E_R(r_1) - E_R(r_2)}{T} = -\frac{E(s_1) - E(s_2)}{T}$$

將此式代回原來的計算：

$$= e^{-[E(s_1)-E(s_2)]/kT} = \frac{e^{-E(s_1)/kT}}{e^{-E(s_2)/kT}}$$

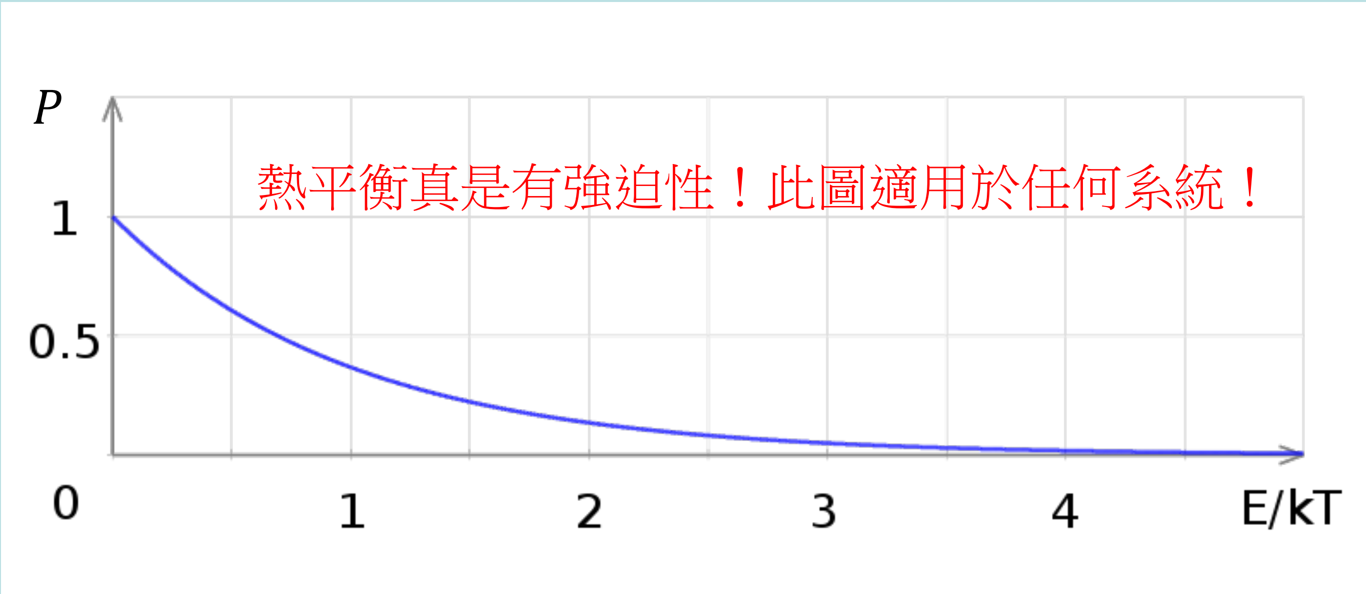
Boltzmann Distribution分布或Canonical Distribution正則分布

$$\frac{P(s_1)}{P(s_2)} = e^{-\Delta E/kT} = \frac{e^{-E(s_1)/kT}}{e^{-E(s_2)/kT}}$$

或 $P \propto e^{-E/kT}$ Boltzmann Factor

一個系統在定溫下，出現在一特定狀態的機率正比於Boltzmann Factor。

完全由溫度與能量就可以決定！和其他細節完全無關。



$E \gg KT, \qquad P \ll 1$

古典的情況：我們可以由此機率計算出能量的平均值：

以在一維空間中的自由粒子的運動為例，在溫度為T的環境，

粒子運動速率為v的機率$P(v)$正比於 $\propto e^{-\frac{mv^2}{2kT}}$

$$P(v) = P_v = \frac{e^{-E_v/kT}}{\sum_{v'} e^{-E_{v'}/kT}} \rightarrow \frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2kT}}} \cdot e^{-\frac{mv^2}{2kT}}$$

注意速率v是連續變化的，因此求和變為積分！

動能的平均值：

$$\left\langle \frac{1}{2}mv^2 \right\rangle_{\mathrm{av}} = \int_0^\infty dv \cdot \frac{1}{2}mv^2 \cdot P(v) = \frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2kT}}} \int_0^\infty dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2kT}}$$

定義一個新的符號：$\beta \equiv \frac{1}{kT}$

$$\frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2}\beta}} \int_0^\infty dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2}\beta} = -\frac{1}{\int_0^\infty dv' \cdot e^{-\frac{mv'^2}{2}\beta}} \frac{\partial}{\partial\beta} \int_0^\infty dv \cdot e^{-\frac{mv^2}{2}\beta}$$

分子積分中的mv^2可以由對β的微分得到！

注意分子與分母中的積分是一樣的，因此可寫成對數的微分：

$$= -\frac{\partial}{\partial\beta} \ln \int_0^\infty dv \cdot e^{-\frac{mv^2}{2}\beta}$$

$$\frac{\partial}{\partial x} \ln f = \frac{1}{f} \cdot \frac{df}{dx}$$

接著變換變數：

$$x \equiv \sqrt{\frac{m\beta}{2}}\, v \qquad dv \equiv \sqrt{\frac{2}{m\beta}}\, dx$$

如此，注意式中的β可以與積分分離。
積分可以不用算。

$$= -\frac{\partial}{\partial\beta} \ln \sqrt{\frac{2}{m\beta}} \int_0^\infty dx \cdot e^{-x^2} = -\frac{\partial}{\partial\beta} \left[\ln \sqrt{\frac{1}{\beta}} + \ln \sqrt{\frac{2}{m}} \int_0^\infty dx \cdot e^{-x^2} \right]$$

$$= -\frac{\partial}{\partial\beta} \ln \sqrt{\frac{1}{\beta}} = \frac{1}{2}\frac{\partial}{\partial\beta} \ln \beta = \frac{1}{2}\frac{1}{\beta} = \frac{1}{2}kT$$

$$\left\langle \frac{1}{2}mv^2 \right\rangle_{\text{av}} = \frac{1}{\int_0^\infty dv \cdot e^{-\frac{mv^2}{2kT}}} \int_0^\infty dv \cdot \frac{1}{2}mv^2 \cdot e^{-\frac{mv^2}{2kT}} = \frac{1}{2}kT$$

注意此結果完全與質量m無關！

因此彈力位能的計算也得到完全相同結果：

$$\left\langle \frac{1}{2}Kx^2 \right\rangle_{\text{av}} = \frac{1}{\int_0^\infty dx \cdot e^{-\frac{Kx^2}{2kT}}} \int_0^\infty dx \cdot \frac{1}{2}Kx^2 \cdot e^{-\frac{Kx^2}{2kT}} = \frac{1}{2}kT$$

任何能量只要是動力座標的平方都會得到相同結果！

一個系統中，任一個可以儲存能量的型式，只要是動力座標的平方，

在頻繁的熱作用（混亂的能量交換）

達到熱平衡後，都會得到同樣的平均能量： $\frac{1}{2}kT$

計算一頻率為ν的量子彈簧在溫度為T的環境中的能量平均值

$$P_n = \frac{e^{-E_n/kT}}{\sum_{i=0}^{\infty} e^{-E_i/kT}}$$

$$E_n = nh\nu$$

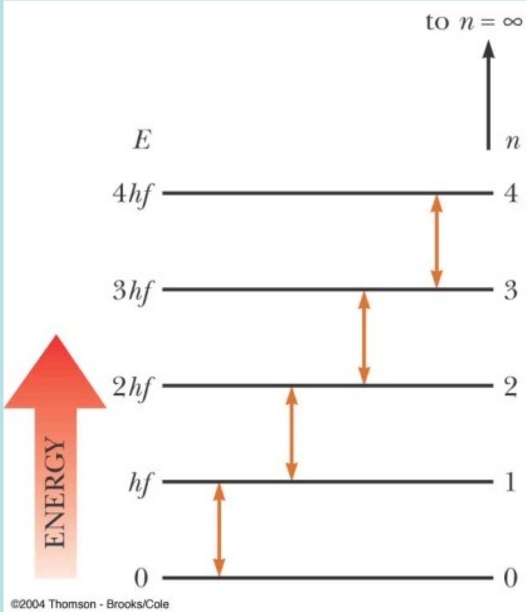
$$\langle E \rangle = \sum_{n=0}^{\infty} E_n P_n = \frac{\sum_{n=0}^{\infty} E_n e^{-E_n/kT}}{\sum_{i=0}^{\infty} e^{-E_i/kT}}$$

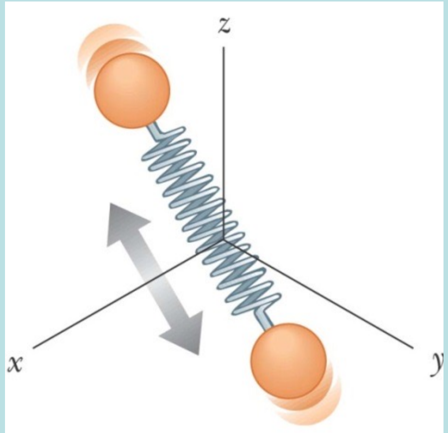
定義一個新的符號： $\beta \equiv \frac{1}{kT}$

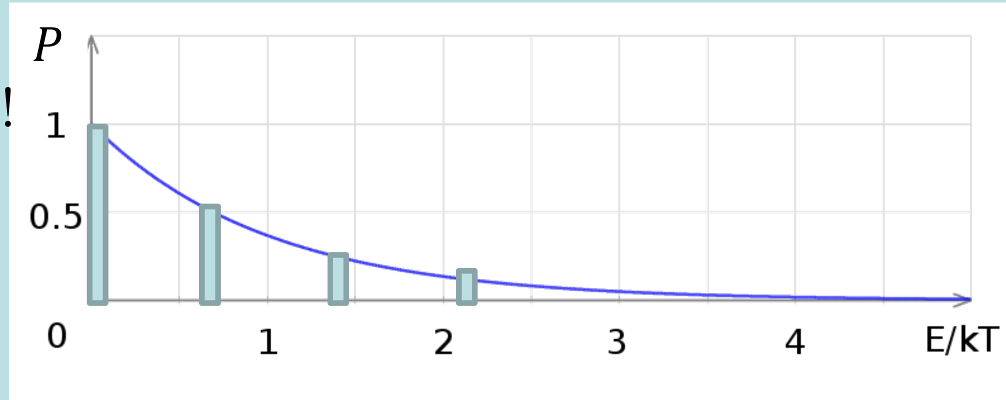
$$= \frac{\sum_{n=0}^{\infty} (nh\nu) \cdot e^{-\beta nh\nu}}{\sum_{i=0}^{\infty} e^{-\beta ih\nu}}$$

分子積分中的nhf可以由對β的微分得到！

$$= -\frac{1}{\sum_{i=0}^{\infty} e^{-\beta ih\nu}} \frac{\partial}{\partial \beta} \sum_{n=0}^{\infty} e^{-\beta nh\nu}$$







注意分子與分母中的求和是一樣的，因此可寫成求和的對數的微分：

$$\frac{\partial}{\partial x} \ln f = \frac{1}{f} \cdot \frac{df}{dx}$$

$$= -\frac{\partial}{\partial \beta} \ln \sum_{n=0}^{\infty} e^{-\beta nh\nu}$$

現在求和只是一個等比級數！

$$\langle E\rangle = \sum_{n=0}^{\infty} E_n P_n = \frac{\sum_{n=0}^{\infty}\left(nh\nu \cdot e^{-\beta nh\nu}\right)}{\sum_{i=0}^{\infty} e^{-\beta ih\nu}} = -\frac{1}{\sum_{i=0}^{\infty} e^{-\beta ih\nu}}\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty} e^{-\beta nh\nu}$$

現在求和只是一個等比級數！

$$= -\frac{\partial}{\partial\beta}\ln\sum_{n=0}^{\infty} e^{-\beta nh\nu}$$

$$= -\frac{\partial}{\partial\beta}\ln\frac{1}{1-e^{-\beta h\nu}}$$

$$= \frac{\partial}{\partial\beta}\ln\left(1-e^{-\beta h\nu}\right) = \frac{h\nu \cdot e^{-\beta h\nu}}{1-e^{-\beta h\nu}}$$

$$= \frac{h\nu}{e^{\beta h\nu}-1} = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1}$$

量子彈簧在溫度為T的環境中的能量平均值

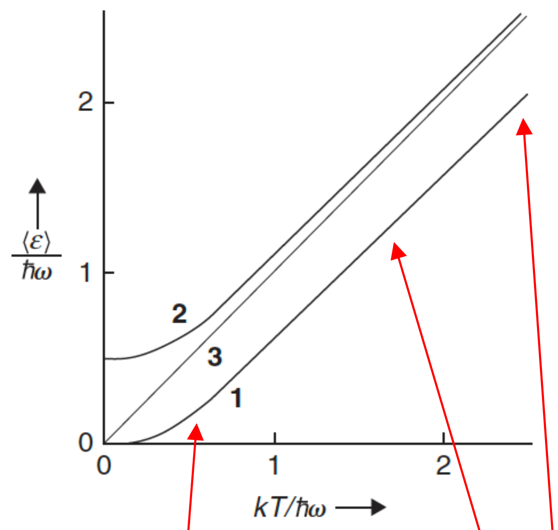
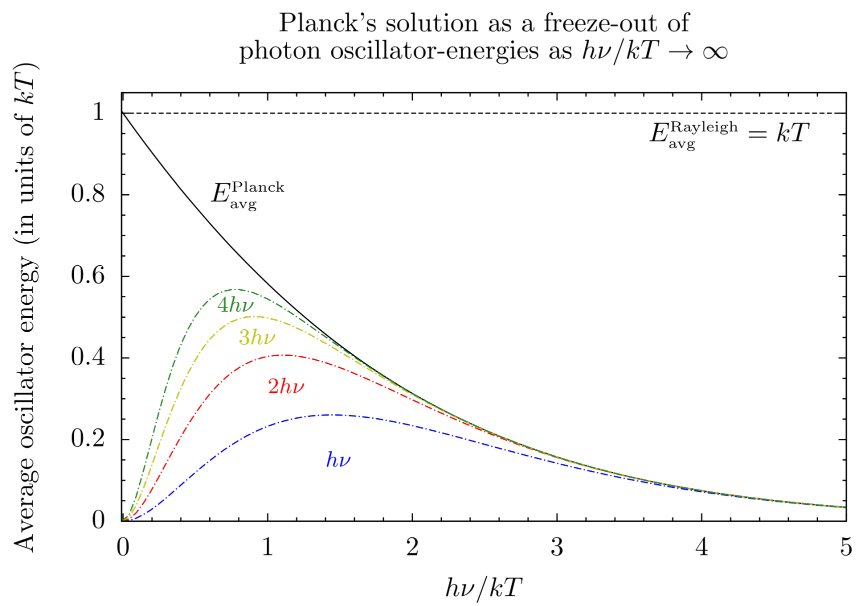


FIGURE 3.4 The mean energy ⟨ε⟩ of a simple harmonic oscillator as a function of temperature. 1, the Planck oscillator; 2, the Schrödinger oscillator; and 3, the classical oscillator.



$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1} \to 0 \quad kT \ll h\nu$$

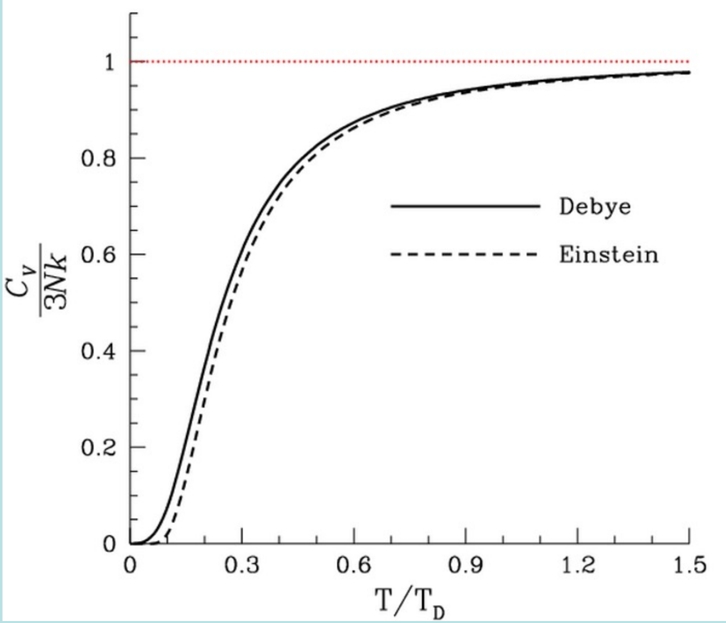
溫度低時，量子彈簧無法激發得到能量。

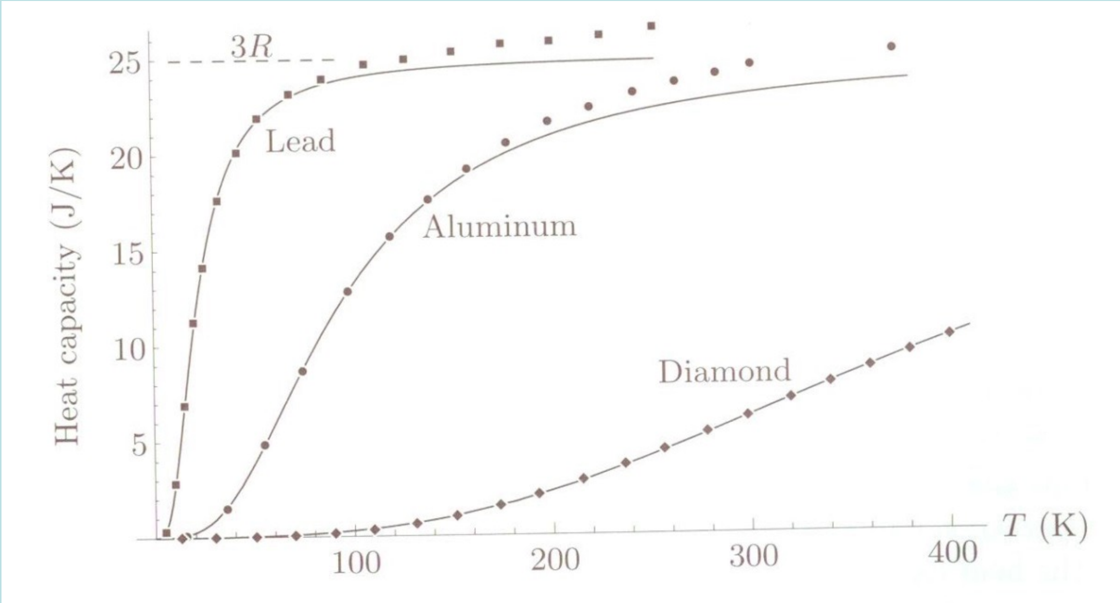
$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1} \to \frac{h\nu}{1+h\nu/kT-1} \to kT \qquad kT \gg h\nu$$

