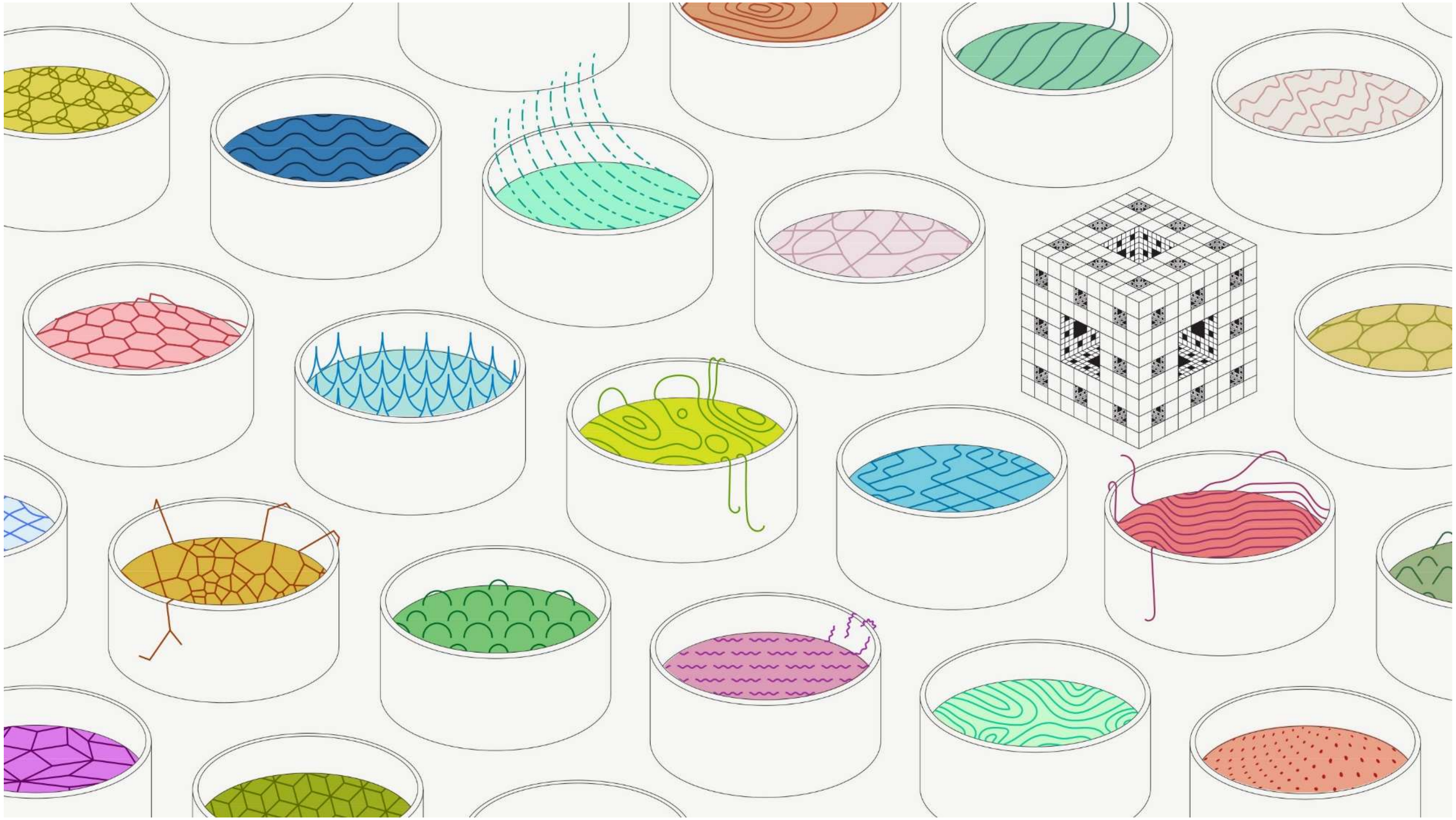
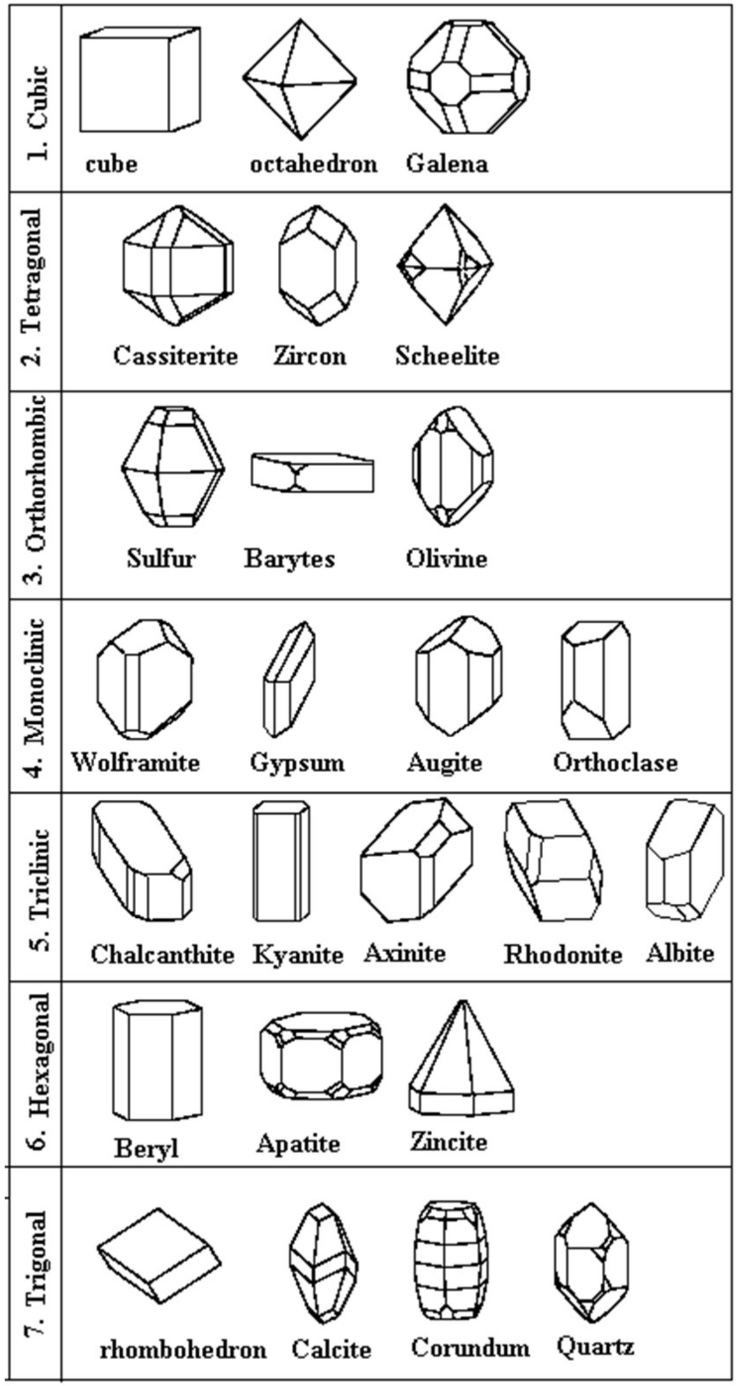
Classification of topological insulator and topological superconductor