$$e^{x} \to 1+x, \qquad x \ll 1$$

溫度高時，量子彈簧激發得到兩個自由度的能量，與古典一致。

固體的比熱　在高溫時，能量均分原則適用，在低溫時，必須考慮量子效應。





愛因斯坦大膽假設空腔內的駐波態對應的是量子彈簧！

單位體積內，頻率介於ν及$\nu + d\nu$之間的駐波狀態總數為

$$\frac{8\pi\nu^2}{c^3}d\nu$$

每一個駐波模式就對應一個量子彈簧，上式就是彈簧總數。

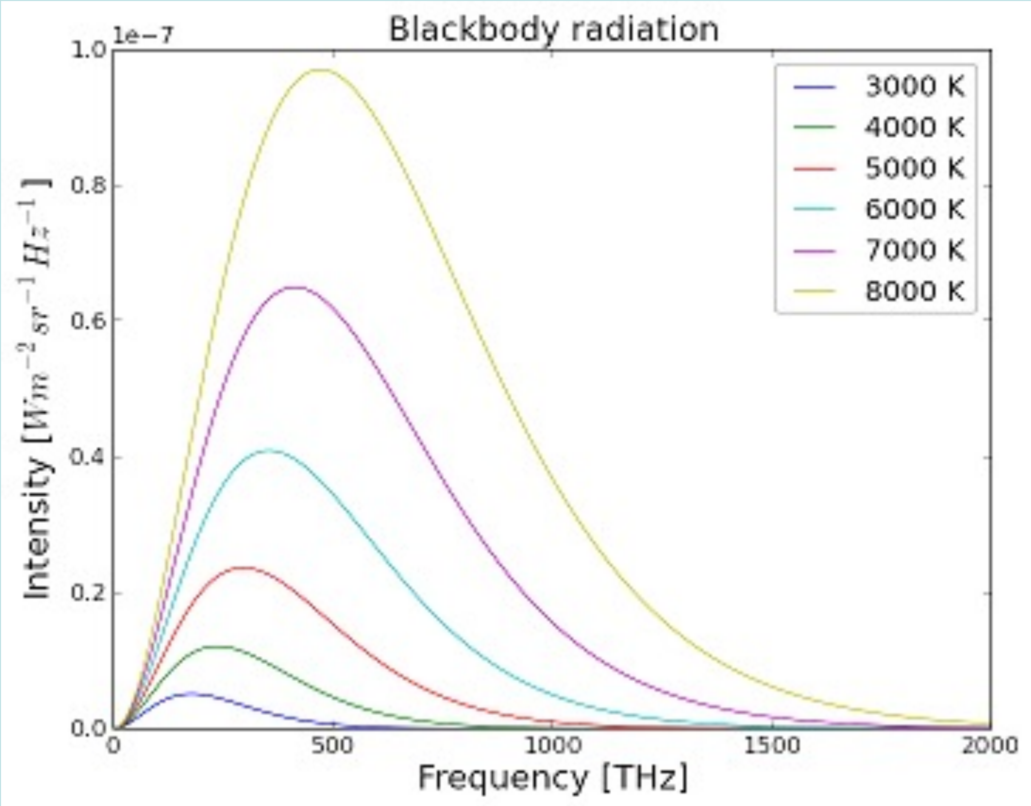
一個量子彈簧在溫度為T的環境中的能量平均值：

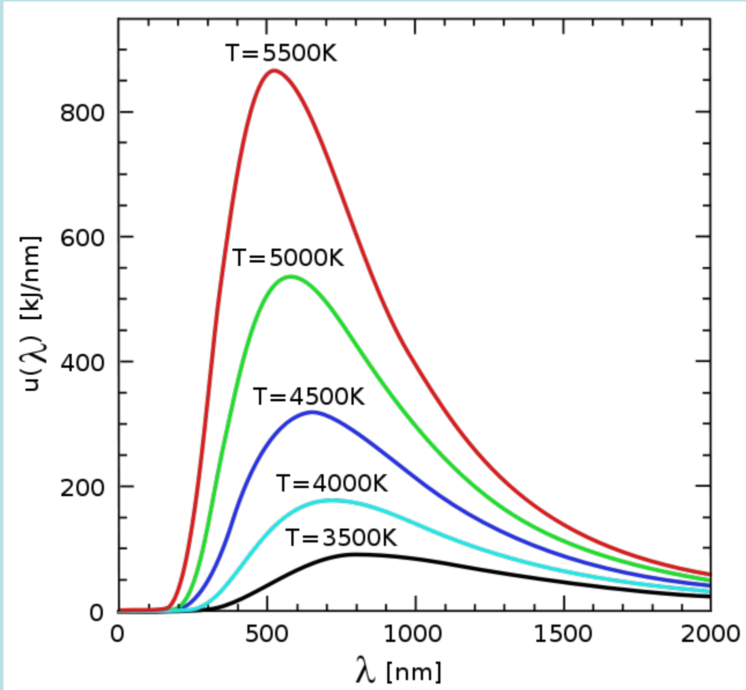
$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

相乘即為頻率介於ν及$\nu + d\nu$之間的量子彈簧即駐波的總能量為：

這對應頻率介於ν及$\nu + d\nu$之間的單位體積空腔輻射能量。

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$





$$E_{\text{黑、空}}(\nu, T) = \frac{c}{4} \cdot u = \frac{2\pi h \nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

Planck Blackbody Formula

二維空腔

單位體積內，頻率介於ν及$\nu + d\nu$之間的駐波狀態總數為

$$\frac{2\pi}{c^2}\nu d\nu$$

每一個駐波模式就對應一個量子彈簧，上式就是彈簧總數。

一個量子彈簧在溫度為T的環境中的能量平均值：

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

相乘即為頻率介於ν及$\nu + d\nu$之間的量子彈簧即駐波的總能量為：

這對應頻率介於ν及$\nu + d\nu$之間的單位體積空腔輻射能量。

$$u(\nu) = \frac{2\pi h}{c^2}\frac{\nu^2}{e^{h\nu/kT} - 1}$$

台灣聯合大學系統111學年度碩士班招生考試試題

類組：物理類　科目：近代物理(2003)　共 2 頁 第 1 頁

一、單選題：答案請填於答案卡。一題五分，答錯倒扣一分，整題不作答不給分也不扣分。倒扣至本大題(即單選題)0分為止。

1. The emission through a small hole on the surface of a cavity can simulate the black body radiation. If we assume that the cavity is a two dimensional cavity and everything else is the same, what would be the corresponding Stephan-Boltzman law? (T is the temperature, λ represents the wavelength) (A) total emissive power $\propto T^4$ (B) total emissive power $\propto T^3$ (C) total emissive power $\propto T^2$ (D) the wavelength of the maximum emissive spectrum satisfies $\lambda_{max}T =$ constant (E) the wavelength of the maximum emissive spectrum satisfies $\lambda_{max}T^2 =$ constant.

1. The total radiation energy per unit volume in the cavity is 變換變數，使積分變數無單位！

$$U(T) = \frac{8\pi h}{c^3} \int_0^\infty d\nu \, \frac{\nu^3}{e^{h\nu/kT} - 1} = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty dx \, \frac{x^3}{e^x - 1} \tag{1-7}$$

$$x \equiv \frac{h\nu}{kT} \qquad \nu = \frac{kT}{h} x$$

The integral can be evaluated

$$\int_0^\infty dx \, \frac{x^3}{e^x - 1} = \int_0^\infty dx \, x^3 e^{-x} \sum_{n=0}^\infty e^{-nx} = \sum_{n=0}^\infty \frac{1}{(n+1)^4} \int_0^\infty dy \, y^3 e^{-y}$$

$$= 6 \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{15}$$

and the result is the Stefan-Boltzmann expression

$$U(T) = aT^4 \tag{1-8}$$

with $a = 7.5662 \times 10^{-16}$ J/m^3·K^4. The T^4 dependence had been derived earlier by Boltzmann, using general thermodynamic arguments, but the numerical value, depending as it does on the value of h, was beyond the reach of classical physics.

2. The value of b in eq. (1-3) can be calculated and agrees with experiment.

$$\lambda_{max} = \frac{b}{T}$$

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

First find the ν_{max} to maximize $u(\nu)$. Defind $\frac{h\nu}{kT} = x$.

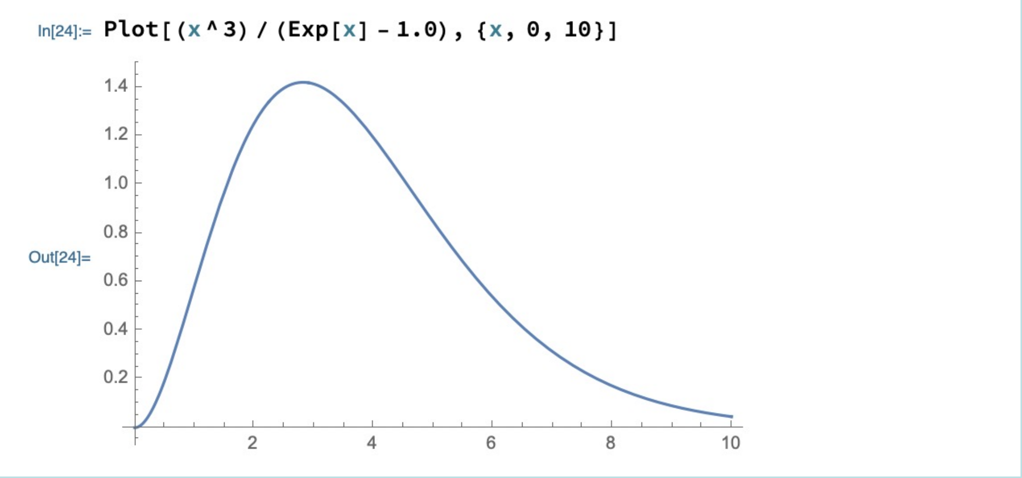
We only need to maximize:

$$x^3\frac{1}{e^x - 1}$$

$$3x^2(e^x - 1) - x^3e^x = 0$$

$$3(1 - e^{-x}) - x = 0$$

$$x_{max} = \frac{h\nu_{max}}{kT} \sim 2.822$$



2. The value of b in eq. (1-3) can be calculated and agrees with experiment.

$$\lambda_{\max} = \frac{b}{T}$$

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

The key is $\lambda_{\max}$ is to maximize $w(\lambda)$ instead of $u(\nu)$.

$$u(\nu, T) = w(\lambda, T)\cdot\left|\frac{d\lambda}{d\nu}\right| = w\left(\frac{c}{\nu}, T\right)\cdot\frac{c}{\nu^2}$$

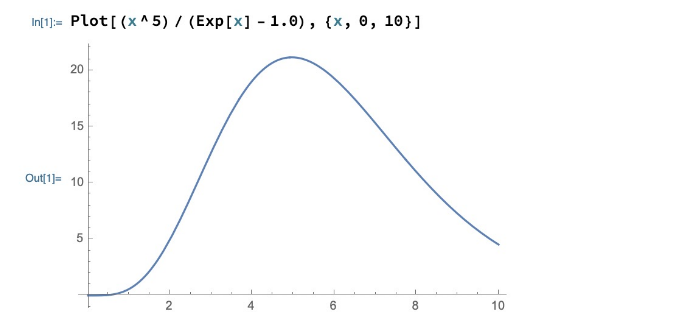
$$w(\lambda, T) = u(\nu, T)\cdot\frac{\nu^2}{c} = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{k\lambda T}} - 1} = 8\pi hT^5c\cdot\frac{1}{(\lambda T)^5}\frac{1}{e^{\frac{hc}{k\lambda T}} - 1}$$

Define: $\frac{hc}{k\lambda T} = x$. We only need to maximize:

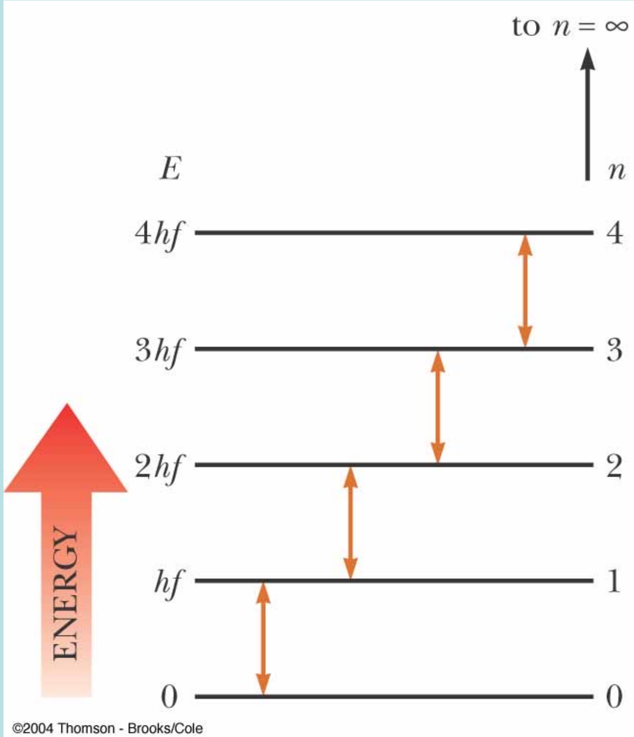
$$x^5\frac{1}{e^x - 1}$$

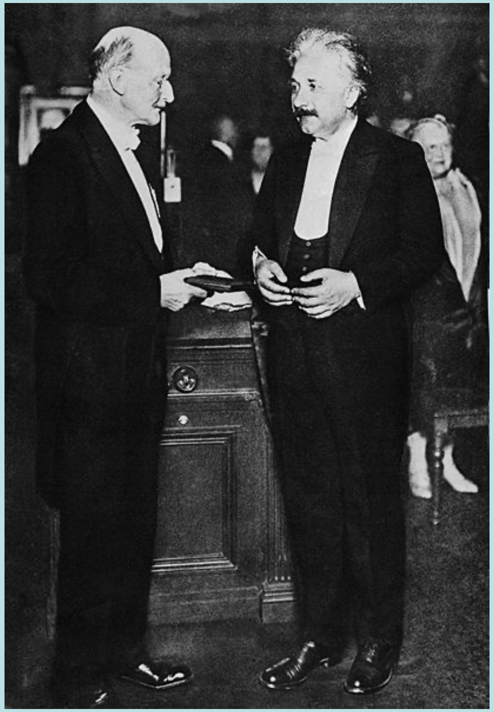
$$5x^4(e^x - 1) - x^5e^x = 0$$

$$5(1 - e^{-x}) - x = 0$$



$$\lambda_{\max}T = \frac{hc}{x_{\max}k} \equiv b$$

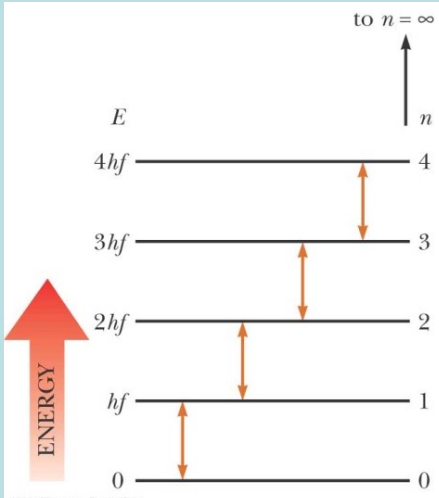




Max Planck 1900

量子 Quantum (Quanta)

$$E_n = n \cdot hf$$



在微觀世界中有許多物理量的值只能是一個最小量或此最小量的整數倍。

這個最小量稱為量子，此物理量即稱被量子化 Quantized。

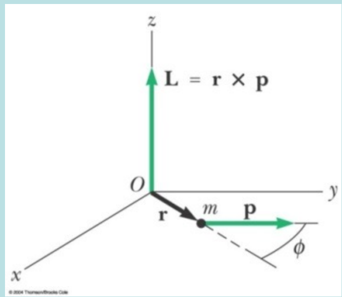
此物理量可以想像是由整數個量子所組成。

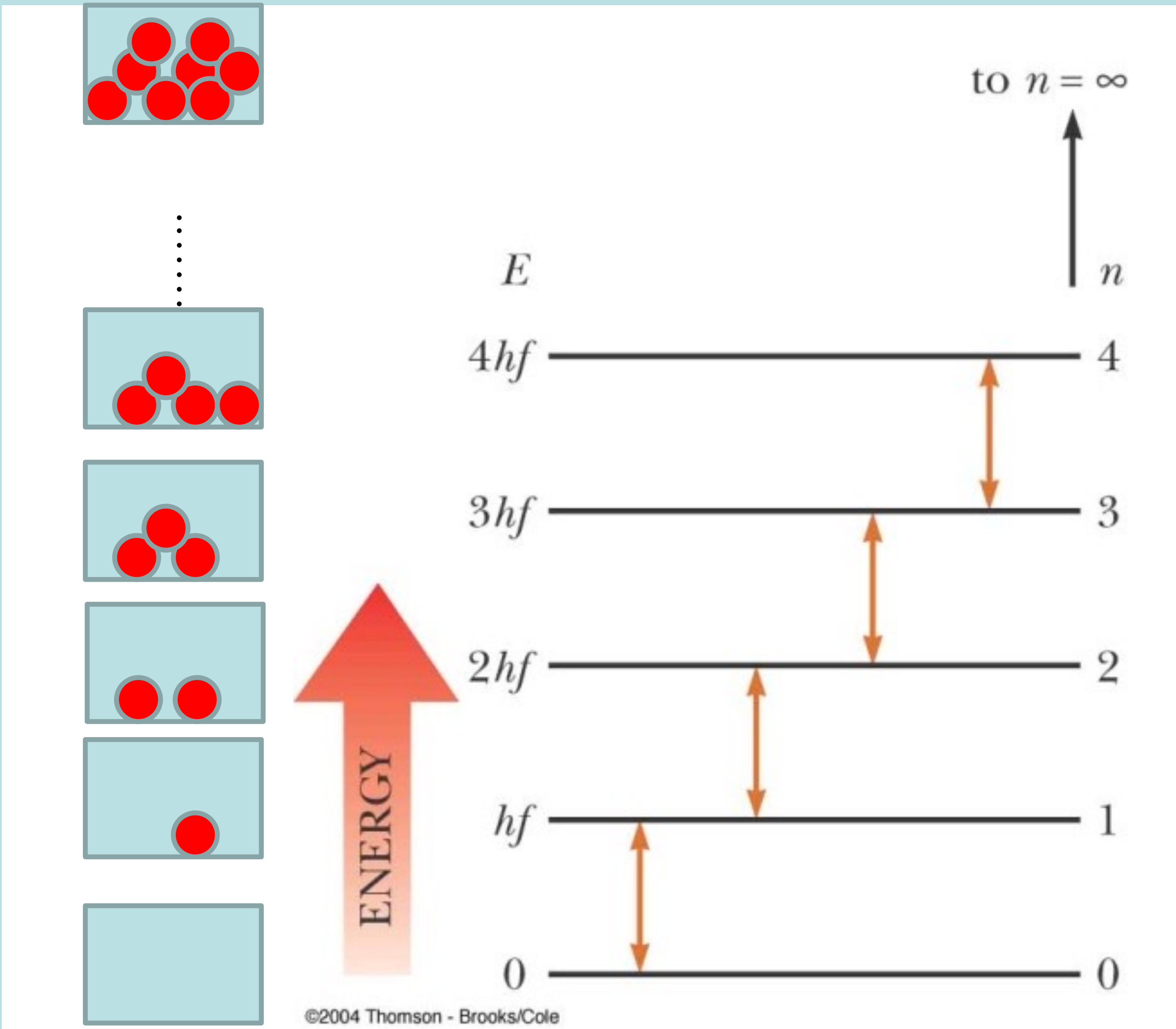
而量子是無法分割的。　這在古典物理中是完全無法想像的！

有時量子化的型式會比較複雜一點：如角動量的大小

$$L^2 = l(l+1)\hbar^2$$

但基本上量子化的測量值指的是非連續的、離散的。





這個量子彈簧的能階像極了在盒子中一個一個裝入能量相同的粒子。

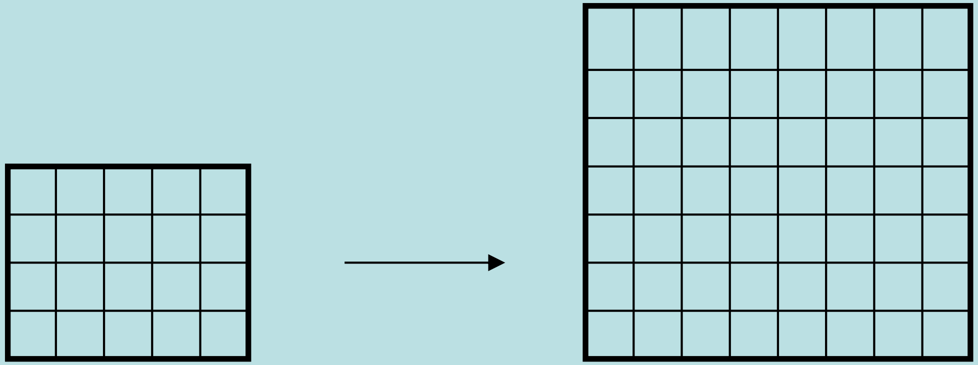
量子 → 粒子　　粒子數 $n = \dfrac{E}{hf}$

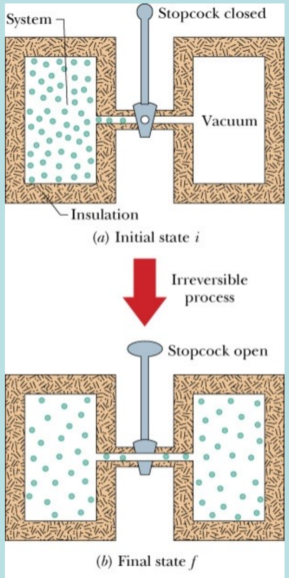
針對空腔中的輻射駐波，由光譜可以計算出輻射的熵值，

愛因斯坦發現如果假設粒子數為： $N = \dfrac{E}{hf}$

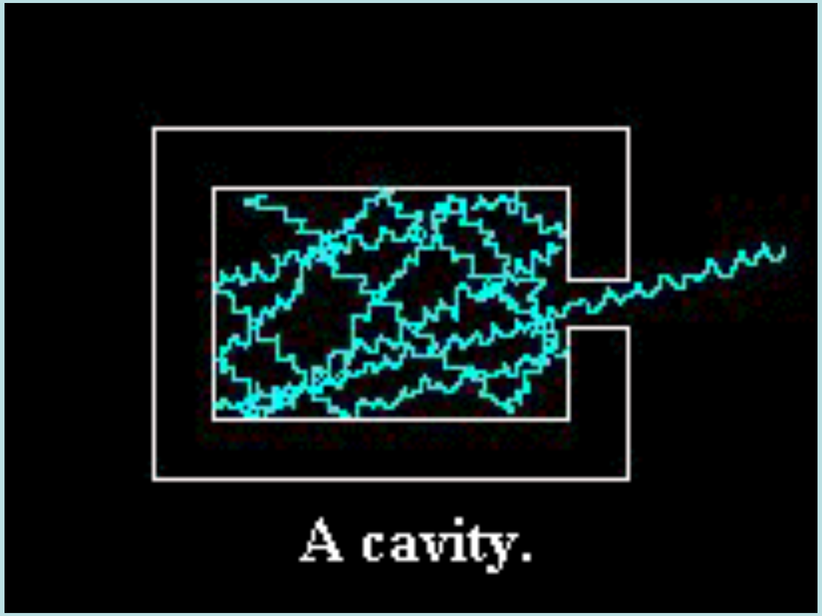
愛因斯坦發現量子彈簧組成的空腔輻射的熵正是理想氣體的熵值：

$$\Delta S = Nk\ln\left(\frac{V_f}{V_i}\right)$$

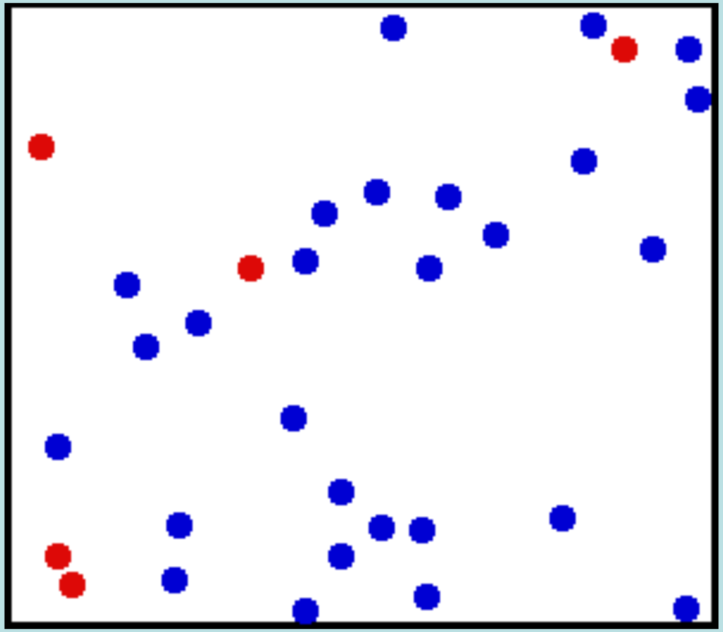




空腔中的輻射量子可以視為粒子！



空腔中的輻射量子可以視為粒子！



只是這個粒子不彼此碰撞

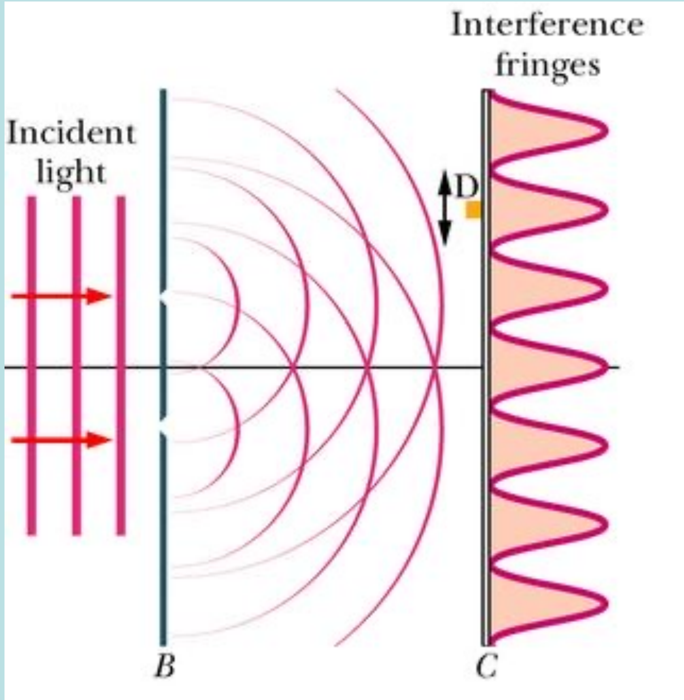
光是什麼？

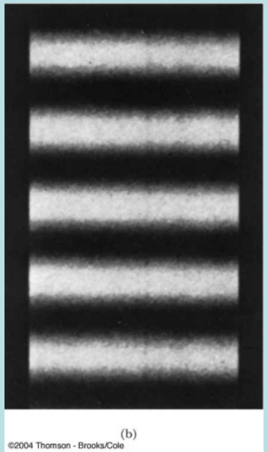
光是直進！就如同粒子一樣！

光的粒子說

我希望我們能用相同的（力學）原則推導出自然界所有的現象。

達文西，惠更斯：光傳播如此之快，不可能是粒子。



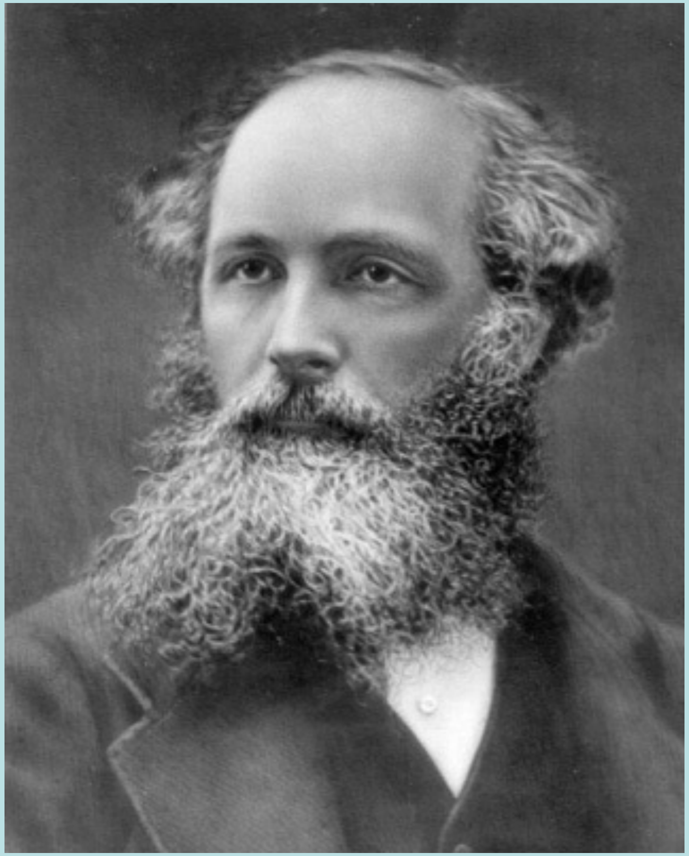


因為光有干涉現象，光是波！　　楊 1804

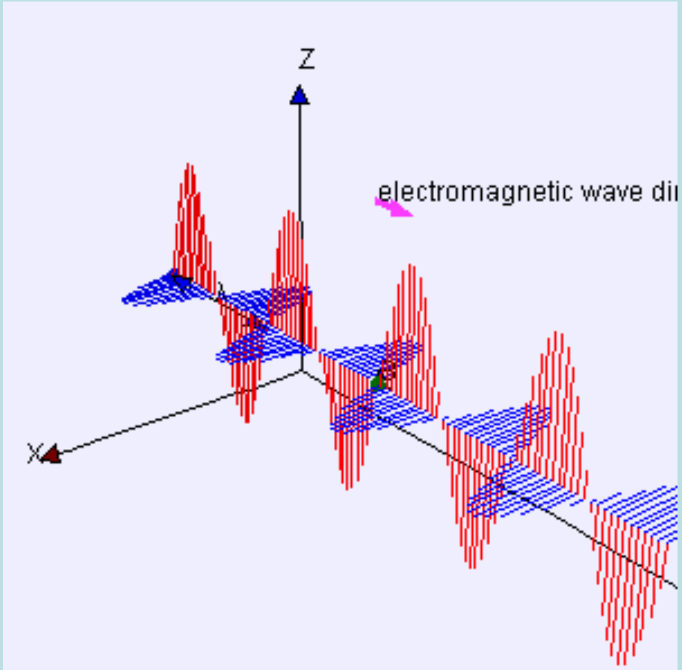
馬克斯威爾推導出光是電磁場的波，將電磁場的擾動傳播到遠方的波。

傳播的獨立的電磁場就是一種波！

電磁波：電與磁的自動化

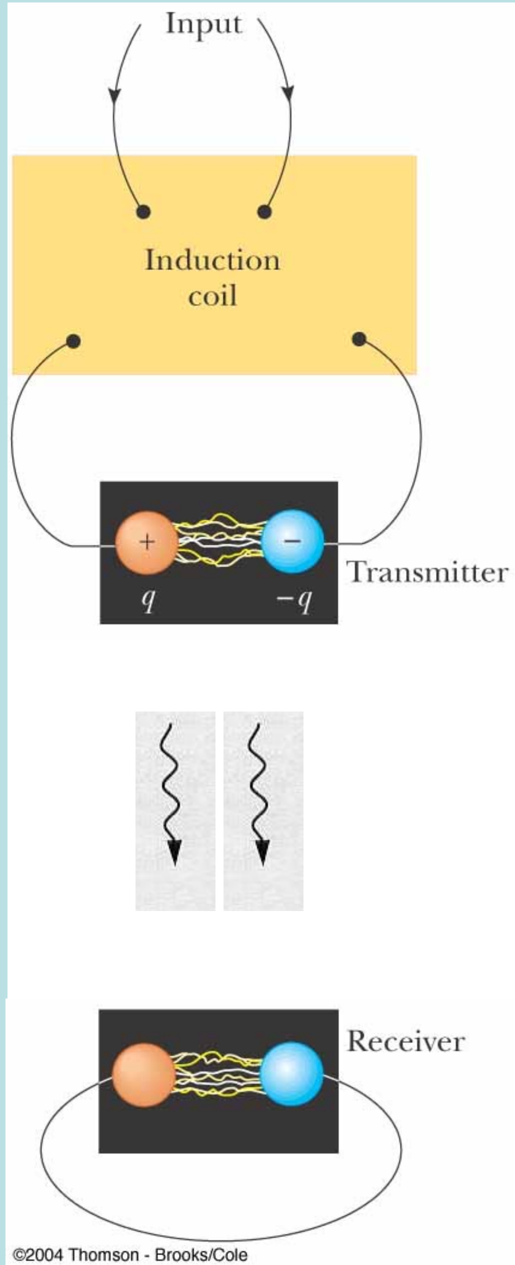


James Clerk Maxwell (1831–1879)