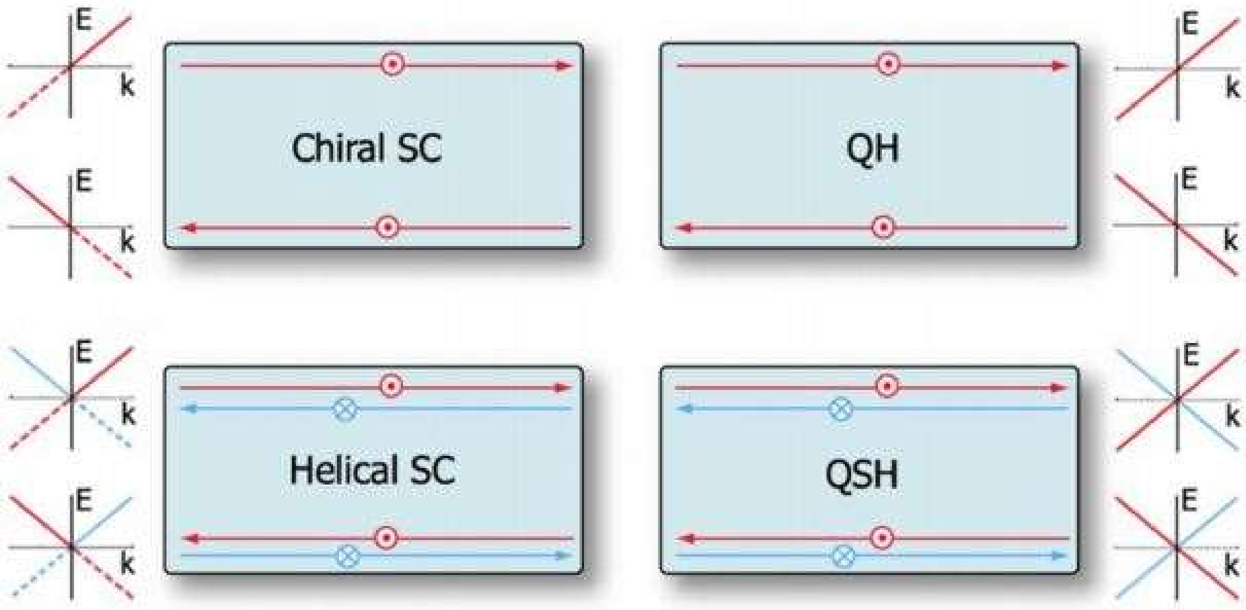
Symmetry as an organizing principle

➡ Topology as an organizing principle

Crystal form of the seven crystal systems



Topological systems we have studied so far:



	1d	2d	3d	T	P	S	Lect
Quantum Hall insulator	0	Z	0	0	0	0	5
Topological insulator	0	Z_2	Z_2	-1	0	0	7,8
Chiral superconductor	Z_2	Z	0	0	1	0	12,13
Helical superconductor	Z_2	Z_2	Z	-1	1	1	14,15

Symmetries of a Hamiltonian

- Unitary symmetry (translation, rotation, reflection …)

  Decompose H to irreducible blocks

- Beyond unitary symmetry

(1) Time-reversal symmetry (anti-unitary)

$$T H_k T^{-1} = H_{-k}, \quad T = U_T K$$

$$\text{TRS} = \begin{cases} 0 & \text{no TRS} \\ +1 & \text{TRS with } T^2 = 1 \quad \text{(integer spin)} \\ -1 & \text{TRS with } T^2 = -1 \quad \text{(half-integer spin)} \end{cases}$$

(2) Particle-hole symmetry (anti-unitary)

$$P H_k P^{-1} = -H_{-k}, \quad P = U_P K$$

$$\text{PHS} = \begin{cases} 0 & \text{no PHS} \\ +1 & \text{PHS with } P^2 = 1 \quad \text{(odd parity: p-wave)} \\ -1 & \text{PHS with } P^2 = -1 \quad \text{(even parity: s-wave)} \end{cases}$$

(3) TRS x PHS = Chiral symmetry (unitary)

$$TPH_k (TP)^{-1} = -H_k$$

S=TP

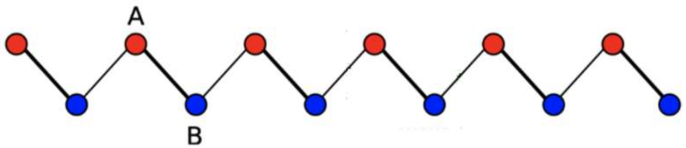
S^2=1 ,-1 (chose +1)

Unitary, but not the usual one

• Any unitary operator that anticommutes with the band Hamiltonian, $SH(k)S^{-1} = -H(k)$, qualifies as a chiral symmetry.