many thanks to Prof. Fu-Kwun Hwang

赫茲在實驗室中第一次人為地製造出電磁波。



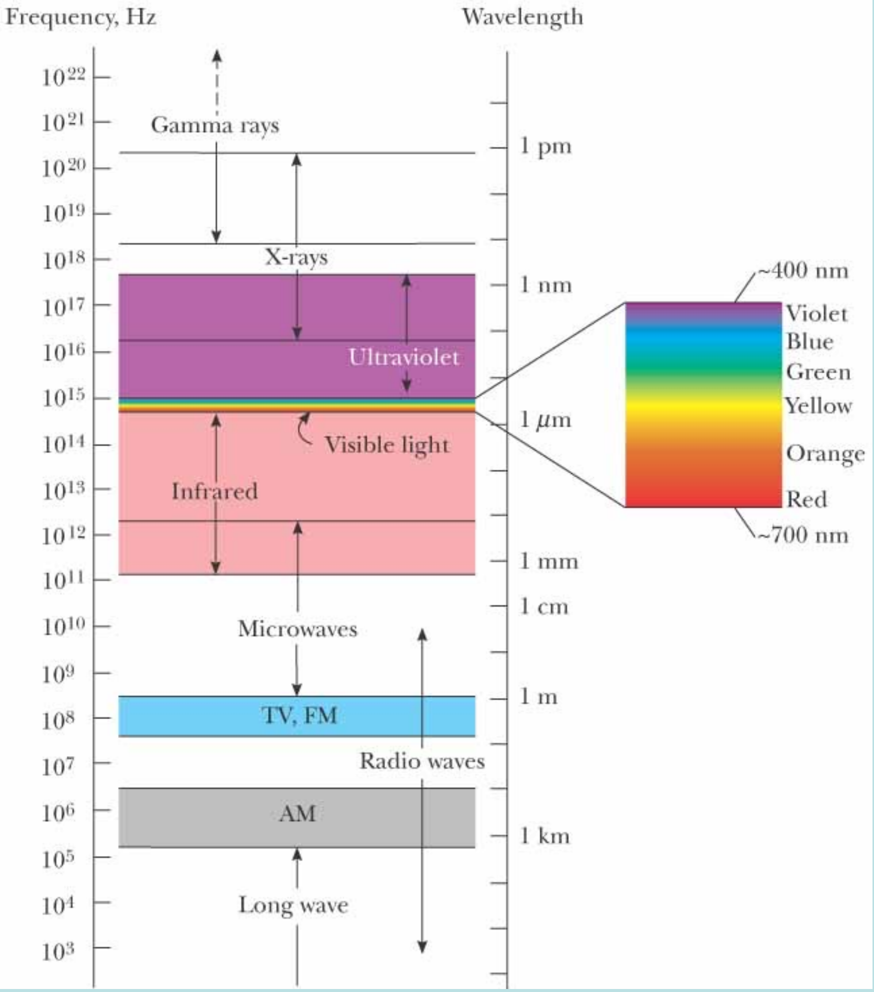
要產生電磁波，就要製造變化的電場或磁場。

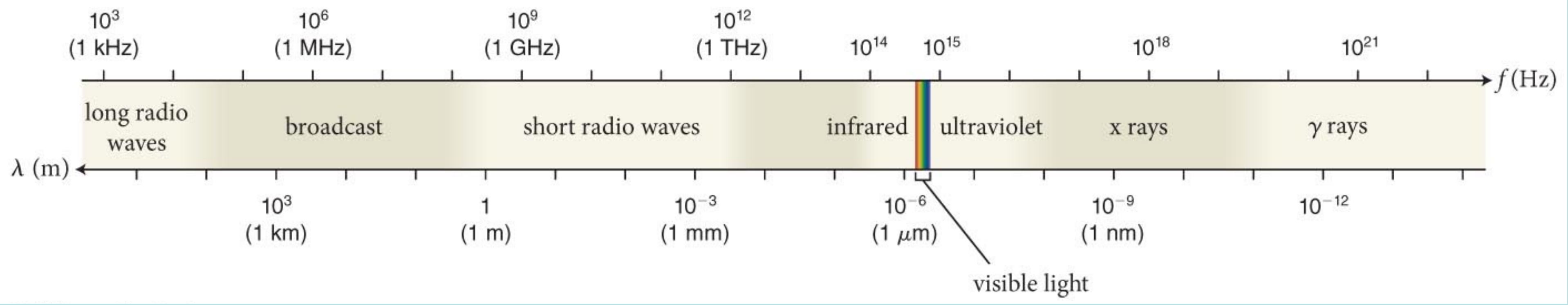
最簡單的方法就是製造變化的電流。LC電路。

電磁波以頻率或波長為特徵

$$\lambda f = c$$

天線的大小大致與波長相當。

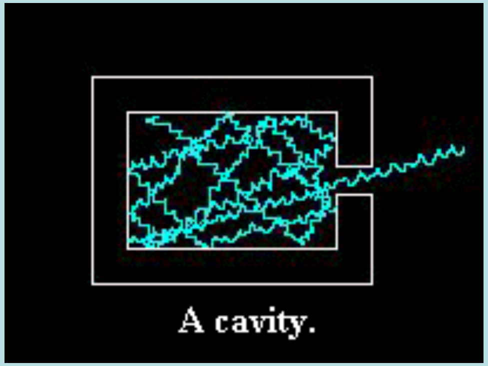


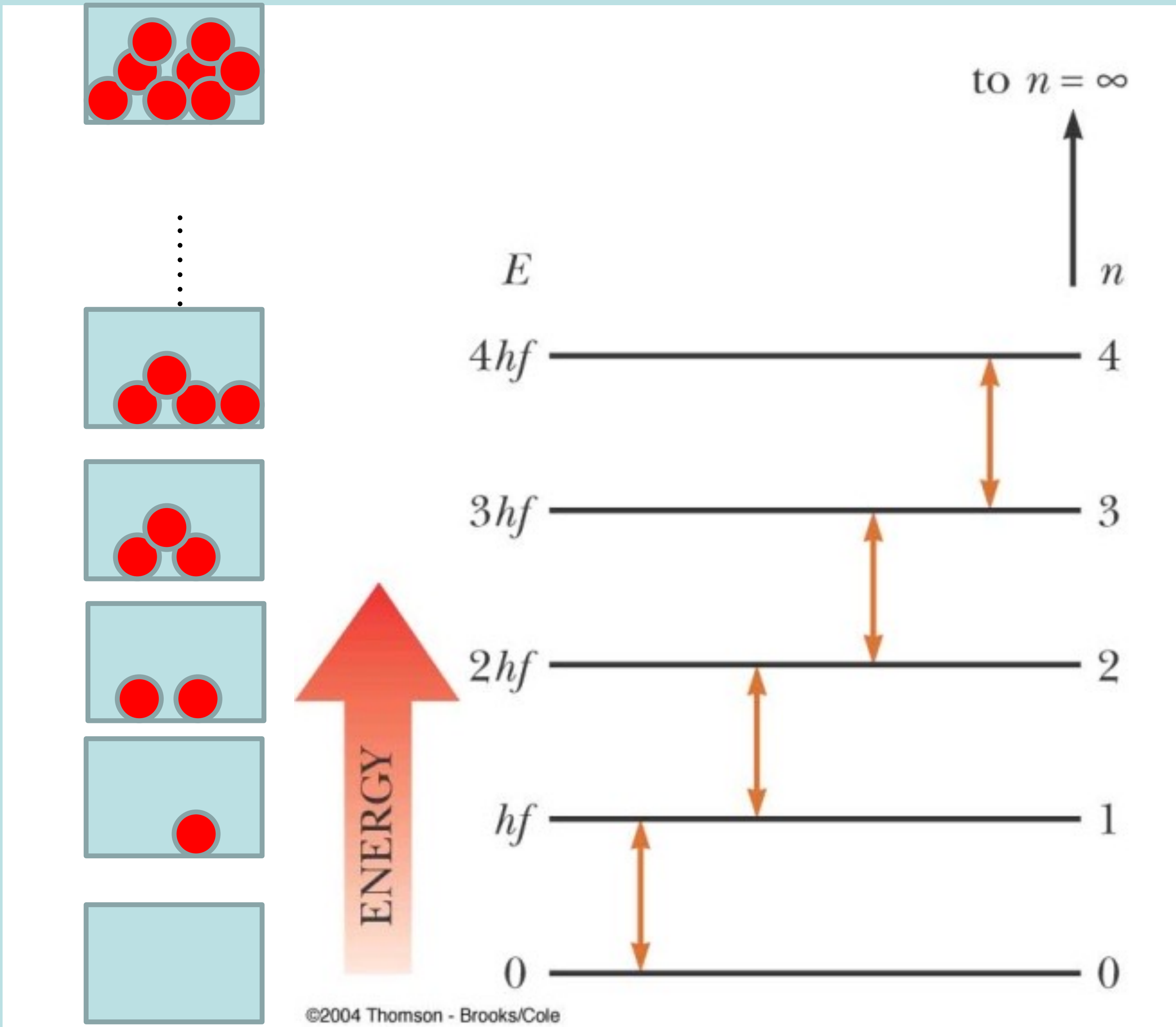


波浪的能量由振幅決定

連續地調整振幅，即可連續地調整能量

電磁波應該也是如此。





但空腔中的電磁波能量卻是量子化的！

難道只有空腔中的電磁波才有量子化的能量？

3 On a Heuristic Point of View about the Creation and Conversion of Light†

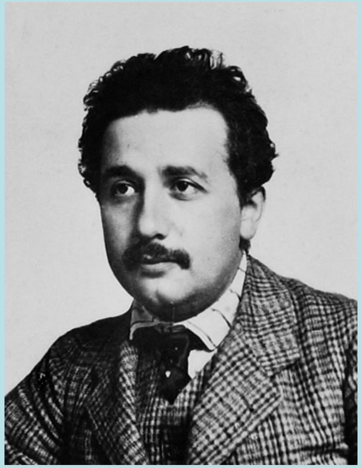
A. EINSTEIN

THERE exists an essential formal difference between the theoretical pictures physicists have drawn of gases and other ponderable bodies and Maxwell's theory of electromagnetic processes in so-called empty space. Whereas we assume the state of a body to be completely determined by the positions and velocities of an, albeit very large, still finite number of atoms and electrons, we use for the determination of the electromagnetic state in space continuous spatial functions, so that a finite number of variables cannot be considered to be sufficient to fix completely the electromagnetic state in space. According to Maxwell's theory, the energy must be considered to be a continuous function in space for all purely electromagnetic phenomena, thus also for light, while according to the present-day ideas of physicists the energy of a ponderable body can be written as a sum over the atoms and electrons. The energy of a ponderable body cannot be split into arbitrarily many, arbitrarily small parts, while the energy of a light ray, emitted by a point source of light is according to Maxwell's theory (or in general according to any wave theory) of light distributed continuously over an ever increasing volume.

The wave theory of light which operates with continuous functions in space has been excellently justified for the representation of purely optical phenomena and it is unlikely ever to be replaced by another theory. One should, however, bear in mind that optical observations refer to time averages and not to

† *Ann. Physik* **17**, 132 (1905).

愛因斯坦不這樣想！



In fact, it seems to me that the observations on "black-body radiation", photoluminescence, the production of cathode rays by ultraviolet light and other phenomena involving the emission or conversion of light can be better understood on the assumption that the energy of light is distributed discontinuously in space. According to the assumption considered here, when a light ray starting from a point is propagated, the energy is not continuously distributed over an ever increasing volume, but it consists of a finite number of energy quanta, localised in space, which move without being divided and which can be absorbed or emitted only as a whole.

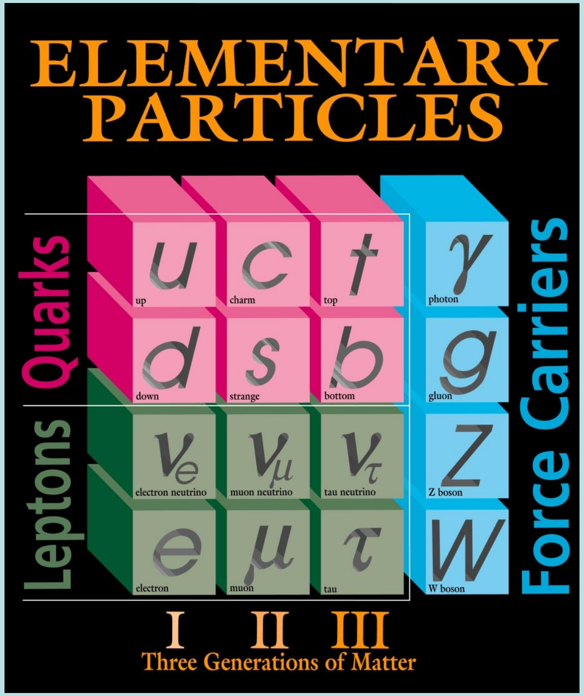
8. On the Production of Cathode Rays by Illumination of Solids

The usual idea that the energy of light is continuously distributed over the space through which it travels meets with especially great difficulties when one tries to explain photo-electric phenomena, as was shown in the pioneering paper by Mr. Lenard.[3]

According to the idea that the incident light consists of energy quanta with an energy $R\beta\nu/N$, one can picture the production of cathode rays by light as follows. Energy quanta penetrate into a surface layer of the body, and their energy is at least partly

光電效應 Photoelectric Effect

光子 **Photon** γ

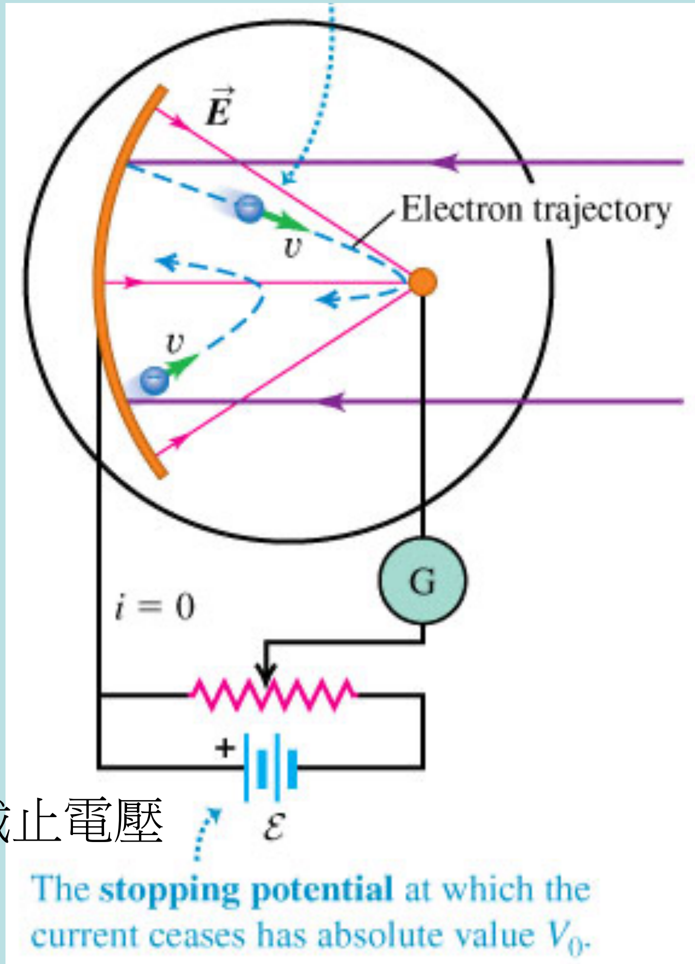


$$E = hf$$

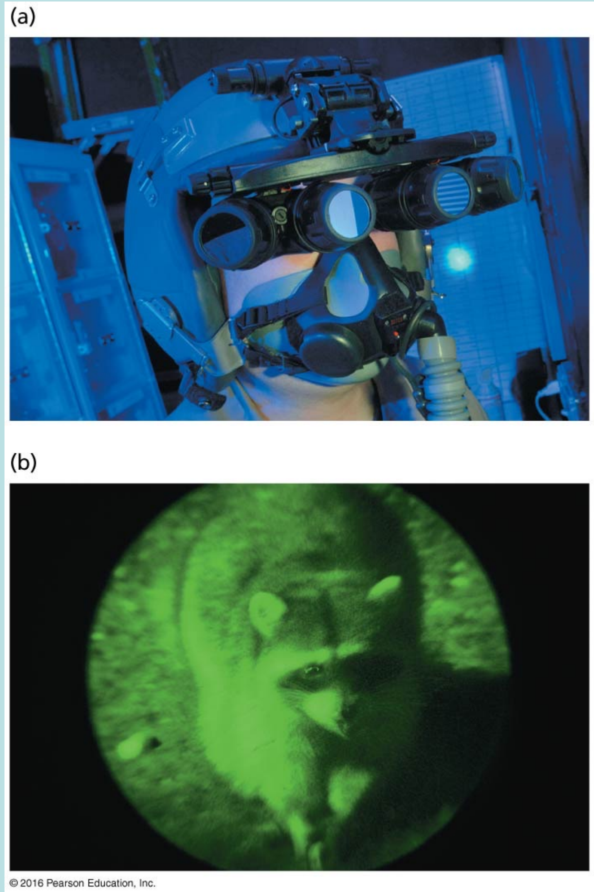
$$p = \frac{h}{\lambda}$$

$$h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s} = 4.136 \times 10^{-15}\ \mathrm{eV \cdot s}$$

光電效應 Photoelectric Effect



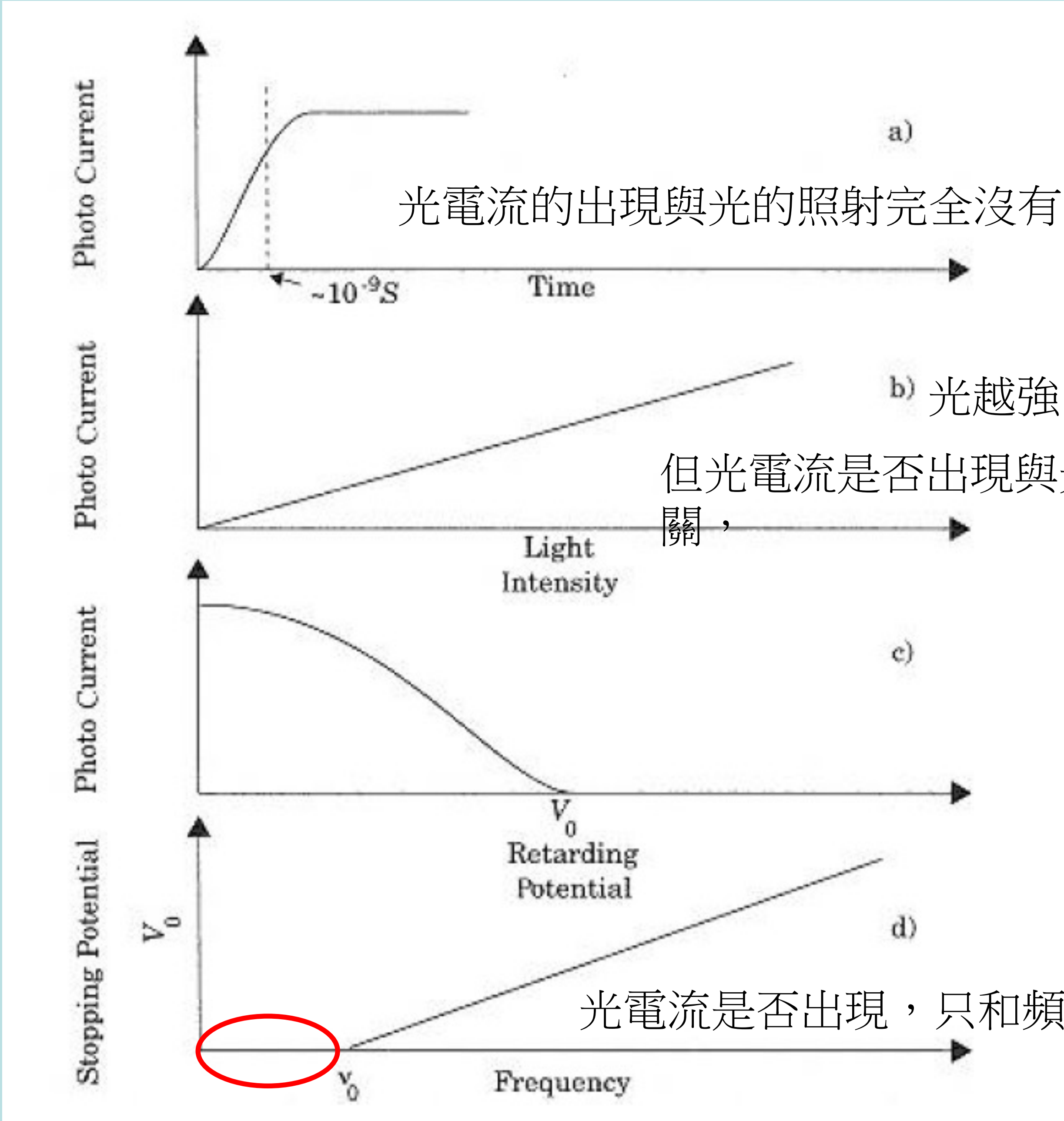
截止電壓



光照射於電極上，可以撞擊出電子產生光電流。

進一部的實驗加上一個截止電壓以抑制光電流，

測得的截止電壓就等於電子的動能！

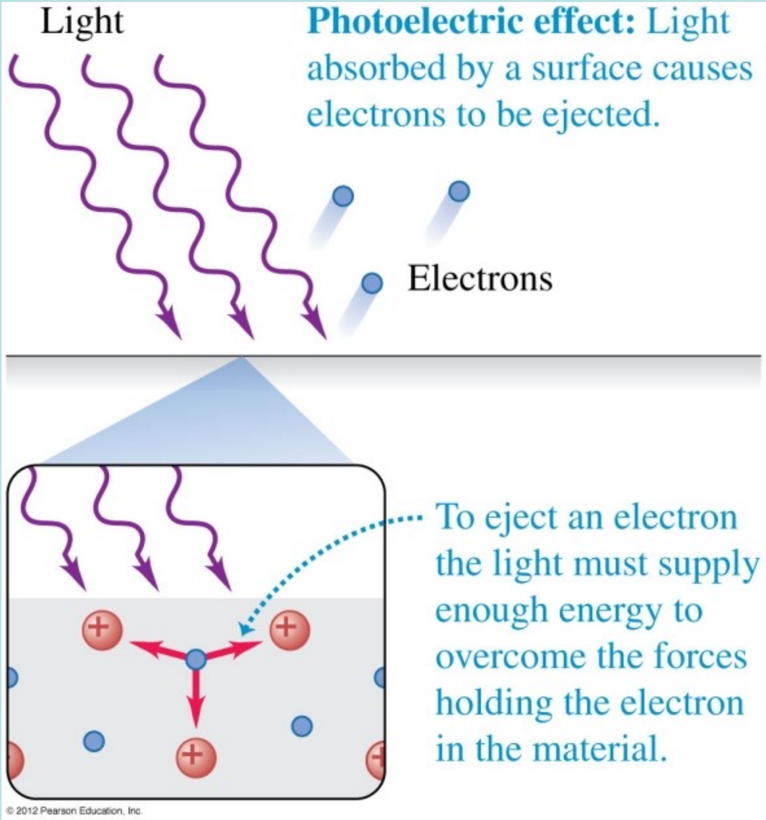


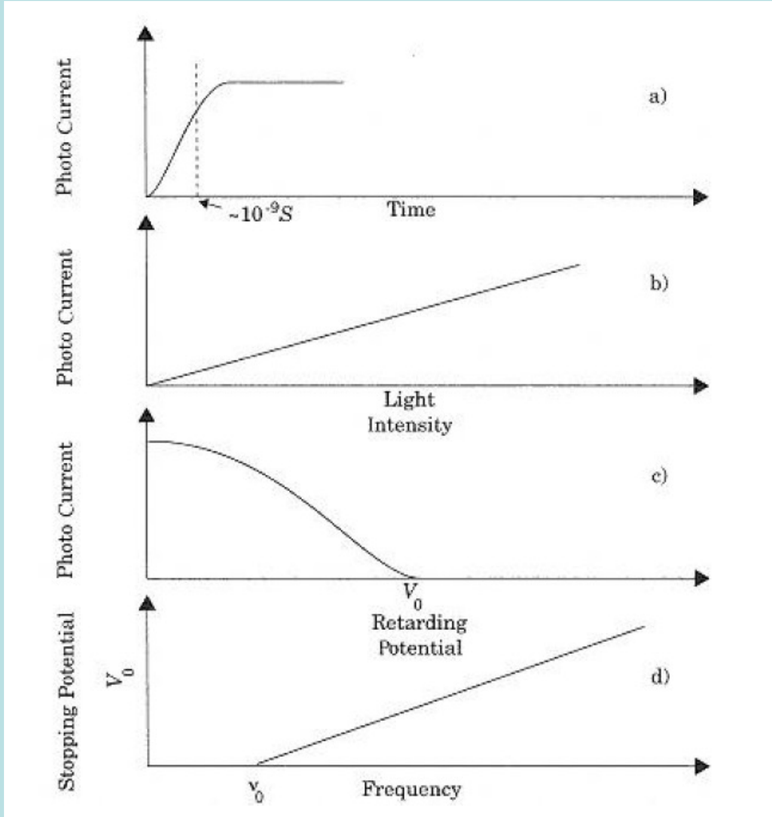
光電流的出現與光的照射完全沒有時間差。

光越強，光電流越強。

但光電流是否出現與光的強度無關，

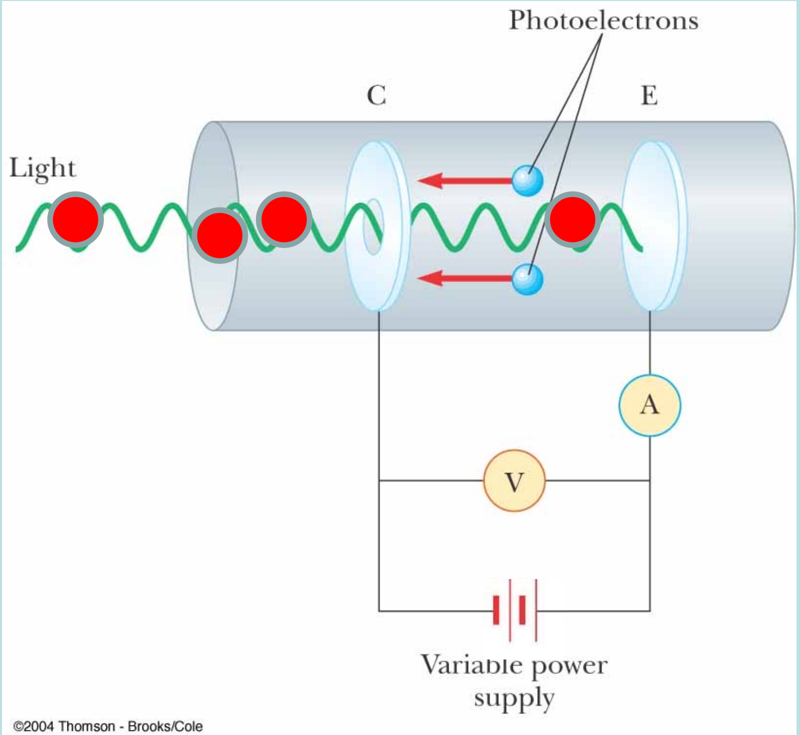
光電流是否出現，只和頻率有關。

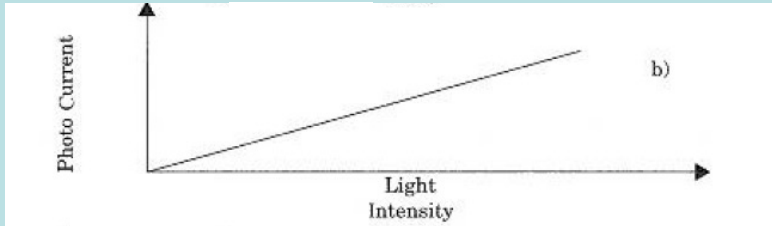


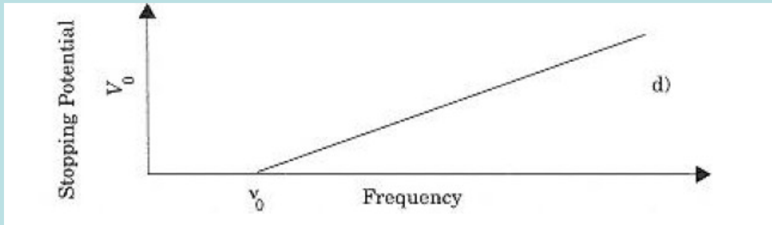


光電流是否出現與光的強度無關，只和頻率有關。

這樣的結果，連續的波動理論是很難解釋的







但如果假設光與物質交換能量時是以一次一顆固定粒子的形式進行：

當一個光量子的能量不足以克服離開電極所需的能量，能量就完全不被吸收。

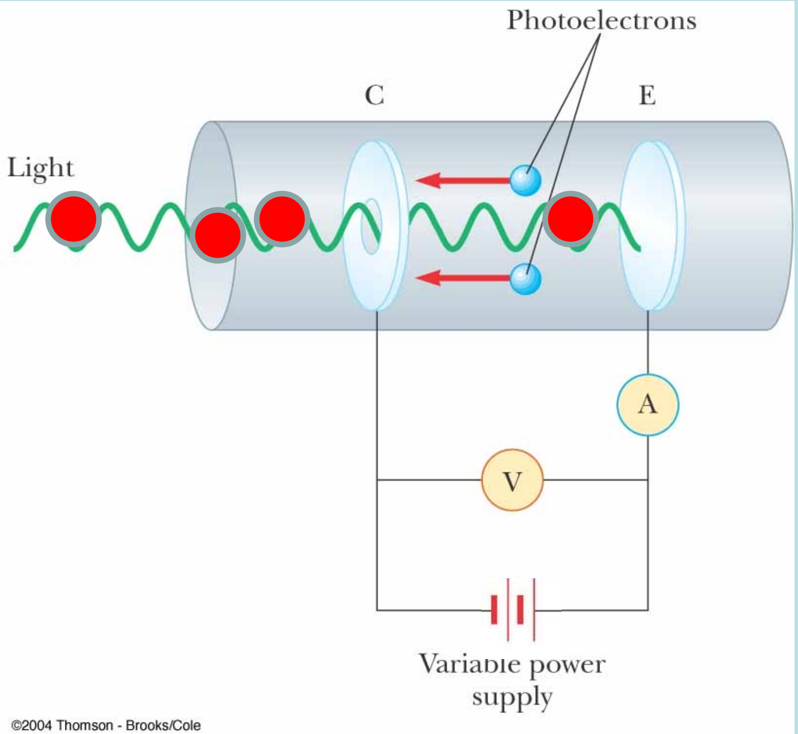
光電流就不會出現，這樣的光即使再強，都無法產生電流。

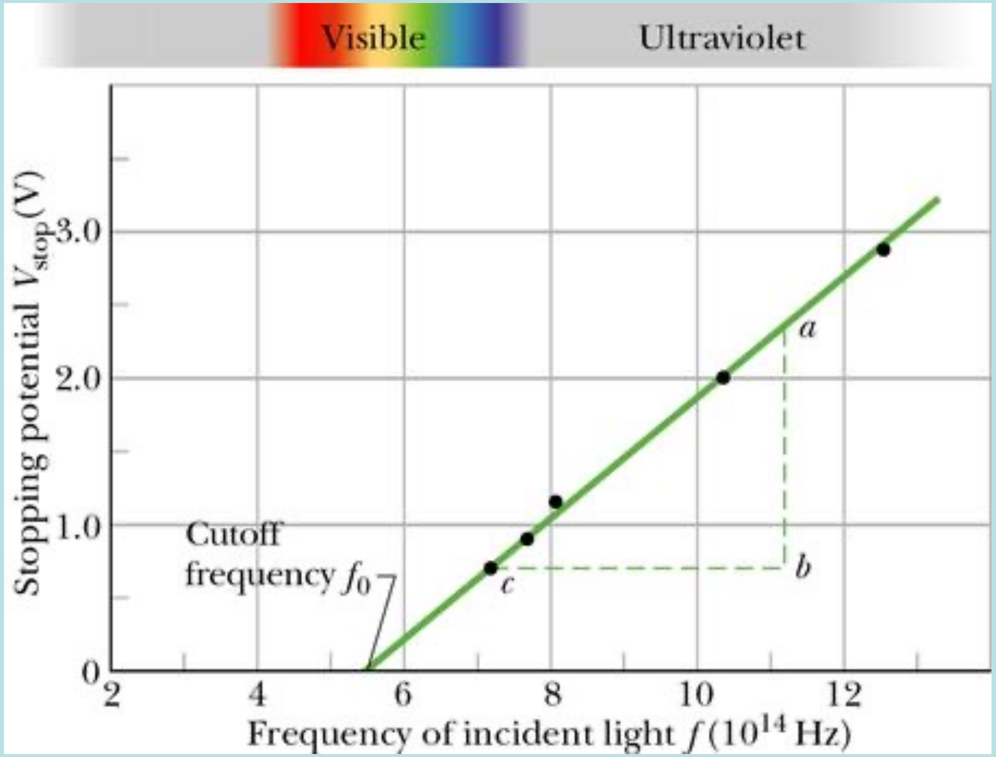
光的強度只決定光粒子的數量，只會影響光電流一旦出現時的大小。

若一個光量子的能量由頻率決定，光電流是否出現，自然只和頻率有關。

$$E = hf$$

將截止電壓 V_{stop} 對光的頻率作圖，呈線性關係！ eV_{stop} 即是一個光電子的最大動能。





$$eV_{stop} = E_e = hf - W$$

光交給電子的能量是一個固定不可分割的值，稱為光量子 $E = hf$

W是光電子離開電極所需克服的最小位能：Work Function。 $W = hf_0$

一個光量子的能量由頻率決定，因此光量子能量低於W即無法打出電子。

直線的斜率即是 Planck常數h。這是最容易測蒲朗克常數的辦法。

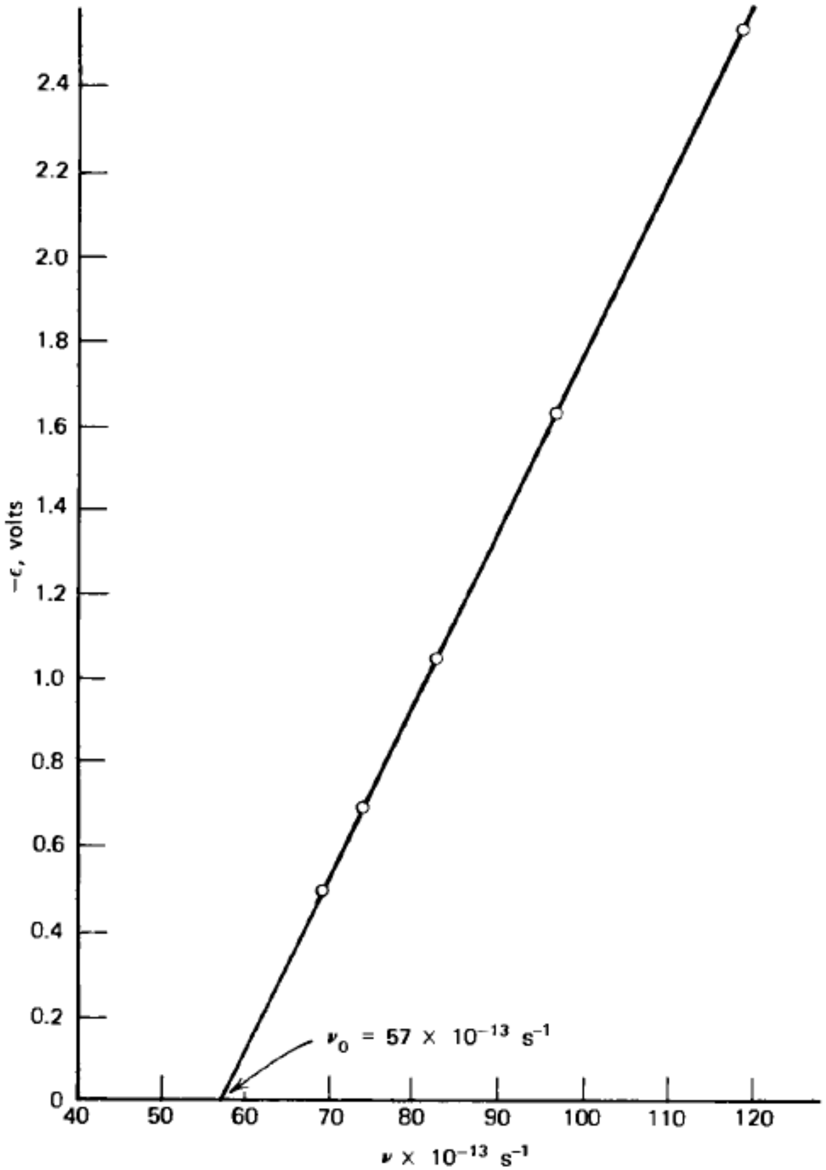
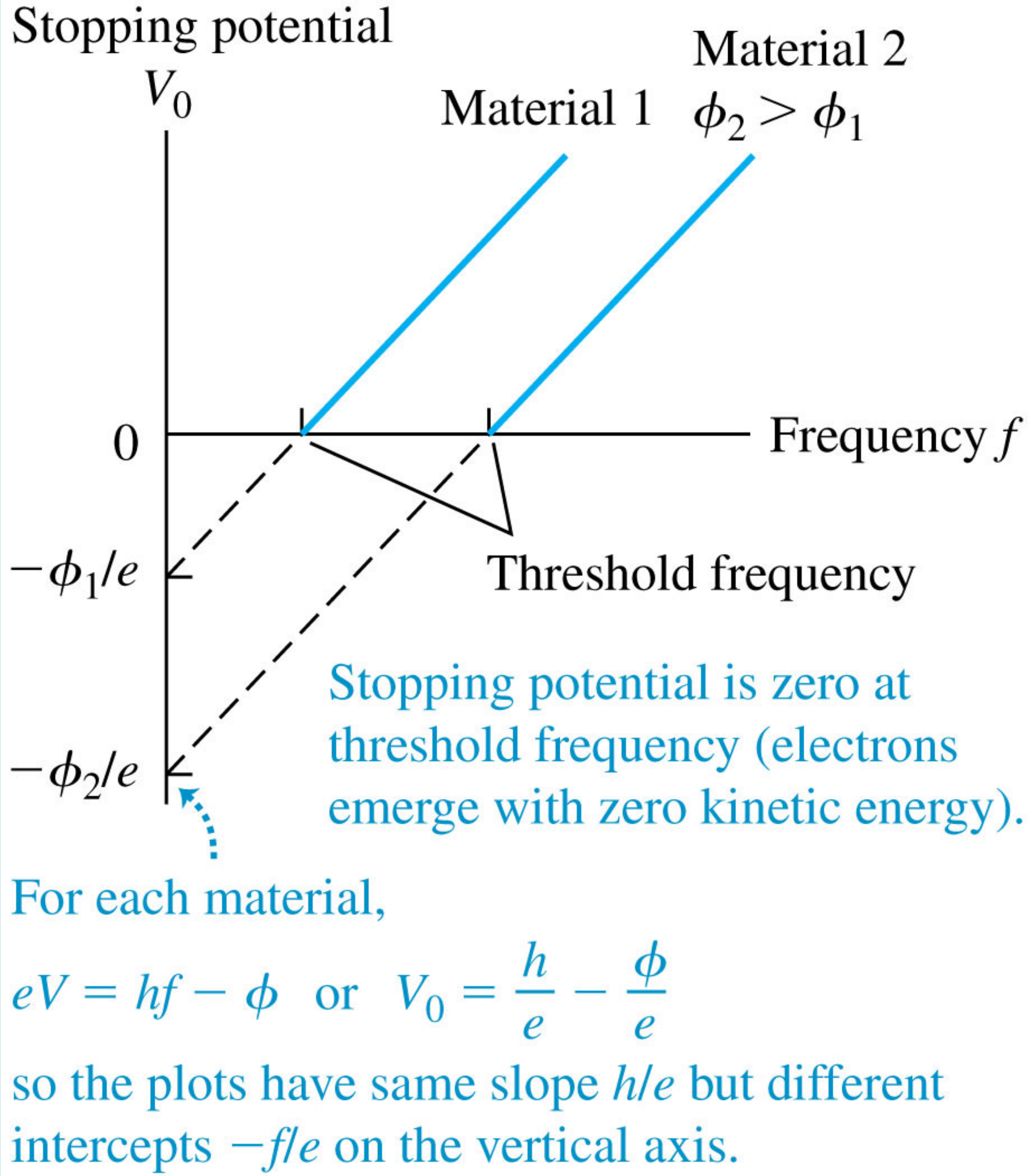
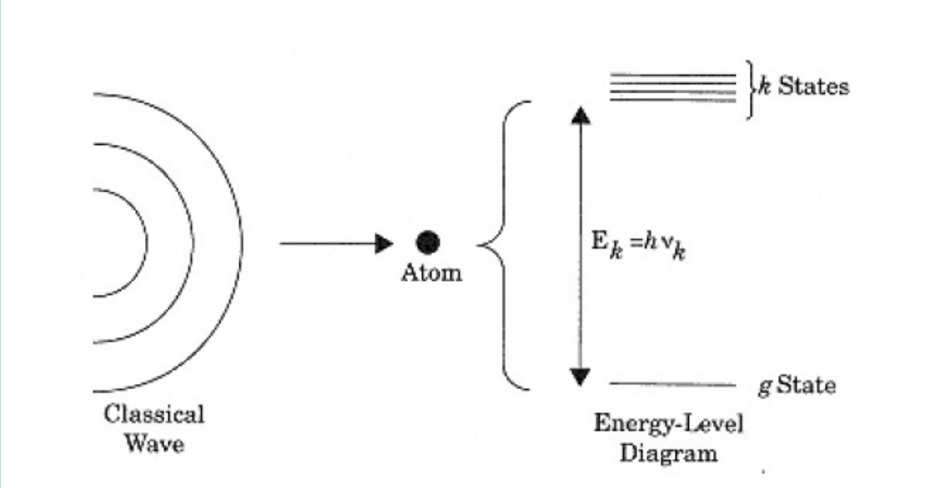


Figure 1-3 Photoelectric effect data showing a plot of retarding potential necessary to stop electron flow from a metal (lithium), or equivalently, electron kinetic energy, as a function of frequency of the incident light. The slope of the line is *h*/*e*.