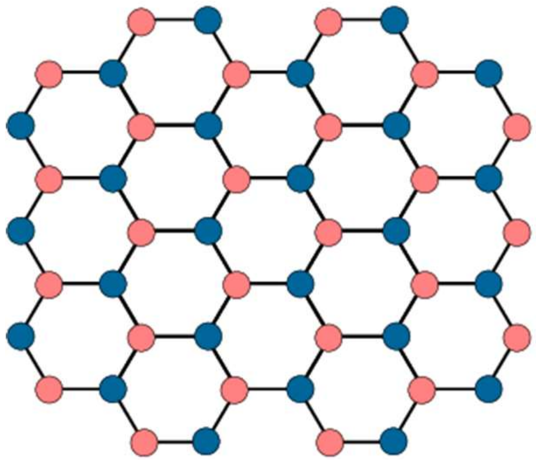
- Bipartite lattices with NN coupling only has the chiral symmetry

SSH model

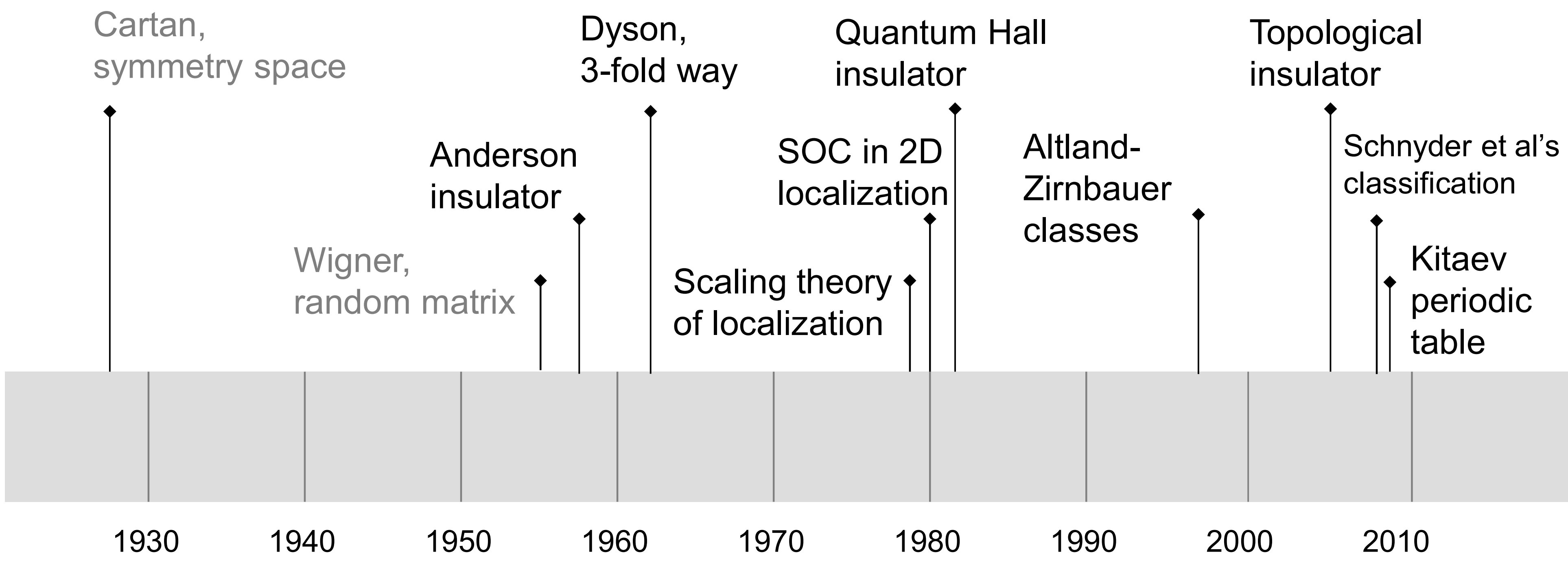


$$H_k = \begin{pmatrix} 0 & h_k \\ h_k^\dagger & 0 \end{pmatrix}$$

graphene



A time-line of the periodic table for non-interacting fermions

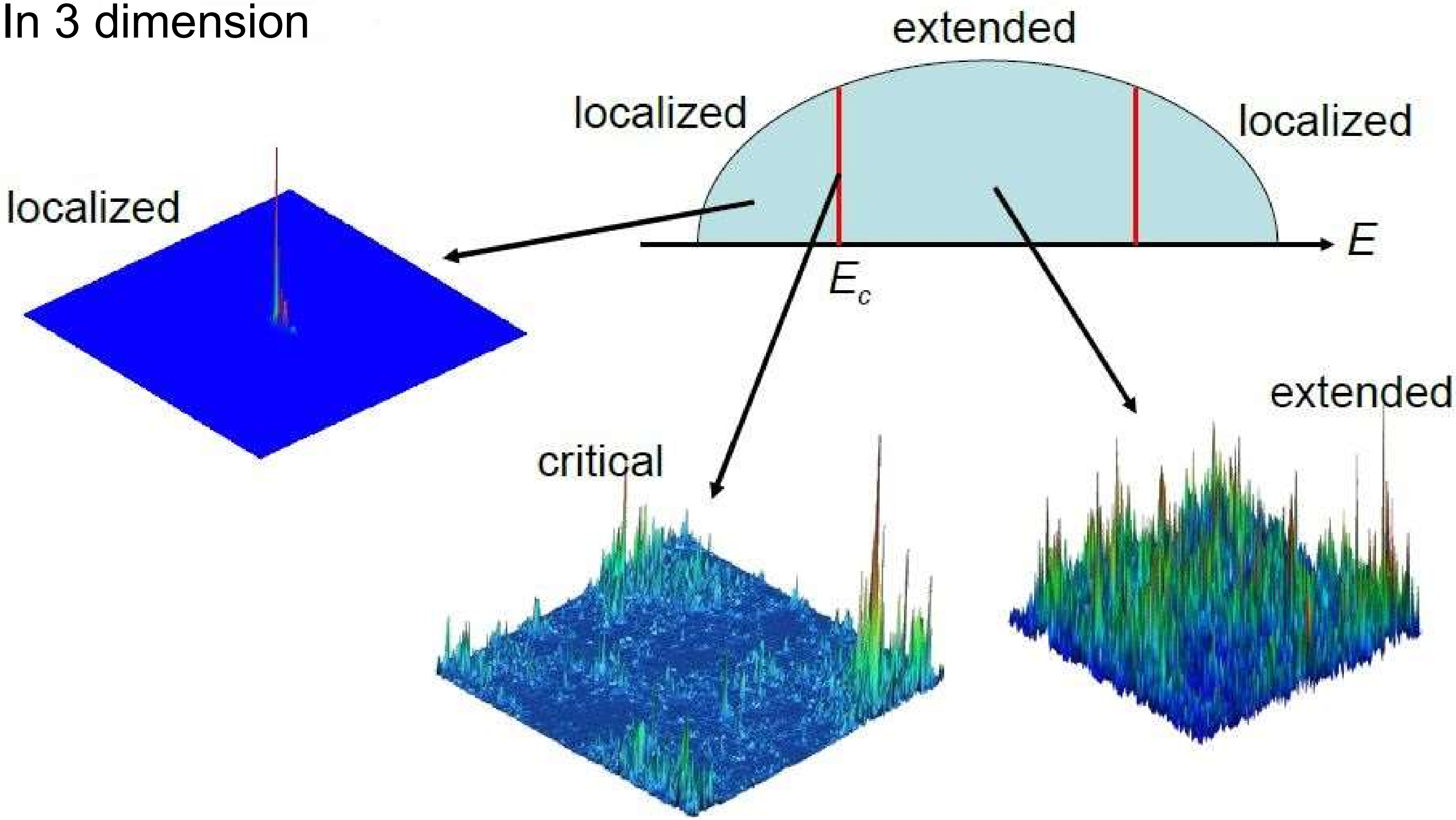


Connection with the classification of disordered systems

First, Anderson localization (Anderson, 1958)

> In the modern literature, the phenomenon of exponential decay of eigenfunctions of a quantum system in a disordered environment is called Anderson localization,

In 3 dimension



VOLUME 42, NUMBER 10 PHYSICAL REVIEW LETTERS 5 MARCH 1979

Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions

E. Abrahams
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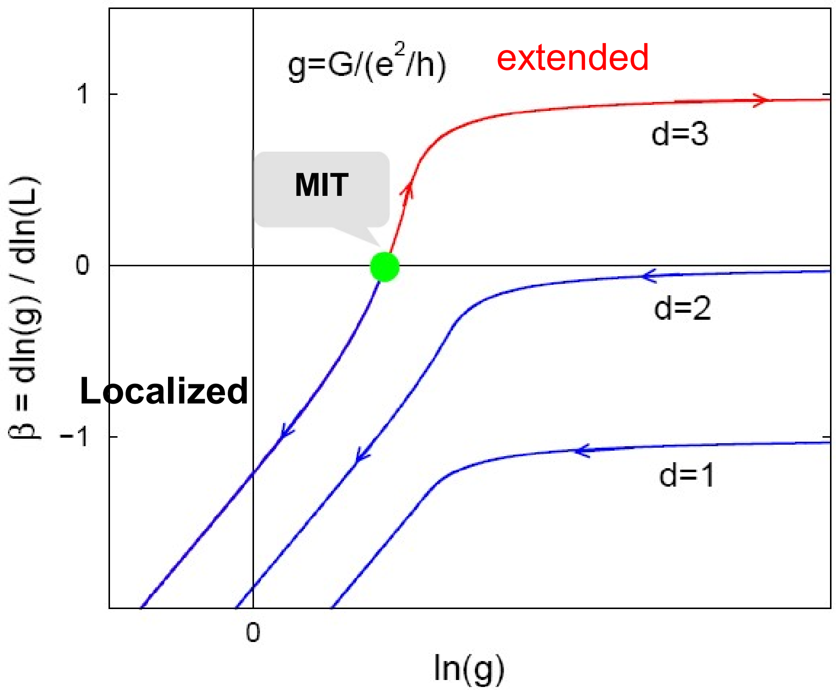
and

P. W. Anderson,[a] D. C. Licciardello, and T. V. Ramakrishnan[b]
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540
(Received 7 December 1978)

Arguments are presented that the $T=0$ conductance G of a disordered electronic system depends on its length scale L in a universal manner. Asymptotic forms are obtained for the scaling function $\beta(G)=d\ln G/d\ln L$, valid for both $G \ll G_c \simeq e^2/\hbar$ and $G \gg G_c$. In three dimensions, G_c is an unstable fixed point. In two dimensions, there is no true metallic behavior; the conductance crosses over smoothly from logarithmic or slower to exponential decrease with L.