Stopping potential
V_0
Material 1
Material 2
$\phi_2 > \phi_1$
0
Frequency f
Threshold frequency
$-\phi_1/e$
$-\phi_2/e$
Stopping potential is zero at threshold frequency (electrons emerge with zero kinetic energy).
For each material,
$eV = hf - \phi$ or $V_0 = \frac{h}{e} - \frac{\phi}{e}$
so the plots have same slope h/e but different intercepts $-f/e$ on the vertical axis.



Thus all four aspects of the photoelectric effect are neatly accounted for by Einstein's theory. But Lamb and Scully showed that this theory, while feasible, is not the only possible one. They were able to find an entirely different theory of the photoelectric effect, one that did not invoke the concept of the particle nature of light at all. Their conclusion was that the photoelectric effect does not constitute proof of the existence of photons.

The theory of Lamb and Scully treated atoms quantum-mechanically, but regarded light as being a purely classical electromagnetic wave with no particle properties. In such a "semiclassical" theory, the atom was quantized in the usual way into energy levels according to the Schrödinger equation. These energy levels were simplified to a ground state g and a series of free-electron states k that formed a continuum (see Figure 2-2). The atom interacted with a classical time-varying electromagnetic field, which they wrote as a single-frequency sinusoidal wave (monochromatic light),

$$E = E_0 \cos \omega t \qquad (2.2)$$

This electromagnetic wave was treated as a perturbation, whose interaction potential with the atom was given in the dipole approximation by

$$V(t) = -eE\,x(t) \qquad (2.3)$$

We recognize $-eE$ as the force on the electron that causes its ejection; $V(t)$ is the time-dependent potential associated with that force.

Using standard methods of time-dependent perturbation theory in quantum mechanics, Lamb and Scully found the following expression for the probability that the perturbing field causes a transition from the ground state g to an excited state k—i.e., that the incident light ionizes the atom and liberates the electron:

$$P_k(t) = \frac{4\left|X_{kg}\frac{eE_0}{2\hbar}\right|^2 \sin^2\left[\left(\frac{E_k}{\hbar}-\omega\right)\left(\frac{t}{2}\right)\right]}{\left(\frac{E_k}{\hbar}-\omega\right)^2} \qquad (2.4)$$

Here X_{kg} is the matrix element of x between the two states and E_k is the energy of the kth state measured relative to that of the ground state.

This result represents the resonance condition for excitation; excitation only occurs when the incoming frequency ω closely matches that required by the energy-level separation, $\omega_k = E_k/\hbar$. As can be seen, the denominator in Equation (2.4) becomes zero for this value; until the light frequency reaches ω_k, no electron will be ejected, while above that frequency electrons will appear. In this way one can account for the threshold phenomenon, which is just the work function of the metal used. Equation (2.1), therefore, can be thought of as a natural consequence of the resonance condition for excitation by an electromagnetic wave, rather than a reflection of microscopic energy conservation for light, as proposed by Einstein.

The second aspect of the photoelectric effect, that the photocurrent is proportional to the light intensity, is similarly accounted for. The intensity of the light is proportional to E_0^2. But from Equation (2.4), the probability of electron emission is just proportional to E_0^2.

The first property of the photoelectric effect, that electrons are emitted immediately after the onset of illumination, is treated as follows. Equation (2.4) represents the probability for a transition to *one* of the continuum levels k. But the probability of emission of a photoelectron is the probability of a transition to *any* such level, which is the sum of Equation (2.4) over all k. Lamb and Scully did this sum, and showed that the transition probability was proportional to the time. This implied that the transition rate was constant, so that even at short times electrons would be emitted.

Thus all aspects of the photoelectric effect were accounted for without resort to Einstein's Nobel Prize-winning argument. Moreover, the random, unpredictable character of individual quantum-mechanical events was properly preserved, but in this model, was due not to the quantum nature of light, but to that of matter.

電磁波及光都帶有能量也帶有動量！

$$p = \frac{E}{c}$$

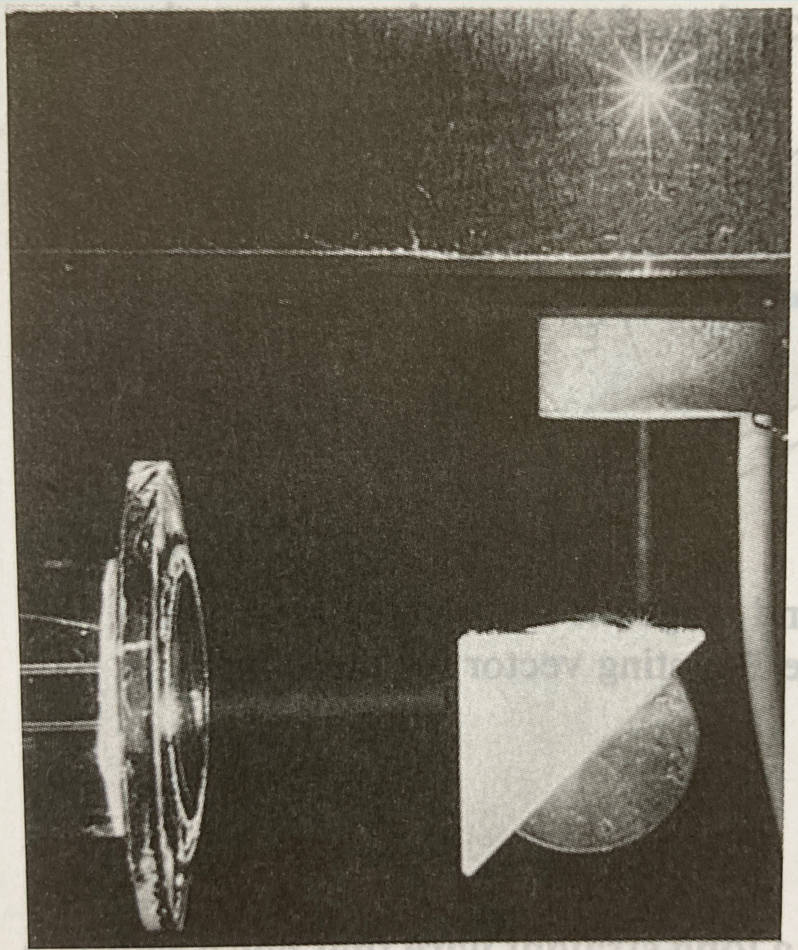


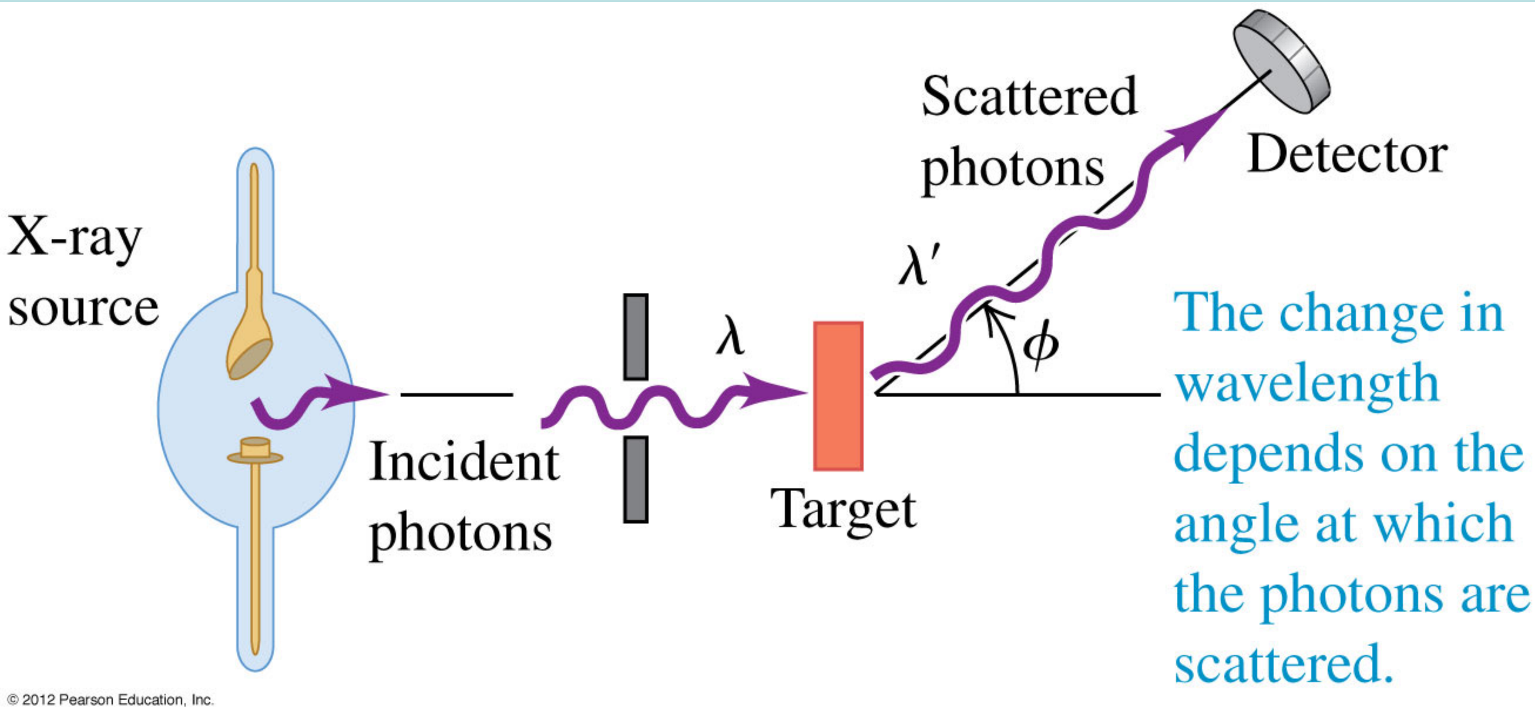
FIGURE 34.10 A tiny particle suspended by the pressure of light from a laser.

如果光是粒子應該可以碰撞。

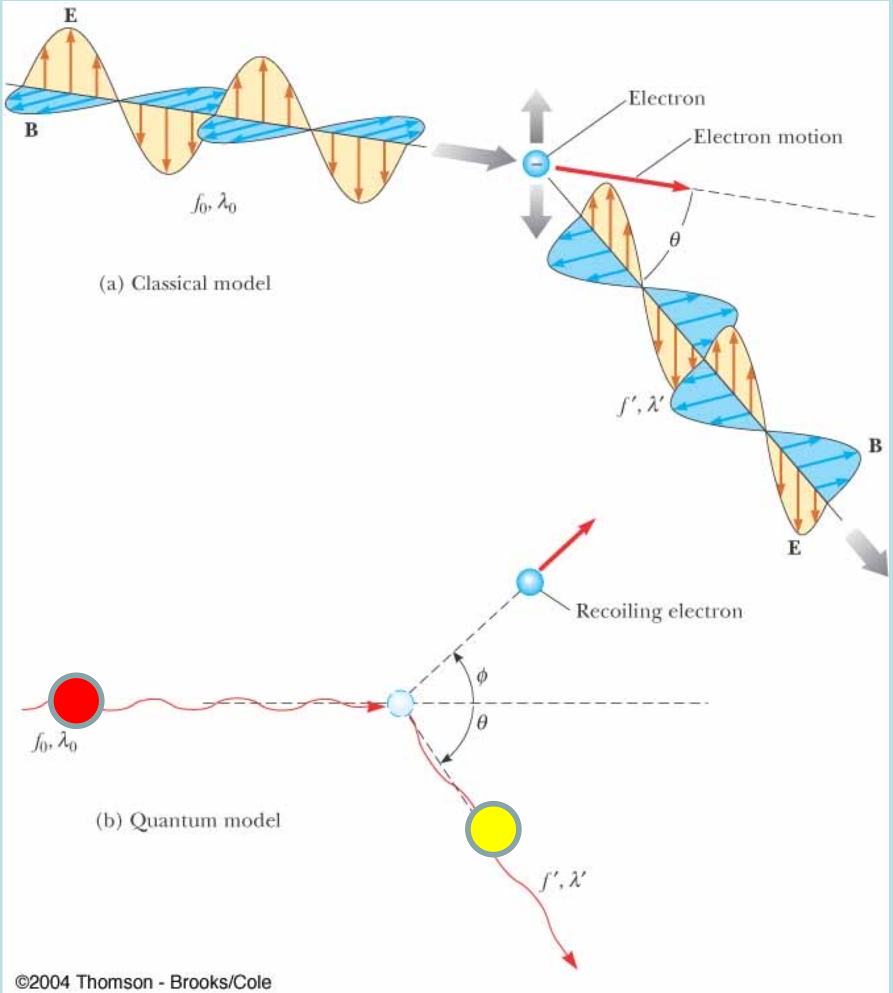
光量子如果像一個粒子，那麼就具有動量。

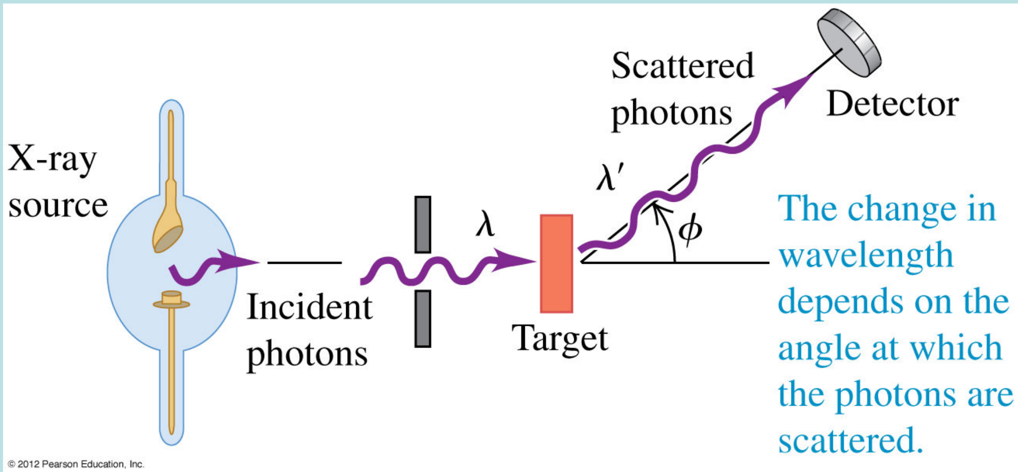
$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$ 光子動量與波長成反比。

在與電子碰撞後，動量會改變。



若是波，散射後頻率不會變

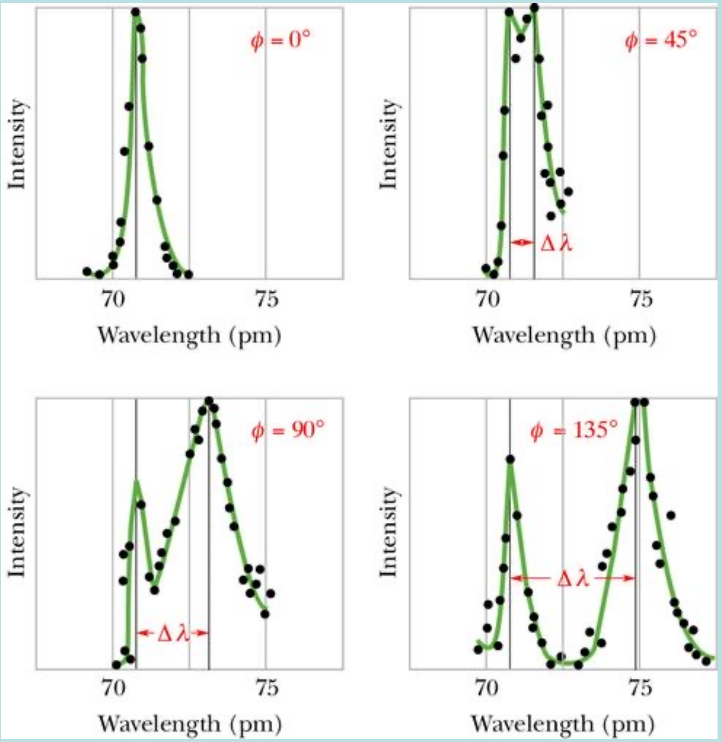


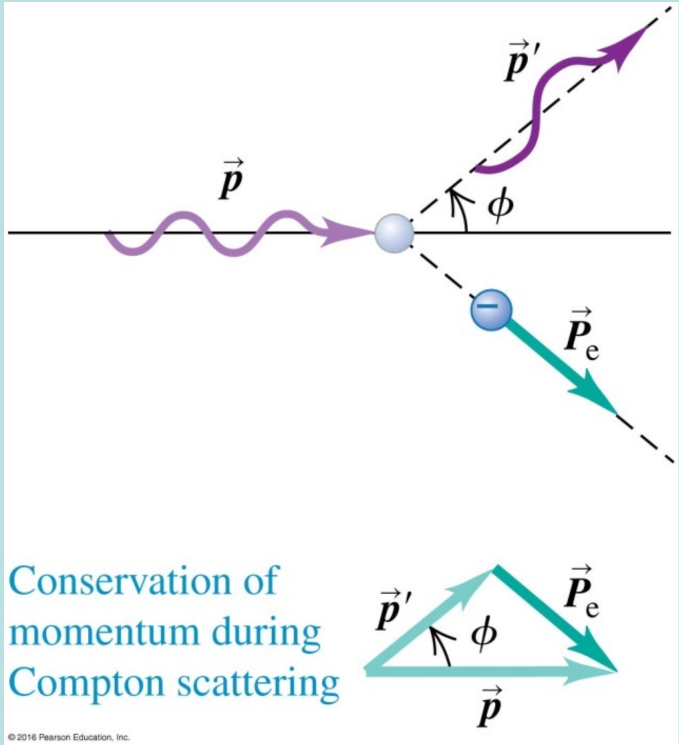


若是量子，動量會改變， 在與電子碰撞後，動量會改變。因此波長會改變。

動量的改變與散射角ϕ有關，

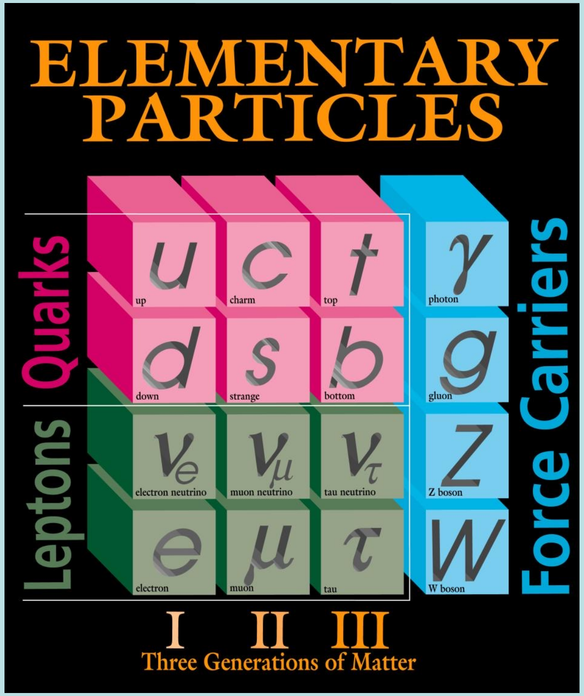
因此波長的變化也與ϕ有關。





$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

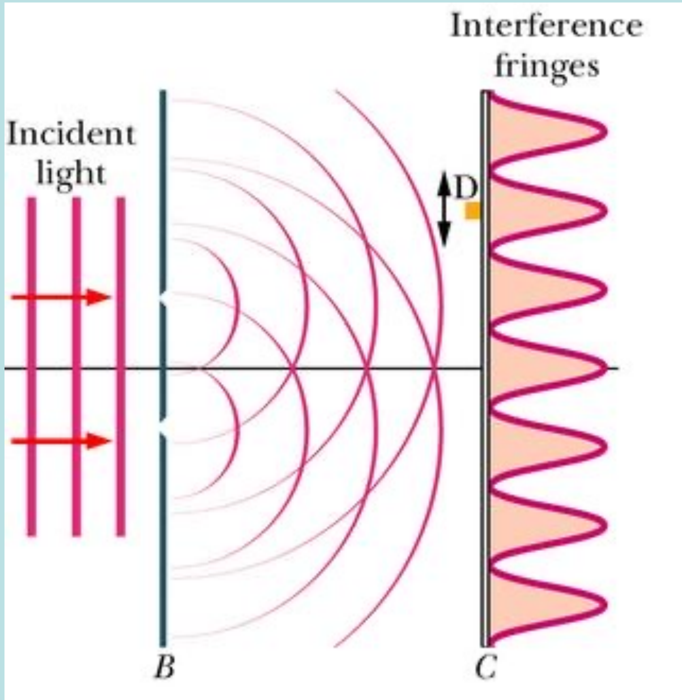
光子 Photon γ

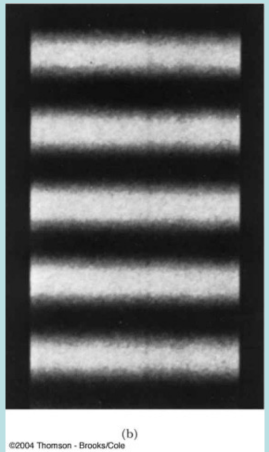


$$E = hf$$

$$p = \frac{h}{\lambda}$$

$$h = 6.626\times10^{-34}\ \mathrm{J\cdot s} = 4.136\times10^{-15}\ \mathrm{eV\cdot s}$$





但光有干涉現象，光的確是波！

PHOTONS: LIGHT WAVES BEHAVING AS PARTICLES 38

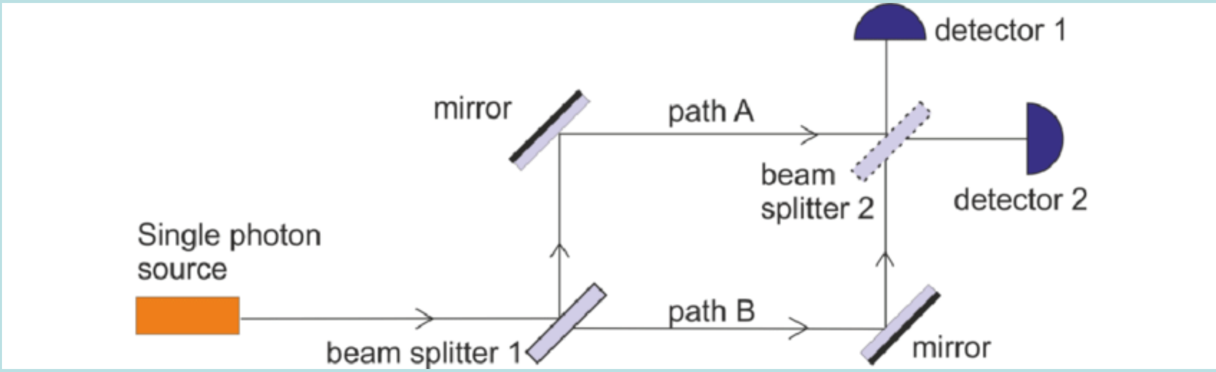
雖然光與電子一樣，有粒子與波的二重性，

但光子的描述比電子複雜得多，

因為光子的數目可以隨時變化，而電子束則是守恆的。

因此不同光子數的狀態或波函數可以疊加，製造出很奇特的狀態！

換句話說，光子擁有更複雜的多重面目。量子光學。



One photon state

EUROPHYSICS LETTERS 15 February 1986
Europhys. Lett., **1** (4), pp. 173-179 (1986)

Experimental Evidence for a Photon Anticorrelation Effect on a Beam Splitter: A New Light on Single-Photon Interferences.

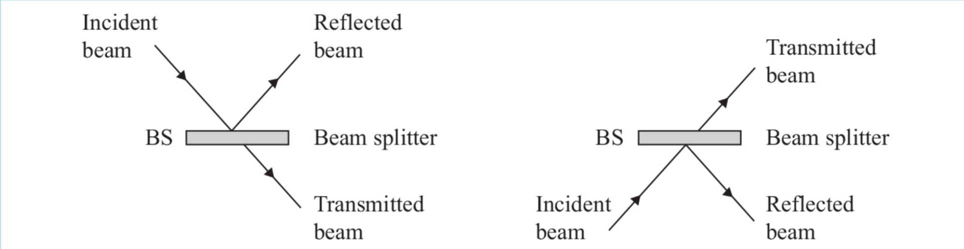
P. GRANGIER, G. ROGER and A. ASPECT (*)
Institut d'Optique Théorique et Appliquée, B.P. 43 - F 91406 *Orsay, France*

(received 11 November 1985; accepted in final form 20 December 1985)

PACS. 42.10. – Propagation and transmission in homogeneous media.
PACS. 42.50. – Quantum optics.

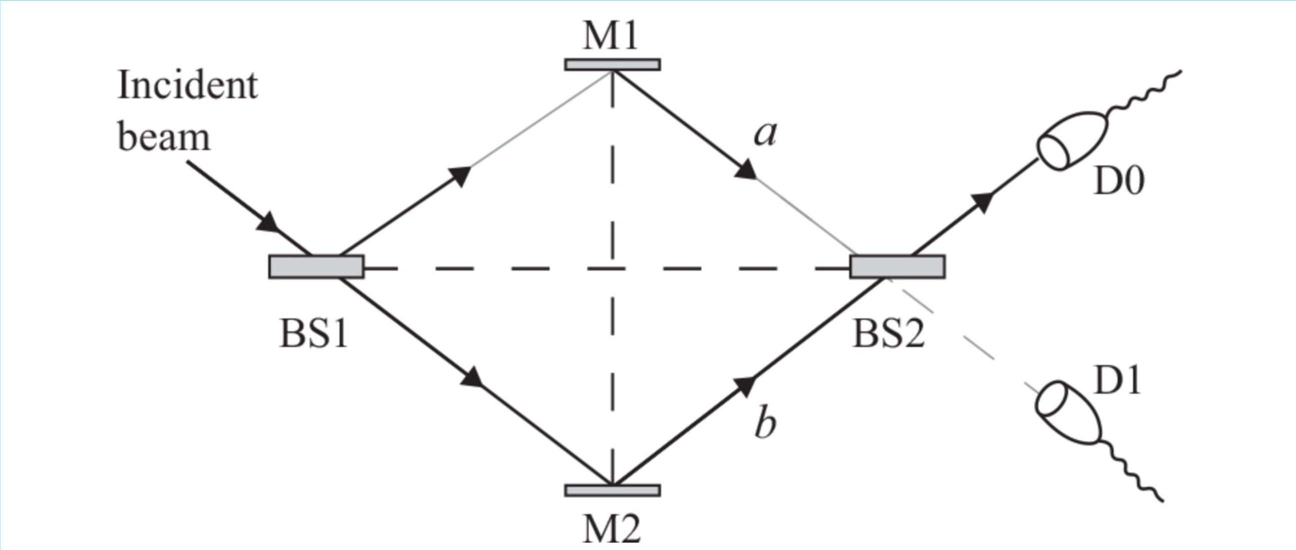
Abstract. – We report on two experiments using an atomic cascade as a light source, and a triggered detection scheme for the second photon of the cascade. The first experiment shows a strong anticorrelation between the triggered detections on both sides of a beam splitter. This result is in contradiction with any classical wave model of light, but in agreement with a quantum description involving single-photon states. The same source and detection scheme

Alain Aspect



BS: Beam Splitter: Devices that split a beam of light into two beams with equal intensity

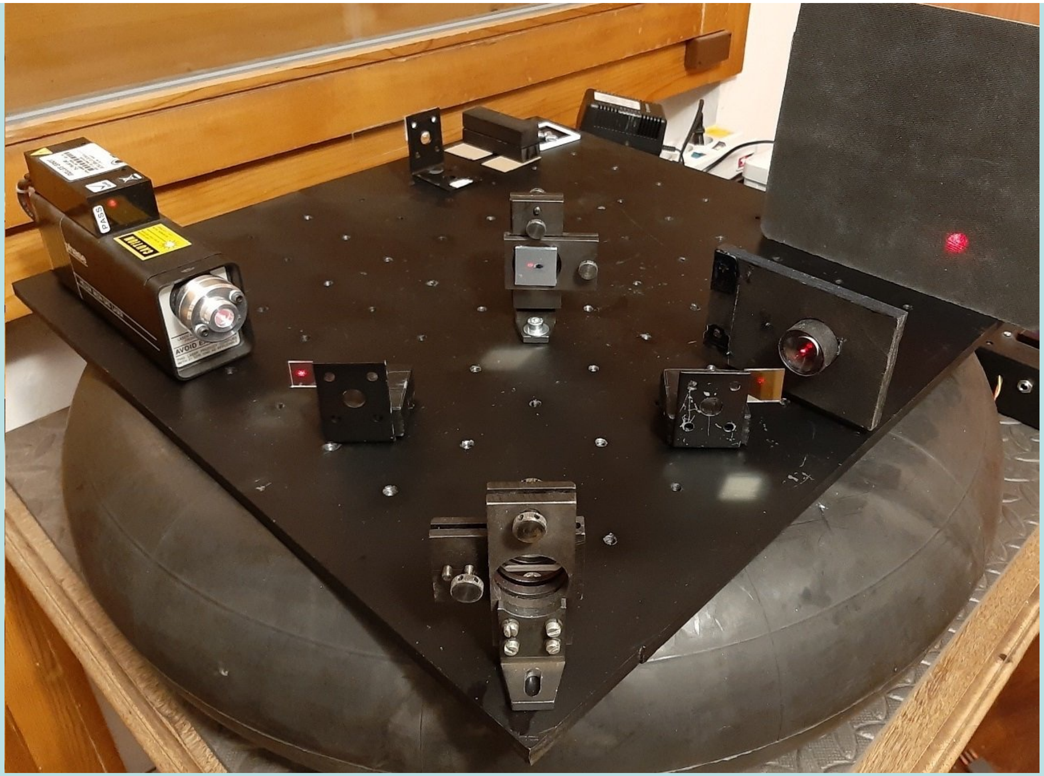
Mach-Zehnder Interferometer

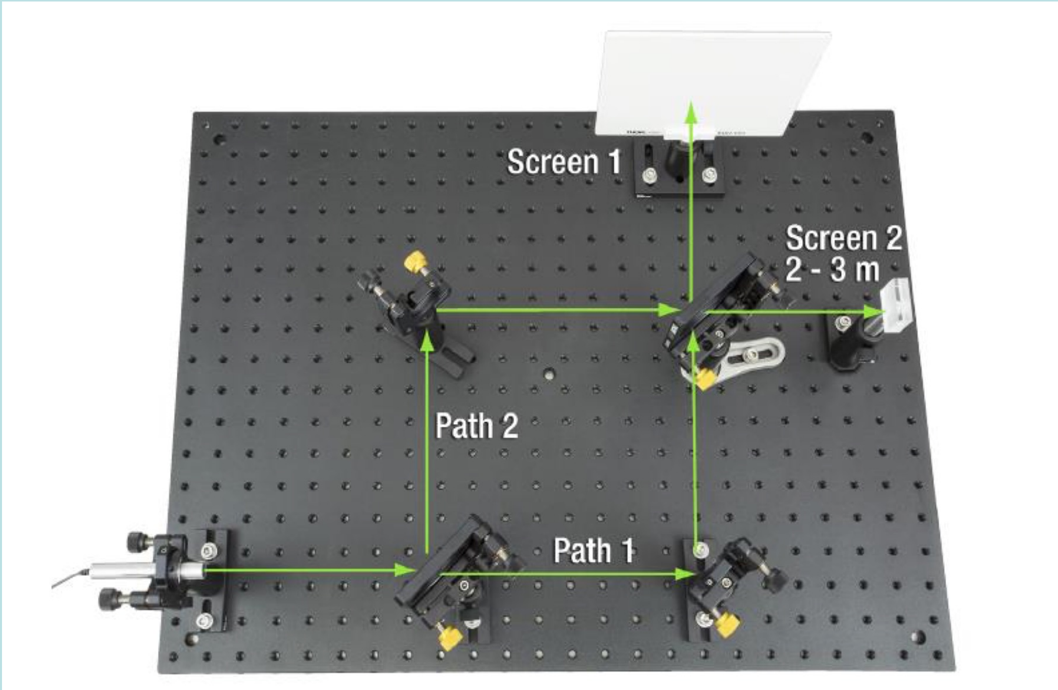


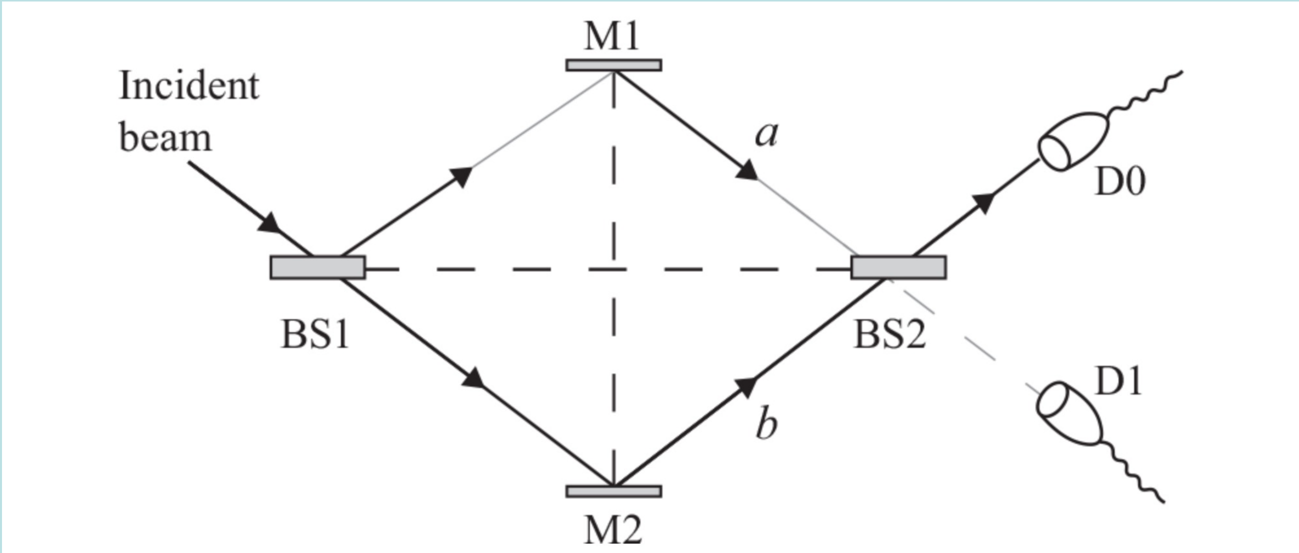
Distances can be adjusted for the phase difference of the two beams at BS2 to change.