Universal scaling function



Flow follows the increase of L

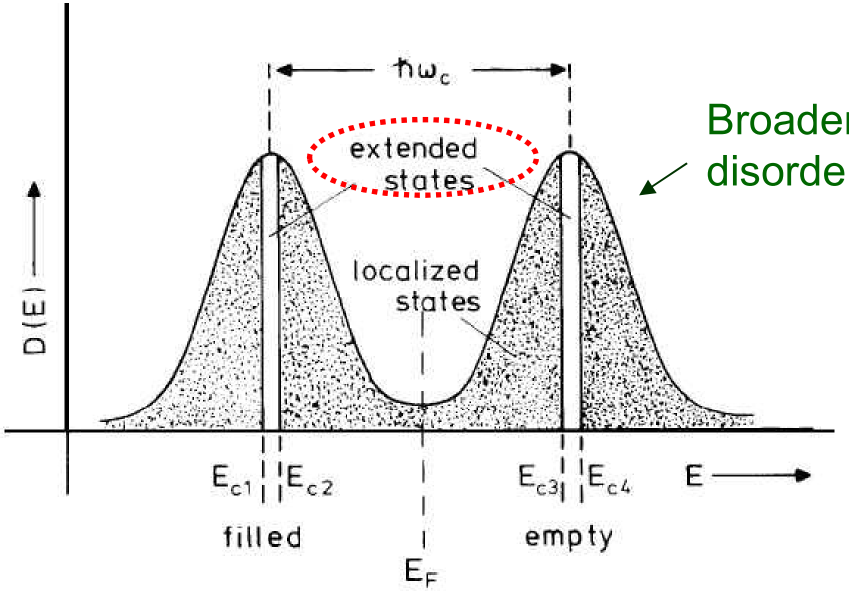
Wave functions of disordered systems in 1D and 2D are localized

• Exception (in 2D): Quantum Hall effect, spin-orbit interaction

[Exception 1]

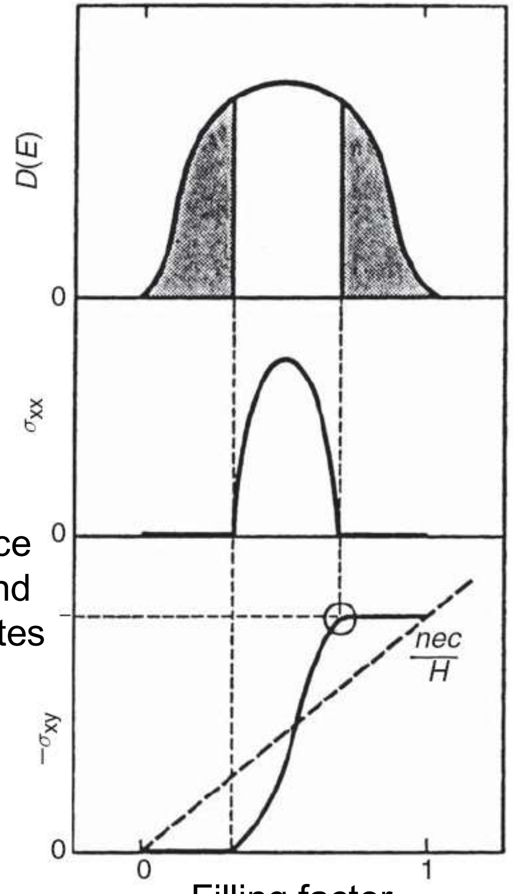
Quantum Hall effect (1980)

Localized states vs extended states



Broadened LL due to disorder

Aoki, CMST 2011



The importance of localized and extended states

Abraham *et al*'s conclusion does not apply (since QHE is in a different universality class)

[Exception 2]

Prog. Theor. Phys. Vol. 63, No. 2, February 1980, Progress Letters

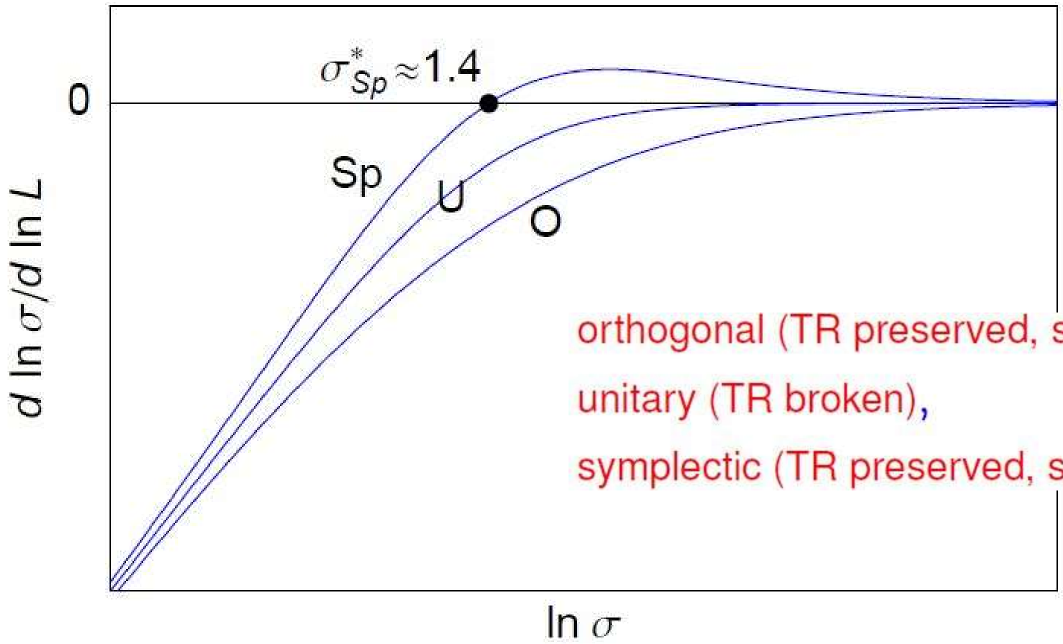
Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System

Shinobu HIKAMI, Anatoly I. LARKIN*) and Yosuke NAGAOKA

Research Institute for Fundamental Physics
Kyoto University, Kyoto 606

(Received November 5, 1979)

Effect of the spin-orbit interaction is studied for the random potential scattering in two dimensions by the renormalization group method. It is shown that the localization behaviors are classified in the three different types depending on the symmetry. The recent observation of the negative magnetoresistance of MOSFET is discussed.



orthogonal (TR preserved, spin preserved),
unitary (TR broken),
symplectic (TR preserved, spin broken)

Ostrovsky's slide

Connection with Random matrix theory

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER 1962

The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

Orthogonal, unitary, symplectic

Energy spectra of complex nuclei

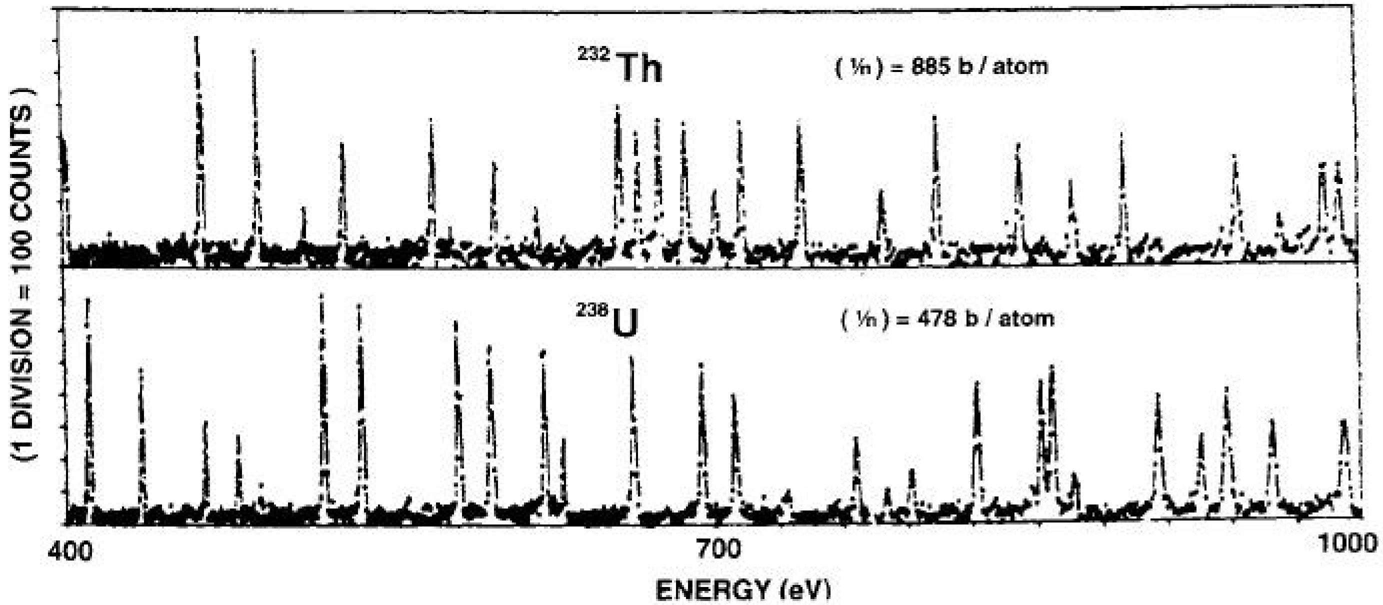


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Problem : excitation spectrum of heavy nuclei
many-body problem; do not know Hamiltonian

Wigner Solution : write Hamiltonian as random matrix

Wigner-Dyson classes

TABLE I. Summary of Dyson's threefold way. The Hermitian matrix $\mathcal{H}$ (and its matrix of eigenvectors U) are classified by an index $\beta \in \{1,2,4\}$, depending on the presence or absence of time-reversal (TRS) and spin-rotation (SRS) symmetry.

β	TRS	SRS	$\mathcal{H}_{nm}$	U		
1	yes	yes	real	orthogonal	AI	**GOE: T^2=1**
2	no	irrelevant	complex	unitary	A	**GUE: T^2=0**
4	yes	no	real quaternion	symplectic	AII	**GSE: T^2=-1**

Spectral distribution of random matrix

Distribution of NN spacing

level repulsion:
P(s<<1) $\sim$ s^β

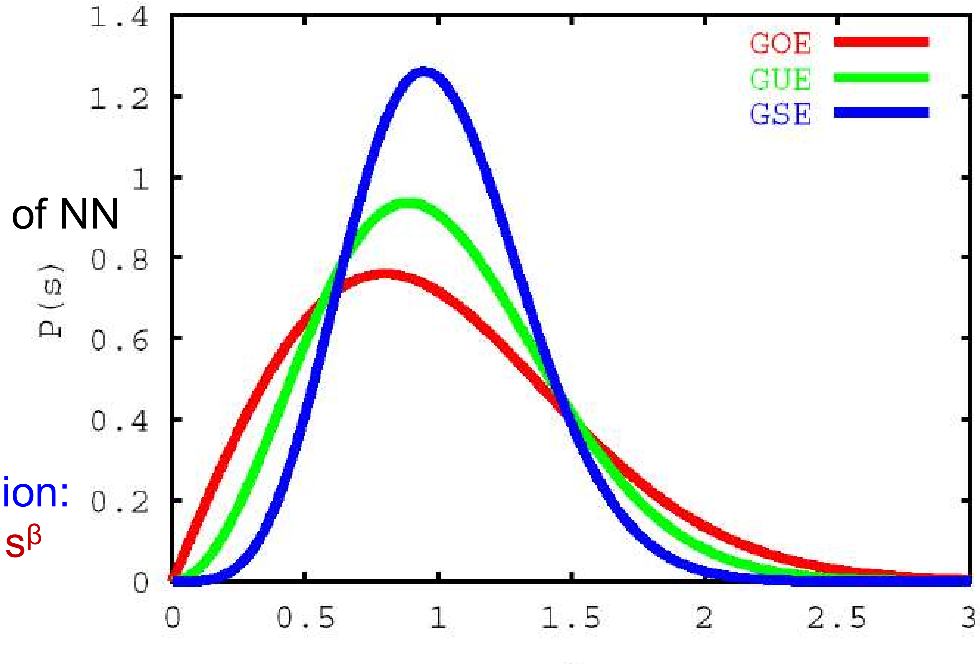


Fig from Altshuler's ppt

Altland-Zirnbauer classes

PHYSICAL REVIEW B VOLUME 55, NUMBER 2 1 JANUARY 1997-II

Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures

Particle-hole symmetry

Alexander Altland and Martin R. Zirnbauer
Institut für Theoretische Physik, Universität zu Köln, Zülpicherstrasse 77, 50937 Köln, Germany

(Received 4 March 1996)

Normal-conducting mesoscopic systems in contact with a superconductor are classified by the symmetry operations of time reversal and rotation of the electron's spin. Four symmetry classes are identified, which correspond to Cartan's symmetric spaces of type *C*, *C*I, *D*, and *D*III. A detailed study is made of the systems where the phase shift due to Andreev reflection averages to zero along a typical semiclassical single-electron trajectory. Such systems are particularly interesting because they do not have a genuine excitation gap but

➡ Classifying topological insulator/superconductor using AZ classes

PHYSICAL REVIEW B **78**, 195125 (2008)

Classification of topological insulators and superconductors in three spatial dimensions

Andreas P. Schnyder,[1] Shinsei Ryu,[1] Akira Furusaki,[2] and Andreas W. W. Ludwig[3]
[1]*Kavli Institute for Theoretical Physics, University of California–Santa Barbara, Santa Barbara, California 93106, USA*
[2]*Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
[3]*Department of Physics, University of California–Santa Barbara, Santa Barbara, California 93106, USA*

(Received 11 April 2008; revised manuscript received 13 September 2008; published 26 November 2008)

- Wigner-Dyson (1951 -1963) : "three-fold way" complex nuclei
- Verbaarschot (1992 -1993) chiral phase transition in QCD
- Altland-Zirnbauer (1997) : "ten-fold way" mesoscopic SC systems

3 internal symmetries

	Cartan's label	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$	
Standard	A (unitary)	0	0	0	-	$\mathbb{Z}$	-	IQH,AQH
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-	
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	2D/3D TI
Chiral	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$	
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-	SSH (with 3
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$	symm)
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-	Kitev chain Chiral p-wave
	C	0	−1	0	-	$\mathbb{Z}$	-	
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	Helical p-wave
	CI	+1	−1	1	-	-	$\mathbb{Z}$	He-3 B

Bogoliubov de Gennes

- (3x3-1)+2=10
- 5 non-trivial classes in each dimension

Periodic table: different approaches

1. Continuous systems: Dirac Hamiltonian, Clifford algebra

   Bernard and LeClair, J Phys A 2002

2. Disordered systems: Surface state localization, random matrix theory, nonlinear sigma model Ivanov, 9911147, Schnyder et al PRB 2008, Ryu et al, NJP 2010

   Bulk-edge correspondence

3. Lattice systems: Homotopy theory Schnyder et al PRB 2008,

   K-theory Kitaev AIP Conf Proc 2009

   Related to classification of symmetric spaces (Cartan 1926-27)

4. Response theory, quantum anomaly Ryu et al, PRB 2012

5. …

Periodic table for topological insulators and superconductors

Hidden order in the classification of topology

Alexei Kitaev

California Institute of Technology, Pasadena, CA 91125, U.S.A.

AIP Conf Proc 2009

Abstract. Gapped phases of noninteracting fermions, with and without charge conservation and time-reversal symmetry, are classified using Bott periodicity. The symmetry and spatial dimension determines a general universality class, which corresponds to one of the 2 types of complex and 8 types of real Clifford algebras. The phases within a given class are further characterized by a topological invariant, an element of some Abelian group that can be 0, $\mathbb{Z}$, or $\mathbb{Z}_2$. The interface between two infinite phases with different topological numbers must carry some gapless mode. Topological properties of finite systems are described in terms of K-homology. This classification is robust with respect to disorder, provided electron states near the Fermi energy are absent or localized. In some cases (e.g., integer quantum Hall systems) the K-theoretic classification is stable to interactions, but a counterexample is also given.

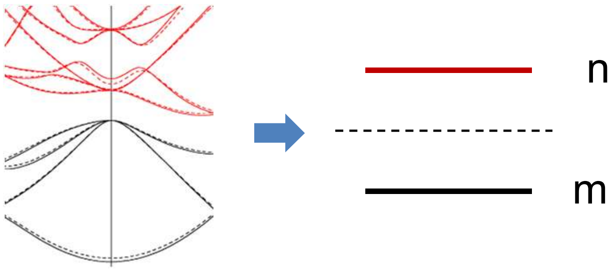
complex class period 2

Real class Period82

	Symmetry			d								
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	IQHE, AQHE
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	**π-flux state**
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	**SSH**
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	Kitaev chain, chiral p-wave (spinless)
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	helical p-wave (spinful), He 3
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	2D/3D TI
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	**d+id, d-id SC**
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	**d_{xy}, d_{x2-y2} singlet SC**

Topology of lattice system

"Flattened" Hamiltonian Q_k



Space of Q_k matrix:
Grassmannian

$$G_{m+n,m}(C) = \frac{U(m+n)}{U(m)U(n)}$$

For chiral system,

$$Q_k = \begin{pmatrix} 0 & q_k \\ q_k^\dagger & 0 \end{pmatrix}$$

TABLE II Altland-Zirnbauer classes

	Cartan's label	T	P	S	1d	2d	3d	Space of Hamiltonians
Standard	A (unitary)	0	0	0	0	Z	0	$\{\mathrm{Q}_k \in G_{m+n,m}(\mathbb{C})\}$
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	0	0	0	$\{\mathrm{Q}_k \in G_{m+n,m}(\mathbb{C}) \mid \mathrm{Q}_k^* = \mathrm{Q}_{-k}\}$
	AII (symplectic)	−1	0	0	0	Z_2	Z_2	$\{\mathrm{Q}_k \in G_{2m+2n,2m}(\mathbb{C}) \mid i\sigma_y \mathrm{Q}_k^*(-i\sigma_y) = \mathrm{Q}_{-k}\}$
Chiral	AIII (chiral unitary)	0	0	1	Z	0	Z	$\{\mathrm{q}_k \in U(m)\}$
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	Z	0	0	$\{\mathrm{q}_k \in U(m) \mid \mathrm{q}_k^* = \mathrm{q}_{-k}\}$
	CII (chiral symplectic)	−1	−1	1	Z	0	Z_2	$\{\mathrm{q}_k \in U(2m) \mid i\sigma_y \mathrm{q}_k^*(-i\sigma_y) = \mathrm{q}_{-k}\}$
BdG	D	0	+1	0	Z_2	Z	0	$\{\mathrm{Q}_k \in G_{2m,m}(\mathbb{C}) \mid \tau_x \mathrm{Q}_k^* \tau_x = -\mathrm{Q}_{-k}\}$
(superconductor)	C	0	−1	0	0	Z	0	$\{\mathrm{Q}_k \in G_{2m,m}(\mathbb{C}) \mid \tau_y \mathrm{Q}_k^* \tau_y = -\mathrm{Q}_{-k}\}$
	DIII	−1	+1	1	Z_2	Z_2	Z	$\{\mathrm{q}_k \in U(2m) \mid \mathrm{q}_k^T = -\mathrm{q}_{-k}\}$
	CI	+1	−1	1	0	0	Z	$\{\mathrm{q}_k \in U(m) \mid \mathrm{q}_k^T = \mathrm{q}_{-k}\}$

The topology is a result of the disconnected pieces of the space X of the Hamiltonian matrix. That is, the topological number counts the disconnected pieces of the mapping from $T^d \to X$. One can start from studying the mapping $S^d \to X$, which is characterized by the homotopy group $\pi_d(X)$. Rigorously speaking, the base space T^d can be replaced by S^d only if $\pi_i(X) = 0$, for all $i < d$ (Avron *et al.*, 1983). So some information could be lost by such a simplification (which means that a lattice system is replaced by a continuous one).

Topological numbers of complex classes

- Class A in *even* dim: Chern number

$$\begin{aligned}\pi_1(G_{m+n,m}(\mathbb{C})) &= 0,\\ \pi_2(G_{m+n,m}(\mathbb{C})) &= Z,\\ \pi_3(G_{m+n,m}(\mathbb{C})) &= 0 \cdots\end{aligned}$$

Note:

$$\pi_3(G_{2,1}(\mathbb{C})) = Z$$

$$C_n = \frac{1}{n!}\int_{T^d} \mathrm{tr}\left(\frac{i\mathsf{F}}{2\pi}\right)^n$$

- Class AIII in *odd* dim (with chiral symm): winding number

$$\begin{aligned}\pi_{d\in \mathrm{odd}}((U(n)) &= Z,\\ \pi_{d\in \mathrm{even}}((U(n)) &= 0,\end{aligned}$$

$$Q_k = \begin{pmatrix} 0 & q_k \\ q_k^\dagger & 0 \end{pmatrix}$$

$$\nu_{2n+1} = \frac{(-1)^n n!}{(2n+1)!}\int_{T^{2n+1}}\left(\frac{i}{2\pi}\right)^{n+1} \mathrm{tr}\left[(\mathsf{q}^{-1}d\mathsf{q})^{2n+1}\right]$$

- The Z numbers in real classes are also related to these two

Overview of topological phases

- **Symmetry-protected topological (SPT) phase**

A phase of matter with a gap and a symmetry G that protect its topology

	Fermion	**Boson**
Non-interacting	• Integer quantum Hall effect • Topological insulator • …	• Photonic systems • Phonons • Magnons • …
Interacting	• … •	• Bosonic TI • Bosonic SC • …

Spin

- Haldane’s odd integer-spin chain
- …

- ## Topological-ordered phase

A phase of matter with degenerate GND state, fractional QP, and long-range entanglement.

Strongly Interacting

- Fractional quantum Hall effect
- Chiral spin liquid
- Z_2 spin liquid (toric code)
- …