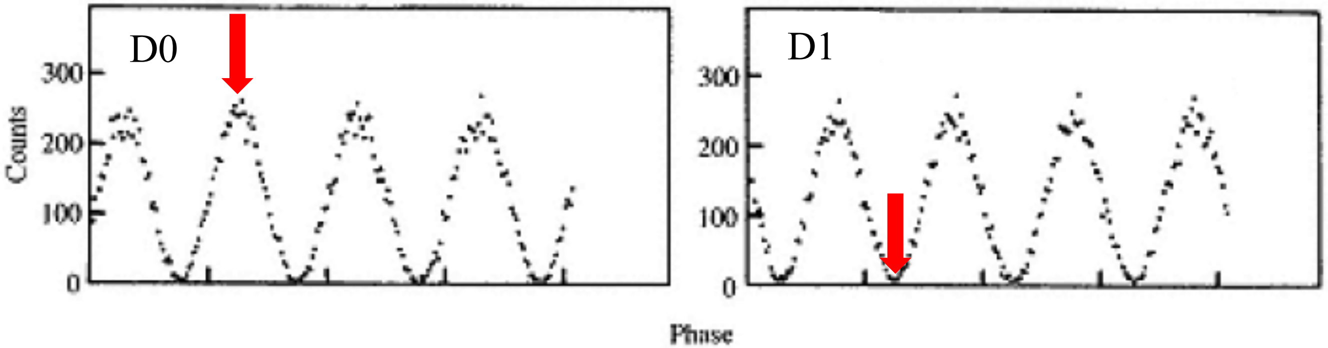
Can arrange so that reflection of *a* at BS2 and transmission of *b* interfere constructively.

Detector D0 receives full incident light beam while D1 receives none.




Screen 1
Screen 2
2 - 3 m
Path 2
Path 1





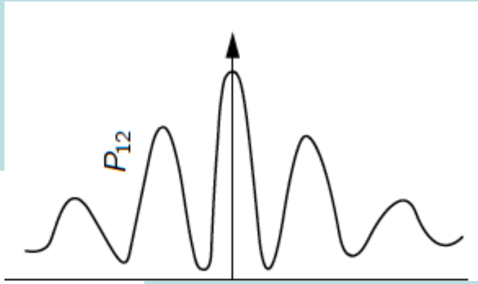
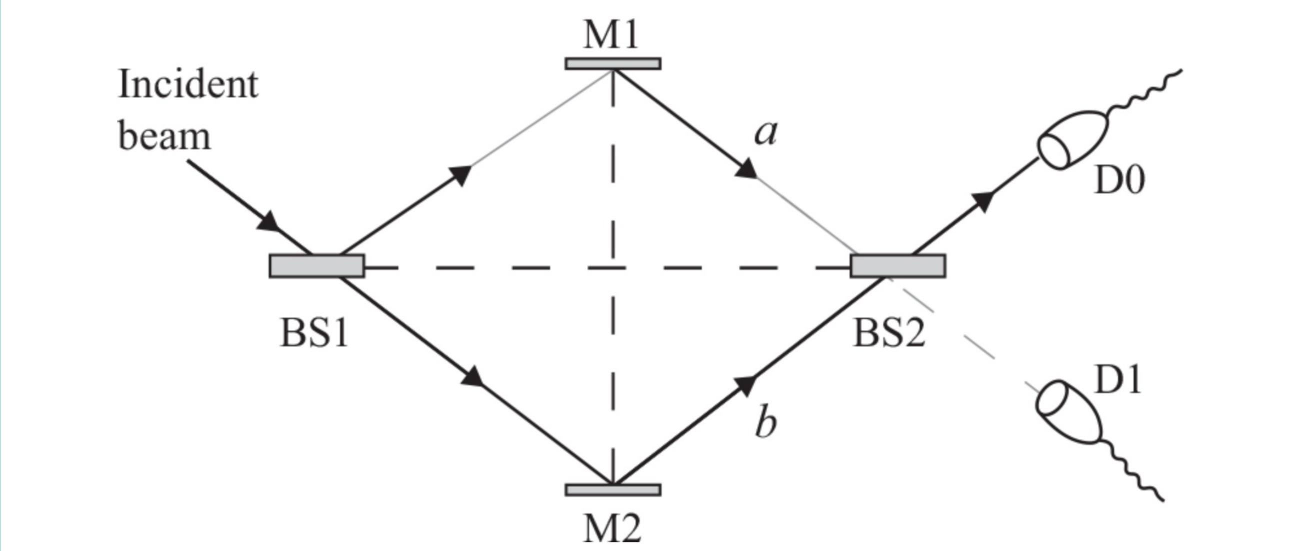
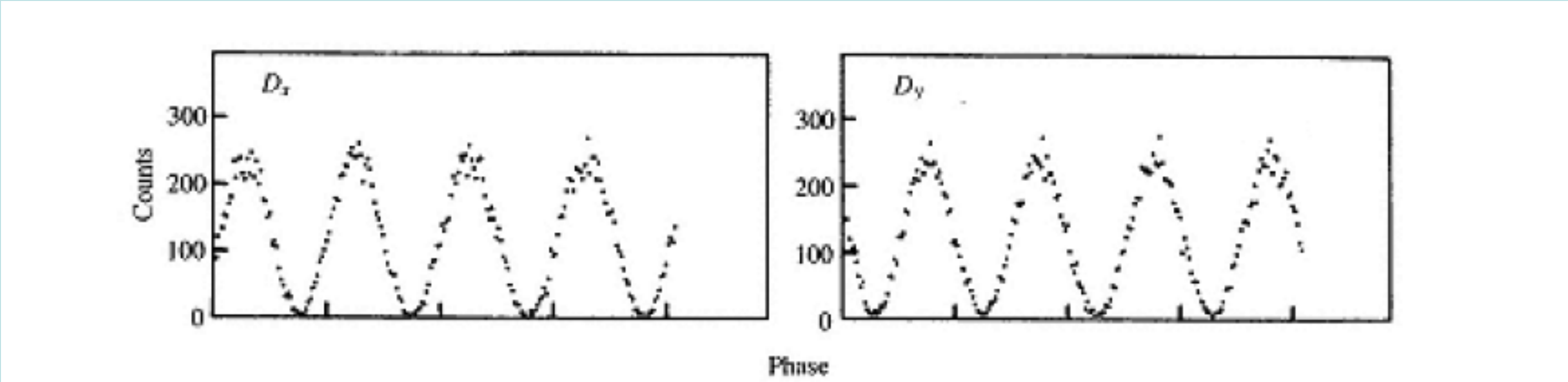


Figure 2-7
Interference of Single Photons (the experimental results of Aspect and co-workers). The number of counts in the two detectors of Figure 2-6 are plotted as a function of the phase difference between the two paths. The photon, which before chose one path, now takes both. SOURCE: Modified with permission from P. Grangier, G. Roger and A. Aspect, "Experimental evidence for a photon anticorrelation effect on a beamsplitter," *Europhys. Lett.,* vol. 1, p. 173. Copyright 1986 Les Editions de Physique.

Mach-Zehnder Interferometer



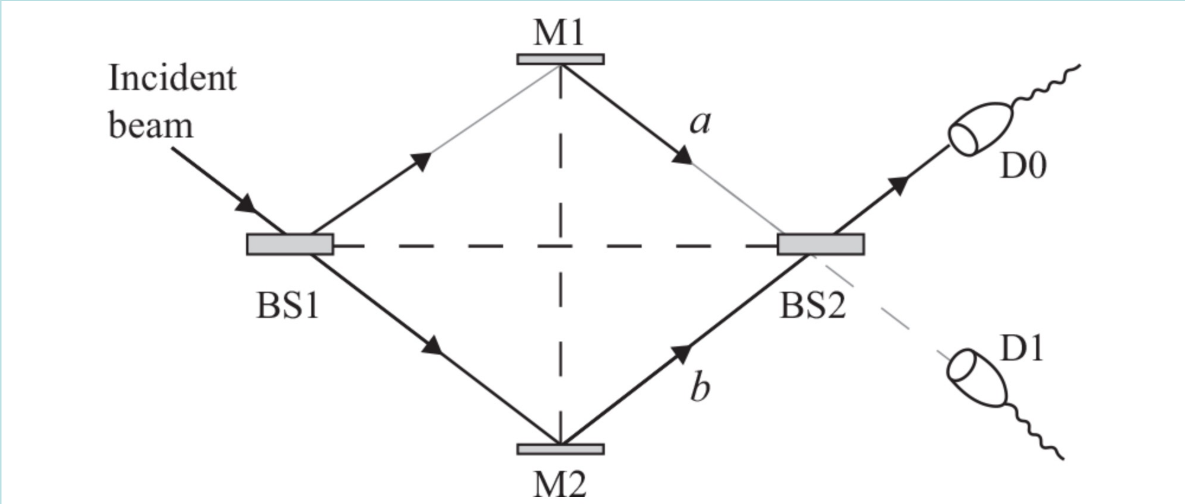


Now reduce incident beam as low as one photon at a time.

The intereference result is still true.

It is not one photon interfere with the other but this photon interfere with itself.

Mach-Zehnder Interferometer



Intereference takes tow routes to work.

So this one photon does the strange thing of going through two routes at the same time.

Going through two routes is impossible for a particle but is noraml for two waves.

We'd say it is a superposition 疊加 of two waves.

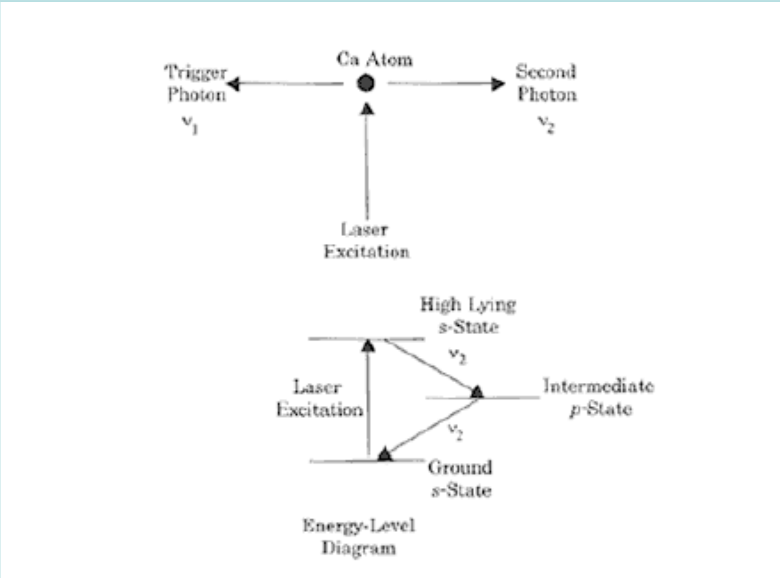
The intereference result shows photons as particles needs to be understood as waves, too.

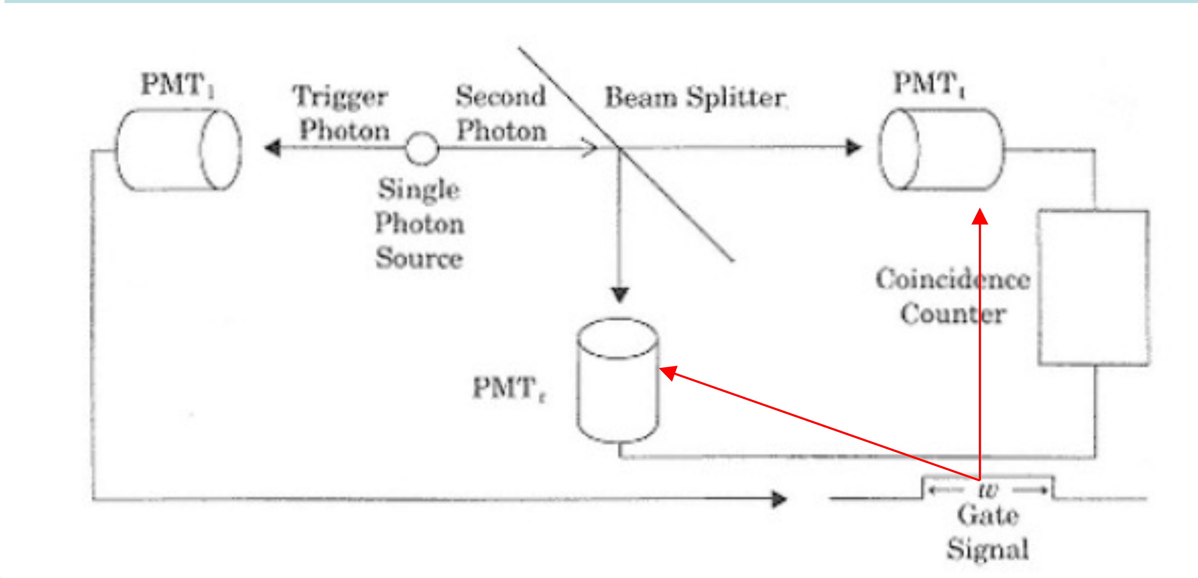
Therefore, each photon does the very strange thing of going through both branches of the interferometer! Each photon is in a superposition of two states: a state in which the photon is in the top beam or upper branch added to a state in which the photon is in the bottom beam or lower branch. Thus, the photon in the interferometer is in a funny state in which the photon seems to be doing two incompatible things at the same time.

As we mentioned before, we speak of wave functions and states as equivalent descriptions of a quantum system. We also sometimes refer to states as vectors. A quantum state may not be a vector like the familiar vectors in three-dimensional space, but it is a vector nonetheless because we can do with states what we do with ordinary vectors: we can add states, and we can multiply states by numbers. Linearity in quantum mechanics guarantees that adding wave functions (or states, or vectors) is a sensible thing to do. Just as any vector can be written as a Screenshot [illegible]er vectors in many different ways, we will do the same with our states. By writing our physical state as sums of other states, we can learn about the properties of our state.

Zwiebach p15

How to achieve single photon?

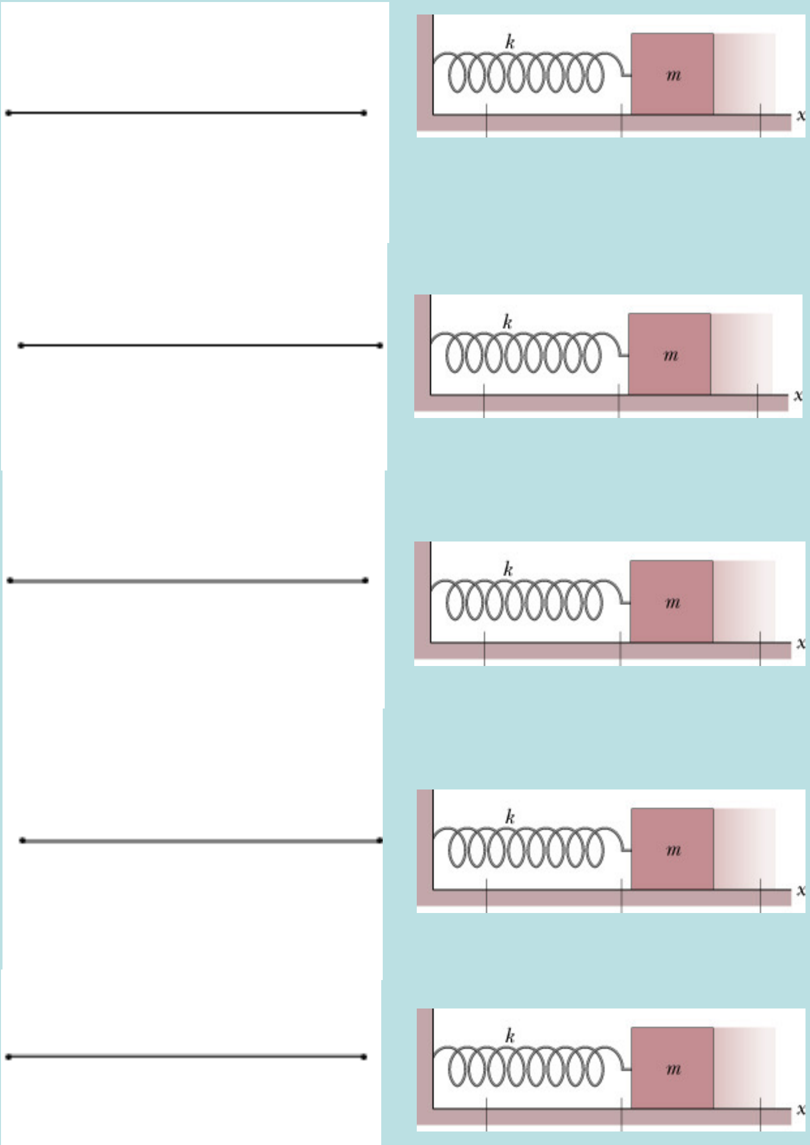




Two photon decay, separated on average around 5 ns.

Detection of the first photon acts as trigger to open the gate for detection at the two PMT’s (detectors).

原子中的電子躍遷通常會製造單個光子的狀態！

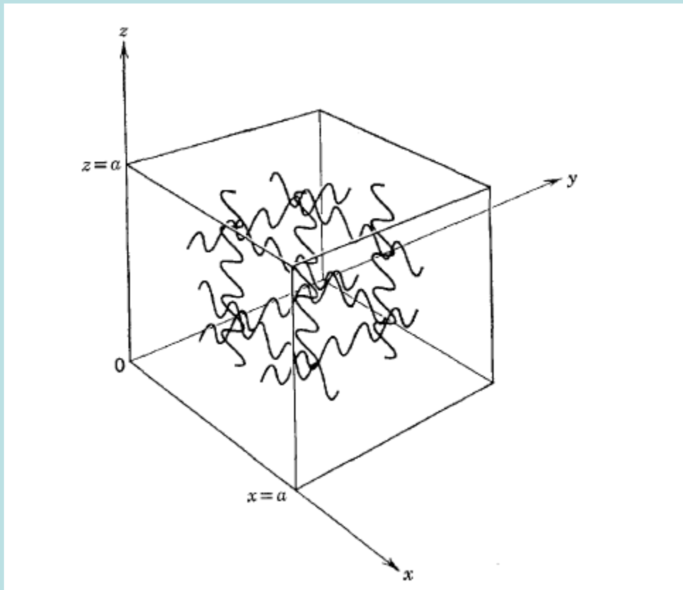


如何製造如電磁振盪的光子態？

例如一維空腔內，一駐波模式對應的電磁波？

$$E(x,t) \sim E_0 \sin kx \cdot \cos \omega t$$

Coherent State 相干態、定調態



$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle.$$

This is a standing wave. To make this clear, we write $\alpha = |\alpha|e^{i\theta}$ and find that

$$\langle \hat{E}_x(z,t)\rangle = 2\mathcal{E}_0 \,\mathrm{Re}(\alpha e^{-i\omega t}) \sin kz \;=\; 2\mathcal{E}_0 \,|\alpha|\, \cos(\omega t - \theta) \sin kz. \qquad (17.4.14)$$

Coherent photon states with large $|\alpha|$ give rise to classical electric fields! In the state $|\alpha\rangle$, the expectation value of the number operator $\hat{N}$ is $|\alpha|^2$. Thus, the above electric field is the classical field associated with a quantum state with about $|\alpha|^2$ photons